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## INFINITE ANTICHAINS OF MATROIDS WITH CHARACTERISTIC SET $\{p\}$

James Oxley, Charles Semple, Dirk Vertigan,  
and Geoff Whittle

*Department of Mathematics and Statistics  
University of Canterbury  
Private Bag 4800  
Christchurch, New Zealand*

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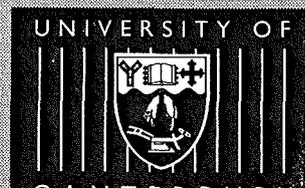
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# INFINITE ANTICHAINS OF MATROIDS WITH CHARACTERISTIC SET $\{p\}$

JAMES OXLEY, CHARLES SEMPLE, DIRK VERTIGAN, AND GEOFF WHITTLE

ABSTRACT. For each prime  $p$ , we construct an infinite antichain of matroids in which each matroid has characteristic set  $\{p\}$ . For  $p = 2$ , each of the matroids in our antichain is an excluded minor for the class of matroids representable over the rationals.

## 1. INTRODUCTION

The *characteristic set* of a matroid  $M$  is the set consisting of the characteristic of every field over which  $M$  is representable. Rado [9] showed that a matroid cannot have characteristic set  $\{0\}$ . However, for every prime  $p$ , it is known [4, 7] that a matroid can have characteristic set  $\{p\}$ .

For each prime  $p$ , Reid [10] conjectured that every matroid that has characteristic set  $\{p\}$  and is an excluded minor for  $\mathbb{Q}$ -representability has at most  $2p+2$  elements. Gordon [6] disproved this conjecture, for all  $p$ , by exhibiting such matroids which have up to  $4p-4$  elements. Furthermore, he showed that, for each  $p$ , there are at least  $2^{p-2}$  matroids that have characteristic set  $\{p\}$  and are excluded minors for  $\mathbb{Q}$ -representability. Recall that a set of matroids is an *antichain* if no member of the set is isomorphic to a minor of another member in the set. In this paper, we prove the following result.

**Theorem 1.1.** *For each prime  $p$ , there is an infinite antichain of matroids each member of which has characteristic set  $\{p\}$ .*

The proof of Theorem 1.1 is constructive in that, for each prime  $p$ , we define an infinite antichain of matroids in which each matroid has characteristic set  $\{p\}$ . For the special case of  $p = 2$ , every matroid in our constructed antichain has the additional property of being an excluded minor for  $\mathbb{Q}$ -representability. Thus the following theorem extends Gordon's result when  $p = 2$ .

**Theorem 1.2.** *There is an infinite antichain of matroids each member of which has characteristic set  $\{2\}$  and is an excluded minor for  $\mathbb{Q}$ -representability.*

We conjecture that the analogue of Theorem 1.2 holds for all other prime characteristics.

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**Conjecture 1.3.** *For each prime  $p$ , there is an infinite antichain of matroids each member of which has characteristic set  $\{p\}$  and is an excluded minor for  $\mathbb{Q}$ -representability.*

The notation and terminology of this paper will follow [8]. In particular, we denote the characteristic set of a matroid  $M$  by  $\mathcal{K}(M)$ . We will assume that the reader is familiar with the basics of matroid representation theory as discussed, for example, in Chapter 6 of [8].

The paper is organized as follows. In the next section, we describe a canonical triple of perfect matchings of the complete graph  $K_{4n}$ . These matchings are fundamental in the construction of each of the antichains that give us Theorem 1.1. In Section 3, we prove Theorem 1.2, and thereby prove Theorem 1.1 for  $p = 2$ . Section 4 contains the proof of Theorem 1.1 for  $p \geq 3$ .

## 2. THREE PERFECT MATCHINGS OF $K_{4n}$

Let  $n$  be a positive integer and consider the complete graph  $K_{4n}$ . Label the vertices of  $K_{4n}$  as  $b_1, b_2, \dots, b_{4n}$ . We shall distinguish three disjoint perfect matchings  $H_1, H_2$ , and  $H_3$  of  $K_{4n}$ , where

$$\begin{aligned} H_1 &= \{\{b_1, b_2\}, \{b_3, b_4\}, \dots, \{b_{4n-1}, b_{4n}\}\}, \\ H_2 &= \{\{b_2, b_3\}, \{b_4, b_5\}, \dots, \{b_{4n}, b_1\}\}, \text{ and} \\ H_3 &= \{\{b_1, b_{2n+1}\}\} \cup \{\{b_2, b_{4n}\}, \{b_3, b_{4n-1}\}, \{b_4, b_{4n-2}\}, \dots, \{b_{2n}, b_{2n+2}\}\}. \end{aligned}$$

Observe that the union of every distinct pair of such matchings induces a Hamiltonian cycle of  $K_{4n}$ . These perfect matchings play an important role in the proof of Theorem 1.1.

