Abstract. We study hereditary properties of convexity for planar harmonic homeomorphisms on a disk and an annulus. A noteworthy class of examples with the hereditary property arises from energy-minimal diffeomorphisms of an annulus, whose existence was established in [9, 11]. An extension of a result by Hengartner and Schober [8] to an annulus is used to deduce the boundary behavior of a harmonic mapping from an annulus into a doubly-connected region bounded by two convex Jordan curves.

1. Introduction

Harmonic mappings, which are complex-valued orientation-preserving univalent functions satisfying Laplace’s equation \( \Delta f = 0 \) on their respective domains in \( \mathbb{C} \), bear some curious features. For example, while harmonic mappings of hyperbolic regions generally do not decrease either the Euclidean metric or the hyperbolic metric (because a result of Heinz [6, Lemma] is optimal—see, e.g., [4, p. 77] or [12, p. 91]), it was shown in [12, Theorem 1.1] that harmonic mappings preserving the unit disk \( \mathbb{D} \) decrease the Lebesgue area measure of concentric disks \( \mathbb{D}_r = \{ z \in \mathbb{C} : |z| \leq r < 1 \} \).

If the image of the unit disk under a conformal mapping is a convex region \( \Omega \), then the image of every disk in \( \mathbb{D} \) is also convex (see, e.g., [3, proof of Theorem 2.11] or [16]). On the other hand, the situation for harmonic mappings is markedly different. The harmonic mapping

\[
f(z) = \Re \frac{z}{1-z} + i \Im \frac{z}{(1-z)^2}
\]
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maps $\mathbb{D}$ onto the half plane \( \{ w : w > -\frac{1}{2} \} \), which is convex, but \( f(\mathbb{D}_r) \) is not convex for \( \sqrt{2} - 1 < r < 1 \) (see, e.g., [2, Example 5.5] or [4, pp. 46–48]). This is related to the fact that unidirectional convexity is not a hereditary property of holomorphic univalent functions (see, e.g., [2, Theorem 5.3; 5; 7]). Hence, convexity is not a hereditary property of harmonic mappings in general. Nevertheless, we obtain sufficient conditions for this hereditary property to be present in harmonic mappings. We also study a related hereditary property of harmonic mappings between doubly-connected regions, which is the main subject of this paper.

2. Connection with the Doubly-Connected Case

For \( 0 < r < 1 \), let \( \mathbb{A}_r \) denote the annulus \( \{ z \in \mathbb{C} : r < |z| < 1 \} \), let \( \overline{\mathbb{A}_r} \) denote its closure, and let \( \mathbb{T}_r \) denote the circle \( \{ z \in \mathbb{C} : |z| = r \} \). We will use \( \mathbb{T} \) to represent the unit circle \( \partial \mathbb{D} \). Harmonic diffeomorphisms will refer to harmonic mappings that are diffeomorphisms.

At first glance, our following result may appear somewhat surprising.

**Theorem 2.1.** Let \( h \) be a harmonic diffeomorphism of \( \mathbb{D} \) into a bounded convex region \( \Omega_0 \) in the finite plane such that the radial limit \( \lim_{r \to 1} h(re^{i\theta}) \) lies on \( \partial \Omega_0 \) for almost all \( \theta \). Suppose that, on \( \mathbb{A}_{\sqrt{2}-1} \),

\[
\Delta \log \left( \frac{1 - \frac{zh_z}{\bar{z}h_z}}{zh_z} \right) = 0,
\]

where \( h_z = \partial h/\partial z \), \( h_{\bar{z}} = \partial h/\partial \bar{z} \), and \( \Delta \) represents the Laplace operator. Then, \( h(\mathbb{D}_r) \) is a strictly convex region for \( 0 < r < 1 \).

**Remark.** If \( h \) is conformal, then \( 1 - \frac{zh_z}{\bar{z}h_z} \equiv 1 \). Hence, its argument function \( \log(1 - \frac{zh_z}{\bar{z}h_z}) \) is a constant integer multiple of \( 2\pi \).

