A CONSTITUTIVE MODEL FOR SANDY SOILS BASED ON A STRESS-DEPENDENT DENSITY PARAMETER

応力に依存する密度定数を用いた砂質土の構成モデル

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ABSTRACT

The results of drained and undrained hollow cylinder tests on clean sand and sand with fines content of 10–15 % by weight constitute the basis for an investigation of the behaviour of sandy soils. An elastic-plastic constitutive model, which account for the combined influence of density and effective stress in a unique way, is developed.

The study addresses many important issues relevant to sandy soil behaviour and represents a unite experimental and theoretical effort to develop a constitutive model for sandy soils comprising balanced features of accuracy, simplicity and versatility. Particular attention in both experimental and theoretical considerations is given to the effects of density and effective stress as well as their combined influence on the sandy soil behaviour.

In general, two series of tests are performed, drained torsional shear tests and undrained torsional shear and torsional simple shear tests. Both series include monotonic and cyclic tests.

The results of the drained torsional shear tests illustrated the effects of density and mean effective stress on the stress–strain and volume change behaviour of sandy soils and indicated that this behaviour is related to the relative initial state of the soil. The significance of the combined effects of density and effective stress is most profoundly illustrated by the measured identical stress–strain characteristics for quite different initial states including combinations of loose sand at low effective stress and dense sand at high effective stress. Another interesting observation is that stress–strain behaviour for the initial states above the threshold void ratio that indicates zero undrained residual strength, is practically independent of both density and effective stress. Thus, the threshold void ratio associated to zero undrained residual strength indicates the softest drained stress–strain behaviour for a given soil and initial fabric.

Based on the experimental evidence obtained from the drained torsional shear tests, a plastic stress–strain relation with density–stress dependent parameters is developed. The parameters of this relation are expressed as functions of an index
parameter established in the framework of the steady state of deformation concept. The parameter employed in the relation is the State Index $I_s$ which characterizes the sandy soil behaviour by accounting for the combined effects of density and effective stress related to a given initial fabric. The validity and the efficiency of the state index for representing drained sandy soil behaviour over a wide range of densities and effective stresses is assessed.

The proposed stress–strain relation is a modified hyperbolic relation with the initial plastic modulus defined as a function of the plastic strain. Thus, it accounts for the greater nonlinearity than that provided by the two–constant hyperbolic relation and improves the accuracy of the stress–strain representation over the entire range of strains. The proposed stress–strain relation is characterized by a single set of coefficients for the entire range of densities and stresses.

An elastic–plastic constitutive model is developed in the framework of the incremental theory of plasticity. The model is defined in a stress space that enables to account for the rotation of principal stresses and comprises: very small, in fact, 'point' yield surface with purely kinematic hardening rule; failure surface which incorporates the effects of density, effective stress and initial anisotropy, and which serves as bounding hardening surface and plastic potential; a set of four hardening surfaces in the memory with a rule for evaluation of the current hardening surface based on an assumption for mixed hardening; and plastic potential formulation providing noncoaxial and nonunique flow for all stress states except for those at failure.

The accuracy and effectiveness of the elastic–plastic constitutive model is assessed through a comparison of the measured and predicted behaviour of sand in monotonic and cyclic, drained and undrained shear tests. The applicability of the model to seismic response analysis is demonstrated through blind–prediction of a seismic response of level ground model obtained from centrifuge test. Results of the element test simulations and blind–prediction of the response in the centrifuge tests have shown that the model is capable of representing drained and undrained, monotonic and cyclic behaviour with high degree of accuracy. Yet, that can be achieved with a single set of strength and deformation parameters over a wide range of densities and effective stresses.
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Chapter 1

INTRODUCTION

1.1 GENERAL REMARKS

The need to solve increasingly sophisticated soil engineering and soil-structure interaction problems has created a demand for better experimental devices and techniques as well as better analytical tools, in order to more clearly understand and represent soil behaviour. In recent years, considerable experimental and theoretical efforts have been made to study and model soil behaviour under various loading conditions, including cyclic loads as caused by earthquakes, traffic and sea wave loads. On the other hand, intensive research has also been done on its monotonic behaviour, providing abundant valuable information on the essence of soil behaviour from very small strains with nearly elastic behaviour up to large deformations and flow of the material.

The rapid development in recent years of advanced computational procedures for nonlinear stress analysis, such as the finite element method, especially emphasized the need for appropriate constitutive models, undoubtedly a crucial element in the numerical analysis. The development of more advanced constitutive models for soils has also been stimulated and thoroughly supported by the increasing supply of high quality experimental evidence and a continuous widening of our knowledge of soil behaviour. A considerable challenge is posed, however, by the fact that soil behaviour is remarkably complex, involving various complicated processes and phenomena both
under monotonic and cyclic loading.

Constitutive modelling of soils confronts two opposing requirements: on the one hand, the model should be able to describe the extreme complexity of soil behaviour, and on the other hand, it should be comprehensible and simple enough to assure easy implementation in numerical analysis and application to practical problems. Whether complexity or simplicity will prevail is the first important decision to be made in the process of the development of a constitutive model. Yet, this judgement and the choice of the relevant aspects of the soil behaviour that should be modeled is probably the most critical for the accuracy, effectiveness and applicability of the model.

This study deals with sandy soils and presents an integrated experimental and theoretical investigation on their behaviour under various loading conditions. The study contains all the necessary steps in evaluation of a constitutive model for sandy soils, including experimental study where the experimental basis for the model is produced, then development of a stress–strain relation, formulation of an elastic–plastic constitutive model, verification of the model, and finally its application to seismic response analysis.

1.2 ON CONSTITUTIVE MODELLING

Without aiming to review present constitutive models for sandy soils at this point, a brief discussion on the general trends in the modelling and basic requirements that a constitutive model has to fulfill will be given. To start with, it is important to recognize the existence and the need of various types of models since there are considerable differences as regards the purpose of the model, the problem that it considers and the required level of accuracy and simplicity among others. The development of advanced constitutive models has already been under way for several decades, and these models have passed well beyond the speculative or qualitative description of soil behaviour. During this period, a great number of constitutive models have been proposed involving various theories and concepts. Those concepts are either of a micromechanical or phenomenological (macromechanical) nature. Even though the principal feature of soils is their particulate structure and thus the ultimate adequate representation of the true nature of soils lies in the understanding of the microscopic behaviour, the complexity of these processes and the diversity of soil
field conditions will definitely restrict the quantitative predictions of micromechanical models and especially their applicability in the foreseeable future. Therefore, it is quite understandable that the majority of the present advanced constitutive models rely on phenomenological or macroscopic observations of soil behaviour. Only these models will be of interest in the present work.

Recent developments in the constitutive modelling of sandy soils are characterized by usage of a variety of approaches and theories including elasticity, viscosity and plasticity, as well as combinations of these theories: visco-elasticity, visco-plasticity and elasto-plasticity. This trend in the modelling is simply due to the fact that soils exhibit behaviour related to these theories. However, as pointed out earlier, it is crucial to select and model only the important aspects of the sandy soil behaviour that a constitutive model should cover, since that is the only approach which will contribute to model simplicity, comprehensibility and applicability. That this has not been always done in recent constitutive models is clearly indicated by the comments of Scott (1988) on the constitutive models presented at the workshop on Constitutive Equations for Granular Non-Cohesive Soils (Saada and Bianchini, 1988):

"Clearly, we cannot go on like this, with the ultimate goal of achieving a thermo-viscoelastic hypoplastic multiple critical bounding capped yielding endochronic state surface model. At some stage, physical intuition has to re-enter the problem ...."

It has to be recognized, on the other hand, that plasticity models are already proved to be capable of capturing the important features of soil behaviour such as dilatancy, nonlinear and irreversible deformations, dependency on stress path and inherent and induced anisotropy among others. In fact, plasticity theory has been the cornerstone of the constitutive modeling of soils for more than two decades. The balance between the simplicity and accuracy of a constitutive model seems to be more or less achievable within the framework of the incremental theory of plasticity.

The requirements that a particular model has to fulfill are strongly dependent on the purpose of the model and in particular on the problem that it aims to solve. However, in general, the model has to satisfy theoretical, experimental and computational requirements. In other words, besides the requirement for theoretical
rigour, it is essential to model the soil behaviour on a physically relevant basis in order to attain a realistic representation of soil behaviour and versatile model. No less important is the fact that the number of the model parameters should be reasonable and that their experimental evaluation can be easily assessed in conventional tests. However, the requirement for evaluation of model parameters by using only conventional triaxial tests appears to be too rigorous if a constitutive model accounting for generalized stress path is needed. As regards the application of a constitutive model to numerical analysis, it is crucial that the theoretical constitutive model be convertible into a form that can be used in a finite element program. Computational efficiency and memory requirements are certainly features that can seriously influence the applicability of a constitutive model.

1.3 ON SANDY SOIL BEHAVIOUR

The evidence on soil behaviour obtained from in–situ observations, case history studies and experiments on soils is more or less conclusive despite the complexity of the behaviour, the diversity of in–situ soil conditions and the difficulties in properly duplicating them in the laboratory. As regards cohesionless soils, most attention has been given to problems related to liquefaction, mainly due to the fact that this phenomenon has been a direct cause of a great deal of severe damage to modern engineering structures. For clarity, it is important to distinguish between two different phenomena that have been identified as liquefaction, both related to undrained response of cohesionless soils. These are: flow failure (field observations) or contractive flow and consequent liquefaction (experimental observations) which are characteristic of very loose cohesionless soils, and can be triggered by both monotonic and cyclic loading; and cyclic mobility (experimental observations and indications in field observations) which can be observed for both loose and dense cohesionless soils and represents typical undrained behaviour during cyclic loading.

Flow failure or liquefaction is characterized by reduction and even complete loss of the strength of the soil accompanied by contractive response and extremely large deformations with liquid–like flow of the material. It is typical for very loose soil and sloping ground conditions. Most of the research on this phenomenon has been done within the framework of the steady state of deformation, which is already established
as the cornerstone for the stability analysis of soil structures.

Cyclic mobility is characteristic undrained cyclic behaviour of soils which takes place after substantial pore pressure has been generated, and the effective stress has been reduced to nearly zero, by the cyclic loading. In the further loading, the effective stress path is characterized with almost steady position and temporary reduction of the effective stress to zero, accompanied by continuous increase in the strain amplitude. Both loose and dense sand can exhibit cyclic mobility behaviour with the growth of strain amplitude rapid in the former and relatively slow in the latter.

It is important to emphasize that liquefaction and cyclic mobility are only two manifestations of the undrained behaviour of sandy soils and have been singled out since they are already recognized as a major hazard to various engineering structures. No less important are the characteristics of the undrained behaviour of sandy soils prior to occurrence of liquefaction and cyclic mobility, since they reveal the progress of the response that eventually leads to liquefaction or cyclic mobility. Thus, for proper representation of soil behaviour it is necessary to account for the progressive failure characteristics of soils, i.e. the transition from the initial elastic or nearly elastic state and small deformations to the failure state involving large deformations. In order to achieve this formidable task, it is necessary to consider appropriately all the factors that affect sandy soil behaviour as density, effective stress, stress state, initial fabric, stress and strain history among others. Constitutive equations for soils should, in fact, provide the solution of the progressive failure problems and definition of the relationship between stresses and strains by accounting for the above mentioned factors.

1.4 PURPOSE AND SCOPE OF THE PRESENT STUDY

Recent research in soil mechanics is directed towards better understanding of soil behaviour and development of advanced constitutive models capable of fulfilling the considerable demands related to geotechnical and earthquake engineering problems. In this study, experimental and theoretical investigation is performed on the behaviour of sandy soils with the ultimate goal of developing a constitutive model capable of simulating the sandy soil behaviour during monotonic and cyclic loading.

Particular attention is given to the effects of density and effective stress in both
experimental and theoretical considerations. Even though these effects have been thoroughly studied in many previous studies resulting in generally conclusive findings and experimental evidence, the outcome from the experiments is practically ignored in current constitutive models for sandy soils. In other words, in spite of the fact that the combination of density and effective stress is relevant to the description of soils, it is prevailing practice in constitutive models to consider these effects separately. The primary objective of this study is to provide a physically more relevant basis for representation of the effects of density and mean effective stress in constitutive models for sandy soils. More specifically, the objectives of the study are:

- To examine and clarify the effects of density and effective stress on sandy soil behaviour by accounting for their combined influence. On the basis of the results of the experimental study to develop a stress–strain model with integrated density–stress effects.

- To study experimentally the characteristics of the drained and undrained behaviour of sandy soils for different initial states in terms of density, mean effective stress, and various loading conditions including cyclic loading and rotation of principal stress directions.

- To develop an elastic–plastic constitutive model for sandy soils based on the results of the experimental study. To verify and validate the model through simulation of element tests including monotonic and cyclic, as well as drained and undrained conditions. Finally, to implement the model in a finite element code and to apply it to seismic response analysis.

1.5 ORGANIZATION OF THIS THESIS

The thesis is composed of nine chapters, the contents of which are briefly summarized in the following.

The introductory part of the study is presented in Chapters 1 and 2. Here Chapter 1 presents background information on constitutive modelling and on the basic concept applied in the current constitutive models for sandy soils. In order to better understand the rest of the study, a comprehensive review of relevant previous studies on sandy soils is given in Chapter 2. The review includes experimental and theoretical
studies on sandy soils and places particular emphasis on the essential features of sandy soil behaviour. The fundamentals of plasticity theory are briefly summarized as well.

The experimental part of the present study is presented in details in Chapters 3, 4 and 5. A detailed description of the apparatus, sample preparation, testing procedures and materials used in the testing are given in Chapter 3. A discussion on the principles of hollow cylinder testing and distribution of stresses and strains in hollow cylindrical samples is also given in the belief that this information is essential for interpretation and validation of the experimental results.

Chapter 4 presents the experimental results from drained torsional shear tests on four different materials including clean sand and sand with fines content of 10–15 % by weight. The effects of density and effective stress and in particular their interaction are carefully investigated for different initial states of the tested soils. These results have been used as the experimental basis for development of the proposed stress–strain model.

Chapter 5 deals with undrained monotonic and cyclic behaviour of sand. The characteristics of torsional simple shear behaviour are studied, particularly the effects of the initial effective stress ratio $K$. The influence of the deformation mode on the undrained behaviour of sand is also illustrated. These data have been further used in the process of verification and validation of the proposed constitutive model.

The theoretical part of the study is presented in Chapters 6, 7 and 8. Modelling concept and basic relations adopted for the present model are elaborated in Chapter 6. Based on the experimental evidence obtained from the drained torsional shear tests, a plastic stress–strain relation with density–stress dependent parameters is developed. The cornerstone of the modeling concept is the state parameter concept in the framework of steady state of deformation.

In Chapter 7, an elastic–plastic constitutive model for sandy soils is formulated. The model is developed in the framework of incremental theory of plasticity and accounts for various aspects of sandy soil behaviour for complex loading conditions including cyclic loading and rotation of principal stress directions. The model is characterized by a single set of parameters for the entire range of densities for a given material.

Chapter 8 covers the verification and validation of the proposed model through
comprehensive examination in element test simulations including drained and undrained, as well as monotonic and cyclic loading conditions. The effectiveness and accuracy of the model in finite element analysis is assessed through the blind-prediction of a seismic response of soil deposit in centrifuge test.

Developments from the present study are briefly summarized in Chapter 9.
Chapter 2

LITERATURE REVIEW

2.1 INTRODUCTION

The objective of this work is to study the behaviour of cohesionless soils with the ultimate goal to develop a constitutive model capable of simulating sandy soil behaviour for monotonic and cyclic, as well as drained and undrained conditions. For that purpose, combined experimental and theoretical investigation on the behaviour of sandy soils is carried out. This chapter is a background information for the present study and contains review of relevant literature so as to better understand and clarify later developments. It comprises results from experimental studies on cohesionless soils with illustration of the influence of the most important factors on their monotonic and cyclic behaviour. Attention is also given to the characteristics of the undrained behaviour of sandy soils and related phenomena such as liquefaction and cyclic mobility. The concept of steady state of deformation has an important role in development of the present constitutive model, and therefore, it is presented in more details. Since the constitutive model of this study is developed within the framework of the flow theory of plasticity, the fundamentals of this theory are briefly introduced. The review addresses to many experimental and theoretical aspects of sandy soil behaviour, and therefore attention is limited to studies which are more closely related to the present work.
2.2 FACTORS INFLUENCING SANDY SOIL BEHAVIOUR

The behaviour of sandy soils is affected by a number of factors including density, effective stress, shear stress, strain history, initial fabric, orientation of principal stress directions and relative magnitude of intermediate principal stress among others. A great many experimental studies have been conducted on sandy soils in order to study the effects of these factors and to quantify their influence on soil behaviour. To demonstrate the influence of the most important factors and to illustrate the essence of the behaviour of sandy soils, some typical data obtained from studies conducted on cohesionless soils under drained conditions will be presented. The undrained characteristics of the cohesionless soils will be discussed in the subsequent section.

2.2.1 Influence of Initial Void Ratio

From the early studies in soil mechanics it has been recognized that the initial density or void ratio and effective stress have most important influence on the stress–strain and volume change characteristics of soils. Thus, traditionally in soil mechanics the general patterns of soil behaviour have been related to parameters associated with density of the soil, such as relative density and void ratio. Characteristics of behaviour of cohesionless soils associated with different initial densities, as those typical for loose sand and dense sand, are now well established and supported by abundant conclusive experimental evidence. Herein, influence of the initial density on behaviour of sand will be illustrated with the data presented by Koerner (1970), obtained from conventional drained triaxial tests.

Fig. 2.1 shows a set of typical stress–strain and volume change relationships obtained in tests with different relative density of sand samples (Koerner, 1970). Apparently, there is significant influence of the relative density on both stress–strain and volume change behaviour. Thus, the stress–strain curves are nonlinear, with gradual change of the stiffness and peak stress ratio depending on the relative density of the soil. Dense soil exhibits a peak in the stress–strain curve and subsequent strain softening post–peak behaviour, while the loose soil is characterized by continuous increase in the stress ratio until the ultimate stress state is reached. However, a unique
residual stress, independent of the relative density, is reached at approximately 30 % axial strain. The stiffness of the stress–strain relationship is greater (or the slope is steeper) with increasing relative density.

The volume change behaviour in these tests is characterized with overall dilative response. There is gradual increase in the compressibility as the initial relative density decreases, with the dense soil being slightly compressive in the initial part of the shearing and strongly dilative as the shearing proceeds. The loose sand, on the other hand, exhibits strong compression and relatively poor dilation. There is abundant
experimental evidence on behaviour of sandy soils with the general patterns as described above.

2.2.2 Influence of Mean Effective Stress

Mean effective stress is another major factor with significant influence on behaviour of sandy soils. It has been subject of numerous studies on cohesionless soils utilizing drained conventional triaxial compression tests (Lee and Seed, 1967; Vesic and Clough, 1968; Fukushima and Tatsuoka, 1984 and Kolymbas and Wu, 1990 among others), triaxial tests on large, squat samples (Hettler and Vardoulakis, 1984), plane strain compression tests (Tatsuoka et al., 1986a) and torsional shear tests on hollow cylindrical samples (Tatsuoka et al., 1986b) among others. Except for the torsional shear tests on hollow cylindrical samples, other conventional tests mentioned above are performed at constant effective confining stress $\sigma'_c$, while the mean effective stress is not constant but rather increasing with the deviator stress. Triaxial tests under constant mean effective stress, e.g. as those reported by Kondner and Zelasko (1963) and Vesic and Clough (1968), have seldom been performed. In general, however, there is conclusive evidence on the influence of the mean effective stress, although some details remain to be clarified. The patterns of the behaviour depending on the mean effective stress are illustrated below.

Fig. 2.2 shows the stress–strain and volume change behaviour in drained triaxial tests on loose sand and dense sand, reported by Lee and Seed (1967). The effective confining stress in these tests is in the range between $\sigma'_c = 1 - 120$ kgf/cm². Lee and Seed concluded that an increasing confining stress has three effects on the behaviour: it reduces the brittleness of the stress–strain curve, it increases the strain to failure or to peak stress ratio, and it decreases the tendency to dilate. They further concluded that the stress–strain characteristics of dense sand at high confining pressures are not unlike those of loose sand at low pressures.

One of the important conclusions in the study of Lee and Seed is that at high confining pressures, the effects of particle crushing on behaviour of cohesionless soils is significant, and therefore, they must be properly considered. Thus, the results of the tests at confining pressures greater than 10 kgf/cm², shown in Fig. 2.2, include the effects of particle crushing. The degree of particle crushing in the considered tests is
Figure 2.2 Influence of Effective Confining Stress on Stress–Strain and Volume Change Behaviour of Sacramento River Sand (after Lee and Seed, 1967)
illustrated in Fig. 2.3 by microphotographs of fresh sand and sand tested at various confining pressures.

Typical results from drained conventional triaxial compression tests on sand (Karslruhe medium sand) in the range of low to moderate confining pressures ($\sigma'_c = 0.5 - 10.0 \text{ kgf/cm}^2$), reported by Kolympas and Wu (1990), are shown in Fig. 2.4. In addition to that, in Fig. 2.5 stress–strain and volume change behaviour observed in plane strain compression tests on Toyoura sand at extremely low pressures ($\sigma'_c = 0.05 - 4.0 \text{ kgf/cm}^2$) presented by Tatsuoka et al. (1986a), are shown. It is interesting to notice that the pattern of the behaviour in tests at low pressures, where particle crushing does not take place, is not unlike the pattern shown in Fig. 2.2.

In evaluation of results from tests at low confining stress or mean effective stress, it is necessary to correct stresses for the effects of membrane forces. Even though considerable efforts have been made to clarify these effects, yet, the accuracy of the results obtained at low effective stress is undoubtedly smaller due to the uncertainties in the correction for the membrane forces and nonuniformities in the stress distribution due to the self–weight of the samples. These effects, together with the effects of particle crushing at high effective stress, are necessary to be considered in order to properly evaluate the influence of the effective stress on sandy soil behaviour.

### 2.2.3 Effects of Intermediate Principal Stress

Influence of the intermediate principal stress $\sigma'_2$ is usually described by the dependence of soil behaviour on the parameter $b$ defined as:

\[
b = \frac{\sigma'_2 - \sigma'_3}{\sigma'_1 - \sigma'_3}
\]  

(2.1)

where $\sigma'_1$, $\sigma'_2$ and $\sigma'_3$ are major, intermediate and minor principal stresses and $b$ is a measure for the relative magnitude of the intermediate principal stress.

Influence of the intermediate principal stress on stress–strain and volume change behaviour of a sand is illustrated in Fig. 2.6 by the data presented by Gutierrez (1989). Fig. 2.6 shows stress–strain and volumetric strain characteristics obtained from drained
Figure 2.3 Degree of Particle Crushing of Sacramento River Sand at Various Confining Pressures (after Lee and Seed, 1967)

Figure 2.4 Influence of Effective Confining Stress on Sand Behaviour in Triaxial Compression (after Kolymbas and Wu, 1990)

Figure 2.5 Influence of Effective Confining Stress of Sand Behaviour in Plane Strain Compression (after Tatsuoka et al., 1986a)
monotonic hollow cylindrical tests on Toyoura sand for various b-values ranged between \( b = 0 - 0.75 \). All these tests have been performed along constant principal stress directions with \( \beta_o = 30^\circ \). Here \( \beta_o \) denotes the angle between the direction of the major principal stress \( \sigma_1 \) and the vertical axis. As shown in Fig. 2.6, both the stress–strain and volume change behaviour are strongly affected by the \( b \)-value or the relative magnitude of the intermediate principal stress.

### 2.2.4 Influence of Inherent Anisotropy

Following the definition of soil anisotropy given by Casagrande and Carrillo (1944) two different forms of anisotropy can be distinguished, namely, inherent anisotropy and induced anisotropy. According to that definition, inherent anisotropy is referred to the non-isotropic soil behaviour related to the physical characteristics inherent in the material, and entirely independent of the applied strains. Thus, it is considered that inherent anisotropy is result of the deposition process and grain characteristics. On the other hand, induced anisotropy is defined as a physical characteristic exclusively related to the strain associated to the applied stresses.

The effects of inherent anisotropy on stress–strain and strength characteristics of cohesionless soils have been shown quite significant (Oda, 1972a, 1972b; Arthur and Menzies, 1972). To make clear the effects of initial fabric on strength and deformation behaviour of sand, Oda (1972a) performed a series of drained triaxial compression tests on sand samples formed in such a way that the angle between original horizontal planes, i.e. horizontal planes during the process of deposition of the material, and the direction of major principal stress \( \sigma_1 \) was different. Schematic illustration of sample preparation with an angle \( \varphi \) between the so-called original horizontal plane and the direction of \( \sigma_1 \) is given in Fig. 2.7. Stress–strain and volume change curves observed for different angles \( \varphi \), for initial void ratio about 0.72 and 0.86, are also shown in this figure. Apparently, the behaviour is remarkably dependent on the angle \( \varphi \), with stiffer response and smaller compressibility as \( \varphi \) increases. The difference in the behaviour is solely due to the depositional anisotropy and in fact demonstrates the significance of the effects of inherent anisotropy on sandy soil behaviour.

Similar conclusions about the effects of inherent anisotropy in sand have been
Figure 2.6 Influence of $b$-value on Behaviour of Sand in Drained Hollow Cylinder Tests (after Gutierrez, 1989)
drawn by Arthur and Menzies (1972). They found differences of well over 200% in the axial strain in order to reach given pre–failure stress ratio, and much smaller variation of the strength, up to 10% of maximum stress ratio, as a result of the inherent anisotropy. In other words, the effects of inherent anisotropy were found remarkable on the stiffness of the stress–strain curve but rather small on the strength of the tested sand.

2.2.5 Summary of the Effects on Drained Strength

In the preceding pages the effects of initial void ratio or density, mean effective stress, relative magnitude of intermediate principal stress and inherent anisotropy on stress–strain and volume change characteristics of cohesionless soils were illustrated. As regards drained strength we can distinguish two reference states: peak strength, defined by the maximum or peak stress ratio attained during shearing, and ultimate strength, associated with the stress ratio at the ultimate state or the critical state. The effects of the considered factors on the peak strength are demonstrated by the summary plots given in Figs. 2.8 – 2.11, while the ultimate strength will be considered in the subsequent sections, when the critical state and steady state concepts will be elaborated in more details.

Fig. 2.8 shows a comparison of Mohr failure envelopes for a sand at four initial densities (Lee and Seed, 1967). The higher position of the failure envelopes as density increases (See Figs. 2.8 and 2.9) shows the effects of the initial void ratio. The effects of the mean effective stress are illustrated, on the other hand, with the curvature of the failure envelopes in Fig. 2.8, and with the decrease of the maximum principal stress ratio with increasing confining pressure in Fig. 2.9. It is interesting to note that the stress ratio reached nearly constant value, independent of the initial void ratio, at large confining pressure.

The effects of b-value on the angle of internal friction $\phi$ observed in several studies are summarized in Fig. 2.10, taken from Gutierrez (1989). The data presented in this figure includes results from studies with true triaxial apparatuses and hollow cylinder apparatuses. The effects of b–value are illustrated on the friction angle $\phi$ normalized with respect to $\phi$ at $b = 0$, where $b = 0$ and $b = 1.0$ are conditions associated with triaxial compression and triaxial extension, respectively (Eq. 2.1).
Figure 2.7 Effects of Inherent Anisotropy on Behaviour of Sand in Drained Triaxial Compression (after Oda, 1972a)
Apparently, the friction angle $\phi$ reaches a peak value somewhere about $b = 0.5$. It is important to notice that remarkably different magnitudes of the effects of $b$–value are obtained in these studies.

In similar manner, the effects of orientation of principal stress directions relative to the vertical, represented by the angle $\beta_\alpha$, on the friction angle normalized with respect to $\phi$ at $\beta_c = 0^\circ$ are shown in Fig. 2.11 (Gutierrez, 1989). The data presented in this figure comprise results from plane strain, triaxial and true triaxial tests using tilted samples (e.g. see Fig. 2.7) and hollow cylindrical tests along different $\beta_\alpha$ directions. In spite of the significant differences between the presented results, there is clear tendency for decrease in $\phi$ with increasing $\beta_\alpha$, with some data indicating lowest resistance for $\beta_\alpha$ in the range between $50^\circ$ to $70^\circ$.

2.2.6 Stress–Path Dependency

It is well known that soil exhibits strongly stress–path dependent behaviour. In fact, as frictional material, soil is by virtue dependent on both normal and shear stresses. It was shown in the preceding pages that stress–strain behaviour and peak strength of cohesionless soils are dependent on $b$–value or on the relative magnitude of the intermediate principal stress as well. Finally, it was demonstrated that due to the inherent anisotropy, soil behaviour is dependent not only on the magnitude of the principal stresses, but also on the orientation of their directions. However, even if all these factors are accounted for, yet, stress–strain behaviour can not be predicted unless the strain history or load history of the soil is not known. Thus, the effects of previously introduced induced anisotropy and rearrangement of soil particles due to the strains induced by the previously applied stresses have to be accounted for. Having all of this in mind, the stress–path dependence of soils and the complexity of their cyclic response can be easily understand.

To illustrate the stress–path dependency of behaviour of cohesionless soils, in Figs. 2.12 and 2.13 data presented by Lade and Duncan (1976) and Lade (1988) are shown. In each of these figures, stress–strain and volume change behaviour of a pair of tests with different stress–paths is compared. The stress–paths, all with varying stress difference and varying confining pressure, involve either only primary loading (Fig. 2.12) or primary loading and unloading (Fig. 2.13). Based on comparison of a
Figure 2.8 Mohr Failure Envelopes for Different Initial Densities of Sacramento River Sand (after Lee and Seed, 1967)

Figure 2.9 Relationship Between Peak Strength and Confining Pressure for Two Sands (after Lee and Seed, 1967)
Figure 2.10 Influence of b-value on Angle of Internal Friction (after Gutierrez, 1989)

Figure 2.11 Influence of $\beta_\sigma$ on Angle of Internal Friction (after Gutierrez, 1989)
Figure 2.12 Stress–Path Dependent Behaviour for Primary Loading Conditions (after Lade, 1988)

Figure 2.13 Stress–Path Dependent Behaviour for Primary Loading and Unloading Conditions (after Lade, 1988)
variety of stress–paths Lade concluded that, for stress–paths involving unloading or reloading, the stress–path with higher average stress level produces larger strains, whereas all stress–paths having the same initial and final states of stress, and involving only primary loading, produce strains of similar magnitudes. Similarly, Tatsuoka and Ishihara (1974a) concluded that, for a given mean effective stress $p'$ and stress ratio $q/p'$, magnitude of the shear strain is uniquely determined irrespective of the stress path along which the stress state has been reached.

In general, soils are inherently anisotropic, and therefore, their response to loading will depend on the orientation of principal stress directions. In other words, soil behaviour depends not only on changes in the magnitudes of the principal stresses, but also on changes in the orientation of their directions. Since rotation of the principal stress directions is a feature of the stress–paths associated with many field loading conditions, and on the other hand, because soils are inherently anisotropic, significant experimental efforts were done in the last decade to clarify the importance of principal stress rotation and to quantify its influence on soil behaviour. (Arthur et al., 1980 and 1981; Symes et al., 1984 and 1988; Miura et al., 1986; Ishihara and Tawhata, 1983 among others). Arthur et al. (1980), using a so–called directional shear cell, concluded that during continuous rotation of principal stresses through a large angle, strain accumulates steadily even only half the static shear strength is mobilized. Similarly, Symes et al. (1984) and (1988) showed that there is compatibility in drained and undrained response involving principal stress rotation, with contraction of the material during drained and rise in the pore pressure during undrained principal stress rotation. In addition to that, Ishihara and Tawhata (1983), Ishihara et al. (1985) and Tawhata and Ishihara (1985) showed the effects of cyclic principal stress rotation on the liquefaction resistance of sand. They concluded that cyclic stress ratio required to cause a certain level of strain is reduced if rotation of principal stresses is involved in the cyclic loading. Thus, a common conclusion from these studies was that if reliable predictions of in–situ behaviour are to be made, then, the influence of the principal stress rotation needs to be accounted for.

Finally, some typical features of sand response subjected to cyclic loading will be illustrated. Fig. 2.14 shows typical results of a drained triaxial test on loose sand with constant cyclic stress ratio $q/p'$, reported by Tatsuoka and Ishihara (1974b). In
the course of the cyclic loading, the sand densifies and volume decreases with a progressively decreasing rate of volumetric compression. The amplitude of the shear strain gradually decreases as the cycling loading proceeds which is associated with reduction in the hysteresis exhibited in a single cycle. Deformations stabilize after approximately nine cycles.

However, cyclic sand response is remarkably different if the stress ratio amplitude gradually increases as the cycles proceed. Thus, as shown in Fig. 2.15, large volumetric contraction and increase of the shear strain amplitude are associated with each subsequent cycle. Hence, both volumetric strain and shear strain gradually increase in the course of cyclic loading. As regards the development of volumetric strain and shear strain, the results presented in Figs. 2.14 and 2.15 show opposite tendencies. It is important to be noted that quantitative aspects of this behaviour are dependent on the density of the material as well as on the magnitude of the cyclic stress ratio amplitude. Having this in mind, the complexity of cyclic sand response due to application of an irregular cyclic load, e.g. earthquake excitation, is apparent.

It was not intention herein to discuss in details the effects of various factors on behaviour of cohesionless soils nor to give an up-to-date review of the reported conclusions in the literature. The present section was aimed only at indicating the most
Figure 2.15 Drained Cyclic Test on Loose Sand with Increasing Stress Ratio Amplitude (after Tatsuoka and Ishihara, 1974)
important factors which influence monotonic and cyclic behaviour of sandy soils. In doing so, the important aspects of soil behaviour that are necessary to be considered in modelling have been introduced.

2.3 UNDRAINED BEHAVIOUR AND ITS CHARACTERIZATION

Soil liquefaction during earthquakes and flow failures triggered either by monotonic or cyclic loads have been identified as a major cause of damage to the ground, earth structures as well as engineering structures resting upon it. Therefore, significant experimental and theoretical efforts have been expended to clarify the undrained behaviour of saturated cohesionless soils and to develop relevant procedures for assessment of the liquefaction potential and safety against occurrence of flow failure. In this section the essence of the undrained behaviour of cohesionless soils will be illustrated, and a brief discussion on the fundamental concepts applied to characterization of this behaviour will be given. Particular emphasis will be placed on the implementation of these concepts for quantitative description of the undrained behaviour of cohesionless soils including a brief introduction of several parameters proposed for characterization of soil behaviour.

2.3.1 Liquefaction and Cyclic Mobility

In order to properly illustrate characteristics of liquefaction and cyclic mobility, first of all the main characteristics of the undrained response of cohesionless soils will be demonstrated. For that purpose, in Figs. 2.16 and 2.17 the stress–strain behaviour and effective stress paths from undrained triaxial tests on Banding sand performed by Castro in 1969 are shown (Mohamad and Dobry, 1986). Two sets of tests are shown in these figures, for samples with different effective confining pressure $\sigma_{3c}$, but with the same void ratio at the end of consolidation $e_c = 0.70$, and for samples consolidated at identical effective confining pressure, $\sigma_{3c} = 4.0$ kgf/cm$^2$, but having different void ratio $e_c$.

As shown in Figs. 2.16 and 2.17, the stress–strain and pore pressure characteristics are strongly dependent on the state of the soil prior to shearing. They change gradually from those typical for dense sand and low confining pressure to
Figure 2.16 Influence of Effective Confining Pressure on Undrained Behaviour of Banding Sand [Mohamad and Dobry, 1986; Data from Castro (1969)]

Figure 2.17 Influence of Void Ratio on Undrained Behaviour of Banding Sand [Mohamad and Dobry, 1986; Data from Castro (1969)]
those typical for loose sand and high confining pressure. It is important to note that the behaviour is consistent with the tendencies described in the previous section when characteristics of the drained response were illustrated. Particular attention should be given to the undrained behaviour exhibiting drop in the shear stress associated with development of large shear strain, such as that observed in Tests 1 and 4 in Figs. 2.16 and 2.17, respectively. In these tests, the applied shear stress \( q \) first increased and then dropped rapidly to a constant, steady state or residual strength \( S_{ur} \). It is important to note that this type of undrained behaviour has been identified under different names in the literature, including actual liquefaction (Casagrande, 1975), liquefaction (Castro and Poulos, 1977; Poulos et al. 1985; Seed, 1979; Vaid et al. 1990 among others), flow deformation or flow failure (Vaid and Chern, 1983; Alarcon and Leonards, 1986) among others.

In order to prevent the confusion by the use of the term liquefaction for entirely different phenomena, Casagrande (1975), Castro and Poulos (1977) and Poulos et al. (1985) defined liquefaction as a phenomenon wherein the shear resistance of a mass of soil decreases when subjected to monotonic, cyclic or dynamic loading and the mass undergoes large strains and flows in a manner resembling a liquid. They attributed this type of behaviour to loose saturated cohesionless soils and stated that only soils that tend to decrease in volume, i.e., contractive soils, can suffer substantial loss of undrained strength. However, they pointed out that even contractive soils are not susceptible to liquefaction unless the driving shear stresses in-situ are large enough. This is illustrated in Fig. 2.18, where the triggering of liquefaction by monotonic and cyclic loading are schematically shown together with driving shear stress and the steady state strength of a soil. Thus, according to this definition, the term liquefaction is associated with certain behaviour of a loose sand, and determination of liquefaction potential is a stability analysis which requires that the shear strength and shear stress in-situ to be determined (Poulos et al. 1985). In the center of the liquefaction evaluation procedure proposed by Poulos et al. (1985) is the concept of steady state of deformation (Poulos, 1981), which represents an extension to the critical void ratio concept developed by Casagrande in 1936.

The term liquefaction, however, is often used to describe a typical behaviour of saturated soils during cyclic loading, reported for the first time by Seed and Lee
(1966). They performed cyclic triaxial compression tests on loose and dense sand, in order to simulate approximately the stress conditions during earthquakes on an element of soil, at some depth below level ground. Since that pioneering work of Seed and Lee, cyclic testing by using triaxial, simple shear, hollow cylinder or other apparatuses has become essential procedure in evaluation of the potential for liquefaction due to cyclic excitation.

To illustrate the typical behaviour of saturated sands during cyclic loading, the data reported by Ishihara (1985), obtained from cyclic torsional shear tests on Fuji river sand, will be briefly discussed. Fig. 2.19 shows the effective stress path and stress strain curve for loose sand with relative density $D_r = 47 \%$, subjected to cyclic stress ratio with amplitude of 0.229. Apparently, the effective confining pressure decreases steadily as the cyclic loading proceeds and after the effective stress path touches the line of phase transformation (Ishihara et al., 1975), the stress path moves right upwards during loading and downwards on left during unloading phase. In other words, once the phase transformation is crossed, loading process for the states below the phase transformation and unloading are associated with strong contractive behaviour and decrease of the effective stress, while loading process for the states above the phase transformation are associated with strong dilation. During the dilation, the effective stress path approaches the failure surface and moves away of the origin,
Figure 2.19 Stress Path and Stress–Strain Curve for Loose Fuji River Sand
Obtained from Cyclic Torsional Shear Test (after Ishihara, 1985)
while on the other hand, during unloading, the effective stress path moves away of the failure surface and towards the origin or zero effective stress state.

Similar behaviour can be observed for dense sand with relative density $D_r = 75\%$, shown in Fig. 2.20. Since the cyclic shear stress ratio amplitude in this test is large, the cyclic mobility starts from the very beginning of loading, in fact, from the first cycle. The stress–strain curve is characterized with continuous increase of the shear strain as cyclic loading proceeds but, yet, strain increase is limited and much smaller than in the case of the loose sand. The strain softening behaviour, as illustrated in the two tests presented in Figs. 2.19 and 2.20, is typical behaviour associated with cyclic mobility. It is important to notice that, in the course of cyclic mobility the stress path reaches nearly zero effective stress state and the sand is said to be "liquified". However, increase in the shear stress would cause dilation or increase of the effective stress, and thus, the soil will regain its strength.

The difference of the liquefaction associated with the flow deformation and liquefaction associated with the cyclic mobility was briefly demonstrated by the typical tests results presented above. In spite of the large dispute about which of these phenomena should be referred to as liquefaction, there is complete agreement among the researchers that these are two different phenomena that should be properly treated in order to perform appropriate stability and deformation analysis.

2.3.2 Characterization of Sandy Soil Behaviour

Characteristics of monotonic undrained behaviour of loose sand are illustrated schematically in Fig. 2.21 with the effective stress path, stress–strain curve and state diagram. Following the peak (point B), there is a drop in the shear stress associated with significant increase in both pore pressure and shear strain until the phase transformation is reached (point A). Following the phase transformation, the soil exhibits dilative behaviour associated with increase in the effective stress and stiffness of the response. This in turn enables increase of the shear stress until eventually the steady state is reached (point C) where the soil mass is continuously deforming at constant volume, constant normal effective stress, constant shear stress, and constant velocity (Poulos, 1981). It is important to notice that the condition of shearing at and around the phase transformation are not unlike those at the steady state. Namely, at
Figure 2.20 Stress Path and Stress–Strain Curve for Dense Fuji River Sand Obtained from Cyclic Torsional Shear Test (after Ishihara, 1985)
and around the point $A$ in Fig. 2.21, relatively large strain developed at nearly constant shear and effective stresses. Since on one hand this state is temporary, and on the other hand it is similar to the steady state of deformation, it is usually defined in the literature as the quasi steady state (Alarcon et al., 1988). The steady state line and the quasi steady state line represent loci of points at which the soil can deform continuously at the steady state and quasi steady state, respectively. The projections of this lines on the state diagram ($e-p'$ plane) are also called the steady state and the quasi steady state lines (Fig. 2.21c).

Monotonic and cyclic undrained behaviour of saturated sand as well as the difference between liquefaction and cyclic mobility can be explained by using the state diagram shown in Fig. 2.22. In this figure the initial states of loose and dense sand are indicated by points C and D, respectively, and the paths in the state diagram during monotonic and cyclic loading are shown by the solid lines. In addition to that, the steady state line for the given soil is displayed. If the loose sand is subjected to monotonic or cyclic loading, liquefaction or flow deformation will occur associated with highly contractive behaviour, reduction in the shear stress and large strain development. The behaviour will be, in fact, exactly as that shown in Fig. 2.18. Following the drop in the shear stress the steady state strength will be reached or the state of the soil denoted by point $A$ on the state diagram.

When a fully saturated dilative sand with the initial state indicated by point D on the state diagram is loaded, however, the path in the stress diagram could move either towards the steady state line or towards the zero effective stress state and cyclic mobility, depending on whether the monotonic or cyclic loading is applied. Hence, the state paths in monotonic and cyclic loading of dilative sand diverge and in fact head towards different final states. The state paths of the loose sand (C–A) and dense sand (D–B) clearly indicate the differences in the behaviour associated with liquefaction and cyclic mobility. Castro and Poulos (1977) summarized this behaviour such that samples with initial state above the steady state can liquefy if the applied load (driving shear stress) is large enough. The further to the right of the steady state line that the initial state is, the greater will be the deformations associated with the liquefaction. It is to be noted that initial states of the soil above point Q (quicksand) (See Fig. 2.21) have zero steady state strength. On the other hand, samples with initial states below
Figure 2.21 Characteristics of Undrained Behaviour of Loose Sand (after Ishihara, 1993)

Figure 2.22 State Paths of Loose and Dense Sand During Monotonic and Cyclic Loading (after Castro and Poulos, 1977)
the steady state will be dilative during monotonic loading and the state path will move to the right. If cyclically loaded than the state path will move to the left towards the zero effective stress state and cyclic mobility.

In order to distinguish between the two types of monotonic undrained behaviour associated either with or without drop in the shear stress during straining, Ishihara (1993) and Verdugo (1992) defined in the state diagram the "initial dividing line". This line demarcates the initial states of a given soil in the state diagram (states of the soil after consolidation) which are associated with drop in the shear stress and occurrence of the quasi steady state, from those states which do not exhibit drop in the shear stress during monotonic loading. Thus, the filled points in Fig. 2.23 indicate the initial states for which the quasi steady state was observed while the empty points designate the initial states without drop in the shear stress. For the material tested in this study (Toyoura sand), the initial dividing line was found close to the steady state line and quasi steady state line with some divergence as the confining stress increases (See Fig. 2.24).

It is clearly indicated in the above discussion that undrained behaviour can be characterized if the initial state of the soil is related to a reference state of the soil such as the steady state, the quasi steady state or the initial dividing line. It is important to note that the critical void ratio concept was developed by Casagrande on the basis of results from drained direct shear tests, and therefore, it is reasonable to expect that these states are relevant reference states for the drained behaviour as well. The first direct use of the critical state of soils as a reference state in order to quantify the response of both clays and cohesionless soils has been proposed by Roscoe and Pookrooshasb (1963). They have investigated the conditions for similarity in the behaviour of small scale soil models and corresponding prototypes. Since model tests are usually carried out on the same soil that is going to be used for the prototype, they concluded that in addition to the geometrical similarity of the stress path for the model and prototype, it is essential that the initial states of the soil in the model and prototype are related in a specific manner if similarity in the strains of the model and prototype are expected. Based on theoretical and experimental considerations, they found that two elements of a given soil will have the same strains if, besides the geometrical similarity in the stress paths, their initial states are on a same distance from their
Figure 2.23 Initial Dividing Line for Moist-Placed Samples of Toyoura Sand (after Verdugo, 1992 and Ishihara, 1993)

Figure 2.24 Characteristic Lines of Toyoura Sand in the $e - \log p$ diagram (after Verdugo, 1992 and Ishihara, 1993)
critical states. Thus, Roscoe and Poorooshab used the projection of the critical state line as a reference line.