## 3. PROOF OF THEOREM 1.2

This section is organized as follows. We describe an infinite set of matroids, show that each of the matroids in this set has characteristic set  $\{2\}$ , and then show that each is an excluded minor for the class of matroids representable over the rationals. It will follow from the last of these proofs that the matroids in the set form an infinite antichain.

Let  $n$  be a positive integer, and consider a geometric representation of  $U_{3,4n}$ . Label the elements of  $U_{3,4n}$  by  $b_1, b_2, \dots, b_{4n}$  and recall the matchings  $H_1, H_2$ , and  $H_3$  from the last section. For each  $i$  in  $\{1, 2, 3\}$ , view the elements of  $H_i$  as 2-point lines of  $U_{3,4n}$  and place a point  $a_i$  on the intersection of all these lines so that  $a_1, a_2$ , and  $a_3$  are collinear. In the resulting configuration, no two distinct lines have more than one common point. Thus this configuration is a geometric representation for a rank-3 matroid, which we denote by  $M_n$ . In particular,  $M_1$  is isomorphic to the Fano matroid. We shall show that  $\{M_n : n \geq 1\}$  is an infinite antichain of matroids, each of which has characteristic set  $\{2\}$  and is an excluded minor for  $\mathbb{Q}$ -representability.

In constructing a representation for  $M_n$ , we shall use the matrix  $B_n$  which equals

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \cdots & b_{2n-1} & b_{2n} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & 1 \\ 0 & 1 & 1 & \alpha_2 & \alpha_2 & \alpha_4 & \alpha_4 & \cdots & \alpha_{2n-4} & \alpha_{2n-2} & \cdots \\ 0 & 0 & \alpha_1 & \alpha_1 & \alpha_3 & \alpha_3 & \alpha_5 & & \alpha_{2n-3} & \alpha_{2n-3} \\ \\ b_{2n+1} & b_{2n+2} & \cdots & b_{4n-4} & b_{4n-3} & b_{4n-2} & b_{4n-1} & b_{4n} \\ 1 & 1 & & 1 & 1 & 1 & 1 & 1 \\ \alpha_{2n-2} & \alpha_{2n-3} & \cdots & \alpha_3 & \alpha_3 & \alpha_1 & \alpha_1 & 0 \\ \alpha_{2n-2} & \alpha_{2n-2} & & \alpha_4 & \alpha_2 & \alpha_2 & 1 & 1 \end{bmatrix}.$$

**Lemma 3.1.** *Let  $\alpha_1, \alpha_2, \dots, \alpha_{2n-2}$  be algebraically independent transcendentals over some field  $\mathbb{F}$ . Then the matroid  $M[B_n]$  is isomorphic to  $U_{3,4n}$ .*

*Proof.* Let  $D$  be a  $3 \times 3$  submatrix of  $B_n$  whose columns are indexed by a subset of  $\{b_1, b_2, \dots, b_{4n}\}$ . We show that the columns of  $D$  are linearly independent. It is easily checked that no two columns are scalar multiples of each other. If the columns of  $D$  are indexed by  $\{b_1, b_2, b_{4n}\}$ , then they are certainly linearly independent. Thus  $D$  has a column containing a transcendental. If, for some  $i$  in  $\{1, 2, \dots, 2n-2\}$ , there is a transcendental  $\alpha_i$  that appears in exactly one column of  $D$ , then, since the members of  $\{\alpha_1, \alpha_2, \dots, \alpha_{2n-2}\}$  are algebraically independent, the columns of  $D$  must be linearly independent. Thus we may assume that every transcendental appearing in a column of  $D$  occurs at least twice in  $D$ . If  $D$  has a transcendental occurring three or more times, then, by our last assumption, this transcendental must be  $\alpha_{2n-2}$  and the columns of  $D$  are indexed by  $\{b_{2n}, b_{2n+1}, b_{2n+2}\}$ . Since these columns are certainly linearly independent, we may now assume that every transcendental  $\alpha_i$  in  $D$  occurs exactly twice. Hence either

- (i)  $\det D$  has a unique term equal to  $\alpha_i^2$ , or
- (ii) both occurrences of  $\alpha_i$  are in the same row or column of  $D$ .

