Curves that change genus can have arbitrarily many rational points

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Let K be a global field of positive characteristic p. In other words, K is a function field in one variable over a finite field of characteristic p. Let C be a projective algebraic curve defined over K. One defines the absolute genus of C, in the usual way, by extending the field to the algebraic closure. We also define the genus of C relative to K to be the integer g_K that makes the Riemann-Roch formula hold, that is, for any K-divisor D of C, of sufficiently large degree, the dimension, $l(D)$, of the K-vector space of functions of $K(C)$ whose polar divisor is bounded by D, is $\deg D + 1 - g_K$. Since K is not perfect, the relative genus may change under inseparable extensions. (See e.g [A] or [T]). We have shown ([V]) that if the genus of C relative to K is different from the absolute genus of C then $C(K)$ is finite. The proof in [V] can be easily adapted to give an upper bound for $\sharp C(K)$, which depends on C. The purpose of this note is to give examples of curves C/K with fixed g_K for which $\sharp C(K)$ is arbitrarily large. The motivation for considering this problem comes from the work of Caporaso et al. [CHM], where they show that a conjecture of Lang implies that, for a number field K, $\sharp C(K)$ can be bounded in terms of g and K only for all curves C/K of genus $g \geq 2$. It is tempting to see this as a way of disproving Lang's conjecture by constructing curves with many points. It is also natural to consider the problem over function fields, but $C(K)$ can be infinite if C is isotrivial. Otherwise $C(K)$ is finite, if $g \geq 2$, by a classical result of Samuel ([S]) and one can ask if $\sharp C(K)$ can be uniformly bounded. We don't know the answer to this question for smooth curves (but see [BV] for a strong bound on $\sharp C(K)$).

Theorem. *Let $p > 2$ be a prime and $q = p^n$. Consider the curve $C_n/\mathbf{F}_p(t)$ defined by $x - (t + t^{q+2} + t^{2q+3} + ... + t^{(p-2)q+p-1})x^p = y^p$. C_n has absolute genus zero but has genus relative to $\mathbf{F}_p(t)$ equal to $(p-1)(p-2)/2$. Furthermore $\sharp C_n(\mathbf{F}_p(t)) \geq p^{2^n/2n}$ and $\sharp C_n(\mathbf{F}_{p^{2n}}(t)) \geq p^{2^n}$.*

Proof: It is clear that C_n has absolute genus zero. The statement about the relative genus holds for any curve $x - ax^p = y^p, a \in K \setminus K^p$ and can be checked by appealing to Tate's analogue of the Hurwitz formula ([T], theorem 2) to the extension of fields $K(x, y, a^{1/p})/K(x, y)$.

We will construct points on C_n whose x-coordinate is of the form $a(t)/(t^{q+1} - 1)$, where $a(t) = \sum_{i=0}^{q-1} \alpha_i t^i$. We will get a point in $C_n/\bar{\mathbf{F}}_p(t)$ if $(t^{q+1} - 1)^{p-1} a(t) - (t + t^{q+2} + t^{2q+3} + ... + t^{(p-2)q+p-1}) a(t)^p$ is a p-th power. Using that $(t^{q+1} - 1)^{p-1} = \sum_{i=0}^{p-1} t^{(q+1)i}$ and comparing coefficients, this condition is equivalent to $\alpha_i = \alpha_{(i+q)/p}^p, i \equiv 0(\mathrm{mod} p), \alpha_i = \alpha_{(i-1)/p}^p, i \equiv 1(\mathrm{mod} p), \alpha_i = 0$, otherwise.

Consider the map $\phi(i)$ defined for positive integers $i, i \equiv 0, 1(\mathrm{mod} p)$ by $\phi(i) = (i + q)/p, i \equiv 0(\mathrm{mod} p), (i - 1)/p, i \equiv 1(\mathrm{mod} p)$. It has the following alternate description, for $i < q$. If $i = \sum_{j=0}^{n-1} \epsilon_j p^j, 0 \leq \epsilon_j \leq p - 1, \phi(i) = \sum_{j=1}^{n-1} \epsilon_j p^{j-1} + \delta p^{n-1}$, where $\delta = 1$ if $\epsilon_0 = 0$ and $\delta = 0$ if $\epsilon_0 = 1$. It follows that if $\epsilon_j \neq 0, 1$ for some j then $\phi^r(i) \not\equiv 0, 1(\mathrm{mod} p)$ for some $r > 0$. On the other hand, if $\epsilon_j = 0, 1$ for all j then $\phi^r(i)$ is defined for all $r > 0$. Moreover it is easy to check that, in this case, $\phi^{2n}(i) = i$.

Returning to our α_i's, we see that $\alpha_i = 0$ if $\epsilon_j \neq 0, 1$ for some j and that $\alpha_{\phi(i)}^p = \alpha_i$ and $\alpha_i^{p^{2n}} = \alpha_i$ if $\epsilon_j = 0, 1$ for all j. If $\alpha_i \in \mathbf{F}_p$ this simply means $\alpha_{\phi(i)} = \alpha_i$. The set of polynomials $a(t) \in \mathbf{F}_p[t]$ satisfying our conditions form a $\mathbf{F}_p$-vector space and each orbit of ϕ contributes one dimension to it. Since each orbit has at most $2n$ elements and there are 2^n distinct $i = \sum_{j=0}^{n-1} \epsilon_j p^j, \epsilon_j = 0, 1$, we obtain at least $2^n/2n$ orbits, hence the count for $\mathbf{F}_p(t)$. In the case of $\mathbf{F}_{p^{2n}}(t)$, an orbit of length r contributes an r-dimensional $\mathbf{F}_p$-vector space. Since $r|2n$, the theorem follows.

Remarks: 1. It can be shown, using the methods of [V], that indeed the points produced in the proof of the theorem are all the rational points of C_n.

2. In the case $p = 3$, C_n is a quasi-elliptic fibration over $\mathbf{P}^1$ in the sense of the classification of surfaces ([BM],[L]) and our result shows that the 3-rank of the group of sections (the "Mordell-Weil" group) can be arbitrarily large. (See [I1,2])

3. The curves C_n are the members of the family of curves $x - tf(u)x^p = y^p$, where $f(u) = \sum_{i=0}^{p-2} u^i$, for $u = t^{p^n+1}$. It follows from the results of [V] that $tf(u)$ is a p-th power

in $\mathbf{F}_p(t)$ for only finitely many $u \in \mathbf{F}_p(t)$, so the curve corresponding to a given $u \in \mathbf{F}_p(t)$ has finitely many points for all but finitely many u's, again by the results of [V]. Following [CHM] we consider the total space of the family, that is, the surface S over $\mathbf{F}_p(t)$ defined by $x - tf(u)x^p = y^p$ and, as is shown in [CHM], the set of rational points of S is Zariski dense, for otherwise, the theorem above would be violated. Since S is unirational, it is not surprising that this holds for some extension of $\mathbf{F}_p(t)$, but since S cannot be covered by $\mathbf{P}^2$ over $\mathbf{F}_p(t)$, it is surprising that this occurs over $\mathbf{F}_p(t)$. Also, S is of general type for $p \geq 7$, thus showing that Lang's conjecture on varieties of general type (see [CHM]) cannot be easily transposed to positive characteristic.

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