The condition for similarity proposed by Roscoe and Poorooshab (1963) is illustrated schematically in Figs. 2.25 and 2.26. Two soil elements with different initial states with respect to the void ratio \( e \) and effective stress \( \sigma \) are considered in these figures. Along their geometrically similar stress paths \( (A_1-Z_1 \text{ and } A_2-Z_2) \), shown in Fig. 2.25, similar stress increments \( m_1-n_1 \text{ and } m_2-n_2 \) are considered. According to the condition for similarity, these increments will produce equal change in the void ratio for the two considered soil elements only if their states are equally distanced from the critical state line (See Fig. 2.26). Hence, two elements when subjected to geometrically similar stress paths will have identical strains only if their states satisfy the following equation:

\[
\frac{\sigma_{o2}}{\sigma_{o1}} = \exp \frac{e_{o1} - e_{o2}}{\lambda}
\]  

(2.2)

where \( e_{o1} \) and \( e_{o2} \), and \( \sigma_{o1} \) and \( \sigma_{o2} \) are initial void ratios and effective stresses, respectively, for the two soil elements, and \( \lambda \) is the slope of the critical state line in the \( e - \log \sigma \) plane.

Been and Jefferies (1985) extended the condition for similarity and formulated the state parameter concept within the framework of the steady state of deformation. They postulated that the behaviour of a sand may be characterized in terms of two variables: a state parameter which combines the influence of void ratio and stress; and a fabric parameter which characterizes the arrangement of the sand grains. The state parameter \( \psi \) proposed by Been and Jefferies (1985), in fact, follow exactly the definition of condition for similarity provided by Roscoe and Poorooshab. Been and Jefferies compiled data from different sands and correlated their behavioural properties with the corresponding values of the state parameter \( \psi \). For example, in Figs. 2.27 and 2.28 the drained angle of shearing resistance for several sands and the undrained peak strength ratio, respectively, are expressed as functions of the state parameter \( \psi \). It is important to note that state parameter \( \psi \) has been introduced by Been and Jefferies as a parameter that characterizes sand behaviour independently of the material. In other
Figure 2.25 Condition for Similarity: Geometrically Similar Stress Paths for Cohesionless Media (after Roscoe and Poorooshab, 1963)

Figure 2.26 Condition for Similarity: Void Ratio Changes of Two Elements Subjected to "Similar" Stress Paths (after Roscoe and Poorooshab, 1963)
Figure 2.27 Angle of Shearing Resistance as Function of the State Parameter $\psi$
(after Been and Jefferies, 1986)

Figure 2.28 Undrained Strength Ratio as Function of the State Parameter $\psi$
(after Been and Jefferies, 1986)
words, the behavioural properties of two different sands are identical if the initial states provide same values for the state parameter \( \psi \).

In a similar manner, Verdugo (1992) and Ishihara (1993) proposed an initial index for characterization of the undrained sandy soil behaviour termed the state index \( I_s \). The state index \( I_s \) is defined by using two reference states, i.e. the quasi steady state and either the threshold void ratio associated with zero steady state strength or the isotropic consolidation line for the loosest state for a given method of sample preparation. Since modelling of the combined influence of density and effective stress on the sandy soil behaviour in the present study is based on the usage of state index \( I_s \), its detailed description will be given in Chapter 6.

2.4 THE FLOW THEORY OF PLASTICITY

It was illustrated in the preceding pages that sandy soil behaviour is affected by many factors, and that in fact, the combination of all these effects is reflected in its drained and undrained response. Therefore, in order to properly describe sandy soil behaviour, it is necessary to consider all relevant factors influencing this behaviour, in a manner which will provide, on one hand, physically sounded model, and on the other hand, simple and versatile model. The flow theory of plasticity has already been established as the cornerstone of the constitutive modelling of soils, and has been accepted as a framework within which soil behaviour can be described providing the balance between the simplicity and accuracy of a constitutive model. Since soil behaviour is rather complex, and the flow theory of plasticity allows large flexibility in the modelling, a great many concepts and models have been proposed within the general framework of this theory. It is the purpose of this section to briefly introduce the main elements of the flow theory of plasticity.

It is customary in the flow theory of plasticity to assume that the total strain increment \( \mathbf{d}e_{ij} \) can be divided into elastic and plastic parts, \( \mathbf{d}e^e_{ij} \) and \( \mathbf{d}e^p_{ij} \), respectively, such as

\[
\mathbf{d}e_{ij} = \mathbf{d}e^e_{ij} + \mathbf{d}e^p_{ij} \tag{2.3}
\]

Essentially, the plastic strains remain after the removal of stresses while the elastic
strains are recoverable. The elastic strains and the plastic strains are calculated separately, by an elastic stress–strain law and by the flow theory of plasticity including yield surface, hardening rule and flow rule as its constitutive elements.

A yield surface defines the limit of elastic behaviour for a material under a combination of stresses. Inside this surface, the behaviour is assumed to be elastic and therefore, strains can be recovered upon the removal of the stresses. The second equally important role of the yield surface is to denote the stress space where plastic or nonrecoverable strains occur. Based on experimental evidence on soil behaviour, various yield surfaces have been proposed, having different size, shape and rules for their evolution. A law governing this aspect of the problem, i.e. the manner of constructing the subsequent yield surfaces, is called the hardening rule.

There are several hardening rules that have been proposed to describe the evolution of subsequent yield surfaces. In general, however, three types of hardening rules have been utilized, and these are: isotropic hardening, kinematic hardening and mixed hardening. In spite of the simplicity of the isotropic hardening, it is well known that this rule can not provide realistic description of sandy soil behaviour subjected to cyclic loading. In order to overcome the shortcoming of the isotropic hardening, kinematic and mixed hardening rules have been used which enable change in the position as well as change in both size and position of the yield surface, respectively.

The above discussion can be summarized, in the following manner. The equation of the yield surface can be written as

\[ f = (\sigma_y, \alpha_y, k) = 0 \]  \( (2.4) \)

where \( \sigma_y \) is the stress state and \( k \) is a hardening parameter giving the size of the yield surface and \( \alpha_y \) is a hardening parameter giving the position of the yield surface. In the case of isotropic hardening, the hardening parameter \( k \) has been defined by: the plastic volumetric strain, as in the Cam–clay models (Schofield and Wroth, 1968) and the cap models (e.g. di Maggio and Sandler, 1971; Sandler et al., 1976); the plastic shear strain (Pis, 1986; Vermeer, 1978); a linear combination of the plastic volumetric strain and shear strain (Nova and Hueckel, 1981); and the plastic work (Lade and Duncan, 1975) among others.

On the other hand, the kinematic hardening rules that have been proposed are
defined through a rule for translation of the yield surface $d\sigma_2$ such as Prager's, Ziegler's or the hardening rule proposed by Phillips and Weng (1975). In order to account for hysteretic effects, more elaborate plastic models based on combination of isotropic and kinematic hardening rules have been proposed, such as the multi-surface models and the two-surface or bounding surface models. One interesting approach in the modelling of behaviour of cohesionless soils is used by Palmer and Pierce (1973) and Prevost (1985) according to which there is no pure elastic domain and soil is, in fact, modelled as a continuously yielding material. The yield surface thus coincides with the stress point and has pure kinematic hardening rule.

Following the above discussion, plastic deformation occurs if the stress state is on the yield surface and attempts to move beyond it, i.e.

$$f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_2} \, d\sigma_2 > 0$$  \hspace{1cm} (2.5)

The elastic behaviour occurs under the conditions such that:

$$f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_2} \, d\sigma_2 < 0$$  \hspace{1cm} (2.6)

Finally, the plastic strain is calculated by the flow rule, defined as:

$$d\varepsilon_2^p = d\lambda \frac{\partial g}{\partial \sigma_2}$$  \hspace{1cm} (2.7)

in which $g$ is the plastic potential function, while $d\lambda$ can be written as:

$$d\lambda = \frac{1}{H_p} \frac{\partial f}{\partial \sigma_{mn}} \, d\sigma_{mn}$$  \hspace{1cm} (2.8)

where $H_p$ is the hardening modulus which depends generally on stress and strain history. It is apparent from the above formulation that magnitude and direction of the plastic strain components depend on the hardening modulus and plastic potential function.
Chapter 3

APPARATUS, TESTING PROCEDURES AND TEST MATERIALS

3.1 INTRODUCTION

One of the primary and probably most difficult goals in soil modelling is to develop a model capable of simulating soil behaviour for a variety of stress paths. For that purpose, high quality laboratory investigations are needed in order to establish appropriate experimental data for evaluation and validation of the model. In addition to that, there is a need to study soil behaviour for a generalized stress path as well as to duplicate the stress conditions that are commonly encountered in the field.

The dependence of soil behaviour on the stress path is widely recognized, and therefore, considerable efforts in the experimental studying of soils have been placed on developing testing devices capable of imposing generalized stress paths on soil specimens. Closely related with these efforts is the development of the hollow cylinder apparatus, which is considered to be one of the few devices, and certainly the most versatile and widely used, that satisfy the above mentioned task. Although, the hollow cylinder apparatus was introduced in soil testing long ago, it is only recently that it has been used to simulate a wide variety of stress paths on soil specimens. Its considerable usage and improvement in this period were mainly as result of numerous studies on soil anisotropy and related effects of the orientation of the principal stresses
including their continuous rotation. A review of the soil studies conducted with hollow cylinder apparatus and its historical background is given by Saada (1988).

The experimental program of the present study was realized using a hollow cylinder apparatus. In general, two types of tests were performed: Monotonic drained torsional shear tests under constant mean effective stress $p'$; and monotonic and cyclic undrained torsional shear and torsional simple shear tests. The drained tests were carried out in order to study the effects of density and mean effective stress on the behaviour of sandy soils and in particular their combined influence on this behaviour. The results of this series of tests were used as the experimental basis for development of a stress–strain model. The undrained tests were used to study the behaviour of sand under simple shear condition and in particular the effects of the initial effective stress ratio $K = \sigma'_h / \sigma'_v$. The validity of the proposed elastic–plastic constitutive model was justified through the simulation of these test results. This chapter presents a detailed discussion on the test apparatus, testing procedures and test materials used in the present study since this information is essential for the interpretation and validation of the experimental results.

### 3.2 TEST APPARATUS

**General Description.** A schematic layout of the hollow cylinder apparatus used in this study is shown in Fig. 3.1. The hollow cylindrical sample with inner radius, $r_i = 30$ mm, outer radius, $r_o = 50$ mm, and height $H = 195$ mm is enclosed laterally in two flexible rubber membranes (inner and outer membrane) and vertically by rigid top and bottom caps. Drainage from the sample is provided by porous discs, built into the caps, which are connected to a 25–cm burette by drainage lines. The burette is in turn connected to the backwater air–pressure line.

Torsional and vertical loads are transmitted to the sample through rough porous stones in which six blades 0.5 mm thick are fixed at regular intervals. The blades enable transmission of the torsional shear stress to the sample up to failure, preventing slippage without causing strain concentration near the ribbed ends (Tatsuoka et al., 1983). The sample is fixed through the bottom cap to the massive pedestal, and torsional and vertical loads are applied at the top cap. The sample as described above is enclosed in an acrylic triaxial cell thus creating separate inner and outer cells.
Figure 3.1 General Layout of the Hollow Cylinder Apparatus
**Loads and Measurements.** The general layout of the pressure control, loading and measurement system is shown in Figs. 3.2 and 3.3. The loading system enables independent control of the torque, the axial load, the inner cell pressure and the outer cell pressure. Quantities that are measured are the torque, the axial load, the effective inner cell and outer cell pressures, the volume changes of the sample and inner cell, the pore water pressure, and the axial and rotational displacements.

The torsional and axial loads are applied using Belloflerom cylinders and are measured by a load–torque cell located out of the cell chamber. The maximum capacity of the load–torque cell is 200 kgf and 200 kgf/cm, for the axial and torsional loads, respectively. Friction in the vertical loading shaft is reduced by using an air–bearing system.

The outer cell air pressure is converted to water pressure in partially filled with water outer cell. The inner cell pressure, on the other hand, is transmitted through a 50–cm burette connected to water–filled inner cell. Hence, the outer and inner pressures are controlled independently. The effective outer cell pressure and the effective inner cell pressure, $p' = p_o - u$ and $p' = p_i - u$, respectively, are accurately measured by two high–capacity differential pressure transducers (HC–DPT), denoted in Fig. 3.2 as DP_o and DP_i. The maximum capacity of the HC–DPT is 3.2 kgf/cm² with a sensitivity of 0.8 gf/cm².

Similarly, the volume changes of the sample and inner cell are measured by two low–capacity differential pressure transducers (LC–DPT), that measure the change in the pressure due to the change in the height of the water in the corresponding burettes (Tatsuoka, 1981). LC–DPT are denoted as $V_s$ and $V_i$ in Fig. 3.2. The maximum capacity of the LC–DPT is set at 300 mm and 500 mm water height, for the sample and inner cell burettes, respectively. Corresponding sensitivities in the measurement of the volume changes are 0.004 cm³ and 0.014 cm³, for the volume change of the sample and volume change of the inner cell, respectively.

The axial displacement is measured by two dial gages located diametrically opposite to each other while the rotational displacement is measured by a potentiometer. The accuracy in these measurements is 1 µm and 0.03°, respectively.
Figure 3.3 Overall View of the Hollow Cylinder Apparatus
3.3 CALCULATION AND REPRESENTATION OF STRESSES AND STRAINS

The advantage of using a hollow cylinder apparatus in soil testing is the possibility to apply generalized stress path on soil specimens, thus subjecting the samples to the stress conditions that often occur in the field. However, for an appropriate interpretation of the results obtained by hollow cylinder apparatus it is particularly important to recognize its shortcomings and limitations. In the following, more detailed discussion on the stress and strain calculation will be given, since the limitations of the hollow cylinder apparatus are mainly due to the non-uniformities of stresses and strains in the wall of the specimen.

Loading condition in the typical hollow cylindrical test is illustrated in Fig. 3.4. A hollow cylindrical sample is subjected to four loading components as follows: Torque $T$, axial load $F$, and outer and inner cell pressures, $p_o$ and $p_i$ respectively. These loads induce vertical normal stress $\sigma_z$, radial normal stress $\sigma_r$, circumferential normal stress $\sigma_\theta$ and shear stress $\tau_{r\theta}$ on an element in the wall of the hollow cylindrical sample as shown in Fig. 3.4. It is to be noted that shear stresses are not induced on the vertical boundaries since $p_o$ and $p_i$ act on flexible membranes, thus $\tau_{r\theta} = \tau_{rr} = 0$. Hence, neglecting the effects of the end restraint, circumferential surfaces through the wall are free of shear stresses, and the radial stress $\sigma_r$ is always a principal stress.

By controlling the four stress components $\sigma_z$, $\sigma_r$, $\sigma_\theta$ and $\tau_{r\theta}$, it is possible to have independent control of the magnitudes of the principal stresses $\sigma_1$, $\sigma_2$ and $\sigma_3$ as well as their directions, as illustrated by the angle $\beta_\theta$ on Fig. 3.5. It is this feature of the hollow cylinder apparatus that gives it special and important place in the arsenal of laboratory testing devices. Needles to say, the behaviour of an anisotropic soil is affected not only because of the change in the magnitudes of the principal stresses but also because of the change in their directions. It is to be noted, that the rotation of the directions of the principal stresses is restricted to the $z$-$\theta$ plane since, because as mentioned above, $\sigma_r$ is always a principal stress.
Figure 3.4 Hollow Cylindrical Sample and the Induced Stress State in Its Wall

Figure 3.5 Rotation of the Principal Stress Directions in the Hollow Cylinder Test
3.3.1 Loads and Displacements

The calculation of the four load components acting on a hollow cylindrical specimen $F_i$, $T$, $p_i$, and $p_o$, and the corresponding displacements $u_z$, $\Delta \theta$, $u_i$ and $u_o$ is described in the following.

The effective axial load is calculated as

$$ F_x = F_a + W_s + \pi \left( p_o r_o^2 - p_i r_i^2 \right) - A_r p_o $$  \hspace{1cm} (3.1)

where $F_a$ is the axial load transmitted through the vertical loading rod, including the weight of the top cap and corrected for the force due to friction; $W_s$ is half of the submerged weight of the sample; and $A_r$ is the cross area of the loading rod.

The torque is calculated as

$$ T = T_r + T_f $$  \hspace{1cm} (3.2)

where $T_r$ is the torque reading and $T_f$ is correction term for the torque due to friction.

As described in the previous section, the inner and outer cell pressures are measured by using high capacity differential pressure transducers. Hence, they are calculated using the calibration curves directly from the DPT readings. Since all the pressures are measured at the bottom of the sample the only correction needed is thus compensation for the height of the water in the sample and inner cell burette.

A basic assumption in the calculation of the displacements is that the specimen remains cylindrical during the shearing. The axial displacement $u_z$ and the rotational displacement $\Delta \theta$ are obtained directly from the measurements without any corrections applied on them. On the other hand, the radial displacements of the inner wall of the sample $u_i$ and outer wall of the sample $u_o$ are calculated based on the measurements of the volume changes of the inner cell and changes of the sample volume as

$$ u_i = \sqrt{\frac{\pi r_i^2 H + \Delta V_i}{\pi (H + u_z)}} - r_i $$  \hspace{1cm} (3.3)

$$ u_o = \sqrt{\frac{\pi r_o^2 H + \Delta V_i + \Delta V_{sp}}{\pi (H + u_z)}} - r_o $$  \hspace{1cm} (3.4)
where $\Delta V_i$ and $\Delta V_{sp}$, are the volume change of the inner cell and the sample, respectively.

### 3.3.2 Average Stresses and Strains

In spite of the versatility of the hollow cylindrical apparatus and its capability to impose generalized stress path on soil samples it is essential for proper implementation and interpretation of the hollow cylinder tests to recognize the shortcomings and limitations of the apparatus. They arise from the nonuniform distribution of the stresses and strains through the wall of the sample. Since recognition of this factor, many studies have been done in order to clarify the distribution of stresses and strains in a hollow cylindrical sample and to establish appropriate criteria for an acceptable level of nonuniformity (Hight et al., 1983; Saada, 1988; Vaid et al., 1990b; Tatsuoka et al., 1986c among others). It has been concluded in these studies that, in general, the degree of stress nonuniformity through the wall of the sample depends on the stress state, sample dimensions and soil's constitutive law.

As regards the dimensions of the sample it has been concluded that the nonuniformities in stresses and strains arise mostly due to the sample's curvature and restraints at the specimen's ends imposed by the rigid top and bottom caps. Therefore it has been recommended that the thickness of the wall be sufficiently large to accommodate the granular material without constraining its deformation and failure mechanism, that the height of the sample be large enough to minimize the nonuniformity caused by the end restraints and that the sample's curvature be reduced by adopting sufficiently large inner and outer radii. It is important to notice that the apparatus used in this study satisfies all of the above mentioned requirements.

For a given material, once the sample dimensions are fixed, the degree of stress nonuniformity depends on the stress state applied to the sample. The nonuniformity increases as the torque increases and in particular when different inner and outer pressure, $p_i$ and $p_o$, are applied. It is therefore common to set restrictions on the stress state within a tolerable limits of difference in the inner and outer cell pressures, which is commonly assumed to be in the range $0.75 \leq p_i / p_o \leq 1.3$ (Hight et al., 1983; Gutierrez, 1989).
Interpretation of hollow cylinder tests is made considering the entire specimen as a single element. However, due to the nonuniformities of the stresses and strains through the wall of the specimen, it is necessary to work with their average values. The average stress components are calculated in this study assuming uniform axial stress distribution $\sigma_z$, linear elastic distribution of the circumferential stress $\sigma_\theta$ and the radial stress $\sigma_r$, and average of elastic and perfectly plastic distributions of the shear stress $\tau_{zh}$. The effective stress components averaged across the sample wall are given below, in terms of the applied loads and the current sample dimensions. The bars and the primes over the stress components to indicate "average" and "effective", respectively, have been dropped in the succeeding equations.

\[
\sigma_z = \frac{F_z}{\pi \left( r_o^2 - r_i^2 \right)} + \Delta \sigma_z
\]

\[
\sigma_r = \frac{p_o r_o + p_i r_i}{r_o + r_i} + \Delta \sigma_r
\]

\[
\sigma_\theta = \frac{p_o r_o - p_i r_i}{r_o - r_i} + \Delta \sigma_\theta
\]

\[
\tau_{zh} = \frac{1}{2} \left[ \frac{3 T}{2 \pi \left( r_o^3 - r_i^3 \right)} + \frac{T \left( r_o - r_i \right)}{\pi \left( r_o^4 - r_i^4 \right)} \right] + \Delta \tau_{zh}
\]

Here $\Delta \sigma_z, \Delta \sigma_r, \Delta \sigma_\theta$ and $\Delta \tau$ are corrections for the membrane forces whose calculation will be discussed in the subsequent section.

Similarly, the average strain components are derived assuming uniform axial displacement $u_z$, linear variation of the radial displacement $u_r$ and uniform angular distortion $\Delta \theta$. The average strain components are calculated based on the above assumptions and the initial sample dimensions, as follows:
\[ \varepsilon_z = - \frac{u_z}{H} \] (3.9)

\[ \varepsilon_r = - \frac{u_o - u_i}{r_o - r_i} \] (3.10)

\[ \varepsilon_\theta = - \frac{u_o + u_i}{r_o + r_i} \] (3.11)

\[ \varepsilon_{z\theta} = \frac{\Delta \theta \left( r_o^3 - r_i^3 \right)}{3H \left( r_o^2 - r_i^2 \right)} \] (3.12)

where the displacements \( u_i \) and \( u_o \) are calculated from Eqs. (3.3) and (3.4), respectively.

### 3.3.3 Corrections for Effects of Membranes

The membranes affect the experimental results from hollow cylinder apparatus in two ways: by the membranes' forces and by the membranes' penetration. The membranes' forces are related to the torsional resistance of the inner and outer membrane. It has been found that these effects are especially significant at low mean effective stress (Tatsuoka et al., 1986c). The membranes' penetration, on the other hand, affects the volume change measurements and pore water pressures in drained and undrained tests, respectively. This influence increases with the particle size of the granular material and especially with the increase of the cell pressure during a test.

The stress corrections for the effects of membranes forces are calculated assuming that both inner and outer membrane retain the shape of a right cylinder and using the theory of elasticity with a Poisson's ratio for the membranes equal to 0.5, as proposed by Tatsuoka et al. (1986c). They are calculated as follows:

\[ \Delta \sigma_z = - \frac{4}{3} \frac{E_m t_m}{r_o^2 - r_i^2} \left\{ r_o \left[ 2(\varepsilon_{zm})_o + (\varepsilon_{\theta m})_o \right] + r_i \left[ 2(\varepsilon_{zm})_i + (\varepsilon_{\theta m})_i \right] \right\} \] (3.13)
\[ \Delta \sigma_0 = -\frac{2}{3} \frac{E_m t_m}{r_o - r_i} \left\{ \left[ (\varepsilon_{zm})_o + 2(\varepsilon_{om})_o \right] + \left[ (\varepsilon_{zm})_i + 2(\varepsilon_{om})_i \right] \right\} \quad (3.14) \]

\[ \Delta \sigma_\theta = -\frac{2}{3} \frac{E_m t_m}{r_o + r_i} \left\{ \left[ (\varepsilon_{zm})_o + 2(\varepsilon_{om})_o \right] - \left[ (\varepsilon_{zm})_i + 2(\varepsilon_{om})_i \right] \right\} \quad (3.15) \]

\[ \Delta \tau_{z\theta} = -2 E_m t_m \frac{r_o^3 + r_i^3}{(r_o^3 - r_i^3)(r_o + r_i)} \gamma_{z\theta} \quad (3.16) \]

where \( E_m = 14.0 \text{ kgf/cm}^2 \) and \( t_m = 0.3 \text{ mm} \) are the Young's modulus and the thickness of the membranes, respectively, and \( (\varepsilon_{zm})_i \) and \( (\varepsilon_{zm})_o \), and \( (\varepsilon_{om})_i \) and \( (\varepsilon_{om})_o \) are the average axial and circumferential strains, in the inner and outer membrane, respectively.

The effects of the membranes' penetration on the volume change of the sample and inner cell are accounted for by using the relationship between the penetration depth and cell pressure derived by Miura (1985) and Gutierrez (1989), given as:

\[ D_m = 0.004673 \, p_c^{0.4359} \quad (3.17) \]

where \( D_m \) is the mean penetration depth in millimeters and \( p_c \) is the cell pressure. Hence, volume change of the sample and inner cell are calculated using the following expressions

\[ \Delta V_{sp} = (\Delta V_{sp})_r - 2 \pi H (r_i D_{mi} + r_o D_{mo}) \quad (3.18) \]

\[ \Delta V_i = (\Delta V_i)_r - 2 \pi H r_i D_{mi} \quad (3.19) \]

where \( (\Delta V_{sp})_r \) and \( (\Delta V)_r \) are the measured sample and inner cell volume changes, respectively.

### 3.3.4 Stress and Strain Representation

In hollow cylinder testing, as mentioned earlier, four stress components can be controlled independently. When the radial strain represents the intermediate principal
stress $\sigma_r = \sigma_z$, the $\sigma_z-\sigma_\theta$ axes can rotate, as illustrated in Fig. 3.5. The angle of the major principal stress $\sigma_1$ relative to the vertical is then defined as:

$$\tan 2\beta_o = \frac{2 \tau_{z\theta}}{\sigma_z - \sigma_\theta}$$  \hspace{1cm} (3.20)

In addition to $\beta_o$, the magnitudes of the principal stresses are controlled. They are calculated as follows:

$$\sigma_1 = \frac{\sigma_z + \sigma_\theta}{2} + \sqrt{\left(\frac{\sigma_z - \sigma_\theta}{2}\right)^2 + \tau_{z\theta}^2}$$  \hspace{1cm} (3.21)

$$\sigma_2 = \sigma_r$$  \hspace{1cm} (3.22)

$$\sigma_3 = \frac{\sigma_z + \sigma_\theta}{2} - \sqrt{\left(\frac{\sigma_z - \sigma_\theta}{2}\right)^2 + \tau_{z\theta}^2}$$  \hspace{1cm} (3.23)

Alternatively, the stress state can be presented by the stress invariants defined as:

$$p = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} (\sigma_z + \sigma_r + \sigma_\theta)$$  \hspace{1cm} (3.24)

$$q = \frac{1}{2} (\sigma_1 - \sigma_3) = \sqrt{\left(\frac{\sigma_z - \sigma_\theta}{2}\right)^2 + \tau_{z\theta}^2}$$  \hspace{1cm} (3.25)

$$b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}$$  \hspace{1cm} (3.26)

where $p$ is the mean effective stress, $q$ is the deviator stress and $b$-value is a measure for the relative magnitude of the intermediate principal stress $\sigma_2$.

The strain components are defined as:
\[
\bar{\varepsilon} = \sqrt{(\varepsilon_z - \varepsilon_\theta)^2 + \gamma^2} \tag{3.27}
\]
\[
\varepsilon_v = \varepsilon_z + \varepsilon_\theta + \varepsilon_r \tag{3.28}
\]

where $\bar{\varepsilon}$ is shear strain, $\gamma = 2\varepsilon_{sv}$ and $\varepsilon_v$ is volumetric strain, with the compression having positive sign.

### 3.4 TESTING PROCEDURES

#### 3.4.1 Sample Preparation

Two methods of sample preparation were employed in this study for reconstituting the specimens: air pluviation and wet plunging method. Generally, sand samples were prepared by air pluviation, while wet plunging was used for reconstituting the samples of sand with fines. These sample preparation techniques were selected in order to ensure uniform samples throughout the whole experimental program, and to avoid the segregation of particles during the reconstitution procedure.

Except for the sample reconstitution stage, a standard procedure was followed in the sample preparation. The main steps of this procedure are briefly described in the following:

**Molds and Membranes Setting.** First, the bottom cap, and the inner split mold and membrane are assembled (Fig. 3.6a). The inner diameter and stretching of the inner membrane are measured, and then, the outer membrane is sealed to the bottom cap (Fig.3.6b). A grid is drawn on the outer membrane in order to get a good picture of the pattern of deformation and failure. Next, the outer mold is assembled and a vacuum of 0.1 kgf/cm² is applied to ensure a good fit of the outer membrane on the outer mold (Fig. 3.6c and 3.6d). Thus, the molds and membranes are set, and prepared for the reconstitution procedure.

**Air Pluviation.** Air-dried sand is continuously pluviated in the mold from a plastic container with a tube attached at its bottom part. The pluviation is carefully done maintaining the height of the pluviation constant until the mold is field with sand. Then, the superfluous material is removed and the upper surface of the specimen is smoothed and formed at the prescribed height of about 195 mm (Fig. 3.7c). The
density of the sample is controlled by adjusting the height of the pluviation (the distance between the end of the tube and the surface of the deposited sand). It is to be noticed that air pluviation procedure under zero height of pluviation is equivalent to the method of dry deposition as described by Ishihara (1993).

**Wet plunging.** In a mixture of sand with fines (or exceptionally clean sand) de-aired water is added and mixed in order to get a uniform material with water content of 3%. The material is poured gently into the mold and compacted by a plunging rod with a foot’s diameter of 12 mm (Fig. 3.7a and 3.7b). The sample is formed in ten layers with a prescribed and identical weight. After compacting the top layer, the superfluous material is removed and the upper surface is levelled carefully (Fig. 3.7c). To ensure a uniform density of the sample, the density of each layer is controlled by adjusting the number of plunging (tamping) strokes as suggested by Tatsuoka et al. (1982).

**Sealing the Sample.** After forming the specimen as described above the top cap is carefully seated on the leveled surface and the membranes are sealed to it (Fig. 3.7d). Then, the top and bottom cap are connected to the drainage lines and a vacuum of 0.1 kgf/cm² is applied to the sample. This provides confinement to the sample and enables dismantling of the split molds. (Fig. 3.8a). Lastly, the dimensions of the sample are measured, thus completing the sample preparation procedure.

### 3.4.2 Test Preparation

After enclosing the sample laterally by inner and outer membranes, and vertically by top and bottom caps, the vertical loading rod is connected to the top cap without introducing any additional load to the sample (Fig. 3.8b). The cell chamber is placed in position and the assembled triaxial cell is centered in the loading frame. Then, carbon dioxide gas is circulated in the inner cell, inner and outer cells are simultaneously supplied with water, and the top of the inner cell is connected to the 50-cm burette making sure that no bubbles are trapped in the inner cell-burette system. The high-capacity differential pressure transducers are then connected to the inner and outer cell drainage lines, and the vacuum in the sample is gradually replaced by cell pressure and axial load, maintaining constant isotropic effective stress on the sample at 0.1 kgf/cm².
Figure 3.7 Reconstituting and Sealing the Sample
Figure 3.8 Assembling and Positioning of the Triaxial Cell and Saturation
Saturation. In order to ensure a high degree of saturation of the samples, carbon dioxide gas is circulated through the sample for approximately one hour for the clean sand samples, and for about two hours for the samples of sand with fines. Then, de-aired water is slowly introduced into the sample (Fig. 3.8c) and after percolation of a sufficient amount of water under very low static height, back pressure of 1.0 kgf/cm$^2$ is applied in increments of 0.025 kgf/cm$^2$, while keeping the effective confining pressure constant at 0.1 kgf/cm$^2$. The sample is then left to saturate overnight. Following the above procedure a $B$-value of at least 0.96 was achieved.

3.4.3 Test Procedures

The experimental program of the present study includes monotonic drained torsional shear tests, cyclic undrained torsional shear tests and monotonic and cyclic undrained torsional simple shear tests. All the tests were performed in a stress-controlled manner following the procedures described below.

Consolidation. In all the drained tests the samples were isotropically consolidated to final mean effective stress of $p' = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0$ or 4.0 kgf/cm$^2$. In the undrained tests, on the other hand, the samples were either isotropically or anisotropically consolidated to a final mean effective stress of $p' = 0.667$ kgf/cm$^2$. The anisotropic consolidation was performed along constant effective stress ratio $K$. In all the tests the consolidation was performed in 9–10 steps and then, the sample was allowed to consolidate for one hour, in the case of clean sand samples, or two hours when sand with fines was tested.

Drained Shearing. After the isotropic consolidation, the samples were sheared under drained conditions by applying torque in small increments. All the normal stress components remained constant during the shearing, i.e. $\sigma'_z = \sigma'_r = \sigma'_\theta = \text{const.}$, and consequently, the mean effective stress was constant at $p' = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0$ or 4.0 kgf/cm$^2$. The samples were sheared until a shear strain of about $\gamma = 15\%$ was attained. Stress changes were not applied until the sample had ceased straining from the preceding stress increment. Depending upon the material, density, mean effective stress level and shear stress level, straining in one increment lasted between 5 and 50 minutes. The stress state in these tests can be summarized as follows:
\[ p' = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0 \text{ or } 4.0 \text{ kgf/cm}^2 = \text{const.} \quad (3.29) \]

\[ q = \tau_{\theta} = \tau, \quad \Delta q = \Delta \tau_{\theta} = \Delta \tau > 0 \quad (3.30) \]

\[ b = 0.5 = \text{const.} \quad (3.31) \]

\[ \beta_\theta = 45^\circ = \text{const.} \quad (3.32) \]

**Undrained Shearing.** After consolidating the samples, torsional shear stress was applied either monotonically or cyclically under undrained conditions. Two types of tests were performed: torsional shear tests with unconstrained lateral deformation during the shearing and torsional shear tests under simple shear condition.

In the undrained torsional shear tests, the shearing was applied by monotonic or cyclic change in the torsional shear stress \( \tau \) while maintaining the normal stress components constant. As regards the testing procedure, the shearing was performed in an identical manner to that used in the drained tests, except for the conditions of drainage. However, the stress components given in Eqs. (3.28) – (3.31) changed during the shearing as a result of the undrained conditions and pore water pressure response.

The simple shear condition is characterized by zero lateral strain increments since it is a plane strain condition and since the normal strain increment in the direction of shearing is zero as well. In the case of undrained shearing, however, since the volume change is zero, vertical strain increment is also zero, and therefore, the shear strain increment \( d\gamma \) is the only non-zero strain increment component, i.e.

\[ de_z = de_r = de_\theta = 0 \quad (3.33) \]

\[ d\gamma \neq 0 \quad (3.34) \]

The simple shear condition during the monotonic and cyclic torsional shearing was achieved as follows:

(a) Since the tests were undrained, the volume of the saturated sample was kept
constant.

(b) By closing the valve of the inner cell the volume of the inner cell was kept constant as well, $\Delta V_i = 0$.

(c) As the torsional shear stress was applied there was a tendency for change in the height of the sample. If there was tendency for the height to decrease then the outer cell pressure was gradually increased in order to prevent any change in the height of the sample. Conversely, when the sample had an expansive tendency, the outer cell pressure was reduced so as to maintain the height of the sample constant. The applied procedure was implemented in small increments. Since the membrane penetration was ignored and slight change in the height of the sample took place during the shearing, the simple shear condition was not fully satisfied. However, these effects on the response are considered to be negligible since the maximum vertical strain was below 0.025 %, and in most of the tests there was reduction in $p'$. It is interesting to note that the inner cell pressure was not controlled and in fact it was measured as a response to the applied loads and deformation conditions. Yet, it was always very close to the outer cell pressure within the ratio $0.95 \leq p_i / p_o \leq 1.05$. A typical record of the fluctuations of the vertical strain as well as effective inner and outer pressures in a cyclic torsional simple shear test are shown in Fig. 3.9.

3.4.4 Repeatability of the Test Results

All the testing procedures described above were performed with utmost care and consistently throughout the whole experimental program. In order to verify the quality of the experimental data, however, many tests were performed under identical conditions, and in fact repeated. Repeatability of the tests results was continuously examined during the whole experimental program. Figs. 3.10 – 3.13 show the comparison of the stress–strain curves and volume change behaviour measured in 'identical' tests conducted for various initial density and stress conditions. As illustrated in these figures, excellent repeatability of the test results have been obtained thus supporting the validity of the tests. In Tables 3.1 – 3.6 identification of the drained and undrained tests performed in this study are listed together with the information about the initial states of the samples, consolidation and loading conditions.
a) Fluctuation of the Vertical Strain $\varepsilon_z$

b) Fluctuation of the Cell Pressure Ratio, $p_i/p_o$

Figure 3.9 Typical Fluctuation of the Vertical Strain and Cell Pressure Ratio $p/p_o$ in the Undrained Torsional Simple Shear Tests
Figure 3.10  Repeatability of the Test Results on Toyoura Sand at $p' = 0.3$ kgf/cm²
Figure 3.11  Repeatability of the Test Results on Toyoura Sand at $p' = 1.0\ \text{kgf/cm}^2$
Figure 3.12  Repeatability of the Test Results on Toyoura Sand at $p' = 2.0 \text{ kgf/cm}^2$
**Figure 3.13** Repeatability of the Test Results on Toyoura Sand at \( p' = 3.0 \ \text{kgf/cm}^2 \)
Table 3.1 Drained Torsional Shear Tests on Toyoura Sand

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_c$</th>
<th>$p'$ (kgf/cm²)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>0.906</td>
<td>1.0</td>
<td>AP</td>
</tr>
<tr>
<td>T2</td>
<td>0.901</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T3</td>
<td>0.897</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
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<td>T4</td>
<td>0.874</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T5</td>
<td>1.006</td>
<td>1.0</td>
<td>Wet Plunging</td>
</tr>
<tr>
<td>T7</td>
<td>0.719</td>
<td>0.3</td>
<td>AP</td>
</tr>
<tr>
<td>T8</td>
<td>0.730</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T9</td>
<td>0.762</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T10</td>
<td>0.899</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T23</td>
<td>0.743</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T24</td>
<td>0.732</td>
<td>3.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T25</td>
<td>0.728</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T26</td>
<td>0.732</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T27</td>
<td>0.877</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T28</td>
<td>0.868</td>
<td>3.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T29</td>
<td>0.800</td>
<td>0.3</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T30</td>
<td>0.804</td>
<td>3.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T31</td>
<td>0.812</td>
<td>1.0</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T32</td>
<td>0.740</td>
<td>2.0</td>
<td>&quot;-&quot;</td>
</tr>
</tbody>
</table>
Table 3.1 Continued

<table>
<thead>
<tr>
<th>Test</th>
<th>(e_c)</th>
<th>(p') (kgf/cm(^2))</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T33</td>
<td>0.739</td>
<td>0.5</td>
<td>AP</td>
</tr>
<tr>
<td>T34</td>
<td>0.727</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T35</td>
<td>0.733</td>
<td>4.0</td>
<td><em>&quot;</em></td>
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<tr>
<td>T36</td>
<td>0.736</td>
<td>5.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T37</td>
<td>0.736</td>
<td>2.0</td>
<td>Unload–Reload Cycle</td>
</tr>
<tr>
<td>T38</td>
<td>0.742</td>
<td>1.0–2.0</td>
<td>Proportional Loading</td>
</tr>
<tr>
<td>T46</td>
<td>0.744</td>
<td>3.0</td>
<td>AP</td>
</tr>
<tr>
<td>T50</td>
<td>0.734</td>
<td>0.3</td>
<td>Used Sand</td>
</tr>
<tr>
<td>T54</td>
<td>0.737</td>
<td>1.0</td>
<td>Cyclic</td>
</tr>
</tbody>
</table>

AP: Air Pluviation
Table 3.2 Drained Torsional Shear Test on Toyoura Sand—TBC 15 %

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_0$</th>
<th>$p'$ (kgf/cm²)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T11</td>
<td>0.785</td>
<td>0.3</td>
<td>WP</td>
</tr>
<tr>
<td>T12</td>
<td>0.733</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T13</td>
<td>0.820</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T14</td>
<td>0.673</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T15</td>
<td>0.640</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T16</td>
<td>0.885</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T17</td>
<td>0.751</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T18</td>
<td>0.709</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T19</td>
<td>0.656</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T20</td>
<td>0.720</td>
<td>3.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T21</td>
<td>0.624</td>
<td>3.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T22</td>
<td>0.680</td>
<td>3.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T39</td>
<td>0.737</td>
<td>2.0</td>
<td>Used Clay</td>
</tr>
<tr>
<td>T40</td>
<td>0.736</td>
<td>1.5</td>
<td>Used Clay</td>
</tr>
<tr>
<td>T41</td>
<td>0.721</td>
<td>0.5</td>
<td>Used Clay</td>
</tr>
</tbody>
</table>

WP: Wet Plunging
Table 3.3  Drained Torsional Shear Test on Toyoura Sand–KAO 15 %

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_c$</th>
<th>$\sigma'$ (kgf/cm²)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T42</td>
<td>0.708</td>
<td>0.5</td>
<td>WP</td>
</tr>
<tr>
<td>T43</td>
<td>0.733</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T44</td>
<td>0.729</td>
<td>0.5</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T45</td>
<td>0.739</td>
<td>1.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T47</td>
<td>0.736</td>
<td>1.0</td>
<td>Unload–Reload Cycle, Swelling Line</td>
</tr>
<tr>
<td>T48</td>
<td>0.735</td>
<td>3.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T49</td>
<td>0.748</td>
<td>3.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T51</td>
<td>0.739</td>
<td>4.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T52</td>
<td>0.733</td>
<td>2.0</td>
<td><em>&quot;</em></td>
</tr>
<tr>
<td>T53</td>
<td>0.752</td>
<td>0.3</td>
<td><em>&quot;</em></td>
</tr>
</tbody>
</table>

WP: Wet Plunging
Table 3.4 Drained Torsional Shear Test on Toyoura Sand—TSS 10 %

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_c$</th>
<th>$p'$ (kgf/cm$^2$)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T59</td>
<td>0.845</td>
<td>0.5</td>
<td>WP</td>
</tr>
<tr>
<td>T60</td>
<td>0.847</td>
<td>4.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T61</td>
<td>0.856</td>
<td>3.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T62</td>
<td>0.864</td>
<td>2.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T63</td>
<td>0.867</td>
<td>1.5</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T65</td>
<td>0.763</td>
<td>0.5</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T67</td>
<td>0.761</td>
<td>1.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>T68</td>
<td>0.761</td>
<td>3.0</td>
<td>-&quot;-</td>
</tr>
</tbody>
</table>

Table 3.5 Drained Torsional Shear Test on Nevada Sand

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_c$</th>
<th>$p'$ (kgf/cm$^2$)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>N1</td>
<td>0.872</td>
<td>1.0</td>
<td>WP</td>
</tr>
<tr>
<td>N2</td>
<td>0.825</td>
<td>1.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>N3</td>
<td>0.757</td>
<td>1.0</td>
<td>-&quot;-</td>
</tr>
<tr>
<td>N4</td>
<td>0.750</td>
<td>0.5</td>
<td>-&quot;-</td>
</tr>
</tbody>
</table>

WP: Wet Plunging
Table 3.6 Undrained Torsional Shear and Torsional Simple Shear Test on Toyoura Sand ($p' = 0.667$ kgf/cm²; Air Pluviation)

<table>
<thead>
<tr>
<th>Test</th>
<th>$e_c$</th>
<th>K</th>
<th>$(\tau/p')_c$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T55</td>
<td>0.755</td>
<td>0.5</td>
<td>0.37</td>
<td>Cyclic Simple Shear</td>
</tr>
<tr>
<td>T56</td>
<td>0.759</td>
<td>0.5</td>
<td>0.30</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T57</td>
<td>0.749</td>
<td>0.5</td>
<td>0.225</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T58</td>
<td>0.801</td>
<td>0.5</td>
<td>0.15</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T64</td>
<td>0.767</td>
<td>0.5</td>
<td>0.225</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T66</td>
<td>0.855</td>
<td>0.5</td>
<td>0.18</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T69</td>
<td>0.861</td>
<td>0.5</td>
<td>0.18</td>
<td>Cyclic</td>
</tr>
<tr>
<td>T70</td>
<td>0.762</td>
<td>0.5</td>
<td>0.225</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>T71</td>
<td>0.899</td>
<td>1.0</td>
<td>0.15–0.30</td>
<td>Double Amplitude</td>
</tr>
<tr>
<td>T72</td>
<td>0.776</td>
<td>1.0</td>
<td>0.225</td>
<td>Cyclic</td>
</tr>
<tr>
<td>T73</td>
<td>0.772</td>
<td>0.75</td>
<td>0.225</td>
<td>&quot;-&quot;</td>
</tr>
<tr>
<td>Test</td>
<td>$e_c$</td>
<td>K</td>
<td>$(\tau/p')_c$</td>
<td>Remarks</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td>---</td>
<td>---------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>T74</td>
<td>0.759</td>
<td>0.5</td>
<td>0.37</td>
<td>Monotonic Simple Shear</td>
</tr>
<tr>
<td>T75</td>
<td>0.751</td>
<td>1.0</td>
<td>-&quot;-&quot;</td>
<td></td>
</tr>
<tr>
<td>T76</td>
<td>0.854</td>
<td>1.0</td>
<td>-&quot;-&quot;</td>
<td></td>
</tr>
<tr>
<td>T77</td>
<td>0.857</td>
<td>0.5</td>
<td>-&quot;-&quot;</td>
<td></td>
</tr>
<tr>
<td>T78</td>
<td>0.776</td>
<td>0.75</td>
<td>0.225</td>
<td>Cyclic Simple Shear</td>
</tr>
<tr>
<td>T79</td>
<td>0.777</td>
<td>0.4</td>
<td>0.225</td>
<td>-&quot;-&quot;</td>
</tr>
<tr>
<td>T80</td>
<td>0.763</td>
<td>0.4</td>
<td>0.225</td>
<td>Cyclic</td>
</tr>
<tr>
<td>T81</td>
<td>0.777</td>
<td>0.9</td>
<td>0.225</td>
<td>Cyclic Simple Shear</td>
</tr>
<tr>
<td>T82</td>
<td>0.864</td>
<td>1.0</td>
<td>0.18</td>
<td>Cyclic Amplitude</td>
</tr>
</tbody>
</table>
3.5 TEST MATERIALS

The tests in this study were conducted on clean Toyoura sand and Toyoura sand with fines contents of 10 – 15 % by weight. The Japanese standard Toyoura sand is classified as a uniform fine sand consisting of subrounded to subangular particles with mostly quartz composition. The physical properties and the grain size distribution curve are given in Table 3.7 and Fig. 3.14, respectively.