In view of condition (2.1), we have stated Theorem 2.1 for harmonic diffeomorphisms in place of harmonic mappings. This is nonetheless hardly a restriction, since a result of Lewy (see, e.g., [4, p. 20] or [14, Theorem 1]) shows that the Jacobian of a harmonic mapping does not vanish at any point.

A harmonic mapping \( f \) of \( \mathbb{D} \) into a bounded region has bounded real and imaginary parts. By Fatou’s Theorem [17, Theorem IV.6], the angular limits of \( f \) exist almost everywhere on \( \partial \mathbb{D} \). Hence, the radial limit assumption in Theorem 2.1 is apparently weaker than customarily requiring either the convexity of \( h(\mathbb{D}) \) or the surjectivity of \( h \).

Given a harmonic mapping \( g \) of \( \mathbb{D} \), where \( g(\mathbb{D}) \) is a convex region, it is known that \( g(\mathbb{D}_r) \) is convex for \( r \in (0, \sqrt{2} - 1) \) (see, e.g., [4, p. 46] or [15, Theorem 1]). This explains the focus on \( \mathbb{A}_{\sqrt{2} - 1} \) in Theorem 2.1. More generally, we prove the following.

**Theorem 2.2.** Let \( h \) be a harmonic diffeomorphism of \( \mathbb{A}_\rho \) into a doubly-connected region \( \Omega \) bounded by two convex Jordan curves in the finite plane such that the
radial limits \( \lim_{r \to 1} h(re^{i\theta}) \) and \( \lim_{r \to \rho} h(re^{i\varphi}) \) lie on \( \partial \Omega \) for almost all \( \theta \) and \( \varphi \).

If (2.1) holds on \( A_\rho \), then \( h(T_r) \) is a strictly convex curve for \( \rho < r < 1 \).

Suppose \( f \) is a bounded harmonic mapping of \( A_\rho \). The compact set \( \partial A_\rho \) may be covered by a finite number of simply-connected neighborhoods \( R_1, R_2, \ldots, R_n \) in \( \overline{A_\rho} \) whose boundaries are Jordan curves. For each integer \( k \in [1, n] \), let \( g_k \) be a conformal mapping of \( \mathbb{D} \) onto the interior of \( R_k \). By Fatou’s Theorem, the harmonic mapping \( f \circ g_k \) has angular limits almost everywhere on \( \mathbb{T} \).

The isogonality of \( g_k \) at each boundary point of \( \mathbb{T} \) (see, e.g., [17, Theorem IX.5 and the subsequent paragraph]) implies that \( f \) has angular limits almost everywhere on \( \partial A_\rho \cap R_k \). Hence, the radial limit assumption in Theorem 2.2 may appear to be weaker than requiring either the boundary components of \( h(A_\rho) \) to be convex Jordan curves or \( h \) to be surjective. While the latter comparison is correct, it will follow from Corollary 4.2 in Section 4 that the radial limit assumption in Theorem 2.2 implies that the boundary components of \( h(A_\rho) \) are convex Jordan curves.

3. ILLUSTRATIVE EXAMPLES

**Definition 3.1.** An orientation-preserving homeomorphism \( h: A_\rho \rightarrow \Omega \) is said to be energy-minimal if \( h \) minimizes the quantity

\[
E(f) = \iint_{A_\rho} |f_z|^2 + |f_{\overline{z}}|^2
\]

among all orientation-preserving homeomorphisms \( f: A_\rho \rightarrow \Omega \) with \( E(f) < \infty \).

**Remark 3.2.** An energy-minimal homeomorphism \( h: A_\rho \rightarrow \Omega \) exists as long as the conformal modulus of \( A_\rho \) does not exceed that of \( \Omega \) (see, e.g., [9, Theorem 1.1] or [11, Theorem 1.1]).

**Remark 3.3.** Energy-minimal homeomorphisms are diffeomorphisms [10, Theorem 1.2]. This can also be seen as a consequence of their harmonicity (see, e.g., [14, Theorem 1] and the next remark). Henceforth, we refer to them as energy-minimal diffeomorphisms.

**Remark 3.4.** Since Poisson modification decreases the quantity in (3.1) (see, e.g., [1, proof of Lemma 7] or [9, Lemma 4.2]), energy-minimal diffeomorphisms are necessarily harmonic [9, Proposition 8.1 and Theorem 2.3].