In the first case,  $\det D$  is clearly non-zero. Therefore, we may assume that (i) fails and (ii) holds for every transcendental  $\alpha_i$  occurring in  $D$ . If  $D$  has a column with two copies of the same transcendental, then this column must be  $b_{2n+1}$  and, since (i) fails for each  $i$ , the other two columns of  $D$  must be in  $\{b_1, b_2, b_{4n}\}$ . It follows that, in this case,  $\det D \neq 0$ . Thus we may assume that no column of  $D$  contains two copies of the same transcendental. Hence, by (ii), either  $b_3$  or  $b_{4n-1}$  is a column of  $D$ , or each column of  $D$  contains two distinct transcendentals. In each case, we easily obtain a contradiction by using (ii) and the structure of  $B_n$ . We conclude that  $M[B_n] \cong U_{3,4n}$ .  $\square$

Let  $A'$  be the  $3 \times 3$  matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

**Lemma 3.2.** *Let  $p$  be a prime and let  $\alpha_1, \alpha_2, \dots, \alpha_{2n-2}$  be algebraically independent transcendentals over  $GF(p)$ . Then, over  $GF(p)(\alpha_1, \alpha_2, \dots, \alpha_{2n-2})$ , the matrix  $[A'|B_n]$  represents  $M_n$  when  $p = 2$  and represents  $M'_n$  when  $p > 2$ , where  $M'_n$  is obtained from  $M_n$  by relaxing the circuit-hyperplane  $\{b_1, b_{2n+1}, a_3\}$ .*

*Proof.* By Lemma 3.1,  $M[B_n] \cong U_{3,4n}$ . The remaining details of the proof are straightforward and are omitted.  $\square$

**Lemma 3.3.** *Let  $n$  be a positive integer. Then  $\mathcal{K}(M_n) = \{2\}$ .*

*Proof.* By Lemma 3.2,  $\mathcal{K}(M_n)$  contains 2. To complete the proof, let  $\mathbb{F}$  be a field and  $D_n$  be an  $\mathbb{F}$ -representation of  $M_n$ . We shall show that  $\mathbb{F}$  has characteristic two. First observe that  $M_n|_{\{a_1, a_2, a_3, b_1, b_2, b_{4n}\}}$  is isomorphic to  $M(K_4)$ . Since  $M(K_4)$  is binary, it is uniquely representable over  $\mathbb{F}$  [5]. Therefore we may assume that the columns of  $D_n$  corresponding to the elements  $a_1, a_2, a_3, b_1, b_2$ , and  $b_{4n}$  are identical to their counterparts in  $[A'|B_n]$ . By successively using the circuits  $\{a_2, b_2, b_3\}, \{a_1, b_3, b_4\}, \{a_2, b_4, b_5\}, \dots, \{a_2, b_{4n-2}, b_{4n-1}\}, \{a_1, b_{4n-1}, b_{4n}\}$  and then the circuits  $\{a_3, b_3, b_{4n-1}\}, \{a_3, b_4, b_{4n-2}\}, \dots, \{a_3, b_{2n}, b_{2n+2}\}$ , we deduce that there are elements  $d_1, d_2, \dots, d_{2n-2}$  of  $\mathbb{F}$  such that  $D_n$  can be obtained from  $B_n$  by replacing  $\alpha_i$  by  $d_i$  for all  $i$  in  $\{1, 2, \dots, 2n-2\}$ . Since  $\{b_1, b_{2n+1}, a_3\}$  is a 3-circuit of  $M_n$ , it now follows that  $\mathbb{F}$  must have characteristic two. This completes the proof of Lemma 3.3.  $\square$

The next lemma follows easily from the symmetry of  $M_n$ .

**Lemma 3.4.** *For each  $i$  in  $\{1, 2, \dots, 4n\}$ , there is an automorphism of  $M_n$  that maps  $b_i$  to  $b_j$  for some  $j$  in  $\{1, 3n+1, 3n+2, \dots, 4n\}$ .*

Theorem 1.2 will follow by combining Lemma 3.3 with the next result.

**Lemma 3.5.** *For all  $n$ , the matroid  $M_n$  is an excluded minor for the class of matroids representable over  $\mathbb{Q}$ .*

*Proof.* Every single-element contraction of  $M_n$  has rank two and so is representable over  $\mathbb{Q}$ . The proof of Lemma 3.5 will be completed by showing that every single-element deletion of  $M_n$  is representable over  $\mathbb{Q}$ . Let  $B = \{b_1, b_2, \dots, b_{4n}\}$ .

There are two cases to consider depending upon whether we are (i) deleting some  $a_i$  from  $M_n$ , or (ii) deleting some  $b_j$  from  $M_n$ . We give geometric arguments in both cases.

To prove (i), again recall the three distinguished perfect matchings of  $K_{4n}$  defined in Section 2. Since every distinct pair of such matchings induces a Hamiltonian cycle of  $K_{4n}$ , it follows that the matroids  $M_n \setminus a_1$ ,  $M_n \setminus a_2$ , and  $M_n \setminus a_3$  are isomorphic. Thus it suffices to show that  $M_n \setminus a_3$  is representable over  $\mathbb{Q}$ . We do this by finding, for all  $n$ , a set  $T_n$  of points of the projective plane  $PG(2, \mathbb{Q})$  such that  $M_n \setminus a_3$  is isomorphic to  $PG(2, \mathbb{Q})|T_n$ .