The fines used were Isogo clay (Tokyo Bay clay), kaolin and silt obtained from Toyoura sand by crushing it by a ball-mill. The grain size distribution curves and the physical properties of these soils are given in Fig. 3.14, and Tables 3.8 and 3.9.

<table>
<thead>
<tr>
<th>Table 3.7 Physical Properties of Toyoura Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Gravity, $G_s$</td>
</tr>
<tr>
<td>Mean Diameter, $D_{50}$ (mm)</td>
</tr>
<tr>
<td>Uniformity Coefficient, $U_c$</td>
</tr>
<tr>
<td>Maximum Void Ratio, $e_{\text{max}}$</td>
</tr>
<tr>
<td>Minimum Void Ratio, $e_{\text{min}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3.8 Physical Properties of Isogo Clay (Tokyo Bay Clay)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Gravity, $G_s$</td>
</tr>
<tr>
<td>Liquid Limit, $w_l$ (%)</td>
</tr>
<tr>
<td>Plastic Limit, $w_p$ (%)</td>
</tr>
<tr>
<td>Plasticity Index, $I_p$ (%)</td>
</tr>
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</table>
Table 3.9 Physical Properties of Kaolin

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Gravity, $G_s$</td>
<td>2.72</td>
</tr>
<tr>
<td>Liquid Limit, $w_L$ (%)</td>
<td>79.2</td>
</tr>
<tr>
<td>Plastic Limit, $w_p$ (%)</td>
<td>30.6</td>
</tr>
<tr>
<td>Plasticity Index, $I_p$ (%)</td>
<td>48.6</td>
</tr>
</tbody>
</table>

Figure 3.14 Grain Size Distribution Curves
Chapter 4

DRAINED BEHAVIOR OF SANDY SOILS AT CONSTANT MEAN EFFECTIVE STRESS

4.1 INTRODUCTION

The stress–strain and volume change characteristics of sandy soils depend on a number of factors including density, effective stress, shear stress, fabric, strain history, orientation of principal stresses and relative magnitude of intermediate principal stress, among others. Most of these factors are closely related to the essential characteristics of cohesionless soils arising from the fact that, as granular materials, they are composed of solid particles and voids or pores, and that, as frictional materials, they do obey the effective stress principle. From the early studies in soil mechanics special emphasis has, not surprisingly, been placed on the importance of density and effective stress since these two factors are most obviously and primarily related to the above mentioned characteristics of cohesionless soils. Moreover, it has been recognized that in fact the combination of density and effective stress is relevant to the description of soil behavior and that has been incorporated in some fundamental concepts of soil mechanics well known as the critical void ratio concept, the critical state concept and the steady state concept.

The experimental evidence on the behavior of cohesionless soils shows that even the normalized behavior (the behavior represented in terms of either the principal stress ratio or shear stress normalized by the effective stress) is affected by the mean
effective stress, resulting in non-unique stress ratio–shear strain relationship and curved failure surface in the $q-p'$ plane for a given density. In spite of this evidence it is common practice in modelling to either assume a unique stress ratio–shear strain relationship for a given density or to consider the effects of density and effective stress by separate and independent quantities. However, despite the remarkable simplification of the stress–strain and failure surface representation, the first assumption appears to lead to some serious consequences related to the combined effects of density and mean effective stress. In fact, this assumption ignores the importance of the influence of mean effective stress and postulates that density alone is relevant to the description of sand behaviour. In the second approach, on the other hand, due to the decoupling of density and effective stress the combined effects are still not necessarily consistent with the observed behavior.

The two aspects of the behavior of cohesionless soils introduced above, i.e., the combined effects of density and effective stress, and the influence of the effective stress on the normalized behavior are the main subject of interest in this chapter. For this purpose an extensive series of monotonic drained torsional shear tests on hollow cylindrical samples of clean sand and sand with fines is conducted. The tests are planned in order to clarify the interaction between density and effective stress in the range of densities and stresses commonly encountered in the field and eventually to develop a constitutive model based on the observations from these tests. The modelling concept and the evaluation of the model are given in Chapter 6, while a concise two-dimensional formulation of the proposed elastic–plastic constitutive model is presented in Chapter 7. Before introducing the experimental results obtained in the drained torsional shear tests, a brief review of the earlier studies and some information about the drained tests performed in the present study are given.

Most of the experimental data on the influence of the effective stress on the strength and deformation of sandy soils have been obtained from conventional drained triaxial compression tests. In these studies, the behavior of sands has been investigated over a wide range of confining pressures, from an extremely low level of $\sigma'_c = 0.02 \text{ kgf/cm}^2$ (Fukushima and Tatsuoka, 1984), up to a high level of around 100 kgf/cm$^2$ (Lee and Seed, 1967) including very high pressures of several hundreds of kgf/cm$^2$ (Vesic and Clough, 1968). It is interesting to note that in the conventional
drained triaxial compression test, the effective confining stress is constant ($\sigma'_c = \text{const.}$), while the mean principal effective stress increases with the deviator stress ($\Delta p' > 0$, $p' = \text{const.}$).

Besides the conventional triaxial tests, the influence of the effective stress on the behavior of sandy soils has been studied using various testing devices. Thus, in order to reduce errors in testing and improve the accuracy of the measurements, Hettler and Vardoulakis (1984) used a large triaxial apparatus with extremely squat samples with initial height of 28 cm and diameter 78 cm. Tatsuoka et al. (1986c) and Fukushima and Tatsuoka (1982) used drained torsional shear tests on hollow cylindrical samples of Toyoura sand at mean effective stress of $p' = 0.3 - 2.0$ kgf/cm$^2$, and Tatsuoka et al. (1986a) used plane strain compression tests, also on Toyoura sand, to study its strength and deformation characteristics at extremely low pressures of $\sigma'_c = 0.05 - 4.0$ kgf/cm$^2$. They compared favorably the results of the torsional shear tests and plane strain compression tests for nearly identical density, stress and anisotropy conditions, and showed that the dependency of strength and stress–strain relationship on the effective stress was very similar in these tests.

The above mentioned studies, and many others not described in the present thesis, have conclusively shown the influence of the effective stress on the strength and deformation characteristics of sands. However, there are some differences, especially in the experimental evidence and interpretation of the influence of the effective stress on the normalized behavior of sands; there is even some conflicting data (Hettler and Vardoulakis, 1984 and Kolymbas and Wu, 1990).

In most of the above mentioned studies based on drained tests on sandy soils, the influence of the effective stress has been investigated for loose as well as dense sand. Hence, the combined effects of density and effective stress have been perceived and taken into consideration. However, Roscoe and Porooshashb (1963) have most explicitly shown the significance of the combined effects of density and effective stress on the behavior of both clays and cohesionless media. They have illustrated the importance of these effects and the need to appropriately consider them if quantitative results from model test results on soils are required. Based on their theoretical considerations supported by the results of drained simple shear tests on steel balls, they proposed a simple criterion to achieve identical strains in the model.
and prototype. This criterion, which describes the behaviour of the soil based on its initial state, in terms of void ratio and effective stress, relative to the critical state of the soil, particularly emphasizes the importance of the interaction of density and effective stress on the soil behaviour. It is worth noting that this aspect of the behaviour of sandy soils has been of considerable interest in some studies in the framework of the steady state of deformation which rely on experimental results obtained in conventional undrained triaxial compression tests.

With all of this in mind a series of monotonic drained torsional shear tests was conducted on hollow cylindrical samples of Toyoura sand and Toyoura sand with fines content of either 10 or 15% by weight. The fines used were Isogo clay (Tokyo Bay clay), kaolin and Toyoura silt. Accordingly, the following materials were tested:

1. Toyoura sand
2. Toyoura sand with 15% Isogo clay (TBC 15%)
3. Toyoura sand with 15% kaolin (KAO 15%)
4. Toyoura sand with 10% of its silt (TSS 10%)

A detailed description of the materials, sample preparation methods and test procedures used in the monotonic drained torsional shear tests is given in Chapter 3.

Contrary to the conditions in the conventional drained triaxial tests, in the present series of tests the mean effective stress was constant during the shearing \((p' = \text{const.})\). This condition implies two important features of the drained response: firstly, the volume change of the samples during the shearing is due to the dilatancy alone; and secondly, it allows distinction of the effects of void ratio or density, and effective stress in tests with various combinations of the initial density and effective stress. Since the purpose of the tests was to assess the effects of mean effective stress as well as its combined effects with density, the basic principle in the testing was as follows: in several tests to achieve an identical void ratio of the samples after isotropically consolidating them to different final effective stresses, \(p'\); and then, to shear the samples under constant mean effective stress \(p'\). The tests were conducted on clean sand samples with relative density between \(D_r = -7.6\%\) \((e = 1.006)\) and \(D_r = 67.6\%\) \((e = 0.719)\) and on samples of sand with fines with void ratio between \(e = 0.624 - 0.885\) over the range of mean effective stress between 0.3 and 4.0 kgf/cm\(^2\).
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It is interesting to note that in the drained torsional shear tests the relative magnitude of the intermediate principal stress was constant at \( b = 0.5 \), while the direction of the major principal stress relative to the vertical axis was constant at \( \beta_o = 45^\circ \). Hence the effective stress conditions at shearing can be summarized as follows:

\[
p' = \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3} = \text{const.} \tag{4.1}
\]

\[
b = \frac{\sigma'_2 - \sigma'_3}{\sigma'_1 - \sigma'_3} = 0.5 = \text{const.} \tag{4.2}
\]

\[
\beta_o = 45^\circ = \text{const.} \tag{4.3}
\]

In accordance with the purpose of the tests and for ease of comparison the results of the tests are presented in terms of the stress ratio \( \tau/p' \). All the void ratios mentioned below refer to the state of the samples after consolidation, i.e. prior to shearing. It should be emphasized that the results are corrected for the effects of membrane forces as explained in Chapter 3. This correction is considered very important since the amount of correction depends on the effective stress \( p' \), and is quite significant for the tests at low effective stress. In the following, first the results on Toyoura sand and Toyoura sand with fines are presented separately, and the effects of void ratio, or density, and mean effective stress are considered separately as well. Then, the effects of void ratio and effective stress on the drained behavior of sandy soils observed in the present series of tests are summarized and compared to the experimental evidence and conclusions derived from other studies. In doing so special emphasis is placed on the illustration and interpretation of the combined effects of density and mean effective stress on the drained behavior of sandy soils.

4.2 EXPERIMENTAL RESULTS ON TOYOURA SAND

A series of monotonic drained torsional shear tests was conducted on samples of Toyoura sand prepared by the method of air pluviation as described in Chapter 3.
The tests were performed on samples with void ratios ranged between \( e = 0.719 - 0.906 \) at mean effective stress of \( p' = 0.3, 0.5, 1.0, 2.0, 3.0 \) and \( 4.0 \) kgf/cm\(^2\). Fig. 4.1 shows the states of the samples of Toyoura sand after consolidation, i.e. prior to shearing, in terms of void ratio and mean effective stress together with the designation of the tests. In this figure, consolidation paths of two tests (T30 and T35) are shown by the solid lines in order to illustrate how the displayed initial states were achieved. Numerical values of void ratios of the samples and mean effective stresses after consolidation are listed in Table 3.1. It is to be noted that tests T5, T38 and T50 differ from the 'standard test' described above either in the sample preparation, loading condition or material used in the test. Thus, in test T5 the sample was prepared by the moist placement method in order to get a very loose sample; in test T38 proportional loading was applied to investigate the effects of mean effective stress in loading condition other than \( p' \) constant; and in test T50 the sample was prepared with used sand instead of fresh sand so as to quantify the effects of crushing of particles in the observed behaviour. In order to check the reproducibility of the samples and test results a number of tests were performed for practically identical initial conditions as discussed in Chapter 3. These tests are also indicated in Fig. 4.1. Since the behaviour in the tests under identical conditions was actually the same, from each group of the reproducibility tests the results of only one test are presented and discussed.

### 4.2.1 The Effects of Void Ratio

The effects of void ratio on the drained torsional shear behavior of Toyoura sand will be illustrated by the results of the tests conducted on samples with void ratio about \( e = 0.74, 0.80 \) and \( 0.90 \). In order to demonstrate the effects of void ratio at different effective stresses, three sets of results for the tests at mean effective stress of \( p' = 0.3, 1.0 \) and \( 3.0 \) kgf/cm\(^2\), respectively, are presented. Thus the relationship between stress ratio \( \tau/p' \), shear strain \( \gamma \) and volumetric strain \( \varepsilon_v \), in the tests at \( p' = 0.3 \) kgf/cm\(^2\) are shown in Fig. 4.2. These data illustrate the effects of void ratio on the stress–strain and volume change characteristics at low mean effective stress. As shown in Fig. 4.2, as the void ratio increases, or the density decreases, the stress–strain curve becomes flatter and the sand becomes more compressible. Hence, the stress–strain curve for the loosest sample with void ratio \( e = 0.899 \) is the softest, and
Figure 4.1 States of the Samples Prior to Shearing
the corresponding strength is the lowest. The volumetric strain in this test is almost entirely contractive. On the other hand, the stiffest response with the highest strength is obtained for the densest sand with void ratio $e = 0.728$. Contrary to the volume change behavior of the loose sand, volumetric strain of the dense sand is characterized by strong dilation. It is to be noted that since the tests are carried out at constant mean effective stress, the volumetric strain is due to the dilatancy alone.

The results of a similar series of tests carried out at somewhat higher mean effective stresses of $p' = 1.0$ kgf/cm$^2$ and $p' = 3.0$ kgf/cm$^2$ are shown in Figs. 4.3 and 4.4, respectively. The pattern is similar to that observed in the tests at low mean effective stress, except that as the mean effective stress increases, the stress–strain curve tends to be flatter with lower peak stress ratio, while there is a greater tendency to compress. Thus, the dilation of the samples in the tests at $p' = 3.0$ kgf/cm$^2$ is not very pronounced even for the dense sand with void ratio $e = 0.744$.

The dilatancy characteristics of the tests discussed above are illustrated by the stress–dilatancy relationship in Fig. 4.5. In this relationship the point where $d\varepsilon_v = 0$, i.e. $-d\varepsilon_v/d\tau = 0$, corresponds to the phase transformation (Ishihara et al., 1975), and represents the state where the volumetric strain changes from contractive to expansive. It is to be noted that the stress–dilatancy relationship is expressed in terms of the plastic shear strain increment $d\gamma_p$, and since the volumetric strain in these tests is due to the dilatancy alone, it represents the characteristics of the plastic response of Toyoura sand. The general tendency of the stress–dilatancy relationship is to move upwards as the void ratio increases, or the density decreases. Hence, the stress ratio at the phase transformation $(\tau/p')_{PT}$ increases with increasing void ratio $e$. However, it appears that the effect of void ratio on the stress–dilatancy relationship is small for the stress ratio $\tau/p'$ greater than 0.2. Below this stress level there is considerable scattering of the data, and therefore, it is difficult to analyze the effects of density on the stress–dilatancy relationship at low shear stress level.

Finally, the effects of void ratio on the principal strains are illustrated in Fig. 4.6. The data presented in this figure were obtained in the tests at $p' = 1.0$ kgf/cm$^2$. A similar pattern, however, was observed in the tests with other mean effective stresses. As shown in Fig. 4.6 the void ratio affects only the magnitude of the principal strains, while the tendency for either contraction or expansion of the
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Figure 4.2 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand at $p' = 0.3$ kgf/cm²
Figure 4.3 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand at $p' = 1.0 \text{ kgf/cm}^2$
Figure 4.4 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand at $p' = 3.0 \text{ kgf/cm}^2$
Figure 4.5 Influence of Density on the Stress–Dilatancy Relationship of Toyoura Sand at $p' = 0.3$, 1.0 and 3.0 kgf/cm$^2$. 

**a)** Toyoura Sand 
$p' = 0.30$ kgf/cm$^2$ 
- $e = 0.728$ (T25) 
- $e = 0.860$ (T29) 
- $e = 0.889$ (T19) 

**b)** Toyoura Sand 
$p' = 1.0$ kgf/cm$^2$ 
- $e = 0.732$ (T26) 
- $e = 0.812$ (T31) 
- $e = 0.906$ (Tl) 

**c)** Toyoura Sand 
$p' = 3.0$ kgf/cm$^2$ 
- $e = 0.744$ (T46) 
- $e = 0.804$ (T30) 
- $e = 0.868$ (T28)
Figure 4.6 Influence of Density on the Stress-Ratio-Principal Strain Relationship of Toyoura Sand at $p' = 1.0$ kgf/cm².
principal strain components is not influenced by the change in the void ratio. The major and the intermediate principal strains, \( e_1 \) and \( e_2 \) are all positive, indicating contractive behavior, while the minor principal strains, \( e_3 \) are all negative, and are thus related to a dilative behavior. As expected, for a certain stress ratio \( \tau / p' \) the principal strains increase with increasing void ratio, or decreasing density.

4.2.2 The Effects of Mean Effective Stress

In a similar manner to the analysis of the effects of void ratio, the effects of mean effective stress are illustrated through a comparison of the results obtained in the tests at different mean effective stress, on samples with the same void ratio prior to shearing. The influence of the mean effective stress is demonstrated for four densities of the samples with the corresponding void ratios around \( e = 0.74, 0.80, 0.87 \) and 0.90. In addition to the results of the 'standard \( p' \) constant tests', the results of two special tests related to the proportional loading condition and crushing of the particles are presented as well.

The stress–strain curves and the volumetric strain–shear strain relationships for dense sand, with void ratio \( e = 0.74 \), obtained in the tests at mean effective stress of \( p' = 0.3, 0.5, 1.0, 2.0, 3.0 \) and 4.0 kgf/cm\(^2\) are shown in Fig. 4.7. As these data show, the influence of the effective stress is significant both on deformation and strength of the sand. The stiffness of the response decreases with increasing mean effective stress, causing a downward shift of the stress–strain curve associated with a reduction of its maximum stress ratio. As a result of this pattern in the stress–strain behaviour, the stiffest stress–strain curve with the highest strength is obtained for the test at the lowest mean effective stress of \( p' = 0.3 \) kgf/cm\(^2\), while the softest stress–strain behaviour associated with the lowest strength is observed in the test with the highest effective stress of \( p' = 4.0 \) kgf/cm\(^2\). The volumetric strain–shear strain relationship is characterized by a gradual change in the behavior, from being expansive with strong dilation for low mean effective stress to being compressive with weak dilation when the mean effective stress is high.

The corresponding results of the tests on looser samples with void ratio \( e = 0.80, 0.87 \) and 0.90 are shown in Figs. 4.8 – 4.10. The sand is less stiff with lower peak stress ratio and more compressible as the mean effective stress increases. Thus, in
Figure 4.7 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand with Void Ratio $e = 0.74$
Figure 4.8 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand with Void Ratio $e = 0.80$
Figure 4.9 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand with Void Ratio $e = 0.87$
a) Stress-Strain Characteristics

b) Volume Change Behaviour

Figure 4.10 Influence of Mean Effective Stress on the Stress-Strain and Volume Change Behaviour of Toyoura Sand ($e = 0.90$)
general, the pattern is similar to that observed in the tests on dense sand. However, it should be pointed out that as void ratio increases the stress–strain curve becomes flatter and the volumetric strain is more contractive. Hence, the peak strength as represented by the peak stress ratio decreases with increasing void ratio, and the volumetric strain is entirely contractive for \( e = 0.90 \) even when \( p' \) is as low as 0.3 kgf/cm\(^2\) (Fig 4.10b). Another important observation is that the influence of the effective stress on the stress–strain behaviour decreases with increasing void ratio, or as the sand becomes looser.

Stress–dilatancy relationships of the tests discussed above are shown in Figs. 4.11 and 4.12. Disregarding the low stress level because of the scattering of the data when \( \tau/p' \) is smaller than 0.7, a unique stress–dilatancy relationship is obtained in the tests at the same void ratio, thus demonstrating that this relationship is independent of mean effective stress over the range of densities of Toyoura sand considered in the present study.

The influence of the mean effective stress on the principal strains is illustrated in Fig. 4.13. This figure shows the relationship between stress ratio and principal strain components obtained for samples with void ratio \( e = 0.74 \) at mean effective stress of \( p' = 0.3, 1.0 \) and 4.0 kgf/cm\(^2\). Apparently, the effective stress affects only the magnitude of the principal strains, while the tendency for contraction (\( e_1 \) and \( e_3 \)) or expansion (\( e_2 \)) of the principal strains is not influenced. In accordance with the previous observations, for a certain stress ratio, \( \tau/p' \), the magnitudes of the principal strains increase with the mean effective stress, \( p' \).

Since the tests were performed in a stress–controlled manner and the shearing was terminated if a shear band was observed, there is no peak and post–peak strain softening portion in the stress–strain curves. Therefore, the stress–strain curves are characterized by continuous increase of the stress ratio \( \tau/p' \) with their peak values attained at the end of the tests. For this reason, the effects of the effective stress on the strength are illustrated by the relation between the mobilized angle of internal friction, defined as:

\[
\phi_{mob} = \sin^{-1} \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3} \quad (4.4)
\]
Figure 4.11 Influence of Mean Effective Stress on the Stress–Dilatancy Relationship of Toyoura sand with Void Ratio $e = 0.74$ and $0.80$
Figure 4.12  Influence of Mean Effective Stress on the Stress–Dilatancy Relationship of Toyoura Sand with Void Ratio $e = 0.87$ and 0.90
Figure 4.13 Influence of Mean Effective Stress on Stress Ratio–Principal Strain Relationship of Toyoura Sand ($e = 0.74$)

- $e_1$, $e_2$, $e_3$ (%)
- Stress Ratio $t / p$
- Toyoura Sand $e = 0.74$
- $p' = 0.3$ kgf/cm² (T25)
- $p' = 1.0$
- $p' = 4.0$
Figure 4.14 Dependence of $\phi_{mob}$ on Mean Effective Stress at Different Shear Strain
and the mean effective stress, $p'$, at equal shear strains $\gamma$, as shown in Fig. 4.14. The data shown in Fig. 4.14 were obtained from tests on dense samples with void ratio $e = 0.74$. Apparently, $\phi_{mob}$ decreases with increasing $p'$ in identical manner for all equal shear strains, indicating that the influence of the effective stress is similar over the whole considered strain range. Relatively sharp change in the slope of the relationship in the range of $p' = 0.3 - 0.5$ kgf/cm$^2$ might be attributed to the uncertainties in the tests at very low $p'$, arising from the nonuniformity in the stress distribution as a result of the self-weight of the sample and due to the large corrections for the effects of the membrane forces.

**Proportional loading test.** In addition to the $p'$ constant tests a proportional loading test was carried out in order to assess the effects of the mean effective stress under conditions other than $p'$ constant. Proportional loading is defined as a loading condition where the stress ratio $\tau/p'$ is held constant while the magnitudes of $\tau$ and $p'$ increase or decrease. The effective stress path and stress–strain curve of the proportional loading tests is shown by the solid lines and filled points in Figs. 4.15a and 4.15b, respectively. Fig 4.15b also shows the stress–strain curves obtained in the $p'$ constant tests at $p' = 1.0$ and 2.0 kgf/cm$^2$ (dashed lines). All the tests mentioned above were conducted on samples with void ratio $e = 0.74$.

The proportional loading test was carried out as follows: after isotropically consolidating the sample to 1.0 kgf/cm$^2$, the sample was sheared under constant effective stress until a stress ratio of $\tau/p' = 0.46$ was reached (path 1–2). Then, an unload–reload cycle was performed by decreasing the shear stress $\tau$ to zero and increasing it back to $\tau/p' = 0.46$, while $p'$ was held constant at 1.0 kgf/cm$^2$ (paths 2–3 and 3–4). After that, proportional loading was applied by simultaneously increasing the shear stress $\tau$ and effective stress $p'$ until $p'$ reached 2.0 kgf/cm$^2$ (path 4–5), followed by a simultaneous decrease in both $\tau$ and $p'$ back to the initial effective stress of $p' = 1.0$ kgf/cm$^2$ (path 5–6). During the proportional loading the stress ratio was kept constant at $\tau/p' = 0.46$. Finally, the test was concluded by decreasing the shear stress $\tau$ to zero under constant effective stress of $p' = 1.0$ kgf/cm$^2$ (path 6–7).

It is interesting to note that the shear strain obtained during the proportional loading, where $p'$ was increased from 1.0 to 2.0 kgf/cm$^2$ while maintaining constant stress ratio of $\tau/p' = 0.46$, is nearly equal to the shear strain difference between the
Figure 4.15 Applied Effective Stress Path and Measured Stress–Strain Behaviour of Toyoura Sand in Proportional Loading Test
stress–strain curves obtained in the $p'$ constant tests at $p' = 1.0$ and $2.0$ kgf/cm$^2$. In other words, during proportional loading while increasing $p'$ from 1.0 to 2.0 kgf/cm$^2$, the stress point gradually shifted from the stress–strain curve of the $p'$ constant test at $p' = 1.0$ kgf/cm$^2$ to the corresponding stress–strain curve for $p' = 2.0$ kgf/cm$^2$. This demonstrates a similar effect of $p'$ on the stress–strain behaviour in proportional loading and $p'$ constant loading condition.

**Particle crushing.** It is well known that when sands are subjected to high effective stresses, crushing of particles takes place and this in turn affects their behaviour. Therefore, it is necessary to check whether crushing of particles, if any, affects the observed behaviour in the present series of tests. For that purpose, test T50 was performed on a sample prepared from the sand previously used in test T28 which was performed at relatively high effective stress of $p' = 3.0$ kgf/cm$^2$. The test on used sand (T50) was performed for identical conditions to the tests using fresh sand T7, T8 and T25, i.e., on samples with void ratio $e = 0.74$ at mean effective stress of $p' = 0.3$ kgf/cm$^2$. Thus, any possible difference between the observed behaviour in the test on used sand (T50) and corresponding tests on fresh sand (T7, T8 and T25) should be attributed to the crushing of particles.

In Fig. 4.16 the stress–strain and volume change behaviour observed in the test on used sand is compared to that observed in the tests on fresh sand. Apparently, identical behaviour is obtained in the tests on used and fresh sand with a unique stress–strain curve and very similar volumetric strains. It is interesting to note that even the small difference in the volume change behaviour can not be attributed to the effects of particle crushing. Namely, the volumetric strain of the sample of used sand is slightly more dilative instead of being more contractive due to the increased compressibility caused by the possible crushing of particles. The above observations clearly indicate that the observed behaviour in the drained torsional shear tests is free of the effects of particle crushing. In further support of this indication is the fact that the maximum mean effective stress used in the present series of tests of $p' = 4.0$ kgf/cm$^2$ is far too small to cause significant crushing of particles of Toyoura sand.
Figure 4.16 Comparison of the Stress–Strain and Volume Change Behaviour from Tests on Fresh and Used Toyoura Sand at $p' = 3.0 \text{ kgf/cm}^2$
4.3 EXPERIMENTAL RESULTS ON TOYOURA SAND WITH FINES

In addition to the tests performed on clean Toyoura sand a series of tests was conducted on samples of Toyoura sand with fines content either of 10 or 15% by weight. The samples were prepared by the method of wet plunging as described in Chapter 3. Fig. 4.17 shows the states of the samples of Toyoura sand with fines in terms of the void ratio and mean effective stress attained after consolidation together with the designation of the tests. As illustrated in Fig. 4.17 the tests were performed on samples with void ratio ranged between $e = 0.624 - 0.885$ at mean effective stress of $p' = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0$ and $4.0$ kgf/cm$^2$. The results of the drained torsional shear tests conducted on Toyoura sand with 15% Isogo clay (Toyoura sand–TBC 15%), Toyoura sand with 15% kaolin (Toyoura sand–KAO 15%) and Toyoura sand with 10% of its silt (Toyoura sand–TSS 10%) are presented and discussed in the following.

4.3.1 Toyoura Sand–TBC 15%

The effects of void ratio are illustrated by the results of three sets of tests carried out at the effective stress of $p' = 0.3, 1.0$ and $3.0$ kgf/cm$^2$, respectively, on samples with various densities. Fig. 4.18 shows the stress–strain and volume change behaviour obtained in the tests at low effective stress of $p' = 0.3$ kgf/cm$^2$ for samples with void ratios ranged between $e = 0.656 - 0.885$. As shown in Fig. 4.18, there is gradual decrease in the stiffness of the stress–strain curve and peak stress ratio with increasing void ratio, or decreasing density. Hence, the stiffest response with the highest peak stress ratio is obtained for the densest sample with void ratio $e = 0.656$. As void ratio increases, or density decreases the stress–strain curve gradually shifts downwards and eventually reaches the lowest stress–strain curve which is obtained in the test on the loosest sample with void ratio $e = 0.885$. Volume change behaviour is characterized by predominantly contractive behaviour of the samples with increasing compressibility as density decreases. It is interesting to notice that nearly identical stress–strain and volume change characteristics are obtained in the tests on loose samples with void ratio between $e = 0.785 - 0.885$. A similar pattern in the stress–strain as well as volume change behaviour is observed in the tests at higher effective stresses of $p' = 1.0$ and $3.0$ kgf/cm$^2$, as illustrated in Figs. 4.19 and 4.20.
Figure 4.17 States of the Samples Prior to Shearing (Initial States)
Figure 4.18 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TBC 15% at $p' = 0.30$ kgf/cm$^2$
Figure 4.19 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TBC 15% at $p' = 1.0 \text{ kgf/cm}^2$
Figure 4.20 Influence of Density on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TBC 15% at $p' = 3.0$ kgf/cm$^2$
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It is interesting to note that the effects of void ratio have an identical pattern in the behaviour to that observed in tests on clean Toyoura sand. However, the effects of fines are clearly manifested by the increased compressibility of the samples of sand with fines and reduced stiffness of the stress–strain curve with lower peak stress ratio as compared to the corresponding behaviour of clean Toyoura sand.

To demonstrate the effects of mean effective stress the results of tests T12, T6 and T20 will be used. These tests were conducted on samples with void ratio \( e = 0.73 \) at effective stress of \( p' = 0.3 \), 1.0 and 3.0 kgf/cm\(^2\). The stress–strain and volume change characteristics observed in these tests are shown in Fig. 4.21. Similarly to the behaviour of clean Toyoura sand, the stress–strain curve is less stiff with lower peak stress ratio and the samples are more compressible, as the mean effective stress increases. It is to be noted, however, that the influence of the effective stress on the stress–strain behaviour is less pronounced than in the case of clean Toyoura sand. This is reflected in the smaller differences of the stress–strain curves for the effective stresses ranged between \( p' = 0.3 – 3.0 \) kgf/cm\(^2\).

The dilatancy characteristics of Toyoura sand with 15% Isogo clay are illustrated by the stress–dilatancy relationship in Fig. 4.22. Here Fig. 4.22 shows the results of all the tests performed on the samples of Toyoura sand–TBC 15% with void ratios between \( e = 0.624 – 0.885 \) and mean effective stress over the range of 0.3 to 3.0 kgf/cm\(^2\). Even though the density of the samples and the applied effective stress vary over wide ranges, a unique stress–dilatancy relationship is obtained. These data demonstrate that neither the void ratio nor the effective stress affects the stress–dilatancy relationship of Toyoura sand with 15% Isogo clay.

In general, the observed behaviour of Toyoura sand with 15% Isogo clay is remarkably similar to that observed in the tests on clean Toyoura sand. Yet, it is important to recognize the effects of fines on both stress–strain and volume change characteristics, which are notably present in the behaviour of Toyoura sand with 15% Isogo clay. These effects are manifested through the decreased stiffness and peak stress ratio of the stress–strain curve and increased compressibility of the volumetric strain. In fact, the results of Toyoura sand–TBC 15% with void ratio \( e = 0.73 \) are not unlike the results of loose Toyoura sand with void ratio about \( e = 0.90 \). It will be demonstrated in Chapter 6 that the similarity in the behaviour regarding the effects of
Figure 4.21 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TBC 15% ($e = 0.73$)
Figure 4.22 Stress-Dilatancy Relationship of Toyoura Sand – TBC 15% for Different Initial States

Toyoura Sand – TBC 15%

\( p' = 0.3 \text{ kgf/cm}^2 \)

\( e = 0.624 - 0.885 \)

\( \frac{d}{d\gamma} \)
density and effective stress is related to the initial state of the soil relative to its reference state defined in the $e-p'$ plane. Since each soil has distinct reference state in the $e-p'$ plane, similarity in the behaviour of different soils is associated with different initial states of these soils.

4.3.2 Toyoura Sand–KAO 15%

The characteristics of the drained torsional shear behaviour of Toyoura sand with 15% kaolin are illustrated by the results of the tests conducted on samples with void ratio $e = 0.73$ at mean effective stress of $p' = 0.5$, 1.0, 2.0, 3.0 and 4.0 kgf/cm$^2$. Stress–strain curves, volume change characteristics and stress–dilatancy relationships of these tests are shown in Fig. 4.23.

Despite the variation of the mean effective stress over a wide range from 0.5 to 4.0 kgf/cm$^2$, a unique stress–strain relationship is obtained. In other words, the stress–strain behaviour of Toyoura sand with 15% kaolin is independent of the effective stress for the considered density and effective stresses. Hence, unlike the stress–strain characteristics of Toyoura sand and Toyoura sand–TBC 15%, there is no reduction in the stiffness and peak stress ratio of the stress–strain curve of Toyoura sand–KAO 15% with increasing mean effective stress. However, regarding previous findings that the influence of the effective stress on the stress–strain behaviour is less pronounced as the density decreases, and that the effects of fines are roughly equivalent to the reduction in density, the observed behaviour of Toyoura sand with 15% kaolin in fact follows the pattern observed in the other materials. As regards the influence of the effective stress on the stress–strain behaviour, this behaviour should be considered as a limit case. Similar behaviour of Toyoura sand with 10 and 20% kaolin has been obtained by Tsujino (1992) in drained triaxial compression tests at constant confining pressure of 0.5, 1.0 and 1.5 kgf/cm$^2$.

Similarly, nearly unique volumetric strain–shear strain relationship is obtained for this tests, with small deviation of the test at $p' = 4.0$ kgf/cm$^2$, thus illustrating that volume change behaviour is hardly affected by the mean effective stress. Although a slight tendency for downward shift of the stress–dilatancy relationship is associated with the increase in the effective stress, the stress–dilatancy relationship can be in fact considered as unique and independent of the effective stress $p'$. 
Figure 4.23 Influence of Mean Effective Stress on Behaviour of Toyoura Sand–KAO 15% with Void Ratio \( e = 0.73 \)
4.3.3 Toyoura Sand–TSS 10%

The effects of mean effective stress on the stress–strain and volume change behaviour of Toyoura sand with 10% of its silt are illustrated with two sets of data obtained in tests on samples with void ratios \( e = 0.76 \) and \( e = 0.86 \).

Fig. 4.24 shows the results of the tests on the medium dense samples with void ratio \( e = 0.76 \) at mean effective stress of \( p' = 0.5, 1.0 \) and \( 3.0 \) kgf/cm\(^2\). As illustrated in this figure, the stress–strain curve is less stiff and the peak stress ratio decreases with increasing mean effective stress, \( p' \). As regards volume change behaviour, it is characterized by contractive response with greater compressibility as the effective stress increases. In general, the pattern in the behaviour is similar to that observed in the tests on clean Toyoura sand and Toyoura sand with 15% Isogo clay.

The results of the tests on the looser samples with void ratio \( e = 0.86 \) at mean effective stress of \( p' = 0.5, 1.5, 2.0, 3.0 \) and \( 4.0 \) kgf/cm\(^2\) are illustrated in Fig. 4.25. Except for the test at \( p' = 0.5 \) kgf/cm\(^2\), a unique stress–strain curve is obtained. It is interesting to note that the above mentioned test at \( p' = 0.5 \) kgf/cm\(^2\) is conducted on a slightly denser sample with void ratio \( e = 0.84 \). Therefore, it may be concluded that the stress–strain relationship of Toyoura sand–TSS 10% with void ratio \( e = 0.86 \) is hardly affected by the effective stress. A similar conclusion can be derived for the influence of the effective stress on the volumetric strain. Unlike the results on the denser samples with void ratio \( e = 0.76 \), the behaviour of loose Toyoura sand–TSS 10% \( (e = 0.86) \) is remarkably similar to that observed for Toyoura sand–KAO 15%.

The initial density and effective stress state of the samples of Toyoura sand with 10% of its silt were selected in order to encompass the typical stress–strain behaviour of Toyoura sand and Toyoura sand with fines in tests on same material. Thus, the results on Toyoura sand–TSS 10% envisage the link between the observations on different materials and additionally illustrate the importance of the effects of the interaction between the density and effective stress on the drained behaviour of sandy soils.
Figure 4.24 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TSS 10% with Void Ratio \( e = 0.76 \)
Figure 4.25 Influence of Mean Effective Stress on the Stress–Strain and Volume Change Behaviour of Toyoura Sand–TSS 10% with Void Ratio $e = 0.86$
4.4 COMBINED EFFECTS OF DENSITY AND MEAN EFFECTIVE STRESS ON DRAINED BEHAVIOUR OF SANDY SOILS

The experimental results presented and discussed in the preceding pages have shown that the drained torsional shear behaviour of Toyoura sand including fines content up to 15% is affected by the mean effective stress and density in a number of ways. A detailed description of many results with similar patterns of the behaviour was given in order to illustrate even the slight tendencies in the stress–strain and volume change characteristics of the tested soils, over a wide range of densities and effective stresses. The usage of several different materials makes the study more comprehensive, and as will be shown in the following chapters, especially contributes to the widening of the initial relative states of the samples considered in the present study.

A method of quantification of the effects of density and mean effective stress on the stress–strain behaviour of sandy soils will be presented in Chapter 6 where a stress–strain model with density–stress dependent parameters will be developed. Herein the experimental evidence on the effects of density and effective stress obtained in the present study will be summarized and compared to the results and conclusions of other studies. In doing so the emphasis will be placed on the combined effects of density and effective stress on the behaviour of sandy soils and on the necessity to consider the effects of the interaction between them in order to encompass the various stress–strain characteristics associated to different initial states.

One of the common conclusions derived from drained, as well as undrained studies, is that, as the effective stress increases the stress–strain characteristics gradually change from those typical of dense sand to those typical of loose sand. Thus, even dense sand under high effective stress behaves similarly to loose sand. This observation is illustrated in Fig. 4.26 by the typical stress–strain curves of Toyoura sand obtained in the present study. The initial states of the samples corresponding to these curves are illustrated schematically on the e–p' diagram in the same figure. The effects of density and effective stress on the stress–strain behaviour can be summarized as follows:
As void ratio increases, or density decreases, the stiffness and maximum stress ratio of the stress–strain curve decrease. Thus, in Fig. 4.26 the stress–strain curves of the dense sand, are always above the corresponding curves of the loose sand.

Similarly, the stiffness of the stress–strain curve as well as the peak stress ratio decrease with increasing mean effective stress. This is illustrated in Fig. 4.26 by the fact that the stress–strain curves for the tests at high effective stress are always below the corresponding curves of the test at low effective stress. It is to be noted that the change is gradual as indicated by the intermediate position of the curves for the test at the corresponding intermediate effective stress.

The effects of void ratio become less pronounced with increasing mean effective stress. Thus, the difference between the stress–strain curves of the dense and loose sand decreases as the mean effective stress increases.

Similarly, the effects of mean effective stress are less pronounced for loose sand, than for dense sand. This is demonstrated in Fig. 4.26 by the smaller difference between the stress–strain curves of loose sand than the difference between the corresponding curves of the dense sand.

The above observations indicate similarity in the effects of density and mean effective stress on the stress–strain behaviour of sandy soils. Namely, the increase of the mean effective stress affects the stress–strain behaviour in identical pattern as the increase of the void ratio, or the decrease of the density. It will be demonstrated in Chapter 6 that the magnitudes of these effects depend on the initial state of the soil relative to its reference state in the $e-p'$ plane.

With respect to the described characteristics of the stress–strain behaviour it is reasonable to expect similar stress–strain curves for certain combinations of void ratio and mean effective stress. This interesting feature of the stress–strain behaviour is illustrated in Fig. 4.27. Here Fig. 4.27 shows two unique stress–strain curves obtained
Figure 4.26  Combined Effects of Density and Mean Effective Stress on the Stress–Strain Behaviour of Toyoura Sand
in five tests with quite different initial states. The initial states of the tests are schematically indicated on the $e-p'$ diagram shown in the same figure. As can be seen, identical stress–strain curves are obtained in the tests on loose sand at low effective stress and on dense sand at high effective stress. The tendency of the stress–strain behaviour to become similar to that typical for loose sand, with increasing effective stress, is indicated by the gradual change of the initial states of the samples with increasing density and mean effective stress, but still maintaining the same stress–strain characteristics.

The effects of density and mean effective stress on the strength of Toyoura sand, as represented by the peak stress ratio, are shown in Fig. 4.28 on a three-dimensional stress ratio–void ratio–mean effective stress plot ($\tau/p'-e-p'$ plot). Since the peak stress ratio is in fact the end point of the stress–strain curve, each of the conclusions derived for the stress–strain curve holds valid for the strength as well. It is clearly demonstrated in this plot that the effects of void ratio and effective stress are most significant for dense sand under low effective stress. On the other hand, the effects of density and effective stress become negligible as the sand gets looser and the effective stress increases.

As regards the volume change characteristics, the following conclusions can be drawn:

- As void ratio increases the volumetric strain becomes more contractive with very weak dilation, if any. An identical pattern in the volume change behaviour is observed when the mean effective stress increases.

- The effects of density and effective stress are less pronounced for loose sand and high effective stress.

- In general, the stress–dilatancy relationship is hardly affected by density and mean effective stress. Though slight effects of density on the stress–dilatancy relationship of Toyoura sand are observed, a unique stress–dilatancy relationship, independent of $e$ and $p'$, is obtained for Toyoura sand–TBC 15%.
Figure 4.27  Similar Stress–Strain Behaviour for Samples of Toyoura Sand with Different Initial States
• The stress-dilatancy relationships of Toyoura sand and Toyoura sand with 10–15 % fines content are not the same. There is difference among these relationships depending on the particular fines used as well.

The above observations and conclusions regarding the effects of density and mean effective stress on strength and deformation characteristics of sandy soils are in close agreement with the findings presented in some other studies (Tatsuoka et al., 1986a, Tatsuoka et al. 1986c, Fukushima and Tatsuoka, 1984, Kolymbas and Wu, 1990 and Lee and Seed, 1967 among others).

Fig. 4.29 shows the dependence of the angle of internal friction (\( \phi \)) on the relative density \( (D_r) \) and the confining pressure \( (\sigma_c) \) obtained in drained triaxial compression tests on Karlsruhe sand by Kolymbas and Wu, 1990. Apparently, the behaviour is remarkable similar to that observed in the present study, as illustrated in Fig. 4.28.

A summary plot of the dependence of the angle of internal friction of Toyoura sand obtained in drained plane strain compression tests, triaxial compression tests and torsional shear tests by Tatsuoka and his coauthors (Tatsuoka et al., 1986a, Fukushima and Tatsuoka, 1984 and Tatsuoka et al., 1986c) is shown in Fig. 4.30. These data show that the influence of the effective stress on the strength of Toyoura sand is less pronounced for the looser sand with void ratio \( e = 0.8 \). However, since the difference of the void ratio is not significant \( (e = 0.70 \) and \( e = 0.80 \) this influence is not remarkable.

Finally, the data compiled by Lade (1978) from several studies on different cohesionless soils is presented in Fig. 4.31. This figure shows the relationship between the exponential constant, \( m \) of the failure surface of Lade’s model (Lade, 1977) and relative density, \( D_r \), or in other words, the relationship between the relative density and curvature of the failure surface. Thus as \( m \) increases the curvature of failure surface increases, and hence the influence of the effective stress on the strength is more pronounced. On the other hand, the decrease of \( m \) is associated with reduced curvature of the failure surface or less pronounced influence of the effective stress with its limit state for \( m = 0 \) when the failure surface is straight and the stress ratio
Figure 4.28  Summarized Effects of Density and Mean Effective Stress on the Peak Stress Ratio of Toyoura Sand in Drained Torsional Shear Tests

Figure 4.29  Dependence of Friction Angle on Relative Density and Confining Stress, Karlsruhe Sand (after Kolumbas and Wu, 1990)
Figure 4.30 Dependence of Angle of Internal Friction on Mean Effective Stress of Toyoura Sand in Plane Strain Compression, Triaxial Compression and Torsional Shear (after Tatsuoka et al., 1986a)

Figure 4.31 Dependence of the Curvature of Failure Surface on the Relative Density for Sands (data from Lade, 1978)
at failure is in fact independent of the mean effective stress.