It turns out that an energy-minimal diffeomorphism \( h: A_\rho \rightarrow \Omega \) satisfies (2.1). Since \( h \) is an orientation-preserving diffeomorphism,

\[
|h_z| > |h_{\overline{z}}|
\]
on \( A_\rho \). It was shown in [9, Lemma 6.1] that

\[
h_z h_{\overline{z}} = \frac{m}{z^2},
\]
where \( m \) is a real constant. We can rewrite (3.3) as
\[
\frac{\bar{z}h_z}{zh_z} = \frac{m}{|z|^2|h_z|^2},
\]
which, in view of (3.2), yields
\[
-1 < \frac{\bar{z}h_z}{zh_z} < 1,
\]
and thus the function \( 1 - \frac{\bar{z}h_z}{zh_z} \) is real and positive. Consequently, its argument function \( \text{Im} \log \left( 1 - \frac{\bar{z}h_z}{zh_z} \right) \) is a constant integer multiple of \( 2\pi \).

Theorem 2.2 now yields the following result.

**Theorem 3.5.** Let \( h: \mathbb{A}_\rho \rightarrow \Omega \) be an energy-minimal diffeomorphism, where \( \Omega \) is a doubly-connected region bounded by two convex Jordan curves in the finite plane. Then, \( h(\partial \mathbb{T}_r) \) is a strictly convex curve for \( \rho < r < 1 \).

We conclude this section with a family of examples for which \( \text{Im} \log \left( 1 - \frac{\bar{z}h_z}{zh_z} \right) \) is not constant. Define \( h: \mathbb{A}_\rho \rightarrow \mathbb{C} \) by
\[
h(z) = \frac{z + a}{1 + az} - b \log |z|,
\]
where \( a \in (0,1) \) and \( b = a(1-\rho^2)/(1-\rho^2a^2) \log \rho < 0 \). Then, \( h \) is harmonic, and
\[
1 - \frac{\bar{z}h_z}{zh_z} = \frac{2(1-a^2)z}{2(1-a^2)z - b(1+az)^2},
\]
which is meromorphic on \( \mathbb{A}_\rho \). Another elementary computation shows that the Jacobian
\[
|h_z|^2 - |h_z|^2 = \left( \frac{1-a^2}{1+az} \right)^2 \left( 1 - \frac{b(1+az)^2}{1-a^2} \right),
\]
whose last factor on the right-hand side has its minimum on \( \mathbb{A}_\rho \) at \( z = -\rho \). Hence, \( h \) will be a harmonic diffeomorphism of \( \mathbb{A}_\rho \) onto \( \mathbb{A}_\sigma \) satisfying (2.1) for values of \( a \) and \( \rho \) such that this minimum is positive, where we have that \( \sigma = \rho(1-a^2)/(1-\rho^2a^2) \in (0,\rho) \). An instance of this occurs at \( a = \rho = \frac{1}{2} \).

4. Auxiliary Results on Boundary Behavior

Let \( \theta \) and \( \phi \) denote polar angles. In this section, we study the boundary behavior of a harmonic mapping between an annulus \( \mathbb{A}_\rho \) and a doubly-connected region \( \Omega \) bounded by two Jordan curves that is not necessarily surjective, but whose radial limits at \( \partial \mathbb{A}_\rho \) are contained in \( \partial \Omega \).
Proposition 4.1. Let $h$ be a harmonic mapping of $A_\rho$ into a doubly-connected region $\Omega$ bounded by two Jordan curves $C_1$ and $C_\rho$ in the finite plane such that the radial limits $\lim_{r \to 1} h(re^{i\theta})$ and $\lim_{r \to \rho} h(re^{i\varphi})$ lie on $C_1$ and $C_\rho$, respectively, for almost all $\theta$ and $\varphi$. Then, there is a countable set $W \subset \partial A_\rho = \mathbb{T} \cup \mathbb{T}_\rho$ such that the unrestricted limits

$$H(e^{i\theta}) = \lim_{z \to e^{i\theta}} h(z), \quad H(\rho e^{i\varphi}) = \lim_{z \to \rho e^{i\varphi}} h(z)$$

exist on $\partial A_\rho \setminus W$, and are contained in $C_1$ and $C_\rho$, respectively. Moreover, we have the following:

(a) $H$ is both continuous and orientation-preserving on $\mathbb{T} \setminus W$ and $\mathbb{T}_\rho \setminus W$.