Suppose that we can find a set  $\{(x_j, y_j) : j \in \{1, 2, \dots, 4n\}\}$  of points of the affine plane  $AG(2, \mathbb{Q})$ , where  $b_j$  is identified with the point  $(x_j, y_j)$ , so that no three distinct points in this set are collinear and, for each  $i$  in  $\{1, 2\}$ , the elements of  $H_i$  are lines of a single parallel class. Let  $S_n$  be the subset  $\{(1, x_j, y_j) : i \in \{1, 2, \dots, 4n\}\}$  of the point set of  $PG(2, \mathbb{Q})$ . Let  $T_n$  be obtained from  $S_n$  by adding, for each  $i$  in  $\{1, 2\}$ , the point of  $PG(2, \mathbb{Q})$  that is the common point of intersection of all the lines in the parallel class induced by  $H_i$ . Clearly  $M_n \setminus a_3 \cong PG(2, \mathbb{Q})|T_n$ .

We now define a set  $\{(x_j, y_j) : j \in \{1, 2, \dots, 4n\}\}$  of points of  $AG(2, \mathbb{Q})$  that satisfies the initial assumption of the last paragraph. For all  $j$  in  $\{1, 2, \dots, 4n\}$ , let  $b_j = (x_j, y_j)$ . For all  $k$  in  $\{1, 2, \dots, 2n-1\}$ , let  $(x_{2k-1}, y_{2k-1}) = (k^2 - k, (k-1)^2)$  and  $(x_{2k}, y_{2k}) = (k^2 - k, k^2)$ , and let  $(x_{4n-1}, y_{4n-1}) = ((2n)^2 - 2n, (2n-1)^2)$  and  $(x_{4n}, y_{4n}) = ((2n)^2 - 2n, 0)$ . Then the elements of each of  $H_1$  and  $H_2$  are lines of a single parallel class containing the lines  $x = 0$  and  $y = 0$ , respectively.

To complete the proof of (i), we should like to show that no three distinct points of  $B$  are collinear. To avoid a tedious case analysis here, we can argue as follows. Clearly no horizontal or vertical line contains more than two elements of  $B$ . If  $b_1$  or  $b_2$  is collinear with two elements of  $B - \{b_1, b_2\}$ , then, for some small positive rational number  $\varepsilon_1$ , where  $\varepsilon_1 < \frac{1}{10}$ , say, we can add  $(\varepsilon_1, 0)$  to the coordinates for  $b_1$  and  $b_2$  so that there are no longer any lines involving  $b_1$  or  $b_2$  and any two members of  $B - \{b_1, b_2\}$ . This move maintains the fact that the lines containing  $\{b_2, b_3\}$  and  $\{b_1, b_{4n}\}$  are horizontal, and the line containing  $\{b_1, b_2\}$  is vertical. Next consider  $\{b_3, b_4\}$ . There is a positive rational number  $\varepsilon_2 < \frac{1}{10}$  so that we can add  $(\varepsilon_2, 0)$  to the coordinates for  $b_3$  and  $b_4$  so that there are no lines involving  $b_3$  or  $b_4$  and any two members of  $B - \{b_3, b_4\}$ . Moreover, all horizontal or vertical lines containing  $b_3$  or  $b_4$  remain intact. Repeating this process for all the pairs  $\{b_5, b_6\}, \{b_7, b_8\}, \dots, \{b_{4n-1}, b_{4n}\}$  ensures that  $M_n \setminus a_3$  is  $\mathbb{Q}$ -representable. This completes the proof of (i).

To prove (ii), it follows from Lemma 3.4 that it suffices to show that  $M_n \setminus b_t$  is  $\mathbb{Q}$ -representable for all  $t$  in  $\{1, 3n+1, 3n+2, \dots, 4n\}$ . For each such  $t$ , by using a similar argument to that given for (i), we shall prove (ii) by defining a set of  $4n-1$  points of  $AG(2, \mathbb{Q})$ , in which each point is identified with exactly one element of  $B - b_t$ , so that no three distinct points are collinear and, for each  $i$  in  $\{1, 2, 3\}$ , the elements of the set obtained from  $H_i$  by deleting the element containing  $b_t$  are lines of a single parallel class.

For each  $t \neq 1$ , we define such a set  $\{(x_j, y_j) : j \in \{1, 2, 3, \dots, t-1, t+1, \dots, 4n\}\}$  of points of  $AG(2, \mathbb{Q})$  as follows, where  $b_j = (x_j, y_j)$  for all  $j$ . Set  $b_2 = (0, 0)$ . The