Lade compiled data on 15 different cohesionless soils for which \( m \) varied in the range between \( m = 0 - 0.435 \). In Fig. 4.31 the relationships of four sands are shown, since these data contain information for the parameter \( m \) at different relative densities. Apparently, as relative density decreases the curvature of the failure surface decreases as well. In other words, the influence of the effective stress decreases as the sand becomes looser; this is in accordance with the conclusions derived in the present study. It is worth noting that from the data including 15 different materials with various densities, in only one case does \( m \) equals zero (apparently shown in Fig. 4.31). This indicates the difficulty of achieving, with most clean sands, the limit state where \( m = 0 \) or where the stress ratio at failure is independent of the effective stress; this limit state was observed in the present study in the tests on Toyoura sand–KAO 15\% and Toyoura sand–TSS 10\%.

4.5 SUMMARY

A series of drained torsional shear tests was conducted on Toyoura sand and Toyoura sand with 10–15\% fines content by weight in order to study the effects of density and mean effective stress on the deformation and strength characteristics of sandy soils. The experimental results demonstrated the influence of these factors on the behaviour of sandy soils and in particular the importance of the combined effects of density and effective stress.

The gradual change of the stress–strain characteristics from those typical for loose sand and high effective stress to those typical for dense sand and low effective stress was observed for certain initial states of the tested materials. It was found, however, that stress–strain and volume change behaviour is dependent on the relative state of the soil expressed in terms of density and effective stress. The experimental data covers a wide range of densities and effective stresses as well as relative states of the soils, as demonstrated by the different magnitude of the effects of density and mean effective stress on the stress–strain and volume change behaviour of the tested soils.

In general, it was found that the pattern of change in the behaviour depending on the density and effective stress is similar for clean sand and sand with 10–15\%
fines content. However, this similarity is neither related to the void ratio nor to the relative density of the soil, but rather to its relative state expressed in terms of density, or void ratio, and mean effective stress. The results and conclusions of the present study were found in good agreement with other published data and findings presented in other studies.
Chapter 5

SOME ASPECTS OF UNDRAINED BEHAVIOUR OF SAND

5.1 INTRODUCTION

One of the major subjects in geotechnical engineering studies that has stimulated extensive experimental research and that has challenged the analytical methods and constitutive models for sandy soils, is certainly the undrained behaviour of sand and related phenomena such as contractive flow, liquefaction and cyclic mobility. The diversity of the field conditions and complexity of the undrained response including simultaneous change in the absolute and relative magnitudes of the principal stresses as well as their directions, have resulted in the development and application of various testing devices and techniques so as to study and duplicate this behaviour in the laboratory. Still, most of the experimental evidence on the behaviour of sand has been obtained from conventional triaxial tests and from simple shear devices based on the NGI-type simple shear apparatus (Bjerrum and Landva, 1966), Cambridge-type simple shear apparatus (Roscoe, 1953) and ring shear apparatus (Yoshimi and Oh–oka, 1973) among others.

It is considered that in many in-situ conditions soil exhibits simple shear deformation, and therefore, it is particularly important to duplicate this mode of deformation in the laboratory. Besides the simple shear devices mentioned above it is interesting to notice the usage of torsional simple shear tests on hollow cylindrical samples by Ishihara et al. (1977), Tatsuoka et al. (1982) and (1986c), and Pradhan et
al. (1988) and (1989c). These studies and especially the latest have demonstrated the versatility of the hollow cylinder apparatus and have shown that simple shear deformation can be accurately simulated in the torsional simple shear tests where both stresses and strains are rather uniform and easily measured.

In order to study the undrained behaviour of sand a series of monotonic and cyclic undrained torsional simple shear tests on Toyoura sand is conducted for different densities of the samples, initial ratios between the horizontal effective stress and the vertical effective stress $K = \sigma'_h / \sigma'_v$ (referred as effective stress ratio in the following) and cyclic stress amplitudes. A series of torsional shear tests with unconstrained lateral deformation is also performed so as to illustrate the difference in the undrained response depending on the mode of deformation. All the tests are performed on samples of Toyoura sand prepared by air–pluviation, following the sample preparation and testing procedures described in Chapter 3. The initial states of the samples including initial void ratio, mean effective stress and effective stress ratio $K$ are listed in Table 3.6 together with the identification of the tests and description of the loading conditions. The discussion in this chapter will be restricted to soil elements which have no initial shear stresses on the horizontal planes, a condition typical for soil elements under level ground or beneath structure along its axis of symmetry, as illustrated schematically in Fig. 5.1, with the element $A$ in the free field, and element $B$ underneath the structure.

5.2 TORSIONAL SIMPLE SHEAR BEHAVIOUR

The characteristics of the undrained torsional simple shear behaviour will be illustrated by both monotonic and cyclic tests on medium dense and loose Toyoura sand with void ratio, $e = 0.76$ and 0.86, respectively. The stress–strain behaviour and effective stress path, including rotation of principal stress directions and change in the relative magnitude of intermediate principal stress, will be closely examined for stable, as well as unstable undrained behaviour of sand involving contractive flow. In addition to that, the effects of the initial effective stress ratio $K$ on monotonic and cyclic torsional simple shear behaviour will be demonstrated for initial effective stress ratio in the range between $K = 0.4 - 1.0$. 
Figure 5.1  Schematic Illustration of Soil Elements with Zero Shear Stresses on the Horizontal Planes

Figure 5.2  Soil Element Under Undrained Simple Shear Deformation
5.2.1 Monotonic Loading Behaviour

A schematic representation of a soil element undergoing simple shear deformation under undrained conditions is shown in Fig. 5.2. Apparently, both normal in-plane strain increments are zero \( (d\varepsilon_x = d\varepsilon_y = 0) \) and so is the out of plane normal strain increment \( (d\varepsilon_z = 0) \) since the shearing is under plane strain conditions. Hence, the only non-zero strain increment is the shear strain increment in the plane of the shearing \( dy \neq 0 \), and therefore, the direction of the total principal strain increment is fixed at 45° relative to the vertical.

Fig. 5.3 shows the effective stress path and stress–strain curve obtained in monotonic torsional simple shear test on anisotropically consolidated \( (K = 0.5) \) sample of Toyoura sand with void ratio \( \varepsilon = 0.759 \). This typical result for medium dense sand shows that, initially, the shear stress and pore water pressure increased progressively, with the effective stress path moving upwards to the left in Fig. 5.3a. This part of the shearing is characterized by very small strain development. This trend continued until the phase transformation (PT) state was reached, where the dilation started with the effective stress path moving upwards to the right on the effective stress diagram. The state of the phase transformation is equally notable on the stress–strain behaviour as well. Namely, the initial small strain development is followed by large increase in the shear strain around and at the phase transformation where the stress–strain curve becomes flat. As shearing proceeds, the soil regains its stiffness, as illustrated by the decreasing rate in the shear strain following the phase transformation state. It can be seen in Fig. 5.4a on the other hand, that the phase transformation corresponds to the point of maximum curvature on the normalized stress–strain relation, and actually denotes the point at which the rate of shear strain starts to increase relative to the stress ratio.

Fig. 5.4b shows the effective stress path of the considered test in the normalized torsional shear stress versus normalized stress difference plot. In this figure the radial lines from the origin denote constant principal stress direction, with the horizontal axis designating \( \beta_o = 0° \) or coincidence of the major principal stress direction with the vertical, and the angle between the horizontal axis and the radial directions being twice the angle between the major principal stress direction and the vertical, \( 2\beta_o \). Consequently, the vertical axis in this plot denotes an angle of \( \beta_o = 45° \) between the
Figure 5.3 Measured Effective Stress Path and Stress–Strain Curve of Medium Dense Toyoura sand in Undrained Torsional Simple Shear Test
a) Normalized Stress-Strain Curve

b) In-Plane Effective Stress Path

Figure 5.4 Normalized Stress-Strain Curve and Effective Stress Path in Torsional Shear versus Stress Difference Plot
major principal stress direction and the vertical. It is important to notice that the radii
of the circles with their center in the origin of the axis in Fig. 5.4b specify the
normalized deviator stress, \( q/p' = (\sigma_1 - \sigma_3)/(2p') \).

Since the sample was anisotropically consolidated along effective stress ratio
\( K = 0.5 \), its initial stress state lies on the \( X \)-axis in Fig. 5.4b, with the direction of \( \sigma_1 \)
prior to torsional shear stress application being coincident with the vertical axis, i.e.
\( \beta_o = 0^\circ \). Once the torsional shearing started under simple shear deformation, very fast
change in the principal stress directions occurred with continuous rotation of the major
principal stress direction, as demonstrated by the anti-clockwise rotation of the stress
path on the diagram in Fig. 5.4b. It is interesting to note that the rapid change in the
principal stress directions occurred in the initial stage of the shearing before the phase
transformation was reached, while afterwards, very small additional rotation took
place. This is more clearly illustrated in Fig. 5.5a where the angle of orientation of
major principal stress relative to the vertical \( \beta_o \) is plotted versus shear strain \( \gamma \).
Apparently, \( \beta_o \) increases rapidly in the initial stage of the shearing and reaches about
40° at the phase transformation. Afterwards, the rotation is negligible and \( \beta_o \) takes
almost constant value close to 45°.

A similar trend in the behaviour was observed regarding the relative magnitude
of the intermediate principal stress. As shown in Fig. 5.5b, the \( b \)-value changed
significantly, from 0 to about 0.30, in the stage of the shearing prior to the phase
transformation, and afterwards maintained value between 0.25 and 0.30. It may be
concluded that both rotation of the principal stress directions and change in the
\( b \)-value occur mostly during the contractive part of the undrained simple shear
behaviour, i.e. before the state of phase transformation is reached.

Fig. 5.6 shows that the \( b \)-value started to increase slightly after the beginning
of the shearing when \( \beta_o \) was already about 12°, and then, both the \( b \)-value and \( \beta_o \)
increased substantially until the phase transformation was reached. Following the
phase transformation, \( \beta_o \) and the \( b \)-value maintained nearly constant values, as
illustrated by the grouping of the points around the point of phase transformation in
Fig. 5.6. An identical pattern in the behaviour as described above has been observed
by Pradhan et al. (1988) in monotonic torsional simple shear tests on denser Toyoura
sand with void ratio about \( e = 0.70 \).
Figure 5.5  Measured Rotation of the Principal Stress Directions and Change in the $b$-value in Monotonic Undrained Torsional Simple Shear Test
Figure 5.6 Relationship Between $\beta_\sigma$ and the $b$–value in Monotonic Undrained Torsional Simple Shear Test

Typical torsional simple shear behaviour for loose sand is shown in Fig. 5.7. In this figure the effective stress path and stress–strain curve obtained from test on Toyoura sand with void ratio $e = 0.857$ are presented. The sample was anisotropically consolidated along effective stress ratio $K = 0.5$, and then sheared in stress–controlled manner under torsional simple shear mode of deformation. Initially, the shear stress and the pore water pressure developed progressively with very small shear strain development. This trend continued until a state of stress was reached where the sample became unstable. A sudden decrease in the shear stress occurred accompanied by a sharp increase in the pore pressure and rapid development of large strain. This characteristic undrained behaviour for loose sand will be referred to as contractive flow in the following. Since the test was carried out in stress–controlled manner and the flow was extremely rapid, the data were not recorded during the flow, and therefore, the experimental curves shown in Fig. 5.7 are discontinuous. The flow deformation ceased once the phase transformation was passed and the dilation started. Yet, as regards the rotation of principal stress directions and relative magnitude of intermediate principal stress, the behaviour was identical to that observed for medium
dense Toyoura sand. This is well illustrated in Fig. 5.8 where the comparison between the $b$-value versus $\beta_o$ plots for medium dense and loose Toyoura sand is given. It is interesting to note that in all the tests where contractive flow occurred, the unloading effective stress path ended up in the origin of the effective stress path, thus reducing the effective stress to zero upon the removal of the shear stress (Fig. 5.7a).

Instability of the undrained behaviour of loose sand illustrated above, is a very important and unique phenomenon because it involves very large deformations even though the stress state where the flow is triggered is well beyond the failure surface and the maximum stress ratio that the soil can sustain. It is worth noting that once the flow is triggered the large deformation is inevitable, and therefore, it is particularly important to recognize the condition for outset of the contractive flow. Although the data presented in Fig. 5.7 do not actually comprise the particular stress state at which the flow was triggered, this state should be in close vicinity to the peak stress point or the last recorded point prior to the occurrence of the contractive flow. The condition for triggering contractive flow or instability from the undrained response has drawn considerable experimental and analytical efforts of many researchers and is defined as a line or surface in the corresponding stress space, e.g. critical effective stress ratio (Vaid and Chern, 1983 and Vaid et al., 1989), instability line Lade (1992) and collapse surface (Sladen et al., 1985) among others.

Another important aspect of this particular problem is to identify the initial states of the soil for which instability of the undrained response or contractive flow can occur. Since the combination of density and effective stress is relevant to the description of sandy soil behaviour it is important to notice that whether the contractive flow can or can not occur is not solely defined by the density of the soil but also by its initial effective stress. Within the framework of comprehensive characterization of the undrained sandy soil behaviour based on the steady state concept, Ishihara (1993) and Verdugo (1992) defined a line in the void ratio–mean effective stress plane, termed the initial dividing line (IDL), which separates the initial states with or without drop in the shear stress during undrained shearing, i.e. the initial conditions for which contractive flow can or can not occur. More detailed discussion on the study of Ishihara and Verdugo will be given in the subsequent chapter where the modeling concept adopted in the present study will be elaborated.
Figure 5.7 Measured Effective Stress Path and Stress–Strain Curve of Loose Toyoura Sand in Undrained Torsional Simple Shear Test
5.2.2 Cyclic Loading Behaviour

The characteristics of cyclic undrained torsional simple shear behaviour of sand will be illustrated by the results from the tests on medium dense and loose Toyoura sand with void ratio $e = 0.767$ and $e = 0.855$, respectively. The samples were anisotropically consolidated along initial effective stress ratio $K = 0.5$. Hence, the initial conditions in these tests were identical to those of the monotonic tests presented in the preceding pages.

The effective stress path and the stress–strain curve observed in the test on medium dense sand with void ratio $e = 0.767$ are shown in Fig. 5.9. The sample was anisotropically consolidated to final mean effective stress of $p' = 0.667$ kgf/cm$^2$, and then subjected to cyclic load with constant cyclic stress ratio amplitude at $(r/p')_c = 0.225$. Although progressive increase in the pore water pressure or decrease in the mean effective stress $p'$ is associated with each of the shear stress cycles, it is easy to recognize three distinct parts in the effective stress path. Thus, the first cycle is characterized by a relatively large increase in the pore water pressure, followed by a series of cycles with smaller and quite similar progress of the effective stress path.
Figure 5.9 Measured Effective Stress Path and Stress–Strain Curve of Medium Dense Toyoura Sand in Cyclic Undrained Torsional Simple Shear Test
towards the left in Fig. 5.9a. Close examination of this intermediate loading stage can show that the initial decreasing rate of the pore water pressure is reversed later on at an increasing rate as cyclic loading progresses. The final part of the effective stress path is characterized by rapid development of the pore pressure and subsequent reduction of the effective stress to zero in a typical loop that characterizes the cyclic mobility.

It seems appropriate to consider the final part of the response as initial phase of the cyclic mobility with its beginning in the cycle when the phase transformation line is crossed for the first time during the cyclic loading. Once the effective stress is reduced to zero, or nearly zero, the behaviour in further cyclic loading would be characterized by nearly steady effective stress cycles passing through or near by zero mean effective stress state accompanied by continuous increase in the shear strain. This typical behaviour for cyclic mobility was not considered in the present study.

The corresponding stress-strain curve shown in Fig. 5.9b indicates that large shear strain developed in the final stage of cyclic loading when the response entered in cyclic mobility, while on the other hand, rather small shear strain developed in the preceding cycles. In order to examine the stress-strain behaviour in more details, in Fig. 5.10a the fluctuation of the stress ratio $q/p'$ during the cyclic loading is plotted versus the mean effective stress. The points in this figure represent the neutral points or the states where the torsional stress is zero, $\tau = 0$. Since the sample was anisotropically consolidated, it was initially subjected to stress difference $\sigma_s - \sigma_p$, which yields non-zero initial stress ratio, as shown in Fig. 5.10a. Except for the first cycle, the stress ratio is not greater than the initial stress ratio until the 6-th cycle, when progressive increase in the stress ratio starts, especially pronounced in the last two cycles.

This trend of change in the stress ratio during cyclic loading under undrained simple shear condition is caused by the tendency of the soil to achieve isotropic stress state as loading progresses. Namely, in order to constrain the lateral deformation and maintain simple shear deformation during the cyclic loading, the total lateral stress gradually increases and eventually equals the total vertical stress in the moment when the mean effective stress reduces to zero and the soil liquifies. On the other hand, the increase of the pore water pressure during the cyclic loading causes both lateral
Figure 5.10  Stress Ratio $q/p'$, and Total and Effective Stress States in the Course of Cyclic Undrained Simple Shear Test
effective stress and vertical effective stress to decrease, thus reducing the mean effective stress continuously until eventually zero effective stress is achieved. This is illustrated schematically in Fig. 5.10b where the total stress and the effective stress are shown for the states prior to cyclic loading, during the cyclic loading and at the end of it.

Fig. 5.11a shows the implication of this trend on the effective and total stress paths expressed in terms of the deviator stress $q$ and the mean stress $p$. The points on the effective stress path represent the states where the torsional shear stress is zero (neutral points, $\tau = 0$), while the solid line shows the fluctuation of the deviator stress $q$ due to the cyclic application of the torsional shear stress. The corresponding total stress path is given only schematically for clarity of the presentation. Each of the presented neutral points on the effective stress path has its corresponding point along the total stress path shown in Fig. 5.11a. It can be seen that, starting from the initial state where total and the effective stresses are coincident, the total stress path and effective stress path diverge from each other. The tendency of both total and effective stress path towards isotropic state is demonstrated by the gradual decrease of the deviator stress $q$ at the neutral points, as loading proceeds.

Fig. 5.11b shows that both horizontal effective stress components, i.e. the in–plane effective stress $\sigma'_{\theta}$, and out of plane effective stress $\sigma'_{\phi}$, decrease in slower rate than the corresponding vertical effective stress $\sigma'_{z}$, resulting in gradual increase in the effective stress ratio $K$ from its initial value, $K = 0.5$, to $K = 1.0$. Since the rate of decrease of the shear stress $q$ is faster than that of the mean effective stress $p'$, there is continuous reduction of the stress ratio $q/p'$ at the neutral points as cyclic loading progresses. The increase of the stress ratio at the peak of the torsional shear stress ($\tau = \tau_{max}$) following the 5–th cycle, on the other hand, is solely result of the increased contribution of the torsional stress component in the deviator stress $q$.

Following the above discussion, the stress–strain characteristics of the considered test will be now examined. For that purpose in Fig. 5.12 the stress–strain curve is presented enlarged for the first eight cycles. As shown in Fig. 5.12a, following the initial cycle, the maximum shear strain slightly decreased and nearly steady stress–strain loop developed from the 2–nd to the 5–th cycle. A gradual increase of the shear strain started from the 6–th cycle, with each subsequent loading cycle producing larger
**Figure 5.11** Characteristic Stress Paths During Cyclic Undrained Torsional Simple Shear Test
strain than its preceding cycle, as shown in Fig. 5.12b. It is worth noting that even though the stress ratio progressively decreased in the first cycles (Fig. 5.10a) the 'expected' continuous hardening, as expressed by increase in the stiffness and reduction of the hysteresis loop, has not taken place. Thus, it might be speculated that additional shear strain has been produced by a mechanism that is not related to the absolute magnitude of the principal stresses, but rather to the rotation of the principal stresses and possibly due to change in the relative magnitude of the intermediate principal stress.

To illustrate the orientation of the principal stress directions during the cyclic torsional simple shear test in Fig. 5.13 the effective stress path in the normalized torsional shear stress versus stress difference plot is shown. The filled points in Fig. 5.13 denote the peak points where the torsional shear stress is maximum \( \tau = \tau_{\text{max}} \) in the cyclic test, while the empty points stand for the effective stress path of the corresponding monotonic test, described in details previously. Apparently, there is remarkable similarity in the progress of the rotation of principal stress directions in the monotonic and cyclic test, with almost coincident paths of the peak points in the cyclic test and the effective stress states of the monotonic test.

Rotation of the principal stress directions during the cyclic loading is more clearly illustrated in Fig. 5.14a where the angle of orientation of the major principal stress \( \beta_o \) is plotted versus the mean effective stress \( p' \). This figure shows that there is continuous rotation of principal stress directions during the cyclic loading under simple shear condition, with the axis of rotation being coincident with the \( X \)-axis or the vertical. The angle of rotation gradually increases as the effective stress path is approaching the isotropic stress state and has its maximum of about 80° to 85° in the last cycle when the cyclic mobility started. Again, once the phase transformation was reached and the dilation started, the rotation of principal stresses ceased, as demonstrated by nearly horizontal effective stress path with very small change in \( \beta_o \) following the phase transformation points in Fig. 5.14a.

The similarity between the monotonic and cyclic behaviour holds valid for the relative magnitude of the intermediate principal stress as well. This is illustrated in Fig. 5.14b where the \( b \)-value versus \( \beta_o \) plot for the cyclic test is compared to that observed in the corresponding monotonic undrained torsional simple shear test.
UNDRAINED BEHAVIOUR OF SAND

a) Stress-Strain Curve (1-st to 5-th cycle)

b) Stress-Strain Curve (1-st and 6-th to 8-th cycle)

Figure 5.12 Stress-Strain Curve Observed in Cyclic Undrained Torsional Simple Shear Test on Toyoura Sand
Figure 5.13 Effective Stress Paths of Monotonic and Cyclic Tests in the Normalized Torsional Shear versus Stress Difference Plot
Figure 5.14 Rotation of Principal Stress Directions and Change in $b$-value in Cyclic Undrained Torsional Simple Shear Test
Typical effective stress path and stress–strain behaviour during cyclic undrained torsional simple shear test on loose Toyoura sand is shown in Fig. 5.15. The sample was anisotropically consolidated under $K = 0.5$ condition to final consolidation effective stress of $p' = 0.667$ kgf/cm$^2$, and then subjected to undrained cyclic torsional simple shear with constant cyclic stress ratio at $(	au/p')_c = 0.18$. The discussion presented above on the undrained cyclic torsional shear behaviour of medium dense Toyoura sand applies to loose Toyoura sand as well, and therefore, detailed presentation of the experimental results is not needed. However, it is interesting to further investigate the characteristics of the behaviour of loose sand related to the contractive flow or instability of the undrained response.

The development of the pore water pressure and the advance of the effective stress path towards zero effective stress state prior to the occurrence of the contractive flow for loose Toyoura sand, is not unlike from those observed in the test on medium dense Toyoura sand. Thus, following the initial cycle where relatively large pore water pressure was induced, a series of cycles took place with quite similar pore pressure and stress–strain characteristics. The contractive flow occurred abruptly and was manifested by very rapid and rather large strain development, with sharp increase in the pore pressure and drop in the shear stress. The shear strain which was negligible before the contractive flow increased to about 6% during the flow. The contractive flow ceased once the phase transformation was reached and the dilation started. It is worth to emphasize the abruptness of the contractive flow which took place without any indication in the effective stress path and in the stress–strain behaviour.

Another interesting observation is that the contractive flow in the cyclic loading was triggered at the same stress ratio as that in the monotonic test. This is illustrated in Fig. 5.15a where both effective stress paths, for the cyclic and monotonic tests, respectively, are shown together with the stress ratio where the flows were triggered, illustrated in the figure by the broken line. Similarly to the monotonic behaviour, after the contractive flow ceased further increase in the shear stress caused increase in the mean effective stress due to the tendency for dilation. However, unloading with removal of the shear stress reduced the mean effective stress to zero. Hence, it can be concluded, that in both monotonic and cyclic torsional simple shear if contractive flow occurs, then removal of the shear stress or unloading will reduce the mean effective
Figure 5.15 Effective Stress Path and Stress–Strain Curve of Loose Toyoura Sand in Cyclic Undrained Torsional Simple Shear Test
stress to zero and bring the sand to a liquified state. This behaviour has very serious
effect on the conventional cyclic strength or 'liquefaction strength' where the number
of cycles required to achieve zero or nearly zero effective stress is compared to the
applied cyclic stress ratio. Thus, as shown in Fig. 5.15a by the effective stress path
of the monotonic test, even half a cycle is sufficient to trigger the contractive flow and
reduce the effective stress to zero if the torsional shear amplitude is $\tau = 0.2$ kgf/cm$^2$
or the cyclic stress ratio is $(\tau/p')_c = 0.3$.

5.2.3 Effects of the Initial Effective Stress Ratio

The characteristics of monotonic and cyclic undrained torsional simple shear
behaviour have been demonstrated in the preceding pages for anisotropically
consolidated medium dense and loose Toyoura sand with initial effective stress ratio
of $K = \sigma'_u / \sigma'_v = 0.5$. In order to study the effects of the initial effective stress ratio
$K$ on the monotonic and cyclic undrained simple shear behaviour, a series of
monotonic and cyclic tests was conducted on isotropically and anisotropically
consolidated samples along various constant $K$ values.

In Figs. 5.16 and 5.17 the effective stress paths and the stress–strain curves
observed in monotonic tests on isotropically consolidated samples, $K = 1.0$, are
compared to those observed in the corresponding tests on anisotropically consolidated
samples, with $K = 0.50$, for medium dense ($e = 0.76$) and loose sand ($e = 0.86$),
respectively. Apparently, remarkably similar behaviour can be observed for both
densities even though the initial effective stress conditions are quite different. In spite
of the similarity in the undrained torsional simple shear response, however, it is
important to recognize the differences between the behaviour for initial effective stress
ratio $K = 0.5$ and $K = 1.0$, including differences in magnitude of the stress ratio $q/p'$,
rotation of the principal stress directions and relative magnitude of the intermediate
principal stress in the monotonic shearing under simple shear deformation. To
illustrate these differences in Fig. 5.18 the relationships of $q/p'$, $\beta_o$ and the $b$–value
with $\gamma$, obtained in test with $K = 0.5$, are compared to those measured in test with
initial effective stress ratio $K = 1.0$. The comparison is shown only for the initial parts
of the tests, for shear strain $\gamma = 0 - 0.4\%$. The following may be observed:

**Initial Isotropic Condition: $K = 1.0$;** There is continuous increase in the stress
Figure 5.16 Effects of Initial Effective Stress Ratio $K$ on Behaviour of Medium Dense Toyoura Sand in Monotonic Undrained Torsional Simple Shear Tests
Figure 5.17  Effects of Initial Effective Stress Ratio $K$ on Behaviour of Loose Toyoura Sand in Monotonic Undrained Torsional Simple Shear Tests
ratio $q/p'$ as the torsional shear stress is applied while, on the other hand, the principal stress direction and the relative magnitude of intermediate principal stress are constant at $\beta_o = 45^\circ$ and $b = 0.5$, respectively. Hence, it may be concluded that, in this test, the shear strain is solely result of the increase in the magnitude of the shear stress.

**Initial Anisotropic Condition: $K = 0.5$:** On the contrary, in the case of initial anisotropic condition, the magnitude of the stress ratio in the initial part of the shearing is nearly constant, while on the other hand, there is notable change in both $b$-value and orientation of the principal stress directions. As described earlier and as illustrated in Fig. 5.18, both $\beta_o$ and the $b$-value change mostly in the initial part of the torsional loading where the behaviour is characterized by tendency for contraction and relatively small shear strain development. Thus, unlike the isotropic case ($K = 1$), the shear strain produced in the initial part of the shearing of anisotropically consolidated sample, with $K = 0.5$, is mostly due to continuous rotation of the principal stress directions and possibly because of increase in the $b$-value. The contribution to the shear strain development due to slight increase in the magnitude of the stress ratio $q/p'$ is, however, very small. It is interesting to note that relatively small shear strain induced during the rotation of the principal stress directions, in the test with $K = 0.5$, is in good agreement with the experimental evidence on the effect of continuous rotation derived by Symes et al. (1982), Ishihara and Towhata (1983) and Pradhan (1988). It is worth noting that the notable differences between the isotropic and anisotropic stress conditions as regards $q/p'$, $\beta_o$ and the $b$-value in the early stage of the shearing, have rapidly reduced and became rather small or negligible for all these stress parameters, once the phase transformation was approached and the strain increased. This fact as well as the similarity of the induced response in the initial stage of the shearing although by quite different mechanisms for $K = 1.0$ and $K = 0.5$, provide the overall similarity in the pore pressure development and stress-strain behaviour for these tests, as illustrated in Figs. 5.16 and 5.17.

In addition to the monotonic tests, the effects of the initial stress ratio $K = \sigma'_h / \sigma'_v$ will be illustrated by the results from a series of cyclic undrained torsional simple shear tests on medium dense Toyoura sand with void ratio $e = 0.767 - 0.777$ and initial effective stress ratio of $K = 0.4, 0.5, 0.75, 0.90$ and $1.0$. It is to be noted that the value of $K = 0.4 = 0.52e$, corresponds to the $K_o$ value for
Figure 5.18 Comparison of Stress Parameters in the Initial Stage of Monotonic Undrained Torsional Simple Shear for $K = 1.0$ and $K = 0.5$
normally consolidated Toyoura sand (Okochi and Tatsuoka, 1984). The samples were consolidated either isotropically or anisotropically to final mean effective stress of \( p' = 0.667 \text{ kgf/cm}^2 \), and then subjected to cyclic torsional shear stress of \( \tau = 0.15 \text{ kgf/cm}^2 \) or cyclic stress ratio \( \tau/p' = 0.225 \), under simple shear mode of deformation. Hence, these tests defer only in the initial effective stress ratio \( K \).

If the field conditions related to simple shear behaviour are to be duplicated, it seems appropriate to perform \( K_0 \) consolidation and then to obtain different initial \( K_0 \) values by overconsolidating the samples. In that case, however, the undrained torsional simple shear response will include the effects of the overconsolidation as well, and these are shown to be quite significant (Seed and Peacock, 1971; Ishihara and Takatsu, 1979 among others). Thus, in order to investigate the effects of the initial effective stress alone, different \( K \) values were obtained by anisotropically consolidating the samples along constant effective stress ratios \( K \).

Fig. 5.19 shows the effective stress paths obtained in the undrained cyclic torsional simple shear tests with different initial effective stress ratios \( K \). In general, the behaviour is remarkably similar with nearly identical characteristics for the first and final cycles of the loading, and somewhat different progress of the effective stress path in the intermediate stage of the loading as reflected in the different number of cycles required to attain zero effective stress state. Identical number of cycles to zero effective stress, \( N_c = 12 \), is obtained for isotropically \((K = 1.0)\) and anisotropically \(K_0\) consolidated samples \((K = 0.4)\), while the number of cycles decreased to \( N_c = 9.5\) and \( N_c = 8\), for \( K = 0.5 \) and \( K = 0.75 \) and \( 0.90 \), respectively.

The differences in the progress of the pore pressure build up are illustrated in Fig. 5.20 as well, where the pore pressure ratio \( u / \sigma' \), is plotted versus the number of the load increment, for each of the considered tests. There is identical pattern in the change of the pore water pressure with its gradual increase towards the initial value of the effective vertical stress or pore pressure ratio of unity. The fluctuations of the pore pressures which occurred just before the pore pressure ratios increased to unity are associated with the initial part of the cyclic mobility.

The stress–strain curves observed in the tests introduced above are shown in Fig. 5.21. Remarkably similar behaviour can be seen with very small strain development until the response entered into cyclic mobility during which the shear strain increased
Figure 5.19 Effective Stress Paths in Undrained Cyclic Torsional Simple Shear Tests on Toyoura Sand for Different initial $K$ values
Figure 5.20  Pore Water Pressure Development in Undrained Cyclic Torsional Simple Shear Tests on Toyoura Sand for different initial $K$ values
to about 3 – 5 % double strain amplitude in a single cycle.

In order to examine the influence of the initial effective stress ratio \( K \) in more details, in Figs. 5.22 – 5.25 the change in the deviator stress \( q \), stress ratio \( q/p' \), orientation of principal stress directions \( \beta_o \) and the \( b \)-value with the mean effective stress \( p' \) are presented for the considered tests. As shown in Fig. 5.22, the initial deviator stress \( q \) is greater for the lower values of the initial effective stress ratio \( K \) because of the larger initial stress difference. However, due to the tendency to achieve isotropic stress state during the cyclic loading, there is continuous decrease in the stress difference and consequently in \( q \), as cyclic loading progresses. The lowest stress states of the fluctuating stress paths shown if Fig. 5.22 corresponds to the neutral points, with \( \tau = 0 \).

This pattern of change in the deviator stress \( q \) together with the associated reduction in the effective stress \( p' \) provide the typical changes in the stress ratio \( q/p' \) which depend on the initial stress ratio \( K \), as illustrated in Fig. 5.23. Thus, in the case of small initial effective stress ratio or when \( K \) is about \( K_0 \) value, there is reduction in the stress ratio \( q/p' \) in the initial stage of the shearing which is later on reversed to increasing \( q/p' \) values, as the cyclic loading proceeds. On the other hand, in the case of initially isotropic or nearly isotropic stress state with \( K \) value about unity, the peak stress ratio continuously increases with the cyclic loading and eventually reaches its maximum during the dilative part in the cyclic mobility.

As regards the orientation of the principal stress directions during the cyclic loading, typical behaviour related to different values of the initial effective stress ratio \( K \) is illustrated in Fig. 5.24. Except for the isotropic case with \( K = 1.0 \), there is continuous rotation of the principal stresses around the vertical. The angle of rotation continuously increased as the cyclic loading progressed and eventually attained about 90° in the final cycles, when the cyclic mobility started. It is worth noting that the amount of rotation in the initial cycles and intermediate part of the loading is larger for the initial states with larger value of \( K \) (\( K = 0.75 \) and \( K = 0.90 \)). On the other hand, the behaviour of isotropically consolidated sample is characterized by fixed orientation of the major principal stress direction at \( \beta_o = \pm 45^\circ \) and by sudden jump in the orientation of principal stress directions for 90°, as illustrated by the broken line in Fig. 5.24. Consequently, there is no continuous rotation of the principal stress.
Figure 5.21 Stress–Strain Curves Measured in Undrained Cyclic Torsional Simple Shear Tests on Toyoura Sand for Different Initial $K$ values
Figure 5.22 Deviator Stress \( q \) in Cyclic Undrained Torsional Simple Shear Tests on Toyoura Sand for Different Initial \( K \) Values
**Figure 5.23** Stress Ratio $q/p'$ in Undrained Cyclic Torsional Simple Shear Tests on Toyoura Sand for Different Initial $K$ Values
directions when the stress state is isotropic with $K = 1.0$.

Finally, in Fig. 5.25 the effects of the initial stress ratio $K$ on the $b$--value are illustrated. It can be seen that remarkable different fluctuations of the $b$--value are obtained depending on the initial effective stress ratio. For $K$ values close to the $K_0$, there is gradual increase in the $b$--value from $b = 0$ to its final value of about 0.3 to 0.4. It is important to notice that the large fluctuations of the $b$--value during the cyclic mobility are due to nearly isotropic stress state and almost equal values for $\sigma_1$, $\sigma_2$ and $\sigma_3$. Therefore, even small change of the stress state causes large variation in $b$--value. For the same reason, the fluctuation of the $b$--value for $K = 0.9$ is very pronounced. Again, the behaviour for $K = 1.0$ is unique, with constant $b$--value at 0.5 throughout the load application.

The results presented above show that the torsional shear behaviour has different characteristics depending on the $K$ value. The differences are pronounced in the initial and intermediate stage of the loading and are reflected on the magnitude of the stress ratio ($q/p'$), orientation of the principal stress directions ($\beta_\phi$) and relative magnitude of the intermediate principal stress ($b$--value). As the loading progresses and approaches its final stage, the differences gradually reduces and eventually the behaviour becomes nearly unique.

It is interesting to note that remarkably similar behaviour with identical cyclic strength has been obtained for the two most distant conditions as regards the initial effective stress ratio, i.e. $K = 0.4$ and $K = 1.0$, corresponding to $K_0$ and isotropic consolidation, respectively. This result is in accordance with some other experimental evidence (Ishihara et al., 1977; Tatsuoka et al., 1986b).

It may be postulated, on the other hand, that the reduction in the number of cycles to achieve nearly zero effective stress, observed in the tests with $K = 0.75$ and 0.90, is due to the greater angle of continuous rotation of the principal stresses, which is shown to increase the build--up of pore pressure (Shibuya and Height, 1988). Since the behaviour is under complex influence, however, involving simultaneous change in the absolute and relative magnitudes of the principal stresses as well as their directions, and on the other hand, the number of the data is rather limited, any rigorous conclusion seems inappropriate at this point.
Figure 5.24 Rotation of Principal Stress Directions During Cyclic Undrained Torsional Simple Shear Tests with Different Initial $K$ Values
Figure 5.25 Change in the $b$-value During Cyclic Torsional Simple Shear Tests with Different Initial $K$ Values
5.3 EFFECTS OF DEFORMATION MODE

The characteristics of monotonic and cyclic undrained torsional simple shear behaviour presented in the preceding pages have shown the mechanism of the pore pressure build-up and the influence of the initial effective stress ratio on it. In order to demonstrate the effects of the deformation mode, and in particular the effects of the constraint of the lateral strains imposed in the simple shear test, a series of test was performed on samples with identical initial conditions to those applied in the torsional simple shear tests, but with unconstrained lateral strains during the cyclic loading. Herein, the results of the test on anisotropically consolidated sample of Toyoura sand with void ratio $e = 0.76$ and initial effective stress ratio $K = 0.5$, will be presented for the torsional simple shear test (constrained lateral strains) and corresponding torsional shear test (free lateral strains). The mean effective stress prior to shearing and the cyclic stress ratio in these tests were $p' = 0.667$ kgf/cm$^2$ and $(\tau/p')_c = 0.225$, respectively.

In Figs. 5.26 and 5.27, the effective stress path and stress–strain curve obtained in the torsional simple shear test are compared to those obtained in the corresponding torsional shear test. Unlike the behaviour under simple shear condition, the behaviour in the torsional shear test is characterized by very small reduction of the mean effective stress, or pore pressure build-up, during the cyclic loading. Moreover, after somewhat larger pore pressure was induced in the initial loading cycle, the rate of reduction in the effective stress with further cyclic loading gradually decreased, and eventually the effective stress path ceased to progress after several cycles. The corresponding stress–strain curve shows that this behaviour was associated with rather small strain development.

Pore pressure characteristics of the torsional shear test are shown in Fig. 5.28a where the pore pressure ratio $u/\sigma'_v$ is plotted versus the load increment number. It is worth noting that the pore pressure increased only to about 12% of the initial effective vertical stress, and maintained almost constant value in further cyclic loading. The reason for this trend in the behaviour in the torsional shear test is shown in Fig. 5.28b where the effective stress paths for both considered tests are comparatively presented. As described previously, the undrained cyclic torsional simple shear behaviour is characterized by gradual decrease in the deviator stress during the cyclic
Figure 5.26  Effects of the Deformation Mode on the Effective Stress Path
Figure 5.27  Effects of the Deformation Mode on the Stress–Strain Behaviour
loading which is illustrated by the downward movement of the effective stress path towards the origin of the effective stress diagram. This pattern in the behaviour results in gradual increase in the effective stress ratio from its initial value $K = 0.5$ to $K = 1.0$. On the other hand, in the case of torsional shear behaviour, where the lateral strain is unconstrained, the effective stress path during the cyclic loading moves horizontally to the left, thus approaching the failure envelop with decrease in the $K$ value from $K = 0.5$ to $K = K_f$. For this reason, the pore pressure stopped to increase and the effective stress path ceased to progress after only several cycles. It is important to notice, however, that torsional shear behaviour is accompanied by vertical strain $e_v$ which, as shown in Fig. 5.29, is quite significant for loose sand. Therefore, this mode of deformation should be associated with possible large settlements.

5.4 SUMMARY

A series of monotonic and cyclic torsional simple shear tests was conducted on medium dense and loose Toyoura sand in order to study the characteristics of undrained simple shear behaviour of sand. The tests were performed on isotropically and anisotropically consolidated samples along effective stress ratio of $K = 0.4$, 0.5, 0.75, 0.9 and 1.0. In addition to the torsional simple shear tests, a series of cyclic tests with unconstrained lateral strains was performed so as to illustrate the effects of the deformation mode on the undrained behaviour of sand.

The experimental results demonstrated that undrained torsional simple shear behaviour is characterized by simultaneous change in the magnitudes of the principal stresses as well as their directions. In general, the pattern of change in the orientation of principal stress directions, $\beta_o$, and relative magnitude of the intermediate principal stress, the $b$–value, was identical for the monotonic and cyclic tests. Thus, the undrained torsional simple shear behaviour is characterized by gradual and substantial increase in both $\beta_o$ and $b$–value during the contractive phase of the response, i.e. prior to the phase transformation. On the other hand, the dilative phase of the response following the phase transformation is characterized by relatively small change in $b$–value and by nearly constant value of $\beta_o$.

The experimental results from the monotonic tests indicated remarkable similarity of the stress–strain behaviour and pore pressure characteristics obtained in
Figure 5.28  Pore Pressure Characteristics and Fluctuation of Deviator Stress in Cyclic Undrained Torsional Shear Test
Figure 5.29  Vertical Strain versus Shear Strain in Cyclic Undrained Torsional Shear Test
the tests on isotropically consolidated samples \( K = 1.0 \) and \( K_0 \) consolidated samples. In spite of the similarity, however, it was demonstrated that the shear strain and pore pressure response during the initial stage of the shearing (prior to phase transformation) are induced by different causes, i.e. by increase in the stress ratio and by continuous rotation of principal stress directions, for the case of \( K = 1.0 \) and \( K = K_0 \), respectively. Similarity in the response is also due to the fact that the considerable initial differences between \( q/p' \), \( \beta_n \), and \( b \)-value for \( K = 1.0 \) and \( K = K_0 \), rapidly reduced and became negligible well before the phase transformation was reached and large strain developed.

Characteristics of the undrained cyclic torsional simple shear behaviour with respect to the stress ratio \( q/p' \), rotation of the principal stress directions \( \beta_n \), and relative magnitude of the intermediate principal stress, the \( b \)-value, are strongly dependent on the initial \( K \) value. The differences in the behaviour are especially pronounced in the initial phase of the cyclic loading and are reflected on the magnitude and trend in change of the stress ratio \( q/p' \), amount of continuous rotation of principal stress directions in a single cycle and, increase and fluctuation of the \( b \)-value. As loading proceeds and approaches cyclic mobility, these differences significantly reduce and the behaviour becomes nearly unique. For this reason, and because the initial stage of the cyclic loading is characterized by very small strain development, there is remarkable similarity in the stress–strain characteristics for different initial \( K \) values.

As regards the number of cycles needed to reduce the effective stress to zero or nearly zero, it is indicated that lesser number of cycles are needed for the cases when \( K \) is between \( K = K_0 \) and \( K = 1.0 \) because of the greater amount of continuous rotation during the cyclic loading under these conditions. However, since the behaviour is characterized by rather complex and simultaneous changes of several important factors such as \( q/p' \), \( \beta_n \) and the \( b \)-value, more experimental evidence is needed to further verify this indications. On the other hand, identical cyclic strength or number of cycles to initial liquefaction was obtained for isotropically consolidated sample \( K = 1.0 \) and \( K_0 \) consolidated sample \( K = K_0 \).

Finally, it was demonstrated that the characteristics of the undrained simple shear behaviour are induced by the pattern of change in the stress state as a result of the imposed constraint in the lateral strains during the simple shear deformation.
Chapter 6

STRESS–STRAIN MODEL FOR SANDY SOILS WITH DENSITY–STRESS DEPENDENT PARAMETERS

6.1 INTRODUCTION

Based on the observations from the experimental study elaborated in the previous chapters, an elastic–plastic constitutive model for sandy soils will be developed in the following sections. In this chapter, the concept of the model will be presented together with the basic set of equations describing the stress–strain–dilatancy characteristics of sandy soils. The formulation of a two–dimensional elastic–plastic constitutive model as well as its verification and application to seismic response analysis will be given in Chapter 7 and Chapter 8, respectively.

The results of the drained torsional shear tests presented in Chapter 4 are the basis for the development of the stress–strain–dilatancy model. These results have shown that the normalized stress–strain curve of sandy soils, i.e. stress ratio–shear strain relationship, is affected by both density and effective stress, and that the magnitude of these effects depends on the combination of density and mean effective stress, i.e. on the initial state of the soil. On the other hand, it has been found that stress–dilatancy relationships of the tested materials are hardly affected by density and effective stress. It has been further demonstrated that the experimental evidence obtained in the present study is in close agreement with the observations and conclusions drawn in other studies conducted on sandy soils.
In spite of the experimental evidence and the conclusive findings that the combination of density and effective stress are relevant to the description of the behaviour of sandy soils, the current modeling concept applied to the constitutive models for sandy soils basically ignores this fact. According to this concept the parameters of current constitutive models are referenced to a particular void ratio or relative density, and the void ratio does not appear as a variable in these models. For any change in the void ratio a new set of model parameters has to be determined by a corresponding testing procedure and calibration of models. It seems that this modelling concept was generally accepted in the belief that it simplifies the matter considerably and following the tendency to single out the void ratio, or density, as the most important factor that affects the behaviour of sandy soils.