(b) For each $e^{i\theta} \in W$, the one-sided limits

$$H(e^{i\theta}-) = \lim_{\sigma \to \theta^-, e^{i\sigma} \notin W} H(e^{i\sigma}), \quad H(e^{i\theta}+) = \lim_{\sigma \to \theta^+, e^{i\sigma} \notin W} H(e^{i\sigma})$$

exist, belong to $C_1$ and are distinct.

(c) For each $\rho e^{i\varphi} \in W$, the one-sided limits

$$H(\rho e^{i\varphi}-) = \lim_{\sigma \to \varphi^-, \rho e^{i\sigma} \notin W} H(\rho e^{i\sigma}), \quad H(\rho e^{i\varphi}+) = \lim_{\sigma \to \varphi^+, \rho e^{i\sigma} \notin W} H(\rho e^{i\sigma})$$

exist, belong to $C_\rho$ and are distinct.

(d) The cluster sets of $h$ at the points $e^{i\theta} \in W$ and $\rho e^{i\varphi} \in W$ are the line segments joining $H(e^{i\theta}-)$ to $H(e^{i\theta}+)$ and $H(\rho e^{i\varphi}-)$ to $H(\rho e^{i\varphi}+)$, respectively.

A version of the above result for harmonic mappings between $D$ and bounded simply-connected regions with locally-connected boundary was given by Hengartner and Schober [8, Theorem 4.3]. Since the conclusions concern local properties of $h$, we may obtain Proposition 4.1 by covering $\partial A_\rho$ with a finite number of simply-connected neighborhoods $R_1, R_2, \ldots, R_n$ in $A_\rho$ whose boundaries are Jordan curves, and applying Hengartner and Schober’s result to the harmonic mapping $h \circ g_k$, where $g_k$ is a conformal mapping of $D$ onto the interior of $R_k$ for each integer $k \in \{1, n\}$. A noteworthy consequence of Proposition 4.1, besides Corollary 4.2 below, is that $h$ may be extended continuously to $\partial A_\rho$ outside the countable set $W$, and that each boundary point of $h(A_\rho)$ corresponds to a non-empty “pre-image” (with infinitely many points on a boundary line segment of $h(A_\rho)$ associated with a “pre-image” point in $W$ from (d)). These facts will prove their worth in Section 5.
Suppose \( f \) is a harmonic mapping of \( A_\rho \) into a doubly-connected region \( \Omega \) bounded by two convex Jordan curves in the finite plane such that the radial limits \( \lim_{r \to 1} f(re^{i\theta}) \) and \( \lim_{r \to \rho} f(re^{i\varphi}) \) lie on \( \partial \Omega \) for almost all \( \theta \) and \( \varphi \). The radial limits \( \lim_{r \to 1} f(re^{i\theta}) \) and \( \lim_{r \to \rho} f(re^{i\varphi}) \) are contained in distinct boundary components of \( f(A_\rho) \) by virtue of the fact that \( f \) is a homeomorphism (see, e.g., [13, p. 11]), and thus they lie on distinct boundary components of \( \partial \Omega \). It follows from Proposition 4.1 that the boundary of \( f(A_\rho) \) consists of two Jordan curves. Since any inner boundary line segment of \( f(A_\rho) \) has to be a subset of the inner boundary of \( \Omega \), the inner boundaries of \( f(A_\rho) \) and \( \Omega \) must coincide. On the other hand, replacing any outer boundary sub-arc \( \gamma \) of \( \partial \Omega \) with a line segment joining the endpoints of \( \gamma \) results in a convex Jordan curve. Hence, we have the following result.

