Central automorphisms of finite Laguerre planes

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What is a Laguerre plane?

Definition
A finite Laguerre plane $\mathcal{L} = (P, C, G)$ of order $n$ consists of a set $P$ of $n(n+1)$ points, a set $C$ of $n^3$ circles and a set $G$ of $n+1$ generators (where circles and generators are both subsets of $P$) such that the following three axioms are satisfied:

(G) $G$ partitions $P$ and each generator contains $n$ points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be joined by a unique circle.

A finite Laguerre plane of order $n$ is a transversal design $\text{TD}_1(3, n+1, n)$, or equivalently, an orthogonal array of strength 3 on $n$ symbols, $n+1$ constraints and index 1. In case $n$ is odd the Laguerre plane corresponds to an antiregular generalized quadrangle of order $(n, n)$. 
Models of Laguerre planes

All known finite Laguerre planes are ovoidal, that is, they are obtained as the geometry of non-trivial plane sections of a cone, minus its vertex, over an oval in 3-dimensional projective space over a finite field $\mathbb{F}$. In case the oval is a conic one obtains the miquelian Laguerre plane over $\mathbb{F}$. 

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\[
P G(2, \mathbb{F}) \quad \quad P G(3, \mathbb{F})
\]
Derived incidence structures

The derived design at a point $p$ of a finite Laguerre plane of order $n$ is an affine plane of order $n$. Circles not passing through $p$ induce ovals in the projective completion of the affine plane at $p$ by adding the point $\omega$ at infinity of vertical lines that come from generators of the Laguerre plane.

A planar representation of an ovoidal Laguerre plane $L(f)$ has point set $(\mathbb{F} \cup \{\infty\}) \times \mathbb{F}$ and circles are of the form

$$\{(x, af(x) + bx + c) \mid x \in \mathbb{F}\} \cup \{(\infty, a)\}$$

where $a, b, c \in \mathbb{F}$ and $f : \mathbb{F} \to \mathbb{F}$ is parabolic.
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Theorem

- A finite Laguerre plane of odd order with a Desarguesian derivation is miquelian. (Chen, Kaerlein 1973, Payne, Thas 1976)
- A Laguerre plane of order at most ten is ovoidal and, in fact, miquelian except in case of order 8. (S. 1992, 2003)
An automorphism of a Laguerre plane $\mathcal{L}$ is a permutation of the point set that takes generators to generators and circles to circles.

A homothety of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely two points on different generators and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a $\{p, q\}$-homothety if $p$, $q$ are the two fixed points.

A group $\Gamma$ of automorphisms of $\mathcal{L}$ is said to be $\{p, q\}$-transitive if $\Gamma$ contains a subgroup of $\{p, q\}$-homotheties that acts transitively on each circle through $p$ and $q$ minus $p$ and $q$. 
Laguerre homotheties

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Ruth Kleinewillinghöfer investigated the possible configurations $\mathcal{H}$ of all unordered pairs of distinct points $\{p, q\}$ for which the automorphism group of $\mathcal{L}$ is $\{p, q\}$-transitive and found 13 feasible configurations. One says that $\mathcal{L}$ is of type $m$ if $\mathcal{H}$ is as in configuration $m$. 
Kleinewillinghöfer types w.r.t. homotheties

1. $\mathcal{H} = \emptyset$.

5. There are a circle $C$ and a fixed-point-free involution $\phi : C \to C$ such that $\mathcal{H} = \{\{p, \phi(p)\} \mid p \in C\}$.

8. There are two distinct generators $F, G$ such that $\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\}$.

9. Each point of $\mathcal{L}$ is in exactly one pair in $\mathcal{H}$.

11. There is a point $p$ such that $\mathcal{H} = \{\{p, q\} \mid q \in P \setminus \{p\}\}$.

12. There is a generator $G$ such that $\mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\}$.

13. $\mathcal{H}$ consists of all unordered pairs of points on different generators.
A finite ovoidal Laguerre plane has Kleinewillinghöfer type 1, 8, 12 or 13.

The respective types are obtained as $\mathcal{L}(f)$ over $\text{GF}(2^h)$ when

$$f(x) = \begin{cases} 
 x^{1/6} + x^{3/6} + x^{5/6} & \text{where } h \geq 5 \text{ is odd;} \\
 x^6 & \text{where } h \geq 5 \text{ is odd;} \\
 x^{2i} & \text{where } \gcd(i, h) = 1; \\
 x^2 & \text{any } h.
\end{cases}$$
Characterisations and exclusions

Theorem

- A Laguerre plane is of Kleinewillinghöfer type 13 if and only if it is miquelian. (Hartmann, 1982)

- A finite Laguerre plane has Kleinewillinghöfer type 12 if and only if it has even order and is ovoidal over a proper translation oval (not a conic). (Hartmann, 1982, S. 2015)

  \( \mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\} \)

- A finite Laguerre plane of Kleinewillinghöfer type 5 or 9 has odd order. (Kleinewillinghöfer, 1979)

  \( \text{type 5: } \mathcal{H} = \{\{p, \phi(p)\} \mid p \in C\}, \phi \text{ fixed-point-free involution on } C, \)

  \( \text{type 9: each point is in exactly one pair in } \mathcal{H} \)
Theorem

- A finite Laguerre plane that contains a group of automorphisms of Kleinewillinghöfer type 11 is miquelian or ovoidal over a translation oval; the plane then is of type 13 or 12.

\[(\mathcal{H} = \{\{p, q\} \mid q \in P \setminus [p]\})\]

- A finite Laguerre plane of type 8 is an elation Laguerre plane, that is, the plane admits a group of automorphisms that acts trivially on the set of generators and regularly on the set of circles.

\[(\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\})\]

- A finite non-ovoidal elation Laguerre plane has type 1 or 8.