As regards the modelling of the influence of the effective stress in the framework of the above mentioned concept, two approaches are generally used. In the first one, it is postulated that the plastic strains, if represented as functions of the ratio between the shear stress and effective stress (normalized stress–strain relationship), are independent of the effective stress, and therefore they can be described in terms of the stress ratio alone. In other words, it is assumed that there exists a unique stress ratio–shear strain relationship for a given density. A typical representative of this group of models is the ‘rigid granular model’ proposed by Kabilamany and Ishihara (1990 and 1991). Since this approach is rather simple and enables implementation of the conventional Mohr–Coulomb type of failure criterion, it has been followed in many of the proposed constitutive models for sandy soils. However, as seen from the results shown in Chapter 4, it implies very serious restrictions with respect to the combined effects of density and mean effective stress, and in fact, oversimplifies the behaviour of sandy soils. In the second approach, on the other hand, the influence of the effective stress on the normalized stress–strain curve, i.e. on the stress ratio–shear strain relationship, is taken into account (e.g. Lade, 1977; Vermeer, 1978; Gutierrez, 1989 among others). Yet, since the model parameters are referenced to a particular density and the influence of the effective stress is considered by a separate quantity, it is difficult to account properly for the combined influence of density and mean effective stress. Thus, in order to appropriately describe these effects, it is necessary to determine the parameters of the model describing the influence of the effective
stress for each of the considered densities through corresponding testing and calibration procedures. However, since this usually requires considerable experimental effort, it is common to assume that these parameters are independent of density, and actually, contrary to the experimental evidence, it is assumed that the influence of the effective stress on the stress–strain relationship is not related to the density of the soil.

Unlike current constitutive models for sandy soils, the present model aims to describe the stress–strain relationship as a function of both, density and effective stress. For that purpose, a concept developed within the framework of the steady state of deformation will be employed. According to that concept, sand behaviour depends on a state related to the density and the effective stress, and hence many behavioural properties vary as a direct function of this state. Thus, a parameter that represents a direct measure for the state of the soil is used for characterization of sandy soil behaviour. Since this approach captures the influence of both density and effective stress, the modelling concept is simply to develop an appropriate stress–strain relation and to express its parameters as a function of a parameter representing the state of the soil in terms of void ratio, or density, and effective stress. Hopefully, such a concept should provide a simple approach for quantification of the combined effects of void ratio and effective stress on the stress ratio–shear strain behaviour of sandy soils.

The constitutive model of the present study is developed in the framework of the flow theory of plasticity. Therefore, as is customary for this theory, the total strain increments $de_{ij}$ are divided into an elastic component $de^e_{ij}$ and a plastic component $de^p_{ij}$ such that

$$de_{ij} = de^e_{ij} + de^p_{ij}$$  \hspace{1cm} (6.1)

These strain components are calculated separately, the elastic strains by Hooke's law, and the plastic strains by the flow theory of plasticity incorporating the corresponding stress–strain and stress–dilatancy relationships. In the following, firstly the elastic behaviour of sandy soils will be discussed and then the stress–strain and stress–dilatancy relationships will be introduced.
6.2 ELASTIC BEHAVIOUR OF SANDY SOILS

Experimental evidence on the behaviour of cohesionless soils has shown that practically both elastic and plastic deformations occur from the beginning of loading. A similar conclusion has been derived by Mindlin from the theoretical consideration of two elastic spheres pressed together, which states that slippage must occur even for the smallest tangential force (Hardin, 1978; Tatsuoka and Shibuya, 1991). Thus, elastic and plastic deformations are in fact always in parallel, and therefore, it is difficult to isolate them experimentally. Even though calculation of the elastic strains is related to considerable experimental difficulties, and their magnitude is relatively small in the presence of plastic strains, their precise evaluation is still extremely important. Basically, there are three reasons for such a need: first of all, any accurate calculation of the behaviour of sandy soils at very small strains (e.g. less than $10^{-5}$) relies on the elastic parameters since at such small strains the elastic behaviour dominates; on the other hand, in the formulation of the flow theory of plasticity only the plastic strains are involved, and therefore, it is necessary to deduct the elastic contributions from the experimentally obtained strains; finally, constitutive models must include elastic strain components in order to invert the stress–strain relations. This will be demonstrated on the strain controlled version of the present model in Chapter 7 and Chapter 8. Hence, the elastic behaviour has to be appropriately represented in the constitutive model in order to assure accurate simulation of the soil behaviour over the whole required range of strains, from small strains, up to large strains and failure of soil.

Despite the difficulties related to the determination of the elastic parameters there are many experimental procedures for their evaluation, including laboratory tests, field loading tests and field shear wave velocity measurements. As regards the laboratory tests, it is considered that at least a good estimate of the elastic moduli can be obtained from the following: the initial slope of the virgin stress–strain relation (Tatsuoka and Shibuya, 1991); the initial slope of the unloading curve (Cole, 1967; Lade and Nelson, 1987); from the slope of the unloading–reloading curve (Duncan and Chang, 1970); and from proportional loading tests (constant stress ratio) with decreasing effective stress and shear stress (Rowe, 1971) among others. All these methods for determination of the elastic parameters are based on the assumption that soil exhibits purely elastic behaviour or that at least this behaviour is dominant under
those loading conditions. With respect to the above mentioned methods, it appears that the first two can provide much better information about the elastic behaviour of sandy soils, provided that both stresses and strains are accurately measured at very small strain level, while the two later methods provide relatively rough data for the elastic properties since these loading conditions involve plastic strains of comparable magnitude to the elastic ones.

Tatsuoka and his coauthors (Tatsuoka and Shibuya, 1991; Shibuya et al., 1992) conducted a comprehensive study on different soils and rocks, where strains were accurately measured at extremely low strain level of about $10^{-6} - 10^{-5}$, by local deformation transducers (LDT) (Goto et al., 1991), in monotonic and cyclic torsional simple shear (Teachavorasinskun et al., 1989 and 1990), plane strain compression (Dong et al., 1991) and triaxial compression tests. A few of the many important conclusions drawn in the above mentioned study regarding the elastic behaviour of sandy soils, and which are also related to the present study will be discussed herein.

Based on the experimental results from the above mentioned studies Tatsuoka and Shibuya (1991) concluded that the behaviour at very small strains is linear and recoverable and they set a 'limit-strain' at $10^{-5}$ below which the behaviour is considered as linear and elastic. This is illustrated in Fig. 6.1, where the stress-strain relations, with enlarged initial portion, of Toyoura sand and Silver Leighton Buzzard sand are shown. Another important observation by the same authors is that the elastic region does not expand with increase of the applied maximum stress level, and therefore, they suggested using 'kinematic hardening modelling with fixed and rather small size of elastic region' in order to realistically represent the behaviour of sands. As shown in Fig. 6.2, the shear modulus at unloading is equal to the shear modulus measured at the initial part of the curve only at the very beginning of the unloading part, thus demonstrating that the elastic region is very small.

It is generally accepted that for practical purposes the elastic behaviour of sandy soils can be considered as isotropic, and that all anisotropic behaviour should be referenced to the plastic part of the response. This assumption simplifies the matters considerably and requires only two independent parameters for describing the elastic behaviour, and these may be Young's modulus $E$ and Poisson's ratio $\nu$ or the shear
Figure 6.1 Stress–Strain Relations of Toyoura Sand and Silver Leighton Buzzard Sand at Small Strains (after Tatsuoka and Shibuya, 1991)

Figure 6.2 Illustration of the Small Size of the Elastic Region (after Tatsuoka and Shibuya, 1991; Teachavorasinskun, 1989)
modulus $G$ and the bulk modulus $B$.

Experimental results indicate that elastic properties are dependent on the state of the soil, i.e. they may be expressed in terms of void ratio or density and mean effective stress. Thus, it is customary to assume that Poisson's ratio $\nu$ is constant, whereas the Young's modulus $E$, the shear modulus $G$ and the bulk modulus $B$ are represented as functions of void ratio and density. It should be emphasized that the elastic moduli are related to the density of the soil, or to void ratio $e$, rather than to the relative density $D_r$, as pointed out by Iwasaki and Tatsuoka (1977) and Alarcon-Guzman et al. (1989).

Most of the expressions for elastic shear modulus $G$ presented in the literature have the following general form:

$$ G = A \cdot F(e) \cdot p^n $$

(6.2)

where $A$ is a constant, $F(e)$ is a function describing the influence of void ratio, $p'$ is the mean effective stress and $n$ is an exponent which is usually taken as 0.5. Following this general form, Iwasaki and Tatsuoka (1977) and Iwasaki et al. (1978) proposed a set of equations for sandy soils where $G$ is expressed as:

$$ G = A(\gamma) \cdot B \cdot \frac{(2.17 - e)^2}{1 + e} \cdot p^{n(\gamma)} $$

(6.3)

in which $A(\gamma)$ and $n(\gamma)$ vary from 900 to 700, and 0.4 to 0.5, respectively, for the strain range between $10^{-6}$ to $10^{-4}$. The variation of these parameters with strain indicates that even at these small strains, the behaviour was yet not purely elastic. However, for practical purposes the values proposed for the smallest strain at $10^{-6}$ can be assumed to represent the elastic behaviour. As regards the parameter $B$, Iwasaki and Tatsuoka found that for uniform clean sands, $B$ is equal to unity and independent of the grain size in the range between $D_{10} = 0.16 - 3.2$ mm. On the other hand, they concluded that $B$ is strongly affected by the nonuniformity of the sand and by the fines content such that $B$ decreases as sand is more non-uniform and as the presence of fines increases. This is illustrated in Fig. 6.3 where the results of their study with respect to the parameter $B$ are presented.

The calculation of the elastic parameters for the tests presented in Chapter 4
Figure 6.3 Influence of Uniformity and Fines Content on the Coefficient $B$
(after Iwasaki and Tatsuoka, 1977)

was done based on the equation proposed by Iwasaki and Tatsuoka (1977) for clean sands at $\gamma = 10^{-6}$ (Eq. 6.3). As explained above the coefficient $B$ is unity for uniform clean sand, while the coefficient $A$ was evaluated by a back calculation using Eq. (6.3) and values for the initial shear modulus $G$ of Toyoura sand measured in torsional shear tests at very small strain levels, from the data published by Tatsuoka and Shibuya (1991), Shibuya et al. (1992), and Teachavorasinskun et al. (1989 and 1990). Hence, the following expression was obtained for the elastic shear modulus of Toyoura sand:

$$G = 960 \frac{(2.17 - e)^2}{1 + e} p^{0.4}$$  \hspace{1cm} (6.4)

Shear modulus was also calculated from the initial slope of the unloading curve in several of the tests presented in Chapter 4 such as

$$G = \left. \frac{\partial \tau}{\partial \gamma} \right|_{\text{at unloading}}$$  \hspace{1cm} (6.5)
Figure 6.4 Evaluation of the Elastic Shear Modulus from Unloading Curve
This is schematically illustrated in Fig. 6.4 for the tests on Toyoura sand at \( p' = 1.0 \) and 2.0 kgf/cm\(^2\). It is to be noted that the values of shear modulus \( G \) obtained by Eq. (6.5) are smaller than those obtained from Eq. (6.4), since in the former case the slope of the first increment of change was in fact measured \( (G = \Delta \tau/\Delta \gamma) \). Any accurate judgement of the initial slope of the unloading curve is very difficult practically. In the present study purely elastic behaviour was not observed or measured at any loading condition, as was the case with strain and stress level. For this reason, the calculation of the elastic shear moduli of Toyoura sand for different density and stress conditions comprised in the tests presented in Chapter 4, was done by using Eq. (6.4). Since this equation is derived also on the basis of results obtained from tests on Toyoura sand (Iwasaki and Tatsuoka, 1977 and Iwasaki et al., 1978) and the parameter \( A \) was back calculated from results of tests on the same material with accurate measurements at very small strains (Tatsuoka and Shibuya, 1991; Teachavorasinskun et al. 1989 and 1990), it is considered that this equation provides a very good estimate of the elastic shear moduli of Toyoura sand for various density and stress conditions. In the calculation of elastic strains, Poisson's ratio was assumed constant at \( v = 0.2 \).

In a similar manner to the procedure described for Toyoura sand, the shear modulus for Toyoura sand–KAO 15% was determined as \( G = 350 \) kgf/cm\(^2\), for the density and stress conditions given in Fig. 6.5. From the swelling line shown in Fig. 6.5b, the bulk modulus was calculated and then inverted to an equivalent value of Poisson's ratio of \( v = 0.225 \). It is to be noted that the shear modulus of Toyoura sand – KAO 15% is much smaller than the corresponding value of the shear modulus for Toyoura sand. This is in agreement with the effects of fines and nonuniformity on the shear modulus, as described in the previous discussion on parameter \( B \) of Eq. (5.3).

6.3 SHEAR STRESS–PLASTIC STRAIN RELATIONSHIP

6.3.1 Modelling Concept

It was shown in Chapter 4 that the stress–strain behaviour of sandy soils is strongly influenced by the interaction of density and effective stress, and therefore, in order to appropriately describe this behaviour over a wide range of densities and
Figure 6.5 Stress–Strain Curve, Consolidation Line and Swelling Line for Toyoura Sand–KAO 15%
effective stresses it is necessary to account for their combined influence. However, it is difficult to achieve this in the framework of the existing concept applied to the constitutive models for sandy soils since, as mentioned earlier, this concept is based on the postulate that density alone should be used as a reference parameter. Besides the difficulty to properly describe the behaviour by this concept, it is important to recognize its practical shortcoming. Namely, since all model parameters are referenced to, and valid for a given density only, their determination through corresponding experimental and calibration procedures has to be repeated for each of the considered densities of the soil.

Instead of defining the soil behaviour with reference to the density alone, the present model aims to reference sandy soil behaviour to a parameter that accounts for the effects of both density and mean effective stress. For this reason, a parameter for characterization of sandy soil behaviour developed within the steady state of deformation will be employed in the model. Development of a shear stress–plastic strain relation whose parameters are expressed as functions of such a parameter, and thus, are dependent on both density and effective stress, is given in the following.

**Steady State and Quasi Steady State.** Before introducing the parameter employed in the present model, a brief description of the characteristic states of the undrained response of sandy soils will be discussed. Fig. 6.6 shows a typical result for an undrained triaxial compression test on loose Toyoura sand with 10% of its silt (Zlatovic, 1993). The pre peak behaviour is characterized by relatively small axial strain and increase in the pore water pressure as indicated by the effective stress path moving towards the left in Fig. 6.6b. Following the peak there is a drop in the deviator stress associated with a relatively sharp increase in the pore water pressure or decrease in the effective stress. The strain softening behaviour continues until a transient state is reached where the behaviour changes from contractive to dilative. This is reflected in start of the dilation or decrease in the pore pressure with the effective stress path moving towards the right in Fig. 6.6b. This state is denoted in the figure as ‘quasi steady state’ (Alarcon–Guzman et al., 1988) and it is associated with the phase transformation state (Ishihara et al., 1975). It is to be noted that at the quasi steady state the sample has developed minimum strength. Further shearing is associated with dilation and an increase in the deviator stress until the steady state is
Figure 6.6 Characteristic States Observed in Undrained Triaxial Compression Tests on Toyoura Sand–TSS 10% (data from Zlatovic, 1993)
reached where the soil mass is deforming continuously under constant shear stress and constant normal stress (Poulos, 1981).

It is interesting to note that the conditions of shearing at quasi steady state are not unlike the conditions at steady state. Namely, the flat part of the stress–strain curve around the quasi steady state is associated with very small part of the corresponding effective stress path, thus indicating that relatively large deformation took place under nearly constant shear stress and constant normal stress. However, since this state is temporarily developed it has been called by many researchers the 'quasi' steady state. The loci of the points of steady state and quasi steady state in the $e-p'$–$q$ space (void ratio–effective normal stress–shear stress space) define the steady state line (SSL) and the quasi steady state line (QSSL), respectively. Their projections in the $e-p'$ plane, which are also called steady state line and quasi steady state line, are shown in Fig. 6.7a. Here Fig. 6.7a shows the steady state line and the quasi steady state line together with the corresponding states of the test shown in Fig. 6.6.

Fig. 6.7b shows the quasi steady state lines obtained by Zlatovic (1993) from undrained triaxial compression tests on samples of Toyoura sand with 10% of its silt, prepared by two different methods of sample preparation, moist placement and dry deposition (corresponding to air pluviation with zero height of pluviation, as described in Chapter 3). An important feature illustrated in this figure is that the quasi steady state is affected by the method of sample preparation, resulting in different quasi steady state lines for moist placement and dry deposition. It is well known that different methods of sample preparation, or different methods of deposition of the material result in different initial fabric of the samples, and therefore, this figure in fact indicates that the quasi steady state is dependent on the initial fabric.

Based on an extensive series of undrained triaxial compression tests on Toyoura sand prepared by wet tamping (moist placement), dry deposition and water sedimentation, Verdugo (1992) defined the corresponding steady states and quasi steady states which are shown in Fig. 6.8. Verdugo (1992) concluded that the steady state of Toyoura sand is not affected by the method of sample preparation, or initial fabric, while the corresponding quasi steady state is affected by it, as illustrated in Fig. 6.8. Further support of this conclusion was provided by results with identical patterns
Figure 6.7 Steady State Line and Quasi Steady State Line of Toyoura Sand–TSS 10% (data from Zlatovic, 1993)
Figure 6.8 Effects of Sample Preparation Method on the Steady State and the Quasi Steady State Lines of Toyoura Sand (after Verdugo, 1992)
for several other materials in the same study.

It is worth noting that the existence of the quasi steady state is associated with drop in the deviator stress and, as pointed out by Ishihara (1993), the quasi steady state can be considered as a special case in the wider sense of the phase transformation. With respect to this point, reference should be made to the 'initial dividing line' (IDL), defined by Ishihara (1993) and Verdugo (1992) as demarcation line which separates the initial states of the soil either with or without drop in the deviator stress. Hence, if there is no drop in the deviator stress even for the loosest possible state for a given method of sample preparation, then the quasi steady state, in fact, does not exist, as is the case with the samples of Toyoura sand prepared by water sedimentation (Fig. 6.8). Another important observation by Verdugo (1992) is that there exists a threshold void ratio above which the undrained strength is zero. In other words, after reaching the peak, the deviator stress drops to zero and actually the steady state and the quasi steady state are attained simultaneously.

**State Parameter Concept.** The idea of quantitative description of soil behaviour by using the state of the soil in terms of density and effective stress has been introduced by Roscoe and Poonooshab (1963). In conjunction with the principles of similarity of model behaviour and corresponding behaviour of prototype, Roscoe and Poonooshab (1963) concluded that in order to achieve identical strain response of corresponding elements of soil in model and prototype, besides the similarity in the stress paths, it is essential that the initial states of the elements are related in a specific manner. On the basis of theoretical and experimental considerations they defined the condition for similarity in strains of two elements as

\[
\frac{\sigma_{o1}}{\sigma_{o1}} = \exp \frac{e_{o1} - e_{o2}}{\lambda}
\]

where \(e_{o1}, \sigma_{o1}\) and \(e_{o2}, \sigma_{o2}\) are initial states of the corresponding elements in terms of void ratio and effective normal stress, and \(\lambda\) is the slope of the critical state line plotted as a straight line in \(e-\log \sigma\) plane. In other words, two elements of a given soil when subjected to similar stress paths will have the same strains provided that the difference between the initial void ratio and the void ratio of the critical state line at
the same effective normal stress is the same for each element. A schematic illustration of the condition for similarity proposed by Roscoe and Poorooshab is shown in Fig. 6.9. As indicated by the condition of similarity proposed by Roscoe and Poorooshab, the behaviour of soil is characterized by the state of the soil expressed in terms of void ratio and effective stress relative to a reference state of the soil as represented by the critical state line.

Along the same line Been and Jefferies (1985) proposed the state parameter concept in the framework of steady state of deformation, and postulated that the behaviour of sand may be characterized in terms of two variables: a state parameter which combines the influence of void ratio and stress; and a fabric parameter that characterizes the arrangement of the sand particles. These authors proposed the state parameter, $\psi$, whose definition (Fig. 6.10) strictly follows the condition for similarity (Eq. 6.6) proposed by Roscoe and Poorooshab. However, Been and Jefferies (1985) and (1986) pointed out that the state of the soil in terms of the void ratio and effective stress is not the only controlling parameter, and therefore, the condition for similarity
Figure 6.10 Definition of the State Parameter—\( \psi \) (after Been and Jefferies, 1985)

is necessary, but not a sufficient condition for similarity in the behaviour of sand. Another distinction in their concept is that they have taken up as reference state the steady state instead of the critical state, which under the assumption that the steady state and the critical state are the same (Been et al., 1991; Sladen and Oswell, 1988; among others) has practical, or in other words experimental significance only. It is important to note that the state parameter \( \psi \) has been implemented by Been and Jefferies as a parameter that characterizes the behaviour of sand independently of material type.

**State Index, \( I_s \).** Based on an extensive study relying on triaxial compression tests on several sandy soils including Toyoura sand as the principal material, Verdugo (1992) and Ishihara (1993) provided comprehensive interpretation and characterization of the undrained sandy soil behaviour. In the framework of that study, Verdugo and Ishihara proposed a parameter for characterization of undrained sandy soil response termed State Index, \( I_s \). The original definition of the state index is shown in Fig. 6.11,
Figure 6.11 Definition of the State Index $I_s$ (after Verdugo, 1992; Ishihara, 1993)
while its simplified schematic illustration is given in Fig. 6.12. Unlike the condition for similarity and the state parameter $\psi$, the state index $I_s$ comprises two reference states, and these are: the quasi steady state and an upper reference which is associated either with the threshold void ratio above which the residual strength is zero or with the isotropic consolidation line for the loosest state corresponding to a given method of sample preparation. The state index is defined as ratio between the difference of the void ratio of the upper reference and the initial void ratio, at the initial effective stress ($\Delta e_c = e_{ur} - e_i$), and the difference between the void ratios of the upper reference and the quasi steady state line (QSSL) ($\Delta e_{qss} = e_{ur} - e_{qss}$) for the same effective stress

$$I_s = \frac{\Delta e_c}{\Delta e_{qss}} \quad (6.7)$$

This definition of the state index is derived from the similarities of stress–strain characteristics and pore water pressure response in undrained triaxial compression tests (Verdugo, 1992).

It has already been pointed out that in the definition of the state index the quasi steady state is adopted as a reference state. On the other hand, it has been shown by
Verdugo (1992), Ishihara (1993) and Zlatovic (1993), as well as illustrated in Fig. 6.7, that the quasi steady state is affected by the method of sample preparation, and consequently, there is no unique quasi steady state for a given sandy soil. Hence, by using the quasi steady state as a reference state, it is possible to distinguish the effects of the initial fabric associated with given method of sample preparation. Thus, the state index represents a single parameter that combines the effects of void ratio, effective stress and initial fabric. With respect to the previously introduced state parameter concept, the state index, $I_s$ actually incorporates both, the state parameter and the fabric parameter.

Another important difference is that the quasi steady state takes place in the range of moderate strains while the steady state is usually achieved at very large strains, and therefore, the quasi steady state and the steady state characterize the behaviour over a different strain range, up to moderate strains and at large strains, respectively. Finally, by introducing the upper reference in the definition of $I_s$ it is possible to distinguish the typical undrained behaviour with zero residual strength. It is also important to notice the difference in the approaches of the state parameter concept and the state index $I_s$. Namely, Been and Jefferies (1985) and (1986) introduced the state parameter $\psi$, as a parameter that characterizes the behaviour of sandy soils independently of the sand type. On the other hand, the state index $I_s$ introduced by Verdugo (1992) and Ishihara (1993) is defined, as a parameter that characterizes the behaviour of a given soil for particular method of sample preparation or initial fabric. In other words, the state parameter $\psi$ is implemented for general description of the behaviour of sandy soils while the state index $I_s$ captures in details the behaviour of particular sandy soil.

**State Index $I_s$ as Parameter for Drained Behaviour.** The above discussion clearly indicates that the state index $I_s$ is a more appropriate parameter to be implemented in a sophisticated constitutive model where parameters capable of accurate description of the behavioural properties are needed. However, the state index $I_s$ has been defined on the basis of undrained triaxial compression tests, and therefore, it is necessary to check if this parameter can represent the stress–strain behaviour that has been observed in the drained torsional shear tests which were presented in Chapter 4. It should be emphasized again that the state index has been defined based on the
similarities of the stress–strain curves and the effective stress paths in undrained tests (Verdugo, 1992). Thus, it seems reasonable to assume that similar values of the state index $I_s$ are associated with similar normalized stress–strain curves, i.e. stress ratio–shear strain relationships. Based on this assumption, in the following, the stress ratio–shear strain relationships obtained in the drained torsional shear tests will be compared to the corresponding values of the state index.

In Figs. 6.13 – 6.15 the initial states of the samples of Toyoura sand, Toyoura sand with 15% kaolin and Toyoura sand with 10% of its silt, in the drained torsional shear tests presented in Chapter 4, are shown together with the corresponding reference lines necessary for calculation of the state index. The reference lines are defined based on the data from Verdugo (1992) (Toyoura sand), Zlatovic (1993) (Toyoura sand – TSS 10%) and Huang (1993) (Toyoura sand – KAO 15%). It is to be noted that the upper reference for all the considered soils is represented by the threshold void ratio separating the initial states of the soil with zero and non–zero undrained residual strength.

Fig. 6.13 shows two quasi steady state lines of Toyoura sand, one for wet

![Figure 6.13 Reference Lines for Toyoura Sand and Initial States of the Samples in the Drained Torsional Shear Tests](image-url)
tamping (referred as moist placement in some of the previously mentioned studies) and the other for dry deposition (corresponding to air pluviation with zero height of pluviation), respectively. Since the difference between the two quasi steady state lines is not pronounced and the reference lines of the other soils are obtained from samples prepared by wet tamping, the values of \( I_s \) will be evaluated using the reference lines corresponding to the wet tamping method of sample preparation. It is to be noted that Verdugo (1992) pointed out that effects of initial fabric on QSSL are not pronounced for clean sands while, on the other hand, they are significant for silty sands.

By using Eq. (6.7), values of the state index \( I_s \) were calculated for each of the drained torsional shear tests whose initial states are illustrated by the points in Figs. 6.13 – 6.15. Calculated values for the state index are given in Figs. 6.14 and 6.15 as well. It is worth noting that the state index equals zero \( (I_s = 0) \) for the states along the upper reference line, whereas \( I_s \) equals one \( (I_s = 1) \) for the states along the quasi steady state line. Consequently, the initial states above the upper reference are associated with negative values of the state index \( (I_s < 0) \); for the initial state between the upper reference line and the quasi steady state \( I_s \) changes between zero and unity \( (0 < I_s < 1) \); and when the initial state is below the quasi steady state line, the state index takes a value greater than one \( (I_s > 1) \). The pattern in the change of the value of the state index by the initial state is illustrated for Toyoura sand in Fig. 6.16 where contours of equal state index values are shown for certain values of \( I_s \). It is to be noticed that the contours of equal state index values, shown in this figure, closely follow the points denoting the initial states of the drained torsional shear tests for which almost unique stress–strain curves were obtained (Fig. 4.27).

The effectiveness of the state index in representing the stress–strain behaviour observed in the drained torsional shear tests will be demonstrated in the following. Fig. 6.17a shows the stress–strain curves obtained in tests with different combinations of void ratio and mean effective stress, ranged between \( e = 0.73 – 0.90 \) and \( p' = 0.3 – 4.0 \) kgf/cm\(^2\), respectively. Even though the six tests have quite different initial states, two unique stress–strain curves can be observed. This figure also shows the calculated values of the state index \( I_s \) for these tests. Apparently, for the two unique stress–strain curves, very small variation in the state index is obtained, thus indicating that the similarity of the stress–strain behaviour is associated with similar values of \( I_s \). In other
Figure 6.14  Reference Lines for Toyoura Sand–KAO 15% and Initial States of the Samples in the Drained Torsional Shear Tests

Figure 6.15  Reference Lines for Toyoura Sand–TSS 10% and Initial States of the Samples in the Drained Torsional Shear Tests
Figure 6.16 Equi-State Index Lines (after Verdugo, 1992)
Figure 6.17 Validation of the State Index as Parameter for Drained Behaviour
words, equi-state index lines are nearly coincident with the initial states which are characterized with very similar stress–strain characteristics.

Another interesting observation is shown in Fig. 6.17b, where another set of stress–strain curves of Toyoura sand from the drained torsional shear tests is presented. Again, the initial states are remarkably different with the void ratio in the range between \( e = 0.73 - 1.0 \) and mean effective stress \( p' = 0.3 - 3.0 \text{ kgf/cm}^2 \). In addition to the stress–strain curves the values of the state index are given in the figure as well. The gradual change in the characteristics of the stress–strain curves is associated with an appropriate change in the value of the state index. As a rule, a large value of \( I_s \) corresponds to a stiff curve with high peak strength, and conversely, small values of \( I_s \) are associated with soft behaviour and relatively low strength. It is to be noted that this pattern is demonstrated in Fig. 6.17a as well. Except for one case which will be discussed subsequently, it has been found that the state index \( I_s \) represents an extremely suitable parameter for describing the stress–strain characteristics observed in drained torsional shear tests on several sandy soils over a wide range of densities and effective stresses.

The only difference between the observed behaviour in the drained torsional shear tests and that indicated by the state index appears when the initial states are above the upper reference, or when the state index takes a negative value. As illustrated in Fig. 6.16, the equi-state index lines above the upper reference line are characterized by an increase in both void ratio and mean effective stress. In other words, along these lines similar stress–strain characteristics should be expected. However, the data compiled by Lade (1978) for fifteen different sandy soils (Fig. 4.31) as well as the results of the present study and other studies relying on drained tests show that the lines of equal stress–strain characteristics follow either the pattern of the initial states with positive values of \( I_s \) or at most, behaviour corresponding to the upper reference line, with \( I_s = 0 \). The experimental results of the present study indicate that stress–strain behaviour associated with the initial states at and above the upper reference line is hardly affected by the initial state. In other words, the variation of the stress–strain characteristics for the initial states above the upper reference is very small, if any.

Careful examination of the experimental data from the undrained triaxial
compression tests on Toyoura sand presented by Verdugo (1992), shows that the effective stress paths are not affected by the initial state if $I_z$ is smaller than zero, while there is some slight difference in the stress–strain curves, mostly in the softening part of these curves. However, the difference of the normalized stress–strain curves obtained in the undrained tests is rather small, although present, when the initial states are above the upper reference line; for all these states residual strength is zero, and therefore, the geometrical similarity of the stress–strain curves is almost granted. Hence, it can be concluded that if the initial state is above the upper reference line, no significant difference in behaviour will occur in either undrained or drained tests.

In support of this conclusion, it is to be recalled that the stress–strain behaviour of Toyoura sand with 15% kaolin, presented in Chapter 4 (Fig. 4.23a) was characterized by a unique stress–strain curve although the mean effective stress varied from 0.3 – 4.0 kgf/cm$^2$. As shown in Fig. 6.14, the initial states for all of these tests are well above the upper reference. Similarly, in Fig. 4.25a, a unique stress–strain curve was observed for the test on Toyoura sand – TSS 10%, with void ratio $e = 0.86$, at $p' = 1.5, 2.0, 3.0$ and 4.0 kgf/cm$^2$. Apparently, Fig. 6.15 shows that the initial states for all of these tests are either at or very close to the upper reference line. Finally, an interesting observation related with this matter is illustrated in Fig. 6.18, which shows the stress–strain curves of all the tests on Toyoura sand and Toyoura sand with 10–15 % fines content whose initial states are either at or above the corresponding upper reference lines for each of the considered soils, i.e. the state index takes values between 0 and $-4.3$. Even though four different soils are considered and the initial states in terms of void ratio and mean effective stress are remarkably different, almost unique stress–strain behaviour can be observed. It should be emphasized that all these results were obtained in tests on samples prepared by the wet tamping method.

In the preceding pages the validity of the state index $I_z$ as a parameter for representing drained stress–strain behaviour of sandy soils was demonstrated on the test results from the drained torsional shear tests presented in Chapter 4. It was found that the state index $I_z$ captures the combined effects of void ratio and mean effective stress in a unique manner and assures a high degree of accuracy in representing the stress–strain behaviour of sandy soils. By introducing the quasi steady state into the
Figure 6.18 Stress-Strain Behaviour of Toyoura Sand and Toyoura Sand with 10-15% Fines in Drained Torsional Shear Tests on Samples with $I_s \leq 0$.
definition of the state index (Verdugo, 1992; Ishihara, 1993) this parameter comprises even the influence of the initial fabric, and therefore, enables us to distinguish those effects which are known to be quite significant. The upper reference line provides one important reference state for both, undrained and drained behaviour. Thus, undrained behaviour is characterized by zero residual strength, while drained behaviour shows almost unique stress–strain behaviour when the initial state of the soil is above the upper reference line. Even though some small variations in the stress–strain curves are to be expected for these initial states, it seems reasonable to assume that the upper reference line indicates the softest stress–strain behaviour for a given soil. In general, both undrained and drained behaviour are rather insensitive to the change of the initial state when it is above the upper reference state.

6.3.2 Modified Hyperbolic Relationship

Certainly the most widely used function for representing the nonlinear stress–strain curve of soil is hyperbola. Since it was introduced by Kondner (Kondner and Zelasko, 1963), the original hyperbolic relationship or its modified versions have been used in the majority of constitutive models for sands (Duncan and Chang, 1970; Prevost and Hoeg, 1975; Vermeer, 1978; Poorooshah and Pietruszczak, 1986 among others). The main reason for the significant usage of hyperbolic relation as a basis for modelling the nonlinearity of the stress–strain relationships of sands arises from the following features: the form of the relation is relatively simple with only two parameters, both with a clear physical meaning; the determination of the parameters is simple and straightforward; and finally, it has been considered that stress–strain curve of soils can be approximated with a high degree of accuracy by using hyperbolic relation. However, it is now well known that a two–constant hyperbolic relation can not accurately describe the stress–strain behaviour of soils over a wide range of strains that need to be considered in the static and dynamic problems related to soils. Therefore, many alternative forms and modifications of the hyperbolic relation have been proposed (Hardin and Drnevich, 1972; Tatsuoka and Shibuya, 1991 and Shibuya et al. 1991; Griffths and Prevost, 1990 among others). The stress–strain model of the present study is also based on the hyperbolic relation. It is to be noted that the proposed relationship will only be applied to the plastic component of strains.
Kondner proposed the following relationship for modelling the stress–strain curves of soils subjected to monotonic triaxial compression under confining pressure $\sigma_3$:

$$q = \sigma_1 - \sigma_3 = \frac{e_1}{a + b e_1}$$  \hspace{1cm} (6.8)

where $q$ is the deviator stress, $e_1$ is the axial strain and $a$ and $b$ are the two parameters of the hyperbolic relation. In the original hyperbolic relation (OHR) the parameters $a$ and $b$ are constants related to the initial stiffness and strength of the soil such as

$$a = \frac{1}{E_{\text{max}}} , \quad b = \frac{1}{q_{\text{max}}}$$  \hspace{1cm} (6.9)

where $E_{\text{max}}$ and $q_{\text{max}}$ are Young's modulus and compressive strength, respectively. However, as was clearly demonstrated by Tatsuoka and Shibuya (1991), the original hyperbolic relation significantly overestimates the stiffness of the monotonic stress–strain curves of sandy soils and in fact fits the experimental data only for the early stage of shearing when the strains are very small and nearly elastic. This was noticed by Kondner as well, and therefore he proposed a modified hyperbolic relation which is based on determination of the parameters $a$ and $b$ as best fit values of the experimental data, from a plot known as transformed hyperbolic representation, shown in Fig. 6.19. Thus, the two constant hyperbolic relation proposed by Kondner (KHR) is based on the assumption that the relation $e_1/q$ versus $e_1$ plots as a straight line. However, Kondner pointed out that most of the experimental data in the range of small strains deviate downward from the straight line plot while, at extremely small strains, some data tends to approach a horizontal line indicating linear behaviour. It is to be noted that the comments made by Kondner are referenced to the stress–strain behaviour represented in terms of total strains. The speculations related to the linear and elastic behaviour at extremely small strains have been firmly supported by the results of the study by Tatsuoka and his coauthors which was discussed in detail in the previous section (Tatsuoka and Shibuya, 1991).

The data obtained by the drained torsional shear tests which were presented in Chapter 4 were plotted in the transformed hyperbolic representation shown in
Figure 6.19 Transformed Hyperbolic Representation of the Stress–Strain Relation (after Kondner and Zelasko, 1963)

Fig. 6.19. The data were plotted in terms of plastic strains, i.e. after subtracting the elastic strain components from the measured total strains. A typical plot of the experimental data together with the fitted linear relation is illustrated in Fig. 6.20. The deviation of the experimental data as shown in Fig. 6.20 is typical for sandy soils and it indicates greater nonlinearity of the stress–strain relationship than that of the hyperbolic relation. It is important to notice that this characteristic of the stress–strain behaviour should be attributed to the plastic strains alone. As shown in Fig. 6.20b, the deviation from the straight line approximation increases with decreasing strain. Since the vertical intercept at $\gamma_p / (\tau/p)$ is equal to the reciprocal of the normalized initial plastic shear modulus, $1/G_p' = p/f_p$, the tendency for the downward shift of the experimental data in fact indicates an increase in the initial plastic shear modulus. Another important observation is that there is good approximation of the experimental data by the two constant hyperbolic relation proposed by Kondner (KHR) when the strains are relatively large, e.g. greater than 0.5 – 1.0 %.
The above discussion clearly shows that neither the original hyperbolic relation (OHR) nor the two-constant relationship proposed by Kondner (KHR) can precisely simulate the stress-strain curves of sandy soils over a wide range of strains, from nearly elastic behaviour up to large strains associated with failure of the soil. Therefore, a number of different approaches have been proposed in order to improve the stress-strain representation and to eliminate the shortcomings of the two-constant hyperbolic relation. Most of the proposed relations are based either on using coefficients as functions of strain, instead of the constants in the hyperbolic relation, or on change of the form of the hyperbolic relation. However, the detailed analysis of these relations performed by Tatsuoka and Shibuya (1991) has revealed that the majority of the proposed stress-strain relations have met with little success in achieving the considerable task. Thus, except for the relation proposed by Tatsuoka and Shibuya (1991) and Shibuya et al. (1991), models are still valid and successful in the simulation of the stress-strain behaviour of sandy soil only within a limited range of strains.

In the preceding pages a detailed description and discussion on the hyperbolic relation, including the original hyperbolic relation (OHR) and the relation proposed by Kondner (KHR), were given in order to emphasize their advantages and shortcomings, and particularly because these relations clarify the modified hyperbolic relation that will be developed in the following. Basic principles followed in the evolution of the stress-strain relation arise from the necessity to represent the stress-strain behaviour over a wide range of strains, and the requirement that the parameters of the relation have a high degree of correlation with the state index $I_s$. It is to be expected that the latter requirement may be fulfilled only if non-fitting parameters, or parameters with clear physical significance are used in the stress-strain relation. Besides that, it is essential to accomplish these demands with minimum number of parameters whose determination should be simple and straightforward.

By expressing Eq. (6.8) in terms of plastic strains and the corresponding quantities for torsional shear, it may be rewritten as

$$
\tau = \frac{\gamma_p}{\frac{1}{G_p} + \left(\frac{1}{\tau_f}\right) \gamma_p}
$$

(6.10)
Figure 6.20  Transformed Hyperbolic Representation of the Shear Stress–Plastic
Strain Relation of Toyoura Sand in Drained Torsional Shear Test
in which \( \tau \) is shear stress, \( \gamma_p \) is plastic shear strain, and \( G_p \) and \( \tau_f \) are initial plastic shear modulus and shear stress at failure, respectively. Further manipulations yield the normalized stress–strain relation

\[
\frac{\tau}{p'} = \frac{G_p^N \gamma_p \left( \frac{\tau}{p'} \right)_f}{\left( \frac{\tau}{p'} \right)_f + G_p^N \gamma_p}
\]  

(6.11)

where \( G_p^N = G_p/p' \) is the normalized initial plastic modulus and \( (\tau/p')_f \) is the stress ratio at failure.

It is postulated that there is no correction for the stress ratio at failure \( (\tau/p')_f \) and hence, the experimental data for the strength is the actual input in the stress strain relation. This is done in the belief that there is no sound reason to express the strength of the soil as function of strain, and on the other hand, to reduce the number of the parameters of the model. It will be shown later that this approach leads to a simple and straight-forward determination procedure for the stress–strain parameters.

On the other hand, it is assumed that the normalized initial plastic shear modulus \( G_p^N \) is a function of strain

\[
G_p^N = f(\gamma_p)
\]  

(6.12)

The evaluation of this function is based on the plot of the experimental data as shown in Fig. 6.21. The points in this figure represent the best fit values for the normalized initial plastic shear modulus so as to achieve accurate simulation of the stress–strain curve at each of the considered strains by using hyperbolic relation. As shown in Fig. 6.21, in order to attain precise representation of the stress–strain curve by using hyperbolic relation (Eq. 6.11) with constant stress ratio at failure \((\tau/p')_f\), it is necessary that the normalized initial plastic shear modulus \( G_p^N \) changes with the plastic shear strain. This figure clearly shows the reason why models based on a two–constant hyperbolic relation have failed to appropriately describe stress–strain behaviour over the whole required range of strains.
It is interesting to note that the minimum normalized initial plastic shear modulus $G^N_{\gamma_p,\text{min}}$ corresponds to the initial plastic shear modulus derived from the transformed hyperbolic representation as proposed by Kondner, i.e., to the initial plastic modulus of KHR. On the other hand, the maximum normalized initial plastic shear modulus $G^N_{\gamma_p,\text{max}}$ corresponds to the initial plastic modulus of OHR, as indicated in Figs. 6.20 and 6.21. Hence, the plastic moduli of OHR and KHR in fact provide the upper and the lower bounds within which the initial plastic shear modulus should change with plastic strain. Based on the observations as illustrated in Fig. 6.21, the relation between initial plastic modulus $G^N_p$ and plastic shear strain $\gamma_p$ is approximated by exponential function with the following expression:

$$G_p^N = (G^N_{\gamma_p,\text{max}} - G^N_{\gamma_p,\text{min}}) \exp\left(-f \frac{\gamma_p}{\gamma_1}\right) + G^N_{\gamma_p,\text{min}}$$  \hspace{1cm} (6.13)$$

where $\gamma_1$ is limit strain and $f$ is exponential parameter.

It is worth noting that actually $\gamma_1$ and $f$ can merge into a single parameter,
although they are separated for the purpose of clarification of the expression. Thus, 
the limit strain $\gamma_l$ shows the strain above which there is no need for change in the 
initial plastic shear modulus, and therefore, the stress–strain curve can be accurately 
simulated by using constant value of $G_p^N = G_{p,\text{min}}^N$. The plot of the initial plastic 
modulus versus plastic shear strain (Fig. 6.21a), which is typical for all of the tested 
materials in the present study, shows that there is in fact no significant change in $G_p^N$ 
when $\gamma_p$ is greater than 1.0 %. In support of this observation are numerous data 
showing that KHR provides very good approximation for the stress–strain curve when 
the strains are greater than 0.5 – 1.0 %. Hence, it is assumed that for sandy soils the 
limit strain can be set constant at $\gamma_l = 1.0$ % plastic strain. Further examination of the 
Eq. (6.13) shows that, $G_{p,\text{max}}^N$ and $G_{p,\text{min}}^N$ control the value of $G_p^N$ and that parameter 
f has no significant effect on it. It will be demonstrated later that the parameter f can 
be set constant and independent of both density and effective stress.

By introducing the expression for $G_p^N$ given in Eq. (6.13) into Eq. (6.11), the 
final form of the stress–strain relation is derived. The accuracy of the proposed 
relation is illustrated in Fig. 6.22 where together with the experimental data the stress–
strain curves obtained using the original hyperbolic relation (OHR), Kondner's 
hyperbolic relation (KHR) and the proposed relation are given. As shown in Fig. 6.22, 
Eq. (6.13) provides gradual change in the initial plastic shear modulus that is reflected 
in gradual shift of the stress–strain curve from the OHR towards KHR as strain 
increases; this in turn assures a high degree of accuracy in the simulation of the 
observed stress–strain curve for the entire considered strain range.

### 6.3.3 Stress–Strain Parameters as a Function of State Index

The stress–strain relation defined by Eqs. (6.11) and (6.13) has four parameters: 
$(\tau/p')_f$ – stress ratio at failure; $G_{p,\text{min}}^N$ – minimum normalized initial plastic shear 
modulus; $G_{p,\text{max}}^N$ – maximum normalized initial plastic shear modulus; and exponential 
parameter $f$. Before determining these parameters for the drained torsional shear tests, 
the effects of these parameters on the stress–strain relation will be briefly discussed 
together with the procedure for their evaluation.