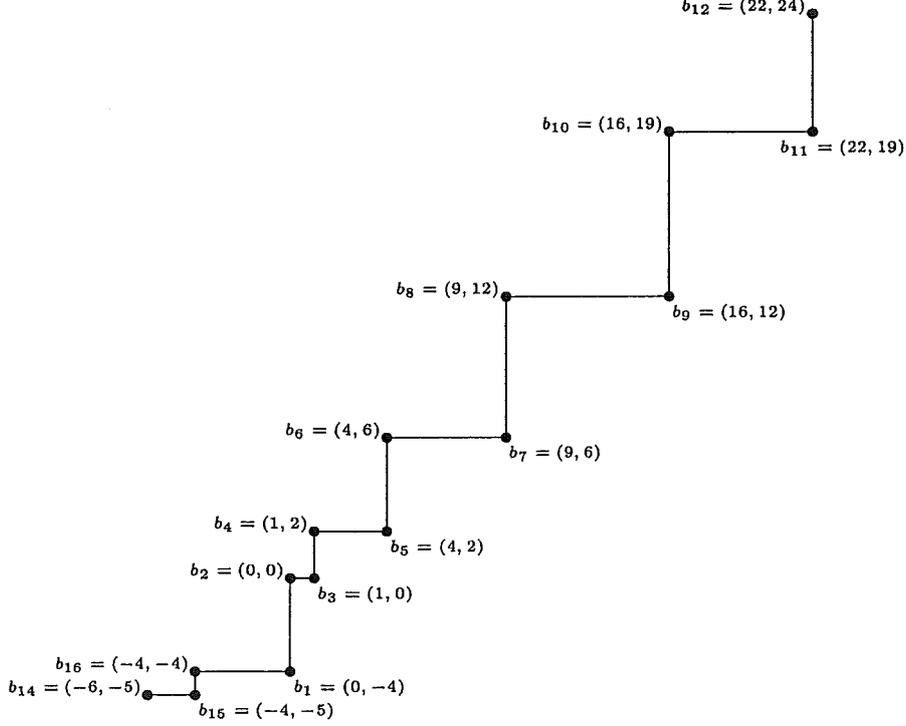


FIGURE 1. The points  $(x_j, y_j)$  in the affine plane  $AG(2, \mathbb{Q})$  before perturbation, where  $n = 4$  and  $t = 13$ .

remaining points are recursively obtained from  $b_2$ :

$$(x_j, y_j) = \begin{cases} (x_{j-1} + j - 2, y_{j-1}) & \text{if } 3 \leq j \leq 2n + 1 \text{ and } j \text{ is odd;} \\ (x_{j-1}, y_{j-1} + j - 2) & \text{if } 4 \leq j \leq 2n + 1 \text{ and } j \text{ is even;} \\ (x_{j-1} + 4n - j + 1, y_{j-1}) & \text{if } 2n + 2 \leq j \leq t - 1 \text{ and } j \text{ is odd;} \\ (x_{j-1}, y_{j-1} + 4n - j + 1) & \text{if } 2n + 2 \leq j \leq t - 1 \text{ and } j \text{ is even;} \\ (0, y_{2n+1} - x_{2n+1}) & \text{if } j = 1; \\ (y_{2n+1} - x_{2n+1}, y_{2n+1} - x_{2n+1}) & \text{if } j = 4n \neq t; \\ (x_{j+1}, y_{j+1} + j - 4n) & \text{if } t + 1 \leq j \leq 4n - 1 \text{ and } j \text{ is odd;} \\ (x_{j+1} + j - 4n, y_{j+1}) & \text{if } t + 1 \leq j \leq 4n - 1 \text{ and } j \text{ is even.} \end{cases}$$

When  $t = 1$ , we use the first four lines of the above to define  $(x_j, y_j)$ , replacing the condition  $2n + 2 \leq j \leq t - 1$  in the third and fourth lines by the condition  $2n + 2 \leq j$ . It is straightforward to check that, for each  $i$  in  $\{1, 2, 3\}$ , the members of the set obtained from  $H_i$  by deleting the element containing  $b_t$  are lines of a single parallel class. In particular, these parallel classes contain the lines  $x = 0, y = 0$ , and  $y = x$ , respectively. For  $n = 4$  and  $t = 13$ , Figure 1 displays the points  $(x_j, y_j)$  in  $AG(2, \mathbb{Q})$ .

We need to show that no three distinct points of  $B - b_t$  are collinear. To avoid a long case analysis, we shall use a modification of the argument given in case (i) whereby we perturb some of the points slightly to destroy any unwanted lines. An additional difficulty that arises here is that these perturbations must be done so as

to maintain three rather than just two parallel classes. We first treat the case when  $t = 1$ . Suppose  $b_2, b_3, b_{4n-1}$ , or  $b_{4n}$  is collinear with two other members of  $B$ . Then, for some positive rational number  $\varepsilon_1 < \frac{1}{10}$ , we can add  $(0, \varepsilon_1)$  to each of  $b_2$  and  $b_3$ , and add  $(-\varepsilon_1, 0)$  to each of  $b_{4n-1}$  and  $b_{4n}$  so as to destroy all lines that contain three elements of  $B$  including at least one element of  $\{b_2, b_3, b_{4n-1}, b_{4n}\}$ . This perturbation maintains the parallel classes associated with  $H_1, H_2$ , and  $H_3$ . We continue this process dealing successively with unwanted lines involving a member of one of  $\{b_4, b_5, b_{4n-3}, b_{4n-2}\}, \{b_6, b_7, b_{4n-5}, b_{4n-4}\}, \dots, \{b_{2n-2}, b_{2n-1}, b_{2n+1}, b_{2n+2}\}$ . At the conclusion of this process, the only remaining unwanted lines must involve all of  $b_{2n}, b_{2n+1}$ , and  $b_{2n+2}$ . Since these three points are not collinear, all unwanted lines have been eliminated.