**Corollary 4.2.** Let \( h \) be a harmonic mapping of \( A_\rho \) into a doubly-connected region \( \Omega \) bounded by two convex Jordan curves in the finite plane such that the radial limits \( \lim_{r \to 1} h(re^{i\theta}) \) and \( \lim_{r \to \rho} h(re^{i\varphi}) \) lie on \( \partial \Omega \) for almost all \( \theta \) and \( \varphi \). Then, the boundary of \( h(A_\rho) \) consists of two convex Jordan curves, of which the inner boundary curve coincides with the inner boundary curve of \( \Omega \).

5. Secant Behavior near the Boundary

Suppose \( h \) is a harmonic diffeomorphism of \( A_\rho \) into a doubly-connected region \( \Omega \) bounded by two convex Jordan curves in the finite plane such that the radial limits \( \lim_{r \to 1} h(re^{i\theta}) \) and \( \lim_{r \to \rho} h(re^{i\varphi}) \) lie on \( \partial \Omega \) for almost all \( \theta \) and \( \varphi \). For each \( \tau > 0 \), let

\[
 f_\tau(re^{i\theta}) = \frac{h(re^{i(\theta + \tau)}) - h(re^{i\theta})}{\tau},
\]

and let \( \psi_\tau(z) = \arg f_\tau(z) \) for all \( z = re^{i\theta} \in A_\rho \). We will establish the following result.

**Lemma 5.1.** The period of \( \psi_\tau - \theta \) is \( 2\pi \), and the single-valued harmonic functions \( \psi_\tau - \theta \) are uniformly bounded on \( A_\rho \) for sufficiently small \( \tau \).

Since the radial limits \( \lim_{r \to 1} h(re^{i\theta}) \) and \( \lim_{r \to \rho} h(re^{i\varphi}) \) are contained in distinct boundary components of \( \partial \Omega \), it follows from Proposition 4.1 that \( h \) has a continuous extension to \( A_\rho \setminus W \) for some countable set \( W \subseteq \partial A_\rho \). The orientation-preserving feature of \( h \) carries over to \( T \setminus W \) and \( T_\rho \setminus W \), and by Corollary 4.2, the boundary components of \( h(A_\rho) \) are convex Jordan curves, one of which is a curve \( \Gamma \) containing a point \( a' \) such that

\[
 |a'| = \sup_{z \in A_\rho} |h(z)|.
\]

We may suppose, without loss of generality, that

\[
 h(T \setminus W) \subseteq \Gamma.
\]
The line $L$ through $a'$ that makes an angle of $\arg a' + \pi/2$ with the positive real axis is a supporting line of $\Gamma$. We let $c'$ be a point on $\Gamma$ that has a parallel supporting line distinct from $L$, and pick distinct points $b'$ and $d'$ on $\Gamma$ that have supporting lines parallel to the line segment $a'c'$. The points $a', b', c', d'$ are chosen so that they follow one another in the positive direction around $\Gamma$, and we denote the angles made by their supporting lines with the positive real-axis by $\alpha$, $\beta$, $\alpha + \pi$, and $\beta + \pi$, respectively, such that

$$ \alpha < \beta < \alpha + \pi < \beta + \pi < \alpha + 2\pi. $$

By virtue of the results obtained in Section 4 (see, e.g., Proposition 4.1 and the subsequent paragraph), we may choose on $T$ four associated “pre-image” points $a, b, c$, and $d$ of $a', b', c', d'$, respectively. Let $A, B, C$, and $D$ be the overlapping open arcs on $T$ from $a$ to $c$, from $b$ to $d$, from $c$ to $a$, and from $d$ to $b$, respectively. We then cover $T$ with a finite number of sufficiently small open disks $D_1, D_2, \ldots, D_n$ such that the following hold:

1. Each disk $D_k$ intersects $T$ in an arc $A_k$ that is contained within at least one of the arcs $A, B, C$, or $D$.
2. The endpoints $a_k, b_k$ of each $A_k$ do not coincide with any of the points $a, b, c$, or $d$.