The examination of each of the parameters of the stress–strain relation is 
illustrated schematically in Fig. 6.23. As mentioned previously, there is no correction
Figure 6.22  Comparison of Proposed Stress–Strain Relation with the Original and Kondner's Hyperbolic Relation

for the stress ratio at failure and its value is obtained directly from the experimental results. Thus, for a known value of $(\tau/p)'_f$ the effects of the other parameters, i.e. $G^{N, min}_{p}$, $G^{N, max}_{p}$, and $f$ on the stress–strain curve are shown in Figs. 6.23a – 6.23c, respectively. It is important to notice that $G^{N, min}_{p}$ and $G^{N, max}_{p}$ control the stress–strain curve over different range of strains, i.e. from moderate to large strains and at small strains, respectively. Another important observation is that the exponential parameter $f$ has very small influence on the stress–strain curve. More specifically, it was found that once the stress ratio at failure is specified, the stress–strain curve for plastic shear strain greater than $\gamma_f$ is solely controlled by $G^{N, min}_{p}$. On the other hand, initial part of the stress–strain curve is mainly influenced by the value of $G^{N, max}_{p}$. The parameter $f$ is only important in that it provides a smooth transition from $G^{N, max}_{p}$ to $G^{N, min}_{p}$. The simulation of stress–strain curves for different sandy soils with various initial states has revealed that the parameter $f$ is independent of both void ratio and effective stress, and therefore, it can be set as constant. By doing so, the parameters of the proposed stress–strain relation are reduced to one constant, $f$, and three parameters $(\tau/p)'_f$, $G^{N, min}_{p}$, and $G^{N, max}_{p}$, dependent on both void ratio and effective stress.
Figure 6.23 Influence of the Parameters of the Proposed Stress–Strain Relation on Stress–Strain Curve
Due to the fact that the initial part of the stress–strain curve and the stress–strain curve at large strains are controlled independently by $G_{p,\text{max}}^N$ and $G_{p,\text{min}}^N$, respectively, the determination procedure for the stress–strain parameters is rather simple and straightforward. Namely, for a given stress ratio at failure $(\tau/p)'_f$ obtained from experimental results, firstly $G_{p,\text{min}}^N$ is defined as best-fit value for $G_p^N$ in Eq. (6.11) so as to accurately simulate the experimental stress–strain curve at strain larger than 0.5 – 1.0 %. Next, $G_{p,\text{max}}^N$ is determined as best fit value for $G_p^N$ at small strain level up to 0.5 %. Due to the insignificant effects of the parameter $f$ on the characteristics of the stress–strain curve, assumption of a constant value for $f$ independent of the initial state of the soil is recommended.

Following the above procedure, the stress–strain parameters $(\tau/p)'_f$, $G_{p,\text{max}}^N$ and $G_{p,\text{min}}^N$ were evaluated for each of the drained torsional shear tests on Toyoura sand and Toyoura sand with 10–15 % fines content by weight, shown in Figs. 6.13 – 6.15. The evaluated stress–strain parameters and the values of the state index for the tests on Toyoura sand are listed in Table 6.1. It is to be noted that $f$ was assumed to be constant at $f = 4$ in the evaluation of the parameters.

The relationship between the stress–strain parameters and the state index $I_s$ for Toyoura sand are shown in Fig. 6.24 together with the best-fit linear regression lines through the data points. Apparently, there is a remarkably good correlation between the stress–strain parameters and the state index, with correlation coefficients of 0.92, 0.93 and 0.96. Hence, the stress–strain parameters of Toyoura sand can be expressed as linear functions of the state index $I_s$ by the following set of equations:

$$\left( \frac{\tau}{p}'_f \right) = \alpha_1 + \beta_1 I_s \quad (\alpha_1 = 0.586, \beta_1 = 0.016) \quad (6.14)$$

$$G_{p,\text{max}}^N = \alpha_2 + \beta_2 I_s \quad (\alpha_2 = 283, \beta_2 = 41) \quad (6.15)$$

$$G_{p,\text{min}}^N = \alpha_3 + \beta_3 I_s \quad (\alpha_3 = 93, \beta_3 = 10) \quad (6.16)$$

Knowing the constants $\alpha_1, \beta_1 \ldots \beta_3$, the stress–strain parameters can be calculated by Eqs. (6.14) – (6.16) for any initial state of Toyoura sand, and by
Table 6.1 Stress–Strain Parameters for Toyoura Sand

<table>
<thead>
<tr>
<th>Test</th>
<th>$I_s$</th>
<th>$(\tau/p')_t$</th>
<th>$G^{N}_{\rho,\text{max}}$</th>
<th>$G^{N}_{\rho,\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>1.04</td>
<td>0.60</td>
<td>400</td>
<td>110</td>
</tr>
<tr>
<td>T4</td>
<td>2.00</td>
<td>0.61</td>
<td>400</td>
<td>110</td>
</tr>
<tr>
<td>T5</td>
<td>-2.71</td>
<td>0.52</td>
<td>240</td>
<td>70</td>
</tr>
<tr>
<td>T9</td>
<td>12.0</td>
<td>0.78</td>
<td>850</td>
<td>250</td>
</tr>
<tr>
<td>T10</td>
<td>2.21</td>
<td>0.63</td>
<td>480</td>
<td>130</td>
</tr>
<tr>
<td>T25</td>
<td>14.4</td>
<td>0.80</td>
<td>950</td>
<td>225</td>
</tr>
<tr>
<td>T26</td>
<td>7.07</td>
<td>0.69</td>
<td>550</td>
<td>165</td>
</tr>
<tr>
<td>T27</td>
<td>3.79</td>
<td>0.68</td>
<td>550</td>
<td>135</td>
</tr>
<tr>
<td>T28</td>
<td>1.17</td>
<td>0.58</td>
<td>275</td>
<td>85</td>
</tr>
<tr>
<td>T29</td>
<td>9.29</td>
<td>0.70</td>
<td>550</td>
<td>130</td>
</tr>
<tr>
<td>T30</td>
<td>2.38</td>
<td>0.61</td>
<td>300</td>
<td>110</td>
</tr>
<tr>
<td>T31</td>
<td>4.21</td>
<td>0.63</td>
<td>400</td>
<td>125</td>
</tr>
<tr>
<td>T32</td>
<td>4.52</td>
<td>0.68</td>
<td>450</td>
<td>150</td>
</tr>
<tr>
<td>T33</td>
<td>10.0</td>
<td>0.74</td>
<td>700</td>
<td>190</td>
</tr>
<tr>
<td>T35</td>
<td>3.18</td>
<td>0.66</td>
<td>320</td>
<td>120</td>
</tr>
<tr>
<td>T46</td>
<td>3.51</td>
<td>0.67</td>
<td>350</td>
<td>130</td>
</tr>
</tbody>
</table>
Figure 6.24 Relationship Between the Parameters for the Stress–Strain Relation of Toyoura Sand and the State Index $I_s$
substituting these parameters in Eqs. (6.13) and (6.11), subsequently, the corresponding stress–strain curve can be evaluated. The accuracy of the proposed relation is demonstrated in Fig. 6.25 where the simulations by the proposed stress–strain relation, shown by the solid lines, are compared to the measured behaviour of Toyoura sand in the drained torsional shear tests, indicated by the marks. As shown in Fig. 6.25, the predicted stress–strain curves are in good agreement with those measured, although both void ratio and effective stress vary significantly over the ranges of \( e = 0.73–1.0 \) and \( p' = 0.3–4.0 \) kgf/cm², respectively. The abilities of the proposed stress–strain relation will be thoroughly tested in Chapter 8 where the performance of the elastic–plastic constitutive model will be verified for a variety of initial states and loading and drainage conditions.

Proposed stress–strain model for sandy soil is summarized in Fig. 6.26 in a form of flow chart. The capability of the model to represent the combined effects of density and effective stress relies on the fact that stress–strain parameters are expressed as functions of the state index \( I_s \). Of prime importance is the fact that the state index has physical significance and is not merely an arbitrary parameter. Due to this fact, it is essential that the stress–strain parameters of the proposed relation have also clear physical significance since that assures good correlation between the stress–strain parameters and the state index. Hence, in general, any other relation that satisfies this requirement can be implemented in the proposed modelling concept. Besides the physical relevance of the concept, its practical benefit is that, once the model parameters are defined, they represent the soil behaviour over a wide range of densities and stresses. Thus, compared to the current modelling concepts applied to the constitutive models for sands which are based on referencing the model parameters to a particular void ratio or relative density, the proposed concept is both physically more relevant and practically more versatile. It is interesting to note that the state index will be used in the elastic–plastic constitutive model as a current variable instead of an initial index parameter as proposed by Verdugo (1992) and Ishihara (1993); this is similar to the usage of the state parameter \( \psi \) by Jefferies (1993).
Figure 6.25  Measured and Predicted Stress–Strain Behaviour of Toyoura Sand
6.4 STRESS – DILATANCY RELATIONSHIP

In the preceding pages a detailed discussion was given on the elastic behaviour and on the relation between the stresses and plastic or irrecoverable shear strains for sandy soils. The stress–dilatancy relation, which represents relation between the stress ratio and plastic strain increments ratio or dilatancy, will be now introduced. This relation is commonly used to describe the dilatancy or the volume change behaviour which takes place during the shearing process in soils.

Since volume change behaviour during shearing is an essential feature of the soil behaviour, significant theoretical and experimental efforts has been made in order to clarify the stress–dilatancy behaviour and to model it for arbitrary stress paths. Herein, the approach used by Moroto (1976) which was subsequently adapted by Momer and Ghaboussi (1982), Kabilamany and Ishihara (1990) and Gutierrez (1989) will be followed. The stress–dilatancy model is based on calculation of volumetric strain increment due to shearing by using the normalized plastic work.

Following the energy consideration by Roscoe (1963), which forms a basis for many stress–dilatancy relations, the dissipated energy per unit volume within an element of soil can be expressed as

\[ dW^p = \sigma_{ij} \, de_{ij}^p \]  

(6.17)

in which \( \sigma_{ij} \) and \( de_{ij}^p \) can be expressed in terms of their respective spherical and deviatoric components, and then Eq. (6.17) may be written:

\[
dW^p = \left( S_{ij} + p \, \delta_{ij} \right) \left( de_{ij}^p + \frac{1}{3} \, de_v^p \, \delta_{ij} \right)
\]

(6.18)

\[
dW^p = p \, de_v^p + S_{ij} \, de_{ij}^p
\]

where \( p \) is mean stress, \( de_v^p \) is plastic volumetric strain increment, and \( S_{ij} \) and \( de_{ij}^p \) are the deviatoric components of \( \sigma_{ij} \) and \( de_{ij}^p \), respectively.

For the general case, the term \( S_{ij} \, de_{ij}^p \) can be expressed as

\[ S_{ij} \, de_{ij}^p = c \, q \, de_v^p \]  

(6.19)
Figure 6.26 Flow Chart of the Proposed Shear Stress–Plastic Strain Relation
The factor $c$ in this equation is a variable depending on the stress path, and accounts for the difference in the dissipated energy due to the noncoaxiality of the principal strain increments and principal stress directions (Gutierrez, 1989). In the case of coaxial behaviour $c$ equals unity. Thus, Eq. (6.17) now becomes:

$$dW^p = p \, d\varepsilon^p + c \, q \, d\varepsilon^p$$  \hspace{1cm} (6.20)$$

Assuming that the volumetric strain increment is due to the dilatancy alone, the normalized shear work increment $d\Omega^p$ is defined by dividing the dissipated energy by the mean stress $p$ such as:

$$d\Omega^p = \frac{dW^p}{p} = d\varepsilon^p_{vd} + c \, \frac{q}{p} \, d\varepsilon^p$$  \hspace{1cm} (6.21)$$

where $d\varepsilon^p_{vd}$ is volumetric strain due to the dilatancy alone. Next, the assumption done by Moroto (1976) that the increment of the normalized plastic shear work $d\Omega^p$ is a unique function of the plastic shear strain increment $d\varepsilon^p$ is introduced:

$$d\Omega^p = \mu \, d\varepsilon^p$$  \hspace{1cm} (6.22)$$

where $\mu$ defines the instantaneous slope of the $\Omega^p$ versus $\varepsilon^p$ plot. Finally, by substituting Eq. (6.22) in Eq. (6.21) the stress–dilatancy relation may be written

$$\frac{d\varepsilon^p_{vd}}{d\varepsilon^p} = \mu - c \, \frac{q}{p}$$  \hspace{1cm} (6.23)$$

A typical plot of the accumulated normalized plastic shear work $\Omega^p$ versus plastic shear strain $\varepsilon^p$ for the drained torsional shear, $p'$ constant test on Toyoura sand presented in Chapter 4, is shown in Fig. 6.27. As shown in Fig. 6.27a, a unique relationship is obtained between $\Omega^p$ and $\varepsilon^p$, independent of the mean effective stress $p'$. The parameter $\mu$, which represents the slope of the curve plotted in Fig. 6.27, is not a constant, but rather it is a function of the plastic shear strain $\varepsilon^p$. This is illustrated in Fig. 6.27b where the initial part of the $\Omega^p$ versus $\varepsilon^p$ plot is presented. Apparently, the slope of the curve or the value for $\mu$ increases as the plastic shear strain increase. It is worth noting that the change in the parameter $\mu$ takes place mostly at small strain level and afterwards the slope of the curve increases gradually with very small and decreasing rate.
Several relations for the parameter $\mu$ as function of the plastic shear strain $\varepsilon^p$ have been proposed (Ghaboussi and Momen, 1982; Kabilamany and Ishihara, 1990; Gutierrez (1989) among others). Herein, the relation proposed by Kabilamany and Ishihara will be employed where $\mu$ is defined as follows

$$\mu = \mu_o + \frac{2}{\pi} (\mu_{\text{max}} - \mu_o) \tan^{-1} \left( \frac{\varepsilon^p}{S_c} \right)$$

(6.24)

in which $\mu_o$ is the initial slope or the slope at very small strain level, $\mu_{\text{max}}$ is the final slope or the slope at large strains and $S_c$ is constant which represents the plastic shear strain at which the average value for $\mu$ is attained. It is interesting to notice that $\mu_{\text{max}}$ will eventually coincide with the stress ratio at steady state $(q/p')_{ss}$.

The factor $c$ which accounts for the effects of the noncoaxiality in Eq. (6.23) is calculated as:

$$c = \cos 2\psi = | \beta_o - \beta_{de} |$$

(6.25)

Substituting Eqs. (6.24) and (6.25) in Eq. (6.23) the final form of the stress–dilatancy relation is derived

$$\frac{de^p_{vd}}{de^p} = \mu_o + \frac{2}{\pi} (\mu_{\text{max}} - \mu_o) \tan^{-1} \left( \frac{\varepsilon^p}{S_c} \right) - \frac{q}{p} \cos 2\psi$$

(6.26)

6.5 SUMMARY

The modeling concept and the basic set of equations for the stress–strain–dilatancy behaviour of sandy soil was developed in this chapter. The relations derived herein describe the essence of the sandy soil behaviour and serve as a basis for the elastic–plastic model that will be developed in the next chapter. Based on the experimental evidence that combination of density and effective stress is relevant to the description of sandy soil behaviour, a modelling concept was adopted in which the stress–strain behaviour is represented by parameters defined as functions of the initial state of the soil. Thus, a shear stress–plastic strain relation for sandy soil is proposed with its parameters defined as a function of an initial index parameter which
Figure 6.27  Normalized Plastic Shear Work versus Plastic Shear Strain
Relationship from Drained Torsional Shear Tests on Toyoura Sand
MODELLING CONCEPT AND BASIC EQUATIONS

characterizes the behavioural properties of sandy soils, in the framework of the steady state of deformation. In accordance with the adopted modelling concept, the elastic parameters are also expressed as a function of density and effective stress.

The index parameter employed in the proposed plastic stress–strain relation is the State Index \( I_s \) (Verdugo, 1992 and Ishihara, 1993) which characterizes the sandy soil by accounting for the combined effects of density and mean effective stress for a given initial fabric. It was demonstrated that the state index \( I_s \) provides a very accurate quantitative description of the stress–strain behaviour observed in the drained torsional shear tests on different materials, for various initial states of the samples. The gradual change in the stress–strain characteristics from those typical of dense sand and low effective stress to those typical of loose sand and high effective stress are manifested in a corresponding gradual change in the value of \( I_s \). It was found that both drained and undrained behaviour are relatively independent of the initial state when the initial states are above the upper reference line. In that respect, it might be considered that the upper reference or the threshold void ratio above which the residual strength is zero indicates the location of the initial states associated with the softest stress–strain behaviour for a given material and initial fabric.

A modified hyperbolic stress–strain relationship is proposed with constant strength parameter and initial plastic shear modulus defined as an exponential function of plastic shear strain. The dependence of the initial plastic shear modulus is derived based on the experimental evidence that nonlinearity in the initial part of the stress–strain curve is larger than that provided by the two–constant hyperbolic equation. It was illustrated that the proposed relation is capable of accurately representing the stress–strain behaviour over a wide range of strains.

The parameters of the proposed modified stress–strain relation were expressed as linear functions of the state index \( I_s \) thus defining a stress–strain relation that describes the combined effects of density and mean effective stress for a given initial fabric. The high degree of correlation between the stress–strain parameters and the state index is due to the fact that both the stress–strain parameters and the state index \( I_s \) have physical significance and are not merely arbitrary parameters. The validity of the proposed stress–strain relationship was verified by comparing the predictions of the model to the observed stress–strain behaviour of the tested soils, for various
combinations of density and effective stress. Besides the physical relevance of the concept, its practical benefit is that a single set of parameters or stress–strain coefficients represents the stress–strain characteristics for a given material and initial fabric over the entire required range of densities and effective stresses.

A stress–dilatancy relation adopted for the present model is presented. This relation is based on the relationship between the normalized plastic shear work and plastic shear strain and accounts for the effects of noncoaxility on the volume change due to the dilatancy.
Chapter 7

FORMULATION OF 2-D ELASTIC–PLASTIC CONSTITUTIVE MODEL FOR SANDY SOILS

7.1 INTRODUCTION

Based on the experimental results and theoretical considerations presented in the previous chapters, a 2-D elastic–plastic constitutive model for sandy soils will be formulated in the following. The model is developed in the framework of the incremental theory of plasticity and accounts for various aspects of monotonic and cyclic behaviour of sandy soils under a wide range of strains, densities and stresses. The principles embodied in classical plasticity theory have been critically reviewed and modified in order to account for the essence of the behaviour of sandy soils. The basic principle followed in the development of the model is that the characteristics of sandy soil behaviour be comprised in a simple and comprehensible model which is, above all, physically relevant. In doing so special emphasis is placed on the applicability and effectiveness of the model in the analysis of soil response problems.

The model is experimentally motivated and has its basis in the results of hollow cylinder tests as well as conventional triaxial tests. By expressing the stress–strain parameters as functions of the state index \( I \), the model has a unique capability of modelling the influence of density and effective stress by considering their interaction related to a given initial fabric. In addition to this, the model accounts for the effects of anisotropy, strain history and stress–path dependence including rotation of the
principal stresses among others. The effectiveness of the model in simulating the
behaviour of sandy soils under various initial states, loads and drainage conditions, as
well as its applicability to seismic response analysis will be demonstrated in
Chapter 8.

7.2 STRESS AND STRAIN REPRESENTATION

A major simplification applied to the constitutive model to be developed in this
chapter is the limitation of the formulation to two dimensions. Strictly speaking even
the simple shear condition, as a special case of plane strain condition, is in fact a	hree-dimensional problem, as demonstrated by the undrained torsional simple shear
tests in Chapter 5 where the variation of the relative magnitude of the intermediate
principal stress as represented by the $b$-value, in fact indicates the change in the out
of plane stress component. However, in both monotonic and cyclic loading condition,
the change of the $b$-value took place mostly when the stress ratio and the strains were
rather small, while at higher stress ratios, which are accompanied by larger strains,
$b$-value was nearly constant around 0.4 to 0.5. Therefore, an assumption of constant
$b$-value at e.g. 0.5, will hold valid for many plane strain problems except for the
initial part of the shearing where the change of $b$-value usually takes place. This
assumption has been implemented and validated by several researchers (Poorooshashb
and Fletcher, 1987; Gutierrez, 1989; Matsuoka and Fukumoto, 1989). It is considered
that the assumption of a two-dimensional condition will avoid complicating the
formulation significantly and yet on the other hand, will be appropriate for idealization
of many in-situ loading conditions within reasonable computational efforts and
required accuracy.

A two-dimensional representation of a soil element with the corresponding stress
components $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$ is shown in Fig. 7.1. It is to be noted that all stresses denote
effective stresses in the subsequent discussion, and therefore, the primes over the
stress components to indicate 'effective' have been dropped in the succeeding
equations. The strain increment components implemented in the formulation are $d\varepsilon_x$,
$d\varepsilon_y$ and $d\gamma_{xy} = 2d\varepsilon_{xy}$.

The model is defined in the $X-Y$ plane where $X$ and $Y$ stress components are
specified as
Figure 7.1 Stress and Strain Components Employed in the Model

\[ X = \frac{1}{2} (\sigma_y - \sigma_x) \]  

(7.1)

\[ Y = \sigma_{xy} \]  

(7.2)

while the corresponding strain increment components are expressed as

\[ de_x = de_y - de_x \]  

(7.3)

\[ de_y = d\gamma_{xy} \]  

(7.4)

Fig. 7.2 shows the stress state as well as the stress and strain increments in the above defined X–Y plane. Unlike the usual representation of the stresses and strains either in the deviatoric and the q–p planes or any other representation in terms of invariants, the plane adopted for the present model enables us to represent the state of stress and strain during rotation of principal stress directions. Thus, for a given state
Figure 7.2 Stress and Plastic Strain Increments in the X–Y plane

of stress by point A in Fig. 7.2, the angle of the major principal stress direction relative to the vertical $\beta_o$, is defined as

$$
\tan 2\beta_o = \frac{2\sigma_{xy}}{\sigma_y - \sigma_x}
$$

(7.5)

Similarly, the angle that vector $AB$, which represents the stress increment, makes with the $X$–axis is given by

$$
\tan 2\beta_{do} = \frac{2d\sigma_{xy}}{d\sigma_y - d\sigma_x}
$$

(7.6)

Vector $AC$ in Fig. 7.2 is proportional to the plastic strain increment, and the angle $2\beta_{de}$ this vector makes with $X$–axis, which is twice the angle of the strain increment relative to the vertical, is defined as

$$
\tan 2\beta_{de} = \frac{d\gamma_{xy}}{d\varepsilon_y - d\varepsilon_x}
$$

(7.7)

The stress invariants implemented in the formulation are the mean effective
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stress $p$ and the shear stress $q$ given by

$$p = \frac{1}{2} (\sigma_x + \sigma_y)$$

$$q = \sqrt{X^2 + Y^2} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2}$$

The stress invariants $p$ and $q$ are equal to the horizontal coordinate of the center and the radius of the Mohr's circle of stresses, respectively, as shown in Fig. 7.3a.

Fig. 7.3b shows the Mohr's circle plot for strain increments where the strain increment invariants are the volumetric strain increment $\delta \varepsilon_v$ and shear strain increment $\delta \varepsilon$ defined as

$$\delta \varepsilon_v = \delta \varepsilon_x + \delta \varepsilon_y$$

$$\delta \varepsilon = \sqrt{(\delta \varepsilon_x)^2 + (\delta \varepsilon_y)^2} = \sqrt{(\delta \varepsilon_x - \delta \varepsilon_y)^2 + \delta \gamma_{xy}^2}$$

Fig. 7.3 shows also the principal stresses and principal strain increment components which are considered in the two–dimensional formulation as well as the angles that $\sigma_i$ and $\delta \varepsilon_i$ make with $X$–axis, or twice the angle they make with the vertical, $2\beta_{\sigma}$ and $2\beta_{\varepsilon}$, respectively. The stress invariants as well as the strain increment invariants can be rewritten in terms of the principal stresses and principal strain increments, respectively, such as

$$p = \frac{1}{2} (\sigma_1 + \sigma_3)$$

$$q = \frac{1}{2} (\sigma_1 - \sigma_3)$$

and

$$\delta \varepsilon_v = \delta \varepsilon_1 + \delta \varepsilon_3$$

$$\delta \varepsilon = \delta \varepsilon_1 - \delta \varepsilon_3$$
Figure 7.3 Stress Invariants and Strain Increment Invariants in the Mohr's Circle Plots
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7.3 INCREMENTAL FORMULATION

To start with, it is assumed that the total strain increment $\mathbf{d}\epsilon_{ij}$ is decomposed into elastic and plastic components $\mathbf{d}\epsilon_{ij}^e$ and $\mathbf{d}\epsilon_{ij}^p$ by a simple superposition:

$$\mathbf{d}\epsilon_{ij} = \mathbf{d}\epsilon_{ij}^e + \mathbf{d}\epsilon_{ij}^p$$  \hfill (7.16)

The elastic and plastic strain increment components are calculated separately, by an elastic stress–strain law of the form

$$\mathbf{d}\epsilon_{ij}^e = D_{ijkl} \mathbf{d}\sigma_{kl}$$  \hfill (7.17)

and by the flow rule

$$\mathbf{d}\epsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}}$$  \hfill (7.18)

where $D_{ijkl}$ is the elasticity matrix, $g$ is plastic potential function and $d\lambda$ is a positive scalar quantity giving the magnitude of the plastic strain increment. The scalar quantity $d\lambda$ can be solved as

$$d\lambda = \frac{1}{H_p} \hat{f}$$  \hfill (7.19)

$$\hat{f} = \frac{\partial f}{\partial \sigma_{ij}} \mathbf{d}\sigma_{ij}$$  \hfill (7.20)

where $H_p$ is plastic hardening modulus, $\hat{f}$ is loading index and $f$ is yield function.

Finally, the general expression for the plastic strain increment can be written as

$$\mathbf{d}\epsilon_{ij}^p = \frac{1}{H_p} \left( \frac{\partial f}{\partial \sigma_{ij}} \mathbf{d}\sigma_{ij} \right) \frac{\partial g}{\partial \sigma_{ij}}$$  \hfill (7.21)

By substituting Eqs. (7.17) and (7.21) in Eq. (7.16), the total strain increment can be calculated as

$$\mathbf{d}\epsilon_{ij} = D_{ijkl} \mathbf{d}\sigma_{kl} + \frac{1}{H_p} \left( \frac{\partial f}{\partial \sigma_{ij}} \mathbf{d}\sigma_{ij} \right) \frac{\partial g}{\partial \sigma_{ij}}$$  \hfill (7.22)

This equation is referred to as the stress–control version of the incremental
formulation, since it provides a solution of the strain increments for a given stress increment. However, when an elastic–plastic model is implemented in a solution technique such as the finite element method, the inverse relation of that given in Eq. (7.22) is needed since the strain increment is usually given and the stress increment must be determined. In order to invert the general stress–strain relation given by Eq. (7.22) the elasticity relation has to be used

\[
d\sigma_{ij} = C_{ijkl} \, d\varepsilon_{kl} ^{e}
\]  

(7.23)

where \( C_{ijkl} = D_{ijkl} ^{t} \). By using Eqs. (7.16) and (7.18), Eq. (7.23) can be rewritten as

\[
d\sigma_{ij} = C_{ijkl} \left( d\varepsilon_{kl} - d\lambda \, \frac{\partial g}{\partial \sigma_{ij}} \right)
\]

(7.24)

Substituting this equation in Eqs. (7.20) and (7.19), subsequently, the following expression is obtained

\[
\frac{\partial f}{\partial \sigma_{ij}} \, C_{ijkl} \left( d\varepsilon_{kl} - d\lambda \, \frac{\partial g}{\partial \sigma_{kl}} \right) - d\lambda \, H_{p} = 0
\]

(7.25)

Solving this equation for \( d\lambda \) and substituting it in Eq. (7.24), yields the strain–controlled version of the incremental formulation

\[
d\sigma_{ij} = \left\{ C_{ijkl} - \frac{C_{ijmn}}{H_{p} + \frac{\partial f}{\partial \sigma_{gr}} \, C_{qrst} \, \frac{\partial g}{\partial \sigma_{st}}} \left( \frac{\partial g}{\partial \sigma_{mn}} \, \frac{\partial f}{\partial \sigma_{qp}} \, C_{optl} \right) \right\} \, d\varepsilon_{kl}
\]

(7.26)

As indicated by the Eqs. (7.18) – (7.21), the calculation of the plastic strain increment components in the framework of the incremental theory of plasticity is based on definition of the yield function, plastic potential function and hardening function. The evaluation of these functions for the present model, with illustration of their relation to the sandy soil behaviour will be given in the following.
7.3.1 Failure Surface

Failure surface is defined as the limiting states of stress forming a surface in stress space that separates the stress states that can be reached from the ones that cannot be reached for a given soil. The results from the drained torsional shear tests presented in Chapter 4 as well as the experimental evidence from other studies on cohesionless soils show that if this definition is adopted then the failure surface is curved for most cohesionless soils and initial states. It was shown in Chapter 6 that the strength of sandy soils can be effectively modelled by the proposed stress–strain relation over a wide ranges of densities and effective stresses, and therefore, this relation will be employed in the model.

In addition to the combined effects of void ratio and effective stress, the effects of the initial anisotropy on the failure characteristics will be modelled, as proposed and used by Gutierrez (1989) and Tsujino (1992), respectively. These authors conducted a series of monotonic tests on hollow cylindrical samples of Toyoura sand (Gutierrez, 1989) and Toyoura sand with fines content up to 25 % (Tsujino, 1992), along different principal stress directions. Typical results from their studies with respect to the effects of the initial anisotropy on the failure characteristics of sandy soils are shown in Figs. 7.4a and 7.4b. It is to be noted that the stress components introduced previously, \( \sigma_x, \sigma_y, \) and \( \sigma_y \), correspond to the components of the stresses for the hollow cylindrical sample, \( \sigma_0, \sigma_\alpha \) and \( \sigma_{\phi} \), respectively. Therefore, these figures actually show plots of the peak stress ratios in the \( X-Y \) plane, for different angles \( \beta_0 \), obtained in tests with constant \( b \)-value at 0.5. Based on this observations Gutierrez (1989) modelled the effects of the initial anisotropy by assuming a circular failure surface in the \( X-Y \) plane with an initial offset of the center along the \( X \)-axis. The two aspects of the failure characteristics of sandy soils introduced above, i.e. the combined effects of density and effective stress and the effects of the initial anisotropy, will be incorporated in the failure surface of the present model.

Failure criterion is defined as:

\[
F = q - r_f p
\]  

(7.27)
Figure 7.4 Effects of the Initial Anisotropy on the Failure Surface of Toyoura Sand (Gutierrez, 1989) and Toyoura Sand–KAO 10% (Tsujino, 1992)
where \( r_f = (q/p)_f \) is the stress ratio at failure. Substituting Eq. (7.9) in Eq. (7.27), an expression for the failure surface in the \( X-Y \) plane is derived as

\[
F = \sqrt{X^2 + Y^2} - r_f p \tag{7.28}
\]

This surface is a circle with its center in the origin of the \( X-Y \) plane, and therefore, it represents isotropic failure characteristics. The anisotropic failure characteristics as illustrated in Fig. 7.4 can be modelled by shifting the failure surface along the \( X \)-axis such that the center of the failure surface will be at \((c_f, 0)\). Thus, the failure surface shown in Fig. 7.5 can be expressed as

\[
F = \sqrt{X'^2 + Y'^2} - r'_f p \tag{7.29}
\]

where \( X' \) and \( Y' \) axes have their origin in the center of the shifted failure surface and \( r'_f \) is the stress ratio at failure \((q'/p)_f \) in which

\[
q' = \sqrt{X'^2 + Y'^2} = \sqrt{(X - c_f p)^2 + Y^2} \tag{7.30}
\]

Then, the expression of the failure surface that accounts for the effects of the initial anisotropy is given by

\[
F = \sqrt{(X - c_f p)^2 + Y^2} - r'_f p \tag{7.31}
\]

Eqs. (7.28) and (7.31) defines the failure surface for initially isotropic and anisotropic material, respectively, where \( c_f \) represents a measure for the effects of the initial anisotropy on the failure characteristics. To model the effects of density and effective stress on the failure surface, the stress–strain relationship developed in Chapter 6 will be employed. Namely, in Eq. (6.14) it was given that the peak stress ratio can be defined as a linear function of the state index \( I_s \) which, if isotropy is assumed, can be written as

\[
r_f = \alpha_1 + \beta_1 I_s \tag{7.32}
\]

where \( r_f = (\tau/p)_f \) since \( \tau = \sigma_y = Y, X = 0 \) and consequently, \( q = \tau \) in the drained torsional shear tests.

It is common to define the quasi steady state line by a straight line in the
$e - \log p$ plot, as shown in Fig. 7.6. Then, the state index $I_s$ can be calculated as

$$I_s = \frac{e_r - e_c}{e_r - e_s} \tag{7.33}$$

where $e_r$ is the void ratio of the upper reference line, $e_c$ is the initial void ratio or the void ratio prior to shearing and $e_s$ is the void ratio of the quasi steady state line, all associated with the initial effective stress $p_c$. For a given reference state on the quasi steady state, e.g. $(e_o, p_o)$, the void ratio $e_s$ can be calculated as

$$e_s = e_o - \lambda_{qss} \log \left( \frac{p_c}{p_o} \right) \tag{7.34}$$

where $\lambda_{qss}$ is the slope of the quasi steady state line in the $e - \log p$ plot (Fig. 7.6).

Substituting this equation in Eq. (7.33) and then in Eq. (7.32) results in

$$r_f = \alpha_1 + \beta_1 \frac{e_r - e_c}{e_r - e_o + \lambda_{qss} \log \left( \frac{p_c}{p_o} \right)} \tag{7.35}$$

By introducing this expression for the stress ratio into the Eq. (7.28) the final form of the failure surface for an isotropic material is obtained. This can be expressed in a more general form as

$$F = \sqrt{X^2 + Y^2} - (\alpha_1 + \beta_1 I_s) p \tag{7.36}$$

This failure criterion has the same attributes as the stress–strain relationship proposed in Chapter 6. Namely, by expressing the stress ratio at failure as a linear function of the state index $I_s$, the combined influence of density and effective stress on the failure surface is taken into account. Thus the failure surface defined by Eq. (7.36) appears as a curved conical surface in the $X-Y-p$ stress space, when the initial state is below the upper reference line or when $I_s$ is positive, whereas it is a circular cone with straight meridian line when the initial state is either at or above the upper reference line, or when $I_s$ is either zero or negative. Typical shapes of the failure surface in the $X-Y-p$, for two arbitrary void ratios associated with non-positive and positive values of the state index $I_s$, respectively, are shown in Fig. 7.7. Recalling the definition of the
Figure 7.5 Modelling the Effects of the Initial Anisotropy on the Failure Surface

Figure 7.6 Reference Lines and State Index $I_s$ in $e - \log p$ plot
reference lines (the upper reference line and the quasi steady state line) given in Chapter 6, it can be concluded that the curvature of the failure surface tends to decrease with increasing effective stress and void ratio that is in agreement with the experimental evidence on the behaviour of cohesionless soils. It is worth noting that the applied concept with the state index \( I_z \) as a parameter for characterization of the sandy soil behaviour enables us to distinguish the effects of the initial fabric associated with a particular method of sample preparation. If the above procedure is applied to the failure surface which takes into account the effects of the initial anisotropy (Eq. 7.31), Eq. (7.36) can be rewritten as

\[
F = \sqrt{(X - c_f p)^2 + Y^2} - \sqrt{[(\beta_1 + \beta_1 I_z) p]^2 + (c_f p)^2} \tag{7.37}
\]

since

\[
r_f' = \sqrt{r_f^2 + (c_f p)^2} \tag{7.38}
\]

where \( r_f \) is the peak stress ratio obtained from drained torsional shear tests as given by Eq. (7.32). It should be emphasized that the measure for the initial anisotropy \( c_f \) is a fabric dependent parameter because the initial fabric is in fact reflected through the initial anisotropy.

Eq. (7.37) represents the final form of the failure surface adopted for the present model. It accounts for the effects of density, effective stress, initial fabric and related initial anisotropy in a unique way based on the concept developed in Chapter 6. It is worth noting that in the case of isotropic material, \( c_f \) equals zero and Eq. (7.37) reduces to its isotropic version given by Eq. (7.36).

### 7.3.2 Yield Surface and Hardening Rule

As explained in Chapter 2, the yield surface serves to define the boundary between the states of stress where both elastic and plastic deformations occur and those where only elastic deformation occurs. By separating the stress conditions of elastic behaviour from those of elastoplastic behaviour, firstly a stress space is defined inside the yield surface where purely elastic behaviour exists, and secondly, a stress condition is set for occurrence of plastic deformations. On the other hand, the
Figure 7.7 Typical Shapes of the Failure Surface in the $X-Y-p$ Space
hardening rule should provide a mechanism for changing the size, shape and location of the yield surface.

If this definition of yield surface is adopted then the yield surface for most of the cohesionless soils would enclose an extremely small stress space and in fact it would shrink to a point. This was clearly indicated in the discussion on the elastic behaviour given in Sec. 6.2, based on both theoretical and experimental considerations including the results of the present study. Namely, as shown in Fig. 6.21, the experimental results obtained in the drained torsional shear tests illustrate the tendency of the plastic shear modulus \( G_p \) to increase with decreasing strain. It can be postulated that the plastic shear modulus tends asymptotically to infinity \( (G_p \to \infty) \), as the strain approaches zero \( (\infty \to 0) \). Hence, due to the extremely large values of \( G_p \) at small strain level, the behaviour would be nearly elastic. In the framework of the above concept, the limit strain of \( 10^{-5} \) set by Tatsuoka and Shibuya (1991) as strain level below which the behaviour of sandy soils is linear and elastic, in fact indicates the strain level below which the magnitude of the plastic strains is not comparable to that of the elastic strains. For this reason, the observed behaviour is 'linear and elastic'. Consequently, the yield surface should enclose an elastic zone in the stress space in which any stress change will induce strain less than \( 10^{-5} \) that practically represents a point in the stress space. As regards the hardening rule, it seems most appropriate and at the same time consistent with the previous discussion, to assume a solely kinematic hardening rule with a fixed and very small size of yield surface (Tatsuoka and Shibuya, 1991).

The yield surface of the present model is defined as a circle in the \( X-Y \) plane by the following expression:

\[
 f = \sqrt{(X - c_x p)^2 + (Y - c_y p)^2} - r_e p \tag{7.39}
\]

where \((c_x, p, c_y, p)\) are the coordinates of the center and \(r_e\) is the radius of the yield surface. In accordance with the previous considerations, it is assumed that the elastic zone is very small, and therefore, a very small and constant value for the radius of the yield surface is adopted such that \(r_e\) tends to zero \((r_e \to 0)\). Due to this assumption and the fundamental requirement expressed through the consistency condition that a current stress point cannot lie outside the yield surface, the coordinates of the center
of the yield surface \((c_x, p, c_y, p)\) always coincide with the \(X\) and \(Y\) coordinates of the current stress point, i.e.

\[
X = c_x p \tag{7.40}
\]

\[
Y = c_y p \tag{7.41}
\]

In other words, since the radius of the yield surface tends to zero \((r_e \to 0)\) and its center is coincident with the current stress point (Eqs. 7.40 and 7.41), the condition \(f = 0\) is always satisfied, and therefore, any change in the stress state will produce both, elastic and plastic deformations. Hence, in the present model elastic and plastic deformations are always accompanied to each other and sandy soil is modelled as continuously yielding material.

By assuming a fixed size for the yield surface through the constant value of its radius, a purely kinematic hardening rule is in fact employed, with \(c_x\) and \(c_y\) being kinematic hardening parameters which locate the center of the yield surface. Since the yield surface must be carried by the stress point, the kinematic hardening rule which specifies the movement of the yield surface is simply given as

\[
dc_x = C \frac{dX}{p} \tag{7.42}
\]

\[
dc_y = C \frac{dY}{p} \tag{7.43}
\]

where \(C\), which gives the amount of movement of the yield surface, can be simplified to \(C \to 1\) as \(r_e \to 0\). Hence, the kinematic hardening rule can be finally written as

\[
dc_x = \frac{dX}{p} \tag{7.44}
\]

\[
dc_y = \frac{dY}{p} \tag{7.45}
\]

Schematic illustration of the yield surface bounded in the \(X\)–\(Y\) plane by the failure surface is shown in Fig. 7.8.

It is interesting to note that the assumption of a yield surface with very small and fixed size ('point' yield surface) diminishes the importance of the yield surface
in the framework of plasticity theory. In other words, such an assumption defines in advance the fact that purely elastic behaviour does not exist and consequently, that plastic deformation is associated with any stress increment. Therefore, the yield surface is not a crucial element of the present model, and more importantly, this approach is in agreement with the experimental evidence on the behaviour of sandy soils and also simplifies the matter considerably.

7.3.3 Hardening Function

The failure surface and the yield surface introduced previously define the limit states of stresses and the elastic region for a given soil, respectively. Next, a hardening surface will be introduced which is defined as a surface of equal plastic hardening modulus $H_p$, which in turn is necessary to calculate the magnitude of the plastic strain increment (Eq. 7.19). Since, in general, the plastic hardening modulus depends on the
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stress state as well as on the strain (load) history, the hardening surface and the rules for its changes serve as a measure for the magnitude of plastic deformation accounting for the complex variation of the plastic hardening modulus during various loading conditions.

In order to overcome the difficulties and complexity of experimental determination of the yield surface (Tatsuoka and Ishihara, 1974b; Nova and Wood, 1978), Vermeer (1978) adopted the plastic shear strain as a suitable hardening parameter, assuming that the yield loci can be most simply approximated to a line of constant shear strain. While this in fact represents a very rough approximation for the yield surface, as can be concluded from the preceding discussion on the yield surface, it appears that the plastic shear strain is a reasonable hardening parameter for the hardening surface (Gutierrez, 1989; Peiris, 1990; Tsujino, 1992). Namely, based on results from monotonic drained torsional tests along different principal stress directions (along radial lines in the X–Y plane) these authors concluded that if the plastic shear strain is adopted as a hardening parameter for the hardening surface, then, the equal plastic shear strain contours represent the lines of equal hardening modulus. Typical results of the above mentioned studies are shown in Figs. 7.4a, 7.4b and 7.9 where it can be seen that the equal shear strain contours in the X–Y plane can be accurately represented by a set of circles concentric with the failure surface. Hence, similarly to the failure surface, the hardening surface can be written as

\[ f_H = q - r \rho \]

(7.46)

where \( r \) is the stress ratio and \( \rho \) is the radius of the hardening surface, respectively. The stress ratio \( r \) can be expressed in the form of the proposed stress–strain relationship (Eq. 6.11) as

\[ r = \frac{G_p^N \bar{\epsilon}_p \rho_f}{\rho_f + G_p^N \bar{\epsilon}_p} \]

(7.47)

where \( r = q/\rho \), \( \bar{\epsilon}_p = (\epsilon_p - \epsilon_d)^2 + \epsilon_d^2 \) is plastic shear strain and \( G_p^N \) is normalized initial plastic shear modulus given by Eq. (6.13). The derivation of the hardening function will be based on the proposed stress–strain relationship in Chapter 6 and on
the assumption for circular distribution of the plastic shear strain in the $X$-$Y$ plane.

Eq. (7.46) can be expressed in a generalized form as

$$ f_H (p, X, Y, r) = 0 \quad (7.48) $$

where $r = f(\varepsilon_p)$ is a function of the plastic shear strain. Next, applying the consistency condition for the hardening surface which means that subsequent point must also lie on the hardening surface, i.e. $df_H = 0$, yields

$$ \frac{\partial f_H}{\partial p} dp + \frac{\partial f_H}{\partial X} dX + \frac{\partial f_H}{\partial Y} dY + \frac{\partial f_H}{\partial r} \frac{\partial r}{\partial \varepsilon_p} d\varepsilon_p = 0 \quad (7.49) $$

where

$$ \hat{f}_H = \frac{\partial f_H}{\partial p} dp + \frac{\partial f_H}{\partial X} dX + \frac{\partial f_H}{\partial Y} dY \quad (7.50) $$

is the loading index whose further derivation will be given in the subsequent section.
From the flow rule (Eq. 7.18)
\[ d\varepsilon_p = d\lambda \frac{\partial g}{\partial q} \tag{7.51} \]

and Eq. (7.50), Eq. (7.49) can be rewritten as
\[ \dot{f} + d\lambda \frac{\partial f_H}{\partial r} \frac{\partial r}{\partial \varepsilon_p} \frac{\partial g}{\partial q} = 0 \tag{7.52} \]
in which the plastic hardening modulus is expressed as
\[ H_p = -\frac{\partial f_H}{\partial r} \frac{\partial r}{\partial \varepsilon_p} \frac{\partial g}{\partial q} \tag{7.53} \]
and hence, Eq. (7.19) is derived.

For further evaluation of Eq. (7.53) the equation of the plastic potential is required to give the derivative \( \partial g/\partial q \). It will be demonstrated later that the plastic potential function adopted for the present model gives \( \partial g/\partial q = 1 \) and hence, Eq. (7.53) reduces to
\[ H_p = p \frac{\partial r}{\partial \varepsilon_p} \tag{7.54} \]

Solving the derivative \( \partial r/\partial \varepsilon_p \) for Eq. (7.47), in which the normalized plastic shear modulus \( G_p^N \) is given by Eq. (6.13), gives the final expression for the plastic hardening modulus under the assumption of isotropic conditions
\[ H_p = \bar{G}_p^N p \left(1 - \frac{r}{r_f}\right)^2 \tag{7.55} \]
where
\[ \bar{G}_p^N = G_p^N - f \frac{\varepsilon_p}{\varepsilon_1} (G_p^N - G_{p_{\text{min}}}) \tag{7.56} \]
with the notation corresponding to Eq. (6.13). Replacing \( r_f \) with \( r_f' \) from Eq. (7.38) results in an expression for the plastic hardening modulus for initially anisotropic conditions such as...
\[ H_p = \bar{G}_p^N \left( 1 - \frac{r}{r_f} \right)^2 \]  

(7.57)

This equation models several aspects of the strain hardening behaviour of sandy soils including the effects of density, effective stress and initial anisotropy related to a given initial fabric. This is achieved by representing its parameters \( G_p^N \) and \( r_f \) as functions of the state index \( I_s \) and by shifting the failure surface in the \( X-Y-p \) space, respectively. It is worth noting that the plastic hardening modulus \( H_p \) varies from \( H_p = G_{p,max}^N \) at \( r = 0 \) and \( \varepsilon_p = 0 \), to \( H_p = 0 \) at failure, when \( r = r_f \).