Now suppose that  $t \neq 1$  and follow the procedure just described until dealing with the 4-set  $\{b_{2i}, b_{2i+1}, b_{4n-(2i-1)}, b_{4n-(2i-2)}\}$  that contains  $b_t$ . In that case, for a suitably chosen small rational number  $\varepsilon_i$ , move  $b_{2i}$  and  $b_{2i+1}$  by  $(0, \varepsilon_i)$  and the member of  $\{b_{4n-(2i-1)}, b_{4n-(2i-2)}\} - \{b_t\}$  by  $(-\varepsilon_i, 0)$ . Then continue dealing with the remaining 4-sets as before. When this process concludes, the only remaining unwanted lines must involve three of  $b_1, b_{2n}, b_{2n+1}, b_{2n+2}$ . Since none of these four points has been moved, it is easily checked that no such line exists and the lemma follows.  $\square$

#### 4. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 for all primes  $p \geq 3$ , and thereby complete the proof of Theorem 1.1.

This proof will use matroids that are defined using the operation of generalized parallel connection [1]. Let  $N_1$  and  $N_2$  be matroids such that  $N_1|T = N_2|T$ , where  $T = E(N_1) \cap E(N_2)$ . Let  $N_1|T = N$  and suppose that  $T$  is a modular flat of  $N_1$ . The *generalized parallel connection*  $P_N(N_1, N_2)$  of  $N_1$  and  $N_2$  across  $N$  is the matroid on  $E(N_1) \cup E(N_2)$  whose flats are those subsets  $X$  of  $E(N_1) \cup E(N_2)$  such that  $X \cap E(N_1)$  is a flat of  $N_1$ , and  $X \cap E(N_2)$  is a flat of  $N_2$ .

For all positive integers  $n$ , recall the construction and labelling of  $M_n$  from the last section. Let  $M_n''$  denote the matroid that can be obtained from  $M_n$  by relaxing the circuit-hyperplane  $\{b_1, b_{2n+1}, a_3\}$  and then placing a point  $a_4$  on the intersection of the lines  $\{a_1, a_2, a_3\}$  and  $\{b_1, b_{2n+1}\}$ . Thus  $M_n'' \setminus a_4$  is the matroid  $M_n'$  defined in Lemma 3.2.

**Lemma 4.1.**  $\{M_n'' : n \geq 1\}$  is an infinite antichain.

*Proof.* Suppose that  $M_j''$  is isomorphic to a minor of  $M_k''$  for some  $j < k$ . As both  $M_j''$  and  $M_k''$  have rank three, there is a map  $\phi : E(M_j'') \rightarrow E(M_k'')$  under which  $M_j''$  is isomorphic to some restriction of  $M_k''$ . Because each of  $M_j''$  and  $M_k''$  has a unique 4-point line, namely  $\{a_1, a_2, a_3, a_4\}$ , this set must be fixed by  $\phi$ .

For each  $i$  in  $\{j, k\}$ , there are two perfect matchings of  $K_{4i}$  associated with  $a_1$  and  $a_2$  such that the union of these matchings is a cycle of length  $4i$ . It follows that  $M_j''$  cannot be isomorphic to a restriction of  $M_k''$ .  $\square$

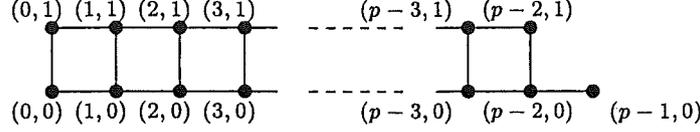


FIGURE 2. The points  $(x_k, y_k)$  in the affine plane  $AG(2, p)$ .