By so doing, we obtain a finite number of simply-connected sets

$$ R_1 = \overline{A_0} \cap D_1, \ R_2 = \overline{A_0} \cap D_2, \ldots, \ R_n = \overline{A_0} \cap D_n $$

in $\overline{A_0}$ whose respective boundaries are Jordan curves, and $R_k \cap T = A_k$ for each $k$. Fix $\delta \in (0, \pi/2)$. For each integer $k \in [1, n]$, let $g_k$ be a conformal mapping of $\mathbb{D}$ onto the interior $R_k^c$ of $R_k$ such that the harmonic measure

$$ \omega(g_k(0), A_k, R_k^c) = 2\delta \quad \text{and} \quad A_k = \{g_k(e^{is}) : -\delta < s < \delta\}. $$

This may be achieved by extending $g_k$ to a homeomorphism of $\mathbb{D}$ and making appropriate choices of $g_k(0)$ and $g_k(e^{is})$ for one particular $s$. If

$$ J = \{s : -\pi < s < -2\delta \} \cup \{s : 2\delta < s < \pi\}, $$

then for all $s \in J$ and $t_k \in (-\delta, \delta)$, we have

$$ \delta < |s - t_k| < \frac{3\pi}{2}, \quad \cos(s - t_k) < \cos \delta, $$

and thus

$$ 1 - 2r_k \cos(s - t) + r_k^2 > 1 - 2r_k \cos \delta + r_k^2 $$

$$ = \sin^2 \delta + (r_k - \cos \delta)^2 \geq \sin^2 \delta. $$
We fix

\[(5.2) \quad \tau \in \left(0, d\left(\{a, b, c, d\}, \bigcup_{k=1}^{n} \{a_k, b_k\}\right)\right),\]

where \(d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}\), and \(d(x, y)\) denotes the Euclidean distance between the points \(x\) and \(y\). The harmonicity of

\[(5.3) \quad u_\tau = \text{Im}(e^{-i\alpha f_\tau})\]

on \(A_\rho\) implies that the compositions \(u_\tau \circ g_k\) are harmonic on \(D\) for all \(k\), and thus

\[(5.4) \quad u_\tau(g_k(r_k e^{is})) = \int_{-\pi}^{\pi} u_\tau(g_k(e^{is})) \cdot P_k(e^{is}) \, ds,
= \left(\int_{-\delta}^{\delta} + \int_{f}\right) u_\tau(g_k(e^{is})) \cdot P_k(e^{is}) \, ds,\]

where

\[P_k(e^{is}) = \frac{1 - r_k^2}{2\pi(1 - 2r_k \cos(s - t_k) + r_k^2)}, \quad 0 \leq r_k \leq 1.\]

If \(A_k \subseteq A\), then the first integral on the second line in \((5.4)\) is non-negative by virtue of the fact that the restriction of \(u_\tau\) to \(A_k\) is non-negative. The second integral, on the other hand, may be estimated as follows. If \(\varepsilon > 0\), then on each set

\[S_k = \left\{r_k e^{is} \in \mathbb{D} : 1 - \frac{\varepsilon}{4|\alpha'|} \sin^2 \delta < r_k \leq 1, -\delta < t_k < \delta\right\},\]

we have

\[\left|\int_{f} u_\tau(g_k(e^{is})) \cdot P_k(e^{is}) \, ds\right| \leq \frac{2|\alpha'|(1 - r_k^2)}{\sin^2 \delta} < \frac{4|\alpha'|(1 - r_k)}{\sin^2 \delta} < \varepsilon.\]

Hence, for each \(\varepsilon > 0\), there is a set \(S_k\) containing \(g_k^{-1}(A_k)\) such that

\[(5.5) \quad (u_\tau \circ g_k) > -\varepsilon.\]

A similar argument applied to each of the other cases

\[A_k \subseteq B, \quad A_k \subseteq C, \quad A_k \subseteq D\]

(with \(\alpha\) replaced by \(\beta, \alpha + \pi, \beta + \pi\), respectively) also yields \(S_k \supseteq g_k^{-1}(A_k)\) such that \((5.5)\) is valid.
Fix \( r \) sufficiently close to 1 such that

\[
\mathbb{T}_r \subset \bigcup_{k=1}^{n} g_k(S_k) \setminus \mathbb{T}.
\]