In addition to the above mentioned factors it is necessary that plastic hardening modulus \( H_p \) represents the effects of strain history as well. For this purpose a mixed hardening law for the hardening surface will be used which has its origin in the multisurface or nested surface models (Mroz, 1967; Iwan, 1967) and bounding surface models (Krieg, 1975; Dafalias and Popov, 1976).

Fig. 7.10a shows surfaces of equal plastic hardening modulus, the failure surface and the yield surface in the \( X-Y \) plane prior to shearing. Since hardening surfaces are concentric to the failure surface during virgin loading, current hardening surface (CHS) or the activated hardening surface (the largest hardening surface with current stress point on it) will appear as uniformly expanding around the axis of the failure surface. The failure surface (FS) acts as a bounding surface (BS) for the hardening surfaces, and therefore, the hardening surface or the surface of equal plastic hardening modulus is either smaller than the bounding surface or coincident with it when failure occurs.

Fig. 7.10b shows a typical set of surfaces at a certain stage of virgin loading. It is interesting to note that since the yield surface (YS) is carried by the stress point (SP), and on the other hand, the stress point always lies on the current hardening surface, the yield surface will always be tangential to the current hardening surface at the current stress point. Another interesting observation is that the largest hardening surface will always be concentric with the failure surface and eventually coincident with it, and hence, it will only be exposed to isotropic hardening. Thus, the largest hardening surface will be called loading surface (LS), which according to the definition of Phillips and Lee (1979) is the "largest surface in stress space produced
Figure 7.10  Virgin Loading: Evolution of Loading Surface
by isotropic expansion from the initial yield surface, and which passes through at least one previous prestress point". Hence, the loading surface provides the criterion for virgin loading. Namely, for a given stress state at the loading surface (e.g. Fig. 7.10b), virgin loading will occur only if the next stress state is either at or outside the loading surface. Mathematically, a stress increment involves virgin loading if \( f_\nu = 0 \) and

\[
\frac{\partial f_\nu}{\partial \sigma_{ij}} d\sigma_{ij} \geq 0
\]  

(7.58)

If the new stress state is inside the loading surface than reversal occurs and unloading takes place. The last stress point on the loading surface or the point from which unloading actually starts is termed the reversal point for the loading surface (RP\(_\nu\)). Fig. 7.11a shows the stage prior to unloading, with the subsequent hardening surfaces inside the loading surface shown by dashed lines. The distribution of the hardening surfaces is typical for the multisurface or nested surface approach with all surfaces that have been traversed during the virgin loading being tangential at the reversal point for the loading surface. It is worth noting, however, that these surfaces are not needed during the process of virgin loading since they have been traversed already. Therefore, instead of memorizing and translating these surfaces during the virgin loading process, an approach is adopted in which each of these surfaces can be constructed by a simple rule obtained by Pereis (1990), Tsujino (1992) and Gutierrez (1989) based on observations from a series of stress probe tests for the considered \( X-Y \) plane, including unloading and reloading conditions. According to their observations, the distribution of the equal plastic hardening surfaces can be accurately represented by circles whose centers lie along the line connecting the reversal point and the center of the circle corresponding to that point. For instance, the centers of the hardening surfaces for the unloading will be along the line connecting the reversal point for loading surface with the center of this surface (Fig. 7.11b). The hardening surfaces, shown by dashed line in Fig. 7.11a, are constructed following this rule. It is worth noting that this rule ensures non-intersection of the surfaces, and hence, it guarantees uniqueness of the formulation.

For any given state of stress in the stage of unloading, e.g. a stress point \((X_p, Y_p)\), the radius of the current hardening surface \(r_\nu\) and the coordinates of its
Figure 7.11 Unloading: Evolution of Unloading Surface
center \((X_{hs}, Y_{hs})\) can be determined from the equation of a straight line passing through the reversal point for the loading surface \((X_{rp}, Y_{rp})\) and the center of the loading surface \((X_{ls}, Y_{ls})\)

\[
\frac{Y_{hs} - Y_{ls}}{Y_{rp} - Y_{ls}} = \frac{X_{hs} - X_{ls}}{X_{rp} - X_{ls}}
\]  

(7.59)

and the equations of circle with its center in \((X_{hs}, Y_{hs})\) passing through the current stress point \((X_p, Y_p)\) and the reversal point \((X_{rp}, Y_{rp})\) such as

\[
r_{hs} = (X_{rp} - X_{hs})^2 + (Y_{rp} - Y_{hs})^2
\]  

(7.60)

\[
r_{hs} = (X_p - X_{hs})^2 + (Y_{hs} - Y_p)^2
\]  

(7.61)

As with the definition of the loading surface, the largest surface in the stress space which is produced during unloading from the loading surface, and which passes through at least one previous stress point and is tangential to the loading surface at its reversal point \((RP_{1s})\), is defined as the unloading surface (US). During the unloading, this surface is coincident with the current hardening surface. Analogously to Eq. (7.58), unloading occurs if \(f_{us} = 0\) and

\[
\frac{\partial f_{us}}{\partial \sigma_{ij}} d\sigma_{ij} \geq 0
\]  

(7.62)

The same pattern is followed if reversal from unloading occurs; this is termed reloading, and there are corresponding definitions for the reloading surface (RS) and reversal point for the unloading surface \((RP_{us})\). A schematic illustration of the reloading surface together with previously defined surfaces is given in Fig. 7.12a. This figure also shows the current hardening surface (dashed line) obtained upon reversal from the reloading surface; this is termed the general surface (GS). There are two general positions of this surface: tangential to the reloading surface – tangential general surface (Fig. 7.12a); or inside the reloading surface – floating general surface (Fig. 7.12b), which occurs upon the reversal from the tangential general surface.

The four hardening surfaces introduced above, i.e. the loading surface, the unloading surface, the reloading surface and the general surface are the only hardening
Figure 7.12  Distribution of the Model Surfaces During Reloading and General Loading
surfaces which are memorized in the model. Therefore, following the rules given previously, distribution of the equal hardening surfaces can be effectively evaluated even for the most complicated loading conditions with small memory requirements. In order to avoid violation of the non-intersection principle, in every case when the stress point goes beyond larger hardening surface which is in the memory, the larger surface is activated and all smaller surfaces are erased from the memory. That is to say, the conditions given by Eqs. (7.58) and (7.62) are not always necessary for occurrence of virgin loading or unloading, respectively.

When the floating general surface is activated then a check should be made whether the stress point is outside the circle, which is tangential to both, current hardening surface and reloading surface, or not (Fig. 7.13). If the stress point is outside this circle, then, a tangential general surface is activated which passes through the current stress point.

It is interesting to note that the failure surface serves as a bounding surface to the hardening surfaces and at the same time as a reference state for calculation of the hardening modulus $H_p$. With respect to that the model is similar to the bounding surface models. Substituting the radius of the current hardening surface and the corresponding plastic shear strain produced from the last reversal point in Eqs. (7.57) and (7.56), respectively will provide the value for the plastic hardening modulus which will reflect the effects of density, effective stress, shear stress, initial anisotropy (related to the initial fabric) and induced anisotropy (related to the load history). Thus, it is reasonable to conclude that it will provide a firm basis for calculation of the plastic deformations of sandy soils even for very complex loading conditions involving cyclic changes of stresses and rotation of principal stress directions.

### 7.3.4 Flow Rule

The flow rule given in its general form by Eqs. (7.18) to (7.21) shows that the magnitude of the plastic strain increment depends on the plastic hardening modulus $H_p$ while the plastic potential function determines the direction of the plastic strain increment, and hence, the relative magnitudes of the plastic strain increment components. The plastic potential function of the present model is based on the formulation for the flow of sand proposed by Gutierrez (1989).
Based on a series of drained tests on hollow cylindrical samples of Toyoura sand, including monotonic tests along fixed principal stress directions, tests with constant magnitudes of the principal stresses involving rotation of principal stress directions and combined radial and rotational tests, Gutierrez concluded that the flow of sand is neither coaxial nor unique. In other words, the directions of the principal plastic strain increment do not coincide with the directions of the principal stresses (noncoaxiality), and on the other hand, the flow does not depend on the current stress state alone but also on the direction of the stress increment, and therefore, it is nonunique. Gutierrez further concluded that the angle of noncoaxiality, the angle between the principal stress directions and plastic strain increments, is strongly dependent on the shear stress level – the higher the shear stress the smaller the angle of noncoaxiality. Consequently, the dependency of the flow on the stress increment decreases with increasing shear stress level. Based on these conclusions, Gutierrez proposed a plastic potential formulation in the $X-Y$ plane which is schematically shown in Fig. 7.14. According to the formulation, the direction of the plastic strain
increment in the $X-Y$ plane is defined by the gradient to the failure surface at the
conjugate stress point $(X_c, Y_c)$, where the conjugate stress point is defined by the point
of intersection of the extended stress increment vector and the failure surface.
Mathematically, this implies that the plastic potential function takes the form
\[ g = g (\sigma_{ij}, l_{ij}) \] (7.63)
where
\[ l_{ij} = \frac{d\sigma_{ij}}{(d\sigma_{kl} d\sigma_{ij})^{1/2}} \] (7.64)
is the stress increment direction.

The plastic strain increments can now be calculated taking the corresponding
derivatives of the plastic potential, i.e. the gradient to the failure surface at the
conjugate point $(X_c, Y_c)$, as follows
\[ de^p_x = d\epsilon^p_x = d\lambda \frac{\partial F}{\partial X}
|_{X = X_c, Y = Y_c} = d\lambda \left( \frac{X_c - c_f p}{q_f} \right) \] (7.65)
\[ de^p_y = d\epsilon^p_y = d\lambda \frac{\partial F}{\partial Y}
|_{X = X_c, Y = Y_c} = d\lambda \left( \frac{Y_c}{q_f} \right) \] (7.66)
where
\[ q_f = r_f p = \sqrt{(X_c - c_f p)^2 + Y_c^2} \] (7.67)
and $d\lambda$ is the scalar quantity given by Eq. (7.19) for which the value of the leading
index is also needed. Substituting Eqs. (7.65) and (7.66) in Eq. (7.11) gives the plastic
shear strain increment
\[ d\epsilon^p = \sqrt{(de^p_x)^2 + (de^p_y)^2} = d\lambda \] (7.68)
It should be noted in passing that this equation proves that the derivative $\partial g/\partial q$ equals
unity ($\partial g/\partial q = 1$) as was assumed in Eq. (7.53) in the derivation of the hardening
function.
Figure 7.14 Plastic Potential Formulation Proposed by Gutierrez (1989)

Next, the stress–dilatancy relationship derived in Chapter 6 will be introduced in the formulation of the flow rule. It can be rewritten as

\[
\frac{d\varepsilon^p_v}{d\varepsilon^p} = \mu - \frac{q}{p} \cos 2\psi
\]  

(7.69)

where \(\psi = |\beta_0 - \beta_d|\) is the angle of noncoaxiality, \(d\varepsilon^p\) is the plastic volumetric strain increment due to the dilatancy alone and \(\mu\) is the slope of the normalized plastic shear work \(\Omega^p\) versus plastic shear strain \(\varepsilon^p\) curve defined by the relationship proposed by Kabilamany and Ishihara (1990). This is given by Eq. (6.24)

\[
\mu = \mu_o + \frac{2}{\pi} \left(\mu_{\text{max}} - \mu_o\right) \tan^{-1} \left(\frac{\varepsilon^p}{S_c}\right)
\]  

(7.70)

in which \(\mu_o\) and \(\mu_{\text{max}}\) are the slopes at small and large strains, respectively (See Fig. 6.27) and \(S_c\) defines the plastic shear strain at which the average value for \(\mu\) is attained.
Substituting Eq. (7.68) in Eq. (7.69) the plastic volumetric strain increment becomes:

\[ d\varepsilon^p_v = d\varepsilon^p_x + d\varepsilon^p_y = d\lambda \left( \mu - \frac{q}{p} \cos 2\psi \right) \]  \hspace{1cm} (7.71)

Using Eqs. (7.65), (7.66) and (7.71) plastic strain increment components can be written as

\[ d\varepsilon^p_x = \frac{d\lambda}{2} \left[ \left( \mu - \frac{q}{p} \cos 2\psi \right) - \left( \frac{X_c - c_f p}{q_f} \right) \right] \]  \hspace{1cm} (7.72)

\[ d\varepsilon^p_y = \frac{d\lambda}{2} \left[ \left( \mu - \frac{q}{p} \cos 2\psi \right) + \left( \frac{X_c - c_f p}{q_f} \right) \right] \]  \hspace{1cm} (7.73)

\[ d\gamma^p_{xy} = d\lambda \left( \frac{Y_c}{q_f} \right) \]  \hspace{1cm} (7.74)

**Loading Index \( \hat{f} \)**

To calculate the magnitude of the plastic strain increment, besides the plastic hardening modulus \( H_p \), it is necessary to determine the loading index \( \hat{f} \) which has the following general form

\[ \hat{f} = \frac{\partial f}{\partial \sigma_{ij}} \, d\sigma_{ij} \]  \hspace{1cm} (7.75)

which for the \( X-Y-P \) stress space reduces to

\[ \hat{f} = \left( \frac{\partial f}{\partial X} \right) \, dX + \left( \frac{\partial f}{\partial Y} \right) \, dY + \left( \frac{\partial f}{\partial P} \right) \, dP \]  \hspace{1cm} (7.76)

The assumption of very small elastic region as represented by the adopted 'point' yield surface always satisfies the condition \( \hat{f} = 0 \), and therefore, plastic strains are produced for any stress increment in the \( X-Y \) plane. Since the yield surface is merely a point, the direction of yielding can not be directly determined. However, it
was pointed out earlier that the yield surface and current hardening surface will always be tangential at the current stress point, and hence, the derivatives of the yield surface can be replaced by the corresponding derivatives of the current hardening surface at the current stress point. Such a formulation will give a non-associated flow rule with different direction of yielding and flow. However, it will make it impossible to simulate the behaviour of sandy soils along a purely rotational stress path with constant magnitudes of principal stresses since, for a tangential stress increment, this formulation provides zero plastic strain, which is contrary to the experimental evidence and the concept of the present model. Therefore, the alternative proposed by Gutierrez will be used, in which the derivatives with respect to \( X \) and \( Y \) stress components are replaced by the corresponding derivatives of the failure surface at the conjugate stress point \((X_c, Y_c)\). Consequently, the flow rule will be associated, giving identical directions of yielding and flow. Hence, the derivatives become

\[
\frac{df}{dX} = \frac{df}{dX} \bigg|_{X = X_c, Y = Y_c} = \frac{X_c - c_f p}{q_f} \tag{7.77}
\]

\[
\frac{df}{dY} = \frac{df}{dY} \bigg|_{X = X_c, Y = Y_c} = \frac{Y_c}{q_f} \tag{7.78}
\]

The remaining derivative is calculated from the derivative of the hardening surface at the current stress point \((X_s, Y_s)\), i.e.,

\[
\frac{df}{dp} = \frac{df}{dp} \bigg|_{X = X_c, Y = Y_c} = -\frac{X_s (X_s - c_s p) - Y_s (Y_s - c_y p)}{q_H p} \tag{7.79}
\]

where \(q_H\) is the radius of the hardening surface and \((c_s p, c_y p)\) are the coordinates of its center. For the virgin loading, since \(c_s = c_y = 0\) and \(q_H = q\), Eq. (7.79) reduces to

\[
\frac{df}{dp} = -\frac{q}{p} \tag{7.80}
\]

Finally, the loading index can be written as

\[
\dot{f} = \left( \frac{X_c - c_f p}{q_f} \right) dX + \left( \frac{Y_c}{q_f} \right) dY - \left[ \frac{X_s (X_s - c_s p) + Y_s (Y_s - c_y p)}{q_H p} \right] dp \tag{7.81}
\]
An alternative expression for the loading index can be obtained by expressing it in terms of \( d\sigma_b \) as

\[
\dot{f} = \frac{\partial f}{\partial \sigma_x} d\sigma_x + \frac{\partial f}{\partial \sigma_y} d\sigma_y + 2 \frac{\partial f}{\partial \sigma_{xy}} d\sigma_{xy} \tag{7.82}
\]

\[
\dot{f} = \left( \frac{\partial f}{\partial \sigma_x} \frac{\partial \sigma_x}{\partial \sigma_x} + \frac{\partial f}{\partial \sigma_y} \frac{\partial X}{\partial \sigma_x} \right) d\sigma_x
\]

\[
+ \left( \frac{\partial f}{\partial \sigma_y} \frac{\partial \sigma_y}{\partial \sigma_y} + \frac{\partial f}{\partial \sigma_{xy}} \frac{\partial X}{\partial \sigma_y} \right) d\sigma_y \tag{7.83}
\]

\[
+ 2 \left( \frac{\partial f}{\partial \sigma_{xy}} \frac{\partial Y}{\partial \sigma_{xy}} \right) d\sigma_{xy}
\]

where the derivatives are defined as

\[
\frac{\partial p}{\partial \sigma_x} = \frac{1}{2} \tag{7.84}
\]

\[
\frac{\partial p}{\partial \sigma_y} = \frac{1}{2} \tag{7.85}
\]

\[
\frac{\partial X}{\partial \sigma_x} = -\frac{1}{2} \tag{7.86}
\]

\[
\frac{\partial X}{\partial \sigma_y} = \frac{1}{2} \tag{7.87}
\]

\[
\frac{\partial Y}{\partial \sigma_{xy}} = 1 \tag{7.88}
\]

The loading index can be now expressed as
\[
\dot{f} = \frac{1}{2} \left\{ - \frac{X_s (X_s - c_x \, p) + Y_s (Y_s - c_y \, p)}{q_H \, p} - \frac{X_c - c_f \, p}{q_f} \right\} \, d\sigma_x \\
+ \frac{1}{2} \left\{ - \frac{X_s (X_s - c_x \, p) + Y_s (Y_s - c_y \, p)}{q_H \, p} + \frac{X_c - c_f \, p}{q_f} \right\} \, d\sigma_y \\
+ \left( \frac{Y_c}{q_f} \right) \, d\sigma_{xy}
\]

Using the expression for the loading index \( \dot{f} \) (Eq. 7.89 or 7.81) and plastic hardening modulus \( H_p \) (Eq. 7.57), the scalar quantity giving the magnitude of the plastic strain \( d\lambda \) can now be calculated and upon substitution in Eqs. (7.72) to (7.74) the plastic strain increment components can be determined.

### 7.3.5 Volumetric Yielding

The formulation of the model given in the preceding pages is confined to the process of yielding and flow of sandy soils during shearing, and therefore all the plastic volumetric strains are due to the dilatancy alone. However, it is well known that plastic volumetric strains also occur during isotropic consolidation or any other stress path that involves virgin loading along the mean effective stress axis under constant shear stress. In the present model the two mechanisms introduced above, i.e. shear and volumetric yielding, are considered separately, and hence, the model belongs to the so-called double hardening models (Prevost and Hoeg, 1975; Lade, 1977; Vermeer, 1978; Banerjee et al., 1992 among others).

The total plastic volumetric strain is calculated as

\[
d\varepsilon^p \varepsilon_{vv} = d\varepsilon^p \varepsilon_{v} + d\varepsilon^p \varepsilon_{vy}
\]

(7.90)

where \( d\varepsilon^p \varepsilon_{v} \) is plastic volumetric strain due to the dilatancy (given by Eq. 7.71) and \( d\varepsilon^p \varepsilon_{vy} \) is plastic volumetric strain due to virgin loading along the mean effective stress. The plastic volumetric strain due to the volumetric yielding may be expressed as

\[
d\varepsilon^p \varepsilon_{vy} = \frac{dp}{B_p}
\]

(7.91)
where $B_p$ is the plastic bulk modulus defined as

$$B_p = p \left( \frac{1 + e_o}{\lambda - \eta} \right) \tag{7.92}$$

in which $e_o$ is the initial void ratio, $\lambda$ and $\eta$ are the slopes of the consolidation and swelling line, respectively, which are assumed as straight lines in the $e - \log p$ plot. The plastic volumetric yielding surface can be written as

$$f_v = p - p_c = 0 \tag{7.93}$$

where $p_c$ is volumetric yielding stress corresponding to the maximum effective stress previously attained. The volumetric yielding occurs only if $f_v = 0$ and $df_v > 0$, or in other words, only if the mean effective stress exceeds the maximum effective stress of the loading history. The volumetric yielding surface given by Eq. (7.93) plots as a vertical line in the $q - p$ plane, passing through the maximum effective stress in the loading history of the soil element.

### 7.3.6 Elastic Strains

Discussion on the calculation of the elastic strains was given in Sec. 6.2 as well as in the introductory part of this chapter. Based on the assumption for isotropic elastic behaviour, the calculation of the elastic strains will be carried out in the present model by the Hooke's law which for plane strain conditions can be written as

$$
\begin{pmatrix}
\frac{de^e_x}{d\sigma_x} \\
\frac{de^e_y}{d\sigma_y} \\
\frac{d\gamma_{xy}^e}{d\sigma_{xy}}
\end{pmatrix} = \frac{1 + \nu}{E} \begin{pmatrix}
(1 - \nu) & -\nu & 0 \\
-\nu & (1 - \nu) & 0 \\
0 & 0 & 2
\end{pmatrix}\begin{pmatrix}
de^e_x \\
de^e_y \\
d\gamma_{xy}^e
\end{pmatrix} \tag{7.94}
$$

For linear isotropic elastic material the Young's modulus can be related to the shear modulus and Poisson's ratio through the standard relation

$$E = 2G (1 + \nu) \tag{7.95}$$

The procedures and the empirical relations for the elastic parameters are given in Sec. 6.2.
ELASTIC–PLASTIC CONSTITUTIVE MODEL

7.4 MODEL PARAMETERS

The parameters of the elastic–plastic constitutive model for sandy soils formulated in the preceding pages will be summarized in the following and the procedures for their determination will be presented. Since the model is developed based on the experimental results from a hollow cylinder apparatus it seems most appropriate to evaluate the parameters using such an apparatus. However, any other apparatus or test procedures which provide sufficient data for determination of the parameters may be used as well.

Elastic Parameters \((A, n, \nu)\)

Due to the assumption of isotropic elastic behaviour, the present model requires only two elastic parameters, as discussed in Sec. 6.2, e.g. elastic shear modulus \(G\) and Poisson's ratio \(\nu\). With respect to the basic concept of the model, according to which model parameters are defined as functions of the state of the soil expressed in terms of density and effective stress, it is preferable to evaluate the elastic shear modulus by using an equation of the type given by Eqs. (6.2) – (6.4) which can be written as

\[
G = A \frac{(2.17 - e)^2}{1 + e} p^n
\]  \hspace{1cm} (7.96)

Thus, by incorporating this equation in the model it is necessary to evaluate three parameters, i.e. the constant \(A\), exponent \(n\) and Poisson's ratio \(\nu\). However, the experimental evidence on the elastic behaviour of cohesionless soils shows that both \(\nu\) and \(n\) can be reasonably approximated by assuming constant values at 0.2 and 0.4–0.5, respectively (Lade, 1987; Iwasaki and Tatsuoka, 1978). The remaining parameter \(A\) should be evaluated from the measured elastic parameters in several tests (4 – 5 tests) for different initial conditions with respect to the void ratio and effective stress. It is to be noted that if a hollow cylinder apparatus is used, then elastic parameters can be accurately evaluated from the measurements of the initial slope of the stress strain curve at very small strain level (Tatsuoka and Shibuya, 1991) in the tests required for determination of the deformation parameters that will be introduced later. In such a case, it is not necessary to perform any additional test for determination of the elastic parameters.
State Index Parameters \( (e_{ur}, \lambda_{ur}, e_{qss}, \lambda_{qss}) \)

According to the definition of the state index discussed in Sec. 6.2, it is necessary to determine the upper reference line as well as the quasi steady state line for its calculation. The simplest representation of these lines appears in the \( e - \log p \) plane where they can be approximated with straight lines, as shown in Fig. 7.6. Hence, the reference void ratios of these lines \( (e_{ur}, e_{qss}) \) together with their slopes in the \( e - \log p \) plot \( (\lambda_{ur}, \lambda_{qss}) \) would be sufficient for evaluation of the state index \( I_s \). An alternative method of calculating the state index is by using tabular representation for the upper reference line and the quasi steady state line. The latter method is recommended since it provides better estimates for the state index over a wide range of density and effective stress. It is most appropriate to evaluate the reference lines in torsional shear mode and for the same method of sample preparation used for evaluation of the stress–strain parameters. However, it will be demonstrated in Chapter 8 that the quasi steady state line obtained from conventional triaxial compression tests can be used as well with negligible loss in the accuracy of the stress–strain representation. To determine the reference lines precisely, 4–5 undrained strain-controlled tests on loose samples are needed, e.g. undrained torsional shear or undrained triaxial compression tests, while a rough estimate can be made by performing a single test for determination of these lines as proposed by Verdugo, (1992).

Failure Parameters \( (\alpha_f, \beta_f, c_f) \)

The failure surface of the present model is defined with the coefficients of the linear relationship of the stress ratio at failure with the state index, \( \alpha_f \) and \( \beta_f \)

\[
r_f = \alpha_f + \beta_f I_s
\]

(7.97)

and with the measure for the initial anisotropy \( c_f \), giving the offset of the failure surface in the \( X-Y \) plane. The coefficients \( \alpha_p, \beta_p, ..., \beta_3 \) or failure and deformation coefficients, should be evaluated from the results of several drained torsional shear \( p \)-constant tests, for different initial conditions \( (e_o, p) \) providing wide range of the state index \( I_s \), from a plot as shown in Fig 6.24. It should be emphasized that the conditions in torsional shear test, regarding the orientation of the principal stress
directions and the relative magnitude of the intermediate principal stress, are rather close to those in simple shear and plane strain compression conditions at larger deformations (See Chapter 5 of this study and Tatsuoka et al., 1986a). For this reason and due to the fact that it is much easier to perform a \( p \)-constant test as a torsional shear test in a hollow cylinder apparatus than in any other conventional apparatus, it is considered most appropriate to use these tests from both theoretical and experimental point of view. The parameter of the initial anisotropy \( c_i \) can be evaluated from the test results, i.e. failure points, obtained in tests along different \( \beta_0 \)-directions, or alternatively, from a pair of drained conventional triaxial compression and extension tests as suggested by Gutierrez (1989).

**Deformation Parameters** (\( \alpha_2, \beta_2, \alpha_3, \beta_3, f \))

The deformation parameters \( \alpha_2, \beta_2, \alpha_3, \beta_3 \) and \( f \) are needed to calculate the plastic shear modulus given as

\[
G_{p,\text{max}}^N = \alpha_2 + \beta_2 \ I_s
\]

(7.98)

\[
G_{p,\text{min}}^N = \alpha_3 + \beta_3 \ I_s
\]

(7.99)

and finally \( G_p^N \) in Eq. (6.13). These parameters have to be evaluated from the same procedure and tests as the above described for the failure parameters, and therefore, the previous discussion applies to these parameters as well. More detailed explanation of the deformation parameters and their physical significance is given in Sec. 6.3.

**Dilatancy Parameters** (\( \mu_o, \mu_{\text{max}}, S_c \))

In the stress–dilatancy equation (Eq. 6.23) the slope of the normalized plastic shear work \( \Omega_p \) versus plastic shear strain \( \bar{\varepsilon}_p \) curve is defined such that

\[
\mu = \mu_o + \frac{2}{\pi} \left( \mu_{\text{max}} - \mu_o \right) \tan^{-1} \left( \frac{\bar{\varepsilon}_p}{S_c} \right)
\]

(7.100)

where \( \mu_o \) and \( \mu_{\text{max}} \) are the initial slope (at small strain level) and final slope (at large strain level), respectively (See Fig. 6.27). The parameter \( S_c \) defines the plastic shear strain at which the average value of \( \mu_o \) and \( \mu_{\text{max}} \) will be attained. While \( \mu_o \) and \( \mu_{\text{max}} \) can
be obtained from the slope of the curve introduced above, the determination of $S_e$
requires trial and error procedure. Usually values between 0.001 and 0.003 provide the
best fit parameter for the experimental data. The determination of dilatancy parameters
may be done by using both drained as well as undrained tests. However, it is
recommended that these parameters be determined based on results from undrained
torsional tests, if the model is to be used in undrained analysis; the reasons for this
will be discussed in the subsequent chapter. On the other hand, due to much smaller
sensitivity of the drained response on the dilatancy parameters their evaluation can be
less rigorous if drained analysis is to be performed.

Consolidation Parameter ($\lambda$)

To evaluate the plastic volumetric strain due to virgin loading along the effective
stress axis the parameter $\lambda$ of the Eq. (7.92) can be determined from the slope of the
consolidation line in the $e$ versus $ln\ p$ plot.

The number of parameters of the present model is comparable to or less than
those of some well-known elastic-plastic models (Lade, 1977; Sandler et al., 1976;
Pastor et al., 1990). Nevertheless, unlike the current concept applied to models for
sandy soils according to which model parameters are referenced to particular void
ratio or density, the parameters of the present model are expressed as functions of the
relative state of the soil in terms of void ratio and effective stress, and therefore, they
represent the behaviour over a wide range of densities and effective stresses. Hence,
in any case where more than one density has to be considered (that is usually the case
for practical problems), the total number of parameters and tests required for current
models will be a product of its number of parameters and number of different
densities. On the other hand, one set of parameters of the present model represents the
behaviour of a given material for particular initial fabric, or mode of deposition of the
soil, over the required densities and effective stresses commonly encountered in the
field. Equally important is the fact that the parameters of the present model have a
clear physical basis.
7.5 SUMMARY

An elastic–plastic constitutive model for sandy soils has been developed on the basis of experimental evidence of soil behaviour obtained in torsional and triaxial tests under various loading and drainage conditions. The modeling concept elaborated in the previous chapter is the cornerstone of the present model providing a unique and above all physically relevant basis for representing the combined effects of density and effective stress on the sandy soil behaviour. By representing the strength and stress–strain parameters as functions of the state index $I_s$, the model is capable of simulating the strength and deformation characteristics of a given soil and initial fabric over a wide range of densities and effective stresses with a single set of parameters. Thus, unlike current models for sandy soils in which void ratio does not appear as a variable and which, in fact, confine the solution to a single state surface for a given initial void ratio, the proposed model considers the multiplicity of these surfaces as a function of void ratio (Tatsuoka and Ishihara, 1974b). Nevertheless, that is achieved without increasing the number of material parameters and procedures required for their determination.

The model is developed within the general framework of the formalism of plasticity theory accounting for the characteristics of the behaviour of sandy soils for complex loading paths including cyclic loadings and rotation of principal stress directions. It is postulated that elastic and plastic deformations occur always in parallel and therefore, purely elastic behaviour cannot be isolated. Consequently, it is assumed that the yield surface has fixed and very small size equivalent to a point in the $X$–$Y$ stress plane, and hence it remains only a mathematical fiction. In accordance with this assumption and consistency condition, a pure kinematic hardening rule is adopted with the yield surface and the current stress point being always coincident.

The failure surface and hardening surface are based on the stress–strain relationship developed in Chapter 6 and on the assumption for circular distribution of plastic shear strain in the $X$–$Y$ plane. The failure surface captures the gradual change of the strength characteristics of sandy soils with change of its initial state and appears in the $X$–$Y$–$p$ space either as a curved conical surface or a circular cone with straight meridian line depending whether the initial state is below or either at or above the upper reference line, respectively. The model combines isotropic and kinematic
hardening laws for the simultaneous expansion and translation of consecutive hardening surface, and traces the loading history with four surfaces in the memory: loading surface, unloading surface, reloading surface and general hardening surface.

The flow rule incorporated in the model is based on the plastic potential formulation given by Gutierrez (1989) where the failure surface at its stress increment conjugate point serves as a plastic potential function. Thus, the flow is not defined by the stress state alone but by the stress increment as well, resulting in noncoaxiality and nonuniqueness of the flow. The plastic potential formulation captures the gradual decrease of the noncoaxiality and nonuniqueness of the flow with increasing shear stress level and eventually provides coaxial and unique flow at failure.
Chapter 8

PERFORMANCE OF PROPOSED MODEL IN ELEMENT TEST SIMULATIONS AND APPLICATION TO SEISMIC RESPONSE ANALYSIS

8.1 INTRODUCTION

To illustrate the abilities and effectiveness of the elastic–plastic model developed in the previous chapters, predictions of the model will be compared to the measured soil behaviour for various loading conditions and initial states. The usage of the state index as a current variable will be illustrated and calculation steps for both stress–control and strain–control analysis will be described. Finally, the model will be implemented in an effective stress method of analysis and applied to a seismic response analysis of a sand deposit.

The experimental results from the drained torsional shear tests and undrained torsional simple shear tests presented in Chapter 4 and Chapter 5, respectively, are used as experimental basis for verification and validation of the model. The capability of the model to represent the gradual change of the stress–strain characteristics from those typical for dense sand and low effective stress to those typical for loose sand and high effective stress will be examined for drained as well as undrained conditions. Cyclic behaviour predicted by the model will be compared to the measured drained torsional shear behaviour and undrained torsional simple shear behaviour in order to demonstrate the accuracy and effectiveness of the model in representing the sandy soil
behaviour for complex loading conditions including principal stress rotation. In addition to the simulation of the element test data, the model will be used to predict seismic response of a model of sand observed in a centrifuge test.

8.2 STATE INDEX $I_s$ AS CURRENT VARIABLE

The definition of the state index proposed by Verdugo (1992) and Ishihara (1993) considers the state index as an initial index parameter which associates the characteristics of soil behaviour with the initial state, i.e. with the state of the soil prior to shearing. However, in the proposed elastic–plastic model, the state index $I_s$ will be employed as a current variable associated with the state of the soil in terms of the current effective stress and the initial void ratio. It is interesting to note that Jefferys (1993) used in similar manner the state parameter $\psi$ which was originally proposed by Been and Jefferys (1985) as an initial index parameter.

The usage of the state index as a current variable in the present model will be illustrated on an example of application of the model to undrained analysis. Fig. 8.1 shows a typical state diagram where the upper reference line and the quasi steady state line are shown together with the initial state, or the state of the soil prior to shearing, denoted by point $A$. The dashed line in this figure schematically shows one typical $e–p'$ path of a sample undergoing monotonic undrained shearing. The initial part of the response is characterized by contractive behaviour as manifested by the increase in the pore pressure or decrease in the effective stress until the phase transformation is reached (PT). After the phase transformation state, the behaviour is dilative, with the $e–p'$ path moving towards right on the state diagram. If the state index is considered as a current variable, then along the undrained path its value will change continuously since

$$
\bar{I}_s = \frac{\Delta e_c}{\Delta e_{qss}}
$$

(8.1)

in which along the undrained path ($e$–constant path) nominator is constant ($\Delta e_c = \text{const.}$), while denominator changes continuously ($\Delta e_{qss} \neq \text{const.}$). By introducing this equation in Eqs. (6.14) – (6.16), the failure and deformation parameters can be written as
Figure 8.1 The State Index as Current Variable

\[
I_s = \frac{\Delta e_c}{\Delta e_{qss}}
\]

\[
I_s = N_{p_{max}} = \alpha_1 + \beta_1 I_s
\]  \(8.2\)

\[
G^N_{p_{max}} = \alpha_2 + \beta_2 I_s
\]  \(8.3\)

\[
G^N_{p_{min}} = \alpha_3 + \beta_3 I_s
\]  \(8.4\)

It is important to note that the initial state \(A\) lies in the \(e-p'\) plane, while on the other hand, the undrained path illustrated by the dashed line is projection on this plane since along this path the shear stress has non-zero value. Therefore, although the current state index \(I_s\) has same value for the initial state \(A\) and the projected point which coincides with the point \(A\), the values of the plastic shear modulus \(G^N_p\) calculated by Eq. (6.13) will be different due to the difference in the plastic shear strain \(e_p\) associated with these points. Hence, the same value of \(I_s\) for two different points along the stress path does not imply the same value of the plastic shear modulus \(G^N_p\) for these points.

Implementation of the state index as a current variable in the model (Eq. 8.1)
is based on the fact that failure and deformation coefficients $\alpha_j, \beta_j \ldots, \beta_j$ are defined from drained $p'$-constant tests and from relations of strength and deformation parameters with the state index $I$, defined as initial index parameter (Fig. 6.24). Hence, these failure and deformation coefficients represent the strength and deformation characteristics of sandy soil with reference to its initial states, for $p'$-constant stress paths involving volumetric strains due to the dilatancy alone. The state index is employed as a current variable in order to account for the influence of the effective stress on the stress–strain characteristics of sandy soils. Namely, using the state index as current variable is equivalent to shifting the current point along a family of $p'$-constant stress–strain curves and taking up the stress–strain parameters corresponding to the state defined by the current effective stress, initial void ratio and the current stress ratio, or plastic shear strain.

8.3 PREDICTION OF DRAINED BEHAVIOUR OF SAND

The drained torsional shear tests presented in Chapter 4 are considered first for purpose of examining the abilities of the model to represent the stress–strain and volume change characteristics of sandy soils in monotonic shearing under constant mean effective stress ($p'$-constant stress path). These tests include various initial states of Toyoura sand the with void ratio and the mean effective stress ranged between $e = 0.73 - 1.0$ and $p' = 0.3 - 4.0$ kgf/cm$^2$, respectively. In addition to the monotonic tests, the predictions of the model are compared to the measured drained behaviour involving cycling and proportional loading as well.

The values of the model parameters for Toyoura sand used in the simulation of the drained torsional tests are listed in Table 8.1. These parameters have been determined following the procedures given in Sec. 7.4 with the failure and deformation parameters obtained from the plot shown in Fig. 6.24. The failure parameter $c_p$, which is related to the initial anisotropy, is defined from the data presented by Gutierrez (1989), shown in Fig. 7.9 as well. The parameters listed in Table 8.1 are independent of density and stress state, and they represent a single set of parameters for Toyoura sand, for the air pluviation method of sample preparation. It should be noted that none of this parameters have dimensions.
Table 8.1  Model Parameters for Toyoura Sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Toyoura sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Deformation</td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td>480</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>$n$</td>
<td>0.4</td>
</tr>
<tr>
<td>State Index</td>
<td></td>
</tr>
<tr>
<td>$e_{aw}$</td>
<td>0.930</td>
</tr>
<tr>
<td>$\lambda_{aw}$</td>
<td>0.000</td>
</tr>
<tr>
<td>$e_{qps}$</td>
<td>0.902</td>
</tr>
<tr>
<td>$\lambda_{qps}$</td>
<td>0.040</td>
</tr>
<tr>
<td>Failure</td>
<td></td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>0.586</td>
</tr>
<tr>
<td>$\beta_t$</td>
<td>0.016</td>
</tr>
<tr>
<td>$c_f$</td>
<td>0.080</td>
</tr>
<tr>
<td>Plastic Deformation</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>283</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>41</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>93</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>10</td>
</tr>
<tr>
<td>$f$</td>
<td>4</td>
</tr>
<tr>
<td>Consolidation</td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.005</td>
</tr>
</tbody>
</table>
The experimental evidence on the stress–dilatancy relationship of Toyoura sand given in Chapter 4, has shown that this relationship is slightly affected by density or void ratio. Therefore, in order to achieve accurate simulation of the element test results it was necessary to use dilatancy parameters which account for the effects of density on the stress–dilatancy relationship. The variation of the dilatancy parameters with the void ratio is given in Table 8.2.

Table 8.2 Dilatancy Parameters for Toyoura Sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Void ratio – e</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.74 – 0.90</td>
</tr>
<tr>
<td>Dilatancy</td>
<td></td>
</tr>
<tr>
<td>$\mu_o$</td>
<td>0.54 – 0.70</td>
</tr>
<tr>
<td>$\mu_{\text{max}}$</td>
<td>0.54 – 0.58</td>
</tr>
<tr>
<td>$S_c$</td>
<td>0.001 – 0.005</td>
</tr>
</tbody>
</table>

8.3.1 Stress Control Analysis

Predictions of drained torsional shear behaviour of Toyoura sand were done utilizing stress–control analysis procedure in which the plastic strain increment was calculated from the model for a given stress increment. As illustrated by Eq. (7.22), in the stress–control analysis the calculation of the elastic and plastic strains is not related to each other, and therefore, elastic strains are not needed for evaluation of the plastic strain components. Flow chart of the calculation of the plastic strain components is given in Fig. 8.2.

The calculation is performed incrementally in the following manner:

1) Firstly, for a given state of stress, strain and initial void ratio, the stress parameters $X$, $Y$, $p$, $q$, $r$ and $\beta_o$ are calculated from Eqs (7.1) to (7.9). Next, the state index is calculated using Eqs. (7.33) and (7.34), and the stress ratio at failure $r_f$ and normalized plastic shear modulus $G_p^N$ are determined from Eqs. (7.32) and (6.13),
Figure 8.2 Flow Chart of Stress-Control Analysis Procedure
respectively. Using Eqs. (7.57) and (7.70) the hardening modulus \( H_p \) and dilatancy parameter \( \mu \) are now calculated.

2) New stress increment is applied, for which the conjugate point is determined \((X_r, Y_r)\) and the angle of noncoaxiality \( \psi \) is calculated. Next, the loading index is calculated using Eq. (7.89). Knowing \( H_p \) and \( \dot{\gamma} \), the scalar quantity \( d\lambda \) is determined from Eq. (7.19).

3) Finally, the plastic strain increment components are calculated using Eqs. (7.72) – (7.74), and stresses and plastic strains are updated for the next increment.

Following the above procedure a simple computer program was written which was later used for prediction of the drained behaviour of Toyoura sand. All predictions were done with the model parameters listed in Tables 8.1 and 8.2.

### 8.3.2 Monotonic Behaviour

Comparison between measured and calculated stress–strain behaviour of Toyoura sand for void ratio in the range between \( e = 0.74 – 1.0 \) and mean effective stress at \( p' = 0.3, 0.5, 1.0, 2.0, 3.0 \) and \( 4.0 \) kgf/cm\(^2\) is shown in Fig. 8.3. The points in this figure represent the measured behaviour and the solid lines represent the predictions of the model.

The predicted stress–strain curves over a wide range of void ratio and mean effective stress agree very well with those measured. Notably, the stiffness and the stress ratio at failure decrease with increasing mean effective stress and void ratio, and the gradual change of the stress–strain characteristics from those typical of dense sand and low effective stress to those typical of loose sand and high effective stress is well accounted for. Because the parameter values were derived from these tests, the good agreement was to be expected.

Volume change characteristics and stress–dilatancy relationship for void ratio \( e = 0.74 \), at different mean effective stresses, are shown in Figs. 8.4a and 8.4b, respectively. The increase in the compressibility with increasing effective stress is correctly predicted by the model providing gradual change in the volumetric strain behaviour from being expansive with strong dilation at low effective stress to being
Figure 8.3 Measured and Predicted Stress–Strain Curves of Toyoura Sand
compressive with weak dilation when the effective stress is high. The deviation of the predicted from the measured behaviour at high shear stress level has to be attributed to the usage of uncorrected stress ratio at failure in the stress–strain relationship, and to the fact that hyperbolic relation approaches this value asymptotically. Therefore, if measured stress ratios at failure are used for determination of the failure parameter \( r_f \), then the predicted stress–strain curve will be always below the measured one when the strains are large, e.g. \( \gamma > 10 \% \). Fig. 8.4b shows that the stress–dilatancy relationship which was used in the prediction of the model compare favorably with the measured stress–dilatancy relationship. The dilatancy parameter was constant at \( \mu = \mu_a = \mu_{\text{max}} = 0.54 \), which corresponds to the measured stress ratio at the phase transformation. It is worth noting that both, stress–strain and volume change characteristics are accurately predicted by the model by using the exact experimental data for the strength (stress ratio at failure) and stress–dilatancy relationship. This demonstrates that the coupling between the stress–strain and stress–dilatancy relationships implemented in the model assures good accurate predictions when measured quantities are used as input parameters for the model.

Similarly, Fig. 8.5 shows comparison between measured and calculated volume change behaviour for looser Toyoura sand, with void ratio around \( e = 0.8 \) and \( e = 0.90 \). Apparently, for both densities volumetric strains are predicted with good accuracy for all considered effective stresses. It can be concluded from the comparisons presented in Figs. 8.3 – 8.5 that both the stress–strain and volume change behaviour observed in the drained torsional shear tests on Toyoura sand are predicted accurately by the model over a wide range of void ratio and mean effective stress. This demonstrates the ability of the model to represent sandy soil behaviour for different initial conditions, with a single set of model parameters.