The infinite antichain  $\{M_n'' : n \geq 1\}$  does not, in itself, prove Theorem 1.1 for, as we shall see, the characteristic set of every  $M_n''$  contains all primes exceeding two. The infinite antichain that will prove the theorem will be obtained by attaching a fixed matroid with characteristic set  $\{p\}$  to every member of  $\{M_n'' : n \geq 1\}$  to form a set of rank-4 matroids. We shall now describe this construction more formally. Let  $p$  be a prime exceeding two. Let  $c_1 = (1, 0, 0)$ ,  $c_2 = (1, 0, 1)$ , and, for all  $k$  in  $\{3, 4, \dots, 2p-1\}$ , let  $c_k = (1, x_k, y_k)$  where

$$(x_k, y_k) = \begin{cases} \left(\frac{k-1}{2}, 0\right) & \text{if } k \text{ is odd;} \\ \left(\frac{k-2}{2}, 1\right) & \text{if } k \text{ is even.} \end{cases}$$

Now view the elements of  $\{(x_k, y_k) : k \in \{1, 2, \dots, 2p-1\}\}$  as points of  $AG(2, p)$ , as shown in Figure 2, and consider the extension of this plane to the projective plane  $PG(2, p)$ . We shall distinguish a set  $A$  consisting of four collinear points  $a_1, a_2, a_3$ , and  $a_4$  of  $PG(2, p)$ , where  $a_1 = (0, 1, 0)$ ,  $a_2 = (0, 0, 1)$ ,  $a_3 = (0, 1, -1)$ , and  $a_4 = (0, 1, 1)$ . We can view each of these points as the common point of intersection of all the lines in a parallel class in  $AG(2, p)$ , these classes containing, respectively, the lines  $y = 0, x = 0, y = -x$ , and  $y = x$ . Let  $N_p$  be the restriction of  $PG(2, p)$  to the set consisting of  $c_1, c_2, \dots, c_{2p-1}$  and all points on the line  $L$  spanned by  $A$ . Let  $L - A = \{a_5, a_6, \dots, a_{p+1}\}$ . Clearly  $L$  is a modular line of  $N_p$ . Let  $M_n'''$  be obtained from  $M_n''$  by freely placing  $(p+1) - 4$  points on the line of  $M_n''$  spanned by  $A$ , labelling these points by the elements of  $L - A$ . Then  $P_L(N_p, M_n''')$  is well-defined. Let  $N_p^n$  be the matroid obtained from  $P_L(N_p, M_n''')$  by deleting  $L - A$ . Clearly the ground set of  $N_p^n$  is the union of  $E(N_p \setminus (L - A))$  and  $E(M_n''')$ .

**Lemma 4.2.** *For all primes  $p$  exceeding two, the set  $\{N_p^n : n \geq 1\}$  is an infinite antichain of matroids.*

*Proof.* Suppose that  $N_p^j$  is isomorphic to a minor of  $N_p^k$  for some  $j < k$ . Then, since  $N_p^j$  and  $N_p^k$  have the same rank, there is a map  $\phi : E(N_p^j) \rightarrow E(N_p^k)$  under which  $N_p^j$  is isomorphic to some restriction of  $N_p^k$ . For each  $i$  in  $\{j, k\}$ , the matroid  $N_p^i/e$  is vertically 3-connected if and only if  $e \notin A$ . Thus  $\phi$  fixes the set  $A$ . Let  $\{s, t, u, v\} = \{1, 2, 3, 4\}$ . Then  $N_p^i/a_s$  has  $\{a_t, a_u, a_v\}$  as a parallel class and is the parallel connection with basepoint  $a_t$  of two rank-2 matroids. Thus, if  $\phi$  does not fix the set  $E(N_p \setminus (L - A))$ , then it maps this set to a subset of  $E(M_k'')$ . But the latter cannot occur since  $N_p \setminus (L - A)$  has a  $(p+1)$ -point line different from  $A$  but  $M_k''$  has no such line. We deduce that  $\phi$  must fix  $E(N_p \setminus (L - A))$ . Thus  $\phi$  maps  $E(M_j'')$  to a subset of  $E(M_k'')$ , so  $M_j''$  is isomorphic to a minor of  $M_k''$ , a contradiction to Lemma 4.1.  $\square$

The next two lemmas will be combined to show that each member of  $\{N_p^n : n \geq 1\}$  has characteristic set  $\{p\}$ .

**Lemma 4.3.**  $\mathcal{K}(N_p \setminus (L - A)) = \{p\}$ .

*Proof.* Order the elements of  $N_p \setminus (L - A)$  as follows: the first eight elements are  $c_1, a_1, a_2, c_4, c_2, c_3, a_3, a_4$  and the remaining elements are  $c_5, c_6, \dots, c_{2p-1}$ . Suppose  $D$  is a matrix representing  $N_p \setminus (L - A)$  over some field  $\mathbb{F}$ . Then, without loss of generality, we may assume that the submatrix of  $D$  indexed by its first four columns is

$$\begin{array}{cccc} c_1 & a_1 & a_2 & c_4 \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