Since \( f_\tau \) is non-zero on the compact set \( \mathbb{T}_r \), there exists \( m_r > 0 \) such that

\[
|f_\tau| \geq m_r
\]
on \( \mathbb{T}_r \). Pick \( k_r > 0 \) satisfying

\[
\varepsilon_r = \arcsin \frac{k_r}{m_r} < \frac{1}{2} \min\{\beta - \alpha, \alpha + \pi - \beta\}.
\]

Since \( u_\tau = \text{Im}(e^{-i\alpha f_\tau}) = |f_\tau| \sin(\psi_\tau - \alpha) \), it follows from (5.7) that on \( g_k(S_k) \cap \mathbb{T}_r \),

\[
\sin(\psi_\tau - \alpha) > -\frac{k_r}{m_r}
\]
if \( A_k \subseteq A \), and thus

\[
-\varepsilon_r < \psi_\tau - \alpha < \pi + \varepsilon_r.
\]

Likewise, on \( g_k(S_k) \cap \mathbb{T}_r \),

\[
\begin{align*}
-\varepsilon_r < \psi_\tau - \beta < \pi + \varepsilon_r & \quad \text{if } A_k \subseteq B; \\
-\varepsilon_r < \psi_\tau - (\alpha + \pi) < \pi + \varepsilon_r & \quad \text{if } A_k \subseteq C; \\
-\varepsilon_r < \psi_\tau - (\beta + \pi) < \pi + \varepsilon_r & \quad \text{if } A_k \subseteq D.
\end{align*}
\]

In particular, on \( \mathbb{T}_r \), we obtain

\[
\psi_\tau(r e^{i(\theta + 2\pi)}) = \psi_\tau(r e^{i\theta}) + 2\pi,
\]
though this may also be seen from the fact that \( h(\mathbb{T}_r) \) is a Jordan curve. Hence, \( \psi_\tau - \theta \) is a single-valued harmonic function on \( \mathbb{A}_\rho \).

In view of (5.8), (5.9), and (5.10), we see that \( \psi_\tau(r e^{i\theta}) - \theta \) is uniformly bounded for all \( \tau \) and \( r \) satisfying (5.2) and (5.6), respectively. A similar argument could be applied to \( \mathbb{T}_\rho \) to obtain a corresponding result when \( r \) is sufficiently close to \( \rho \). This proves Lemma 5.1, since \( \psi_\tau - \theta \) is continuous on \( \mathbb{A}_\rho \).
6. Proof of Theorem 2.2

Let \( \psi(z) = \arg(\partial/\partial \theta)h(z) \) for all \( z = re^{i\theta} \in \mathbb{A}_\rho \). As a consequence of (5.1), the convexity of \( h(\mathbb{T}_r) \) will follow from the inequality

\[
(6.1) \frac{\partial \psi}{\partial \theta} \geq 0
\]

on \( \mathbb{A}_\rho \), with \( h(\mathbb{T}_r) \) being strictly convex if the inequality is strict. Since \( h(\mathbb{T}_r) \) is a smooth (or, more precisely, real-analytic) Jordan curve, we obtain

\[
(6.2) \psi(re^{i(\theta + 2\pi)}) = \psi(re^{i\theta}) + 2\pi,
\]

Observe that

\[
(6.3) \frac{\partial h}{\partial \theta} = i(z h_z - \bar{z} h_{\bar{z}}) = i z h_z \left( 1 - \frac{\bar{z} h_{\bar{z}}}{zh_z} \right).
\]

By (3.2), the quantity in parentheses has positive real part and hence, by (2.1), its argument is a single-valued harmonic function. Since \( h_z \) is holomorphic and non-zero on \( \mathbb{A}_\rho \), it follows from (6.2) and (6.3) that \( \psi - \theta \) is a single-valued harmonic function on \( \mathbb{A}_\rho \). Moreover, it is bounded by virtue of Lemma 5.1 since \( \psi = \lim_{\tau \to 0} \psi_\tau \) on \( \mathbb{A}_\rho \).

Let \( G_z(\zeta) \) denote the Green's function for \( \mathbb{A}_\rho \) with singularity at \( z \equiv re^{i\theta} \), and let \( n = n_w \) be the inward normal at \( w = Re^{i\phi} \in \partial \mathbb{A}_\rho \). We may rotate \( \