### 8.3.3 Cyclic Behaviour

Figs. 8.6 – 8.9 show comparison between measured and predicted stress–strain and volume change behaviour of Toyoura sand in drained cyclic torsional shear test. The sample with void ratio \( e = 0.74 \) was subjected to three cycles with constant peak stress ratio amplitude at \( \tau/p' = \pm 0.4 \) and then monotonically sheared until failure was attained. The purpose of this comparison is to illustrate the ability of the model to
Figure 8.4 Measured and Predicted Volume Change Behaviour and Stress-Dilatancy Relationship for Medium Dense Toyoura Sand
Figure 8.5 Measured and Predicted Volume Change Behaviour for Loose and Very Loose Toyoura Sand
Figure 8.6 Measured and Predicted Stress–Strain Curve of Toyoura Sand (Cyclic Drained Torsional Shear Test)
Figure 8.7 Measured and Predicted Stress–Strain Curve of Toyoura Sand
(Cyclic Drained Torsional Shear Test: Enlarged Cyclic Part)
Figure 8.8 Measured and Predicted Volume Change Behaviour of Toyoura Sand (Cyclic Drained Torsional Shear Test)
Figure 8.9 Measured and Predicted Volume Change Behaviour of Toyoura Sand  
(Cyclic Drained Torsional Shear Test: Enlarged Cyclic Part)
account for the hardening effects associated with cyclic shear stress application with constant or decreasing peak stress ratio. To achieve this, the plastic shear modulus is expressed as a function of the plastic shear work $\Omega_p$ through the parameter $f$ of the proposed stress–strain relationship presented in Chapter 6. It is considered that by using the parameter $f$ it is possible to account only for slight effects of hardening since this parameter can not change significantly the stress–strain curve. Therefore, it would be more appropriate if the maximum plastic shear modulus $G_{p,max}^N$ is expressed as function of $\Omega$, in a similar manner to that used by Ghaboussi and Momen (1982). The current version of the model does not comprise such a function, and in fact, it can not predict hardening associated with the conditions described above.

The stress–strain curves presented in Figs. 8.6 and 8.7 show very good agreement between the predicted and measured behaviour for both the cyclic part of the loading and the subsequent monotonic shearing up to failure. The remarkable similarity of the stress–strain loops shows that shear modulus and damping are quite accurately predicted by the model. In addition to that, Figs. 8.8 and 8.9 illustrate that the volume change behaviour is correctly predicted as well, with continuous contractive volumetric strain during the cyclic part of the loading, followed by a strong dilation which started once the stress ratio at phase transformation was passed during the monotonic increase of the shear stress. It is important to notice that remarkably good quantitative prediction is obtained for both stress–strain and volume change behaviour.

### 8.3.4 Proportional Loading

Finally, the performance of the model in drained cyclic behaviour including proportional loading will be examined. For that purpose, the behaviour observed in the proportional loading test presented in Chapter 4 is compared to the prediction of the model in Figs. 8.10 and 8.11. The predicted stress–strain relationship agree well with that measured for each stage of the loading, including virgin loading, unload–reload cycle, proportional loading and final unloading. However, the plastic strain produced during the proportional loading is smaller than the measured one. It should be noted that this prediction was based on the assumption that curvature of the hardening surface is equal to that of the failure surface. This assumption is supported by the data.
shown in Fig. 4.16, where the effects of the effective stress on the mobilized angle of internal friction are illustrated, and by the fact that the stress ratio during the proportional loading was relatively high at 0.46. Needless to say, if conical hardening surface with straight meridian line is adopted then proportional loading will not produce plastic shear strain.

The comparison of the predicted volumetric strain and that measured in the test on Toyoura sand (See Fig. 8.11) illustrates the accuracy of the prediction for this relatively complex stress path. The similarity of the strain path is accompanied by very good quantitative prediction of both volumetric strain and shear strain. It is worth noting that unload–reload stress path as well as the final unloading are very precisely simulated by the model thus demonstrating that the concept with very small elastic area (point yield surface) and plastic shear modulus as a function of strain, can accurately represent the behaviour in which elastic and plastic response are of comparable magnitudes.

8.4 PREDICTION OF UNDRAINED BEHAVIOUR OF SAND

For further verification and validation of the proposed model, predictions of the model are compared to the measured undrained monotonic and cyclic behaviour of Toyoura sand in the tests presented in Chapter 5. In addition to these tests, the experimental data for the same sand published by Pradhan (1989c) is also used in the comparison in order to widen the range of the initial states and loading conditions considered in the simulation, and also to demonstrate the abilities of the model on data from other sources than this study.

The monotonic undrained torsional simple shear tests considered in the simulation include isotropically and anisotropically consolidated samples with void ratio ranged between \( e = 0.71 \) – 0.85. The data from these tests comprise various aspects of the undrained behaviour of sandy soils and particularly emphasize the effects of density on the stress–strain behaviour and effective stress path. The ability of the model to simulate undrained response from that typical of dense sand to that typical of loose sand including contractive flow or limited flow, and complete lost of the strength will be demonstrated. The accuracy in modelling the rotation of principal stresses will be illustrated as well.
Figure 8.10  Measured and Predicted Stress–Strain Curve of Toyoura Sand During Proportional Loading
Figure 8.11 Measured and Predicted Volume Change Behaviour of Toyoura Sand During Proportional Loading
In a similar manner undrained cyclic simple shear tests are considered for the range of void ratio between \( e = 0.68 - 0.86 \), and various cyclic stress ratios. Besides further verification of the applied concept for modelling the effects of density and effective stress, the complexity of the stress path in these tests presents serious test for each constituent part of the elastic–plastic model.

Except for the plastic deformation constant \( f \), the model parameters used in the simulation of the undrained tests are the same as those used in the simulation of the drained tests. Thus, the parameters listed in Table 8.1 are used with the plastic deformation constant \( f = 1 \). The dilatancy parameters applied in the element test simulations of the undrained behaviour are listed in Table 8.3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_0 )</td>
<td>0.10 – 0.25</td>
</tr>
<tr>
<td>( \mu_{\text{max}} )</td>
<td>0.58 – 0.67</td>
</tr>
<tr>
<td>( S_c )</td>
<td>0.0010 – 0.0027</td>
</tr>
</tbody>
</table>

### 8.4.1 Strain–Control Analysis

In order to implement an elastic plastic constitutive model in a solution technique such as the finite element method, strain–control formulation of the model is needed where the plastic strain increment and the stress increment are calculated for a given increment of total strain. Applicability of the model is therefore largely dependent on the uniqueness and computational efficiency of its strain–control version. The strain–control analysis procedure of the present model is presented in a summarized form by the flow chart given in Fig. 8.12. Based on this procedure a computer program is coded that can be adopted to a two–dimensional finite element program. All predictions of the undrained tests were done using the strain–control
version of the model.

The calculation steps in the strain–control analysis are as follows:

1) For a given stress and strain state, and known value of void ratio, the stress parameters \( X, Y, p, q, r \) and \( \beta_0 \) are determined from Eqs. (7.1) – (7.9) and the state index is calculated using Eqs. (7.33) and (7.34). Next, stress ratio at failure is calculated from Eq. (7.32) and (6.14).

2) After the calculations for the initial state, new total strain increment is applied. In order to calculate the hardening modulus \( H_r \), however, it is first necessary to determine the position of the next stress point with respect to the current hardening surface, and therefore, the stress increment is needed. It is important to notice that in fact, the hardening modulus is independent of the stress increment for a certain loading condition, e.g. virgin loading, unloading, reloading or general loading, unless the loading condition change from one to another. Hence, it is actually not necessary at this point to precisely calculate the stress increment but to only identify which of the above mentioned loading conditions will be activated for the new stress state and in accordance with that to take up the corresponding hardening surface.

The adopted solution to this problem, which is also applied to the solution of the scalar parameter \( d\lambda \) in the later steps of the analysis, is crucial for the computational efficiency of the strain–control analysis. The solution is based on the assumption of validity of the plastic potential formulation adopted for the present model, validity of the superposition principle (\( de = de_e + de_p \)) and coaxiality of the elastic strain increment and the stress increment.

Let us consider an arbitrary case with a given stress state by point \( A \) and total strain increment \( de \), in the \( X-Y \) plane, as shown in Fig. 8.13. If purely elastic behaviour is assumed (\( de = de_e \)), then from the coaxiality condition, the conjugate stress point will be at point \( E \) on the failure surface in Fig. 8.13. On the other hand, if purely plastic behaviour is assumed, then following the adopted plastic potential formulation the conjugate point will be at point \( P \). Hence, in order to satisfy the principle of superposition, for any case involving elastoplastic behaviour, the conjugate point must lie on the arc \( EP \), i.e. between the boundary conjugate points for purely elastic and purely plastic behaviour, as denoted by points \( E \) and \( P \), respectively. The
Figure 8.12 Flow Chart of Strain-Control Analysis Procedure
Figure 8.13 Illustration of the First Assumption for the Conjugate Point

The exact location of the conjugate point will depend on the relative contributions of the elastic and plastic strain components to the total strain increment. Following the above discussion and the experimental evidence that for most loading conditions the plastic behaviour is dominant, it is assumed that the conjugate point is in the vicinity of the boundary conjugate point \( P \). Thus, knowing the directions of the elastic and plastic strain increments as well as the total strain increment vector, it is now possible to calculate the magnitudes of the elastic and plastic strain components. Next, the stress increment is calculated by using the elasticity relations, and hence, the new stress state can be determined. By comparing the position of the new stress state and the current hardening surface, the loading condition is identified and subsequently the corresponding hardening modulus \( H_p \) and dilatancy parameter \( \mu \) are calculated.

3) To calculate the scalar parameter \( d\lambda \) it is necessary to use the inverse relation for the loading index \( \hat{f} \). Namely, in Eq. (7.89), the stress increments are replaced by the elastic strain increments by using the inverse form of the elasticity relation given in Eq. (7.94). Next, using the superposition principle (Eq. 7.16), the elastic strain increments are replaced by the difference between the total and the plastic strain
PERFORMANCE OF PROPOSED MODEL

increments. Finally, by replacing the plastic strain increments with the expressions defined by Eqs. (7.72)-(7.74), \( d\lambda \) is defined in terms of the total strain increments. Thus, \( d\lambda \) is calculated, and from Eqs. (7.72)-(7.74) the plastic strain increment components are determined. It is necessary to perform this calculation in iterations since it is based on assumed conjugate point, and hence, the calculation continues until the assumed and calculated direction of the plastic strain increment coincide in the range of \( \pm 0.5^\circ \). By using the procedure explained in step (2), however, the convergence of the solution is granted in very efficient way with very small number of iterations.

4) Knowing the plastic and total strain increment components, the elastic strain increment components are evaluated by using Eq. (7.16), and then, the stress increment components are calculated from the elasticity relations. Finally, the new stress state is calculated.

8.4.2 Monotonic Torsional Simple Shear

In Figs. 8.14 – 8.21 the predictions of the model are compared to the measured behaviour in the undrained monotonic torsional simple shear tests on Toyoura sand with void ratio between \( e = 0.71 - 0.85 \), including isotropic and anisotropic consolidation conditions. The predicted behaviour is shown by solid lines in these figures along with the measured behaviour, shown either by dashed lines or marks.

Fig. 8.14 shows the comparison between predicted and measured behaviour of isotropically consolidated medium dense Toyoura sand with void ratio \( e = 0.75 \). Apparently, excellent agreement between the predicted and measured behaviour is obtained for both effective stress path and stress–strain curve. Initially contractive response associated with development of pore water pressure and movement of the effective stress path towards left until phase transformation state is reached as well as subsequent dilation are very precisely predicted with an overall coincidence in the stress–strain behaviour. It is worth noting that the large increase in the pore pressure during unloading path is correctly predicted as well.

Similarly, Fig. 8.15 shows the predicted and measured behaviour for similar density and identical mean effective stress conditions as in the former case but for
Figure 8.14  Measured and Predicted Behaviour of Isotropically Consolidated Toyoura Sand
Figure 8.15  Measured and Predicted Behaviour of Anisotropically Consolidated Toyoura Sand
anisotropically consolidated sample with initial effective stress ratio $K = 0.5$. It may be seen that very good agreement between the predicted and measured behaviour is obtained for the anisotropically consolidated sample as well. It is important to notice the difference between the response of the isotropically and anisotropically consolidated samples with respect to the principal stress directions during shearing. Namely, in the case of isotropic consolidation condition the major principal stress direction is fixed at $\beta_o = 45^\circ$ during shearing, while on the other hand, when anisotropically consolidated sample is sheared, there is continuous rotation of the principal stresses with gradual change in $\beta_o$ from $0^\circ$ to slightly less than $45^\circ$ near failure. Fig. 8.16 shows that predictions of rotation of the principal stress directions compare favorably with the observed behaviour where the rotation of principal stress directions takes place during the contractive part of the response until phase transformation state is reached, while afterwards, the principal stress directions remain nearly constant with $\beta_o$ close to $45^\circ$.

The results of the undrained torsional shear tests on loose Toyoura sand are also predicted with good accuracy, as shown in Fig. 8.17. The initial portion up to the peak shear stress, and the subsequent contractive flow or limited liquefaction associated with significant reduction in the effective stress and development of large shear strain are correctly predicted. Since the test was performed in a stress-control manner, and on the other hand, the contractive flow developed rapidly, these part of the response has not been measured, and therefore, the experimental curves are not continuous.

To illustrate the ability of the model to simulate undrained behaviour of very loose sand, in Fig. 8.18 prediction of the model is shown for Toyoura sand with void ratio $e = 0.95$. It may be seen that the contractive flow leads to a complete loss of the strength of the sand with the effective stress path ending up in the origin of the diagram with zero shear stress and mean effective stress. This pattern in the undrained behaviour would be obtained for each of the initial states above the upper reference, defined for Toyoura sand at void ratio $e = 0.93$.

Two monotonic undrained torsional simple shear tests on Toyoura sand with void ratio $e = 0.708$ and 0.792 published by Pradhan (1988) are also used in the element test simulations. As shown in Figs. 8.19 and 8.20, predicted behaviour compare favorably with the measured behaviour in these tests, with slightly lesser
Figure 8.16 Measured and Predicted Rotation of the Principal Stress Directions (Monotonic Torsional Simple Shear Test: K=0.5)

accuracy than in the previous simulations but yet with quite reasonable degree of coincidence between predictions and observations.

Summary plot of measured and predicted stress–strain curves of Toyoura sand for various void ratios in the range between $e = 0.708 - 0.95$ is shown in Fig. 8.21. The model correctly predicts the stress–strain characteristics observed in the undrained torsional simple shear tests on medium dense, loose and very loose sand, providing gradual change in these characteristics from those typical for dense sand, with continuous increase of the shear stress with straining, to those typical for loose sand, with drop in the shear stress associated with development of large strains including the extreme cases with zero residual strength. It should be emphasized again that all the predictions are done using the single set of strength and deformation parameters listed in Table 8.1.
Figure 8.17  Measured and Predicted Behaviour of Loose Toyoura Sand
Figure 8.18  Measured and Predicted Behaviour of Very Loose Toyoura Sand
Figure 8.19  Measured and Predicted Behaviour of Dense Toyoura Sand
Figure 8.20 Measured and Predicted Behaviour of Medium Dense Toyoura Sand
Figure 8.21 Summary Plot of Measured and Predicted Monotonic Undrained Behaviour of Toyoura Sand
8.4.3 Cyclic Torsional Simple Shear

The abilities and effectiveness of the model will be lastly demonstrated through the performance of the model in simulations of cyclic undrained torsional simple shear behaviour of Toyoura sand with void ratio ranged between \( e = 0.68 - 0.86 \). Basic principle applied to the simulations of cyclic undrained response was to use a single set of constant dilatancy parameters for one simulation which provides best overall representation of the cyclic behaviour. In other words, to simulate the virgin loading, intermediate loading and cyclic mobility with the same set of dilatancy parameters so as to achieve accurate prediction of both effective stress path and stress–strain curve in each of the above mentioned stages of the cyclic response.

Figs. 8.22 and 8.23 show the effective stress path and stress–strain curve of measured and predicted behaviour, respectively, for medium dense Toyoura sand with void ratio \( e = 0.78 \). This test represents a soil element under initially isotropic state, with mean effective stress at \( p' = 0.667 \text{ kgf/cm}^2 \), subjected to cyclic shear stress with amplitude \( \tau = \pm 0.15 \text{ kgf/cm}^2 \) corresponding to cyclic stress ratio \( (\tau/p')_c = 0.225 \). As regards the observed effective stress path in the experiment (Fig. 8.22a), at least three distinct stages in the response can be recognized: virgin loading, characterized by relatively large pore pressure increase in a single cycle as compared to other cycles; intermediate loading, with initial decreasing rate in the pore pressure build-up which later on reversed into increasing rate as cyclic loading progresses; and cyclic mobility, as characterized by the typical effective stress loop passing through or nearby the origin of the \( \tau-p' \) diagram.

Comparison of the measured and predicted stress paths in Figs. 8.22a and 8.23a shows that the number of cycles needed to achieve nearly zero effective stress is smaller for the predicted behaviour than for the measured one. It seems that this disagreement results from the fact that in the prediction, the first part of the intermediate loading where the rate of pore water pressure is decreasing, is not accurately represented. More detailed analysis of this part of the behaviour reveals that the increase in the stress ratio is relatively small because of the slow build-up of the pore water pressure, resulting in nearly steady stress–strain curve (See Fig. 5.12). Since the initial stress ratio is relatively small and its rate of increase is very slow during the intermediate phase of the loading, it is possible that this behaviour would
Figure 8.22 Measured Behaviour of Medium Dense Toyoura Sand (e=0.78)
in Cyclic Undrained Torsional Simple Shear Test
Figure 8.23  Predicted Cyclic Undrained Torsional Shear Behaviour of Medium Dense Toyoura Sand (e=0.78)
affect the stress–dilatancy relationship, as demonstrated by Pradhan (1989a) and (1989b). In lack of abundant experimental evidence on the influence of the cyclic loading on the stress–dilatancy relationship, this important feature of soil behaviour is not adequately considered in the present model. As regards the stress–strain curve, there is very good agreement between the prediction and observation, with quite small shear strain development until the response entered into cyclic mobility, when large shear strain developed.

Similar comparison for loose Toyoura sand with void ratio $e = 0.86$ is presented in Figs. 8.24 and Fig. 8.25. Important feature of this behaviour is the sudden occurrence of the contractive flow accompanied by significant decrease in the effective stress and development of very large shear strain. Neither preceding effective stress path nor stress–strain curve indicate the occurrence of the flow. Moreover, the shear strain prior to the flow is remarkably small. Yet, this characteristic behaviour is well represented by the model, with accurate prediction of the stress–strain curve and the effective stress path.

Figs. 8.26 and 8.27 show measured (Pradhan, 1989c) and predicted behaviour of dense Toyoura sand with void ratio $e = 0.68$, subjected to cyclic shear stress amplitude of $\tau = \pm 0.64$ kgf/cm$^2$. Due to the high shear stress amplitude the cyclic mobility occurs after only one cycle. Again, very well agreement can be seen for the predicted and measured effective stress path, while the predicted shear strains during the cyclic mobility are slightly underestimated.

Finally, measured and predicted behaviour of medium dense Toyoura sand with void ratio $e = 0.78$, subjected firstly to cyclic load with small shear stress amplitude, $\tau = \pm 0.15$ kgf/cm$^2$, which is later on doubled, is shown in Figs. 8.28 and 8.29, respectively. The behaviour is characterized with very small increase in the pore water pressure and negligible shear strain response during the cycles with small shear stress amplitude. Once the double stress amplitude is applied, significant increase in the pore pressure occurs moving the effective stress path towards left and eventually reaching zero effective stress after only two cycles. Predicted behaviour compare favorably to that measured with remarkable similarity in both effective stress path and stress–strain curve.

The comparison of the measured and the predicted cyclic undrained torsional
Figure 8.24  Measured Behaviour of Loose Toyoura Sand (e = 0.86) in Cyclic Undrained Torsional Simple Shear Test
Figure 8.25 Predicted Cyclic Undrained Behaviour of Loose Toyoura Sand (e = 0.86)
Figure 8.26 Measured Behaviour of Dense Toyoura Sand (e = 0.68) in Cyclic Undrained Torsional Simple Shear Test
Figure 8.27 Predicted Cyclic Undrained Behaviour of Dense Toyoura Sand (e = 0.68)
Figure 8.28  Measured Behaviour of Medium Dense Toyoura Sand (e = 0.78) in Cyclic Undrained Torsional Simple Shear Test
Figure 8.29  Predicted Cyclic Undrained Behaviour of Medium Dense Toyoura Sand (e = 0.78)
simple shear behaviour of Toyoura sand discussed above further demonstrated the capability of the model to accurately represent cyclic undrained behaviour of sand. All the predictions were done using the set of parameters listed in Tables 8.1 and 8.3.

8.5 APPLICATION OF PROPOSED MODEL TO SEISMIC RESPONSE ANALYSIS

In the preceding sections the performance of the present model was examined through element tests simulations, i.e. considering a single soil element under drained and undrained conditions, and monotonic and cyclic loads. To further test the capabilities of the model and its efficiency and applicability to numerical analysis, the strain–control version of the model was incorporated into a finite element code and used for blind–prediction of a seismic response of level ground observed in a centrifuge test. The prediction was made in the framework of VELACS program (Verification of Liquefaction Analyses by Centrifuge Studies) for model No. 1. The prediction was done prior to the performance of the centrifuge test (Ishihara et al., 1993b). In the subsequent pages, a brief description of the centrifuge test, determination of the model parameters and applied numerical method in the analysis will be given together with the comparison of the measured and predicted seismic response of the level ground.

8.5.1 Description of the Centrifuge Test

Two primary tests and three duplicate tests of VELACS centrifuge Model No. 1 have been performed at Rensselaer Polytechnic Institute (RPI), at University of California at Davis and at University of Colorado at Boulder (Dobry and Taboada, 1993). A detailed comparison of the results of the primary and duplicate tests has revealed general consistency of the experimental data. Therefore, RPI Test 2 will only be considered herein, since RPI is the primary experimenter and Test 2 has been recommended for comparison with the analytical predictions (Dobry and Taboada, 1993).

A sketch of Model No. 1, showing the dimensions of the model and instrumentation, is presented in Fig. 8.30. The container is a laminar box comprised of forty
Figure 8.30  VELACS Centrifuge Model No. 1.  a) Side View of Model No. 1
b) Horizontal Input Motion \((a_{\text{max}} = 0.23\text{g})\)
aluminum alloy rectangular rings separated by roller bearings. A latex membrane is installed inside the laminar box to prevent leakage of water and migration of soil particles to the gaps between rings. The model consists of a 20 cm high, horizontal, uniform layer of Nevada sand, placed at relative density of about 45%, by dry pluviation method. The sand was fully saturated with water, spun at a centrifuge acceleration of 50g, and excited horizontally at the base with the accelerogram shown in Fig. 8.30, having a maximum acceleration of 0.23g. Horizontal and vertical accelerations at the base and in the soil, excess pore water pressures at center point and quarterpoint of the box length, lateral displacements and vertical settlements were measured. The exact locations of the accelerometers, pore pressure transducers and linear variable differential transformers (LVDT) are given in Table 8.4 and in Fig. 8.30. It is considered that, when subjected to base shaking, the soil in the laminar box simulates approximately a semi-infinite layer, and models a prototype 10m thick water-saturated layer having the same properties and compressibility of Nevada sand but which is 50 times more permeable than Nevada sand.

The material used in the test, Nevada sand, is classified as a uniform fine sand with 8% fines content. Its physical properties and grain size distribution curve are given in Table 8.5 and Fig. 8.31, respectively. Permeability of the sand is 0.0066 cm/sec (ETC–Soil Data Report).

8.5.2 Evaluation of Model Parameters

Parameters of the elastic-plastic constitutive model, for Nevada sand, were determined following the procedures given in Sec. 7.4. In addition to the supplied element test data (ETC–Soil Data Report), a series of several undrained triaxial tests and drained torsional shear tests on Nevada sand were carried out for that purpose. Model parameters for Nevada sand are listed in Table 8.6.

As discussed in Sec. 7.4, parameters of the present model are grouped as: elastic, consolidation, state index, failure, deformation and dilatancy parameters. State index parameters were determined based on the results from undrained triaxial compression tests on Nevada sand. In order to define the quasi steady state line more precisely, instead of the slope of this line in the $e - \log p$ plot, it was defined in a tabular form, as discussed in Sec. 7.4. Fig. 8.32 shows both the upper reference line and the quasi
### Table 8.4 Location of Instruments in Centrifuge Model No. 1
RPI Test 2 (after Dobry and Taboada, 1993)

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<tr>
<th>Transducer</th>
<th>Instrument Number (see Fig. 1)</th>
<th>Coordinates (cm)</th>
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<tr>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Accelerometer measuring in vertical direction</td>
<td>AV1</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>AV2</td>
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</tr>
<tr>
<td></td>
<td>AV3</td>
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<tr>
<td></td>
<td>AV5</td>
<td>27.0</td>
</tr>
<tr>
<td>Accelerometer measuring in horizontal direction</td>
<td>AH1</td>
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</tr>
<tr>
<td></td>
<td>AH2</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>AH3</td>
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<tr>
<td></td>
<td>AH5</td>
<td>25.0</td>
</tr>
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<tr>
<td></td>
<td>P2</td>
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</tr>
<tr>
<td></td>
<td>P3</td>
<td>11.5</td>
</tr>
<tr>
<td></td>
<td>P4</td>
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<td></td>
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<td></td>
<td>P8</td>
<td>23.0</td>
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<td>LVDT1</td>
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<td>LVDT6</td>
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Table 8.5 Physical Properties of Nevada Sand

<table>
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<th>Property</th>
<th>Value</th>
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<tr>
<td>Specific Gravity, $G_s$</td>
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<td>Mean Diameter, $D_{50}$ (mm)</td>
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</tr>
<tr>
<td>Maximum Void Ratio, $e_{\max}$</td>
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<tr>
<td>Minimum Void Ratio, $e_{\min}$</td>
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Figure 8.31 Grain Size Distribution Curve of Nevada Sand
Table 8.6 Model Parameters for Nevada Sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nevada Sand</th>
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<td>State Index</td>
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<td>$e_{ur}$</td>
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<td>$\lambda_{ur}$</td>
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<tr>
<td>$e_{qs}$</td>
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<tr>
<td>$\lambda_{qrs}$</td>
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<td></td>
</tr>
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<tr>
<td>$\beta_1$</td>
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</tr>
<tr>
<td>$c_f$</td>
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</tr>
<tr>
<td>Deformation</td>
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</tr>
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</tr>
<tr>
<td>$\beta_2$</td>
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<tr>
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</tr>
<tr>
<td>$f$</td>
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<tr>
<td>Dilatancy</td>
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</tr>
<tr>
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<tr>
<td>$\mu_{max}$</td>
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</tr>
<tr>
<td>$S_c$</td>
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</tr>
<tr>
<td>Elastic</td>
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</tr>
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<td>$G_e$</td>
<td>500</td>
</tr>
<tr>
<td>$\nu$</td>
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</tr>
<tr>
<td>$n$</td>
<td>0.4</td>
</tr>
</tbody>
</table>
steady state line used in the analysis. Deformation and failure parameters were determined from four drained torsional shear tests on Nevada sand. The initial states of the tests are indicated in Fig. 8.32 together with the corresponding values for the state index (Eq. 6.7). The measured stress-strain curves in the drained torsional shear tests are shown in Fig. 8.33. Following the procedure described in Chapter 6 and Sec. 7.4, the relationships between the stress-strain parameters \( r_p, G_{p,\text{ax}}, G_{p,\text{min}} \) and the state index \( I_s \) were defined, and failure and deformation parameters \( \alpha_f, \beta_1, \ldots, \beta_5 \) were determined. Fig. 8.34 shows data points and analytical approximations used for determination of failure and deformation parameters. The effects of the inherent anisotropy were ignored in the analysis, and therefore, the anisotropy parameter was set equal to zero, \( c_f = 0 \). Elastic and dilatancy parameters were evaluated from the results of 1-D consolidation/rebound tests and resonant column tests, and cyclic undrained simple shear tests, respectively, presented in ETC-Soil Data Report. More specifically, dilatancy parameters have been determined from element test simulation of two cyclic undrained simple shear tests (ETC-Soil Data Report), having identical initial density and stress states but different cyclic stress ratio. The dilatancy parameters listed in Table 8.6 were adopted as parameters providing best overall fit of the predicted effective stress path and stress-strain behaviour with the observed ones.

### 8.5.3 Method of Analysis

The strain-control version of the elastic-plastic model was incorporated into a finite element code utilizing the \( u-p \) formulation of the governing equations for two phase medium. The generalized set of the equations governing the behaviour of saturated, porous media under quasi-static and dynamic loads formulated by Biot (1941, 1960), also known as \( u-w-p \) formulation or exact formulation, has been extended to nonlinear problems (Zienkiewicz et al., 1977 and 1980 among others), and various simplifications have been proposed in order to increase its efficiency in numerical transient analysis. In the analysis of the level ground the \( u-p \) formulation (Zienkiewicz et al., 1980 and 1982) is employed. The \( u-p \) approximation of the \( u-w-p \) formulation, is based on the assumption that the acceleration of the water relative to the soil skeleton is negligible, i.e. \( \ddot{w} = 0 \).
Figure 8.32 Reference Lines for Nevada Sand and Initial States in the Drained Torsional Shear Tests

Figure 8.33 Measured Stress–Strain Behaviour of Nevada Sand in the Drained Torsional Shear Tests
Figure 8.34  Relationship Between the Stress–Strain Parameters and the State Index $I_s$ for Nevada Sand
The FEM code used was STADAS (Yoshida and Tsujino, 1992). Velac centrifuge Model No. 1 was idealized in the analysis with a column of 20 four-point isoparametric elements. The analysis was performed assuming that the soil is saturated and under undrained conditions. The governing equations were solved by predictor corrector method, with Newmark's method used as predictor and stress transfer method as corrector. For ensuring stability in the numerical analysis Rayleigh damping was used with $\alpha = 0$ and $\beta = 0.005$.

8.5.3 Description of Predicted and Measured Behaviour

Characteristics of the predicted behaviour will be first illustrated with the effective stress paths and stress–strain curves at 1.25, 2.5, 5.0 and 7.5 m below the surface of the layer as well as with the distribution of the lateral displacements along the depth of the model at 3.0, 6.0, 9.0 and 12.0 sec. Fig. 8.35 shows the predicted effective stress paths at elements 5, 10, 15 and 18 of the FEM model. Apparently, the behaviour is characterized by rapid reduction of the effective stress and occurrence of cyclic mobility after only few cycles. The number of cycles to cyclic mobility was only two for the upper part of the model, while in the bottom part cyclic mobility was attained after about 6 cycles.

The predicted stress–strain behaviour for the same elements is shown in Fig. 8.36. It is characterized by significant reduction in the stiffness of the stress–strain curves as loading progresses, with very flat curves typical for cyclic mobility. The maximum peak to peak shear strain reached about 1 %.

Fig. 8.37 shows that the amplitude of the predicted lateral displacements increased towards the surface of the sand layer. The maximum peak to peak displacement, obtained at the surface of the layer, was about 8 cm.

Comparison of predicted and measured behaviour is shown in Figs. 8.38 – 8.41 in which pore water pressure ratio, acceleration and displacement time histories at four depths of the analyzed profile, are presented. Designations of the transducers for which the comparison is presented corresponds to that given in Fig. 8.30.

The measured and the predicted pore water pressure ratio time histories, shown in Figs. 8.38 and 8.39, respectively, indicate large similarity in the measured and predicted behaviour. Almost identical progress in the measured and predicted pore
Figure 8.35  Predicted Effective Stress Paths at Four Depths of the Model
Figure 8.36  Predicted Stress–Strain Behaviour at Four Depths of the Model
Figure 8.37  Predicted Distribution of Lateral Displacements Along the Depth of the Model
water pressure build-up can be seen for the transducers P5 and P6, while in the bottom part of the sand layer, the predicted behaviour slightly overestimates the increase in the pore water pressure. In general, the model very accurately predicted the pore water pressure development, characterized by relatively fast reduction in the effective stress and occurrence of the cyclic mobility.

The measured and the predicted acceleration time histories for the accelerometers AH3, AH4 and AH5 are compared in Fig. 8.40. Apparently, both measured and predicted behaviours are characterized by significant reduction of the input base motion. Once the effective stress was reduced to zero or nearly zero and the cyclic mobility started, the amplitude of the predicted accelerations reduced below 0.1g. The reduction of the accelerations was even more pronounced in the measured behaviour.

Finally, the measured and predicted lateral displacement time histories are compared in Fig. 8.41. Very well agreement between the measured and the predicted response was obtained, with remarkable similarity in the time histories. The predicted behaviour slightly underestimated the amplitudes of the horizontal displacements in the surface portion of the layer. It is considered that the slight underestimation of the horizontal displacements and overestimation of the accelerations in the predicted behaviour are due to the fact that the model usually develops smaller strains than those observed during the cyclic mobility. In general, however, it can be concluded, that the blind-prediction of the seismic response of Velacs centrifuge Model No. 1 agree very well with the measured response in the centrifuge test.

8.6 SUMMARY

The elastic-plastic constitutive model for sandy soils developed in the previous chapters was thoroughly tested and verified in element test simulations including drained and undrained as well as monotonic and cyclic behaviour. It was shown that the model is capable of simulating many important aspects of sandy soil behaviour for various loading and drainage conditions, and initial states of the samples. Even though all the simulations were performed with a single set of strength and deformation parameters, the combined effects of density and mean effective stress were accurately simulated over a wide range of densities and effective stresses.

The effectiveness of the model and its applicability to seismic response analysis
was demonstrated by the blind prediction of the seismic response of a level ground observed in a centrifuge test. The accuracy of the model was illustrated by the favorable comparison of the measured and predicted behaviour, with very good agreement in all important aspects of the response.

Figure 8.38  Measured Pore Water Pressure Ratio Time Histories (Velacs Centrifuge Model No.1)
Figure 8.39  Predicted Pore Water Pressure Ratio Time Histories (Velacs Centrifuge Model No. 1)
Figure 8.40  Measured and Predicted Horizontal Acceleration Time Histories
(Velas Centrifuge Model No. 1)
Figure 8.41  Measured and Predicted Horizontal Displacement Time Histories (Velacs Centrifuge Model No. 1)
Chapter 9

SUMMARY AND CONCLUSIONS

This study presents an effort to clarify and better understand the behaviour of sandy soils under conditions that can be commonly encountered in the field. An attempt has been made to represent this behaviour on a physically relevant basis and provide a constitutive model comprising balanced features of accuracy, simplicity and versatility. The accomplishments of these aims are briefly summarized in the following.

The study addresses many important issues relevant to sandy soil behaviour and is based on combined experimental and theoretical investigations. In general, two different series of experiments have been conducted to study the behaviour of sandy soils in drained and undrained conditions, respectively. All the tests have been performed on hollow cylindrical samples of clean sand or sand with fines content of 10 – 15 % by weight. Besides the principal purpose of the tests to study the sandy soil behaviour, these test have served as an experimental basis for development of a constitutive model and for its thorough verification and validation through element test simulations.

The theoretical part of the study is focused on development of a proper modelling concept in order to account for the combined effects of density and mean effective stress on the behaviour of cohesionless soils. The influence of various factors on the monotonic and cyclic behaviour of sandy soils is accounted for in an elastic-
plastic constitutive model as a closing result of this work.

More specifically, the major results and conclusions gained in this research are summarized as follows:

EXPERIMENTAL EVIDENCE ON DRAINED BEHAVIOUR

1) Experimental results from the drained torsional shear test on clean sand and sand with 10 – 15 % fines content by weight have shown that sandy soil behaviour is affected by density in a number of ways. As density decreases, the volume change behaviour becomes more contractive while the stress–strain curve is characterized by decrease in both stiffness and peak strength. The effects of density become less pronounced with increasing mean effective stress.

2) The increase in mean effective stress affects the sandy soil behaviour in similar pattern to the decrease in the density. Thus, the stiffness and the peak strength are lower and the volumetric strain is more contractive with increasing mean effective stress. These effects gradually diminish as density decreases. The effects of the mean effective stress observed in the present study are not related to particle crushing.

3) The combined effects of density and mean effective stress are characterized by gradual change in the behaviour from that typical of dense sand and low effective stress to that typical of loose sand and high effective stress. The importance of the combined effects of density and effective stress has been most profoundly illustrated by the measured identical stress–strain behaviour for quite different initial states of the tested soils, including combinations of loose sand at low effective stress and dense sand at high effective stress. These results have indicated the existence of contours in the density–effective stress plane providing initial states with identical stress–strain behaviour.

4) Almost unique stress–dilatancy relationships have been obtained for each of the tested materials in the drained torsional shear tests; the influence of the density and the effective stress on these relationships is very small.

5) In general, an identical pattern in the behaviour of clean sand and sand with fines has been observed. However, the effects of fines have been clearly reflected in the stress–strain and volume change behaviour by reduction of the stiffness and peak
SUMMARY AND CONCLUSIONS

strength, and by increase in the compressibility.

6) The experimental results on the sandy soil behaviour are in close agreement with the experimental evidence obtained from other studies on sandy soils.

EXPERIMENTAL EVIDENCE ON UNDRAINED BEHAVIOUR

1) Both monotonic and cyclic torsional simple shear behaviour on anisotropically consolidated samples of sand are characterized by simultaneous changes in the stress ratio \( q/p' \), orientation of principal stress directions \( \beta_\alpha \) and relative magnitude of intermediate principal stress \( b\)-value). The rotation of the principal stress directions and the change in \( b\)-value take place mostly during the contractive part of the response, while dilation is characterized by nearly constant \( \beta_\alpha \) and \( b\)-value.

2) Remarkably similar stress–strain and pore pressure characteristics have been observed in the monotonic torsional simple shear tests on isotropically and anisotropically samples, with \( K = 1.0 \) and \( K = 0.5 \), respectively. In spite of the similarity, however, it has been illustrated that shear strain and pore pressure response in the initial stage of the shearing are induced by different causes, by increase in the stress ratio and by rotation of principal stress directions, in the case of \( K = 1.0 \) and \( K = 0.5 \), respectively.

3) The undrained cyclic torsional simple shear behaviour is characterized by strong dependency of the stress ratio, rotation of principal stress directions and relative magnitude of the intermediate principal stress on the initial effective stress ratio \( K = \sigma_h' / \sigma_i' \). The differences in the behaviour are especially pronounced in the initial phase of the response and are manifested through different magnitudes and trends of change in \( q/p' \), different angles of continuous rotation of the principal stress directions in a single loading cycle, and different \( b\)-values and degree of their fluctuation. These differences gradually decrease as loading proceeds and eventually the behaviour becomes nearly unique and independent on the initial \( K \) value. Unlike shear strain response which in fact has been found independent of the initial value of \( K \), the pore pressure build-up and consequently cyclic strength are affected by the initial \( K \) value. Due to the complexity and high sensitivity of the behaviour, and on the other hand
because of the limited amount of experimental data, it is difficult to identify whether
or not these effects are due to the effective stress ratio alone.

4) It has been demonstrated that the characteristics of undrained cyclic torsional
shear response as described above are induced by the pattern of change in the stress
state as a result of the imposed constraint in the lateral deformation during the simple
shear test.

5) The characteristics of monotonic and cyclic undrained torsional simple shear
behaviour remain the same for loose sand exhibiting instability or contractive flow.

**STRESS–STRAIN RELATIONSHIP WITH DENSITY–STRESS
DEPENDENT PARAMETERS**

1) The modelling concept adopted for the stress–strain relationship is based on
the experimental evidence that combination of density and effective stress is relevant
to the description of the stress–strain behaviour of sandy soils. For that purpose, a
stress–strain relationship is proposed whose parameters are defined as a function of
an initial index parameter established in the framework of the steady state of
deformation concept.

2) The index parameter employed in the proposed plastic stress–strain
relationship is the State Index $I_s$ proposed by Verdugo (1992) and Ishihara (1993),
which characterizes the sandy soil behaviour by accounting for the combined effects
of density and mean effective stress for a given initial fabric. The state index
represents the state of the soil in void ratio–effective stress plane relative to the quasi
steady state and an upper reference related either to the zero residual strength or the
isotropic consolidation line for the loosest state of the soil.

3) It has been demonstrated that the state index provides a very accurate
quantitative description of the stress–strain behaviour observed in the drained torsional
shear tests on different soils with various initial states. Thus, the contours in the void
ratio–effective stress plane indicating states with identical experimental stress–strain
curves, in fact coincides with the equi–state index lines.

4) Experimental results from the drained torsional shear tests on sand and sand
with fines indicate that the stress–strain behaviour for the initial states above the zero residual strength line is in fact independent of the initial states. In other words, for any initial state above the threshold void ratio which denotes the states with zero residual strength a unique stress–strain curve is obtained. This curve indicates the softest stress–strain behaviour for a given initial fabric.

5) A modified hyperbolic stress–strain relation is proposed with constant strength parameter and initial plastic modulus defined as a function of plastic shear strain. The dependence of the initial plastic modulus on plastic shear strain is derived based on the experimental evidence for a greater non–linearity of the stress–strain curve than that provided by the two–constant hyperbolic relation. The relation is in accordance with the fact that the original two–constant hyperbolic relation provides accurate representation of the stress–strain curve at small strains and that Kondner's hyperbolic relation assures good approximation of the stress–strain curve at relatively large strains. Thus, using the initial plastic shear moduli of the original hyperbolic relation ($G_{p,max}$) and Kondner's hyperbolic relation ($G_{p,min}^N$), as upper and lower bounds for the initial plastic modulus $G_p^N$, respectively, the proposed relation is capable of accurately representing the stress–strain curve over the entire required range of strains. It is essential that stress–strain parameters of the proposed relation have clear physical meaning.

6) The parameters of the modified hyperbolic stress–strain relation are then expressed as functions of the state index. Remarkably good linear correlations are obtained for all the parameters with correlation coefficients between 0.92 and 0.96. A high degree of correlation between the stress–strain parameters and the state index is assured by the fact that stress–strain parameters as well as the state index have physical significance and are not merely arbitrary parameters.

7) The proposed stress–strain relationship is characterized by a single set of coefficients that represents the sandy soil behaviour over a wide range of densities and stresses, for the entire required range of strains.

**ELASTIC–PLASTIC CONSTITUTIVE MODEL**

1) An elastic–plastic constitutive model for sandy soils is developed in the framework of incremental theory of plasticity. The model is experimentally motivated
and has its basis in results of drained and undrained, monotonic and cyclic hollow cylinder tests on different sandy soils.

2) It is postulated that elastic and plastic deformations always occur in parallel and sandy soils are considered as a continuously yielding material. Thus, a yield surface of fixed and very small size is adopted with a purely kinematic hardening rule. The yield surface, in fact always coincides with the current stress point. This concept is in accordance with the experimental observations that the plastic shear modulus tends to increase with decreasing strain, eventually asymptotically approaching to infinity as the strain is reducing towards zero. In other words, it is postulated that at very small strains the magnitude of the plastic strains is not comparable to that of elastic one. Purely elastic behaviour, however, does not exist.

3) Failure surface and hardening function are based on the previously introduced stress–strain relationship, and therefore, they account for the combined effects of density and effective stress. Thus, depending on the state of the soil relative to its reference states in the $e-p'$ plane, i.e. depending on the value of $I_1$, the failure surface gradually changes its curvature and appears as a curved conical surface or circular cone with straight meridian line.

4) The model accounts for both initial and induced anisotropy. The former is modelled by shifting the failure surface along the stress difference axis and the latter by mixed hardening rule for the hardening surface. The model combines isotropic and kinematic hardening law for simultaneous expansion and translation of consecutive hardening surfaces, and traces the load history with four surfaces in the memory: loading surface, unloading surface, reloading surface and general surface. The failure surface serves as a bounding surface for hardening surfaces.

5) The plastic potential formulation proposed by Gutierrez (1989) is adopted for the present model. According to the formulation, the flow is defined by both stress state and stress increment, and consequently the flow is noncoaxial and nonunique. The plastic potential formulation captures the gradual decrease of the non–coaxiality and non–uniqueness of the flow with increasing shear stress and eventually provides coaxial and unique flow at failure.

6) The proposed model captures the effects of density and effective stress for
a given initial fabric in a unique way. Except for the dilatancy parameters, it is characterized by representing the soil behaviour over the entire required range of density, effective stress and strain with a single set of parameters for a given initial fabric.

**PERFORMANCE AND APPLICATION OF THE MODEL**

1) To illustrate the abilities and effectiveness of the proposed model, predictions of the model are compared to the measured behaviour of sand in monotonic and cyclic, drained torsional shear and undrained torsional simple shear tests. For that purpose, stress–control and strain–control versions of the model are developed. The model is further incorporated in a finite element program and applied to seismic site response analysis of level ground.

2) Except for the dilatancy parameters, all element test simulations including drained and undrained, monotonic and cyclic conditions, have been done employing a single set of model parameters.

3) It is demonstrated that the model is capable of accurately representing both drained and undrained behaviour of sand over a wide range of density and effective stress with gradual change in the behaviour from that typical of dense sand and low effective stress to that typical of loose sand and high effective stress. It is illustrated that the model has the ability to correctly simulate the undrained behaviour of loose sand involving instability or contractive flow.

4) Volume change and pore water pressure characteristics predicted for the cyclic drained and undrained tests, respectively, agree very well with the observed behaviour. The same holds valid for the stress–strain behaviour except for the straining during cyclic mobility where predictions by the model usually underestimate the measured strains.

5) The effectiveness of the model and its applicability to seismic response analysis is demonstrated through blind–prediction of the seismic response of level ground model obtained from a centrifuge test.

6) Based on the results of the verification and validation procedures through
extensive element test simulations, it can be concluded that the employed modelling concept in the present model not only captures the effects of density and effective stresses but provides a very precise quantitative description of these effects. Besides the physical relevance of the concept it is important to emphasize the practical benefit arising from the fact that a single set of parameters characterizes the behaviour over a wide range of densities and stresses. Moreover, it is proved that the model is computationally efficient and has negligible computer memory requirements.

7) It is considered that most of the improvement of the model could come from more appropriate representation of the stress–dilatancy relation especially by accounting for the combined effects of density and mean effective stress, and for the effects of cyclic loading on this relation.
REFERENCES


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