If we consider the remaining elements of  $N_p \setminus (L - A)$  in the order specified, it is not difficult to check that  $N_p \setminus (L - A)$  is *sequentially unique* [2], that is, each element lies on the intersection of two lines spanned by points that occur earlier in the sequence. Using this, it follows straightforwardly that, for each element of  $N_p \setminus (L - A)$ , the corresponding column of  $D$  agrees with the coordinates originally assigned to that element of  $N_p \setminus (L - A)$ . But  $c_2, a_4$ , and  $c_{2p-1}$  are collinear in  $N_p \setminus (L - A)$ . Thus  $p = 0$  in  $\mathbb{F}$ . Since  $\mathcal{K}(N_p \setminus (L - A))$  clearly contains  $p$ , the lemma follows.  $\square$

Let  $A''$  be the matrix

$$\begin{array}{cccccc} a_4 & a_5 & a_6 & \dots & a_p & a_{p+1} \\ \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 2 & 3 & \dots & p-3 & p-2 \end{bmatrix}, \end{array}$$

and recall the matrices  $A'$  and  $B_n$  from the last section.

**Lemma 4.4.** *Let  $\alpha_1, \alpha_2, \dots, \alpha_{2n-2}$  be algebraically independent transcendentals over  $GF(p)$ . Then  $[A'|B_n|A'']$  represents  $M_n'''$  over  $GF(p)(\alpha_1, \alpha_2, \dots, \alpha_{2n-2})$ .*

*Proof.* By Lemma 3.2,  $[A'|B_n]$  represents  $M_n'$ , which equals  $M_n''' \setminus \{a_4, a_5, \dots, a_{p+1}\}$ . Thus it suffices to show that, in  $M[A'|B_n|A'']$ ,

- (i)  $\{b_1, b_{2n+1}, a_4\}$  is a line; and
- (ii) each of  $a_5, a_6, \dots, a_{p+1}$  is freely placed on the line spanned by  $\{a_1, a_2\}$ .

Now (i) is easily checked. To check (ii), suppose that it fails. Then  $\{b_i, b_j, a_k\}$  is a circuit of  $M[A'|B_n|A'']$  for some distinct  $i$  and  $j$  in  $\{1, 2, \dots, 4n\}$  and some  $k$  in  $\{5, 6, \dots, p+1\}$ . Thus the matrix

$$\begin{array}{ccc} b_i & b_j & a_k \\ \begin{bmatrix} 1 & 1 & 0 \\ x & y & 1 \\ u & v & k-3 \end{bmatrix} \end{array}$$

has zero determinant. Hence

$$u - v = (k - 3)(x - y). \quad (1)$$

Now  $k - 3 \in GF(p) - \{0, 1, -1\}$ . Thus  $u = v$  if and only if  $x = y$ . But  $b_i$  and  $b_j$  are distinct so  $u \neq v$  and  $x \neq y$ . Moreover, from (1), the number of members of the

multiset  $\{u, v, x, y\}$  that are transcendentals is 0, 2, or 4. In the first case, it follows that  $\{b_i, b_j\} = \{b_2, b_{4n}\}$  and so  $k - 3 = p - 1$ ; a contradiction. In the second and third cases, the structure of  $B_n$  implies that  $u - v = \pm(x - y)$ , so  $k - 3 \in \{1, -1\}$ . This contradiction completes the proof of the lemma.  $\square$

To prove Theorem 1.1, we shall combine the last three lemmas.

*Proof of Theorem 1.1.* We shall show that, for all primes  $p$  exceeding 2, every member of  $\{N_p^n : n \geq 1\}$  has characteristic set  $\{p\}$ . Since  $\{N_p^n : n \geq 1\}$  is an infinite antichain, the theorem will follow.

We take the representations for  $N_p$  and  $M_n''''$  described above and adjoin a row of zeros to each so that the new rows become the first and last rows, respectively. This gives representations for  $N_p$  and  $M_n''''$  over  $GF(p)(\alpha_1, \alpha_2, \dots, \alpha_{2n-2})$  in which  $L$  has a common representation. By a result of Brylawski [3, Proposition 7.6.11], it follows that  $P_L(N_p, M_n''')$  has a  $GF(p)(\alpha_1, \alpha_2, \dots, \alpha_{2n-2})$ -representation. Since  $N_p^n$  is a restriction of  $P_L(N_p, M_n''')$ , the characteristic set of the former contains  $\{p\}$ . But  $N_p^n$  has  $N_p \setminus (L - A)$  as a restriction and the last matroid has characteristic set equal to  $\{p\}$ . Thus  $N_p^n$  also has characteristic set equal to  $\{p\}$  and so the theorem holds.  $\square$

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA

*E-mail address:* `oxley@math.lsu.edu`, `vertigan@math.lsu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CANTERBURY, PRIVATE BAG 4800, CHRISTCHURCH, NEW ZEALAND

*E-mail address:* `c.semple@math.canterbury.ac.nz`

SCHOOL OF MATHEMATICAL AND COMPUTING SCIENCES, VICTORIA UNIVERSITY OF WELLINGTON, WELLINGTON, NEW ZEALAND

*E-mail address:* `whittle@mcs.vuw.ac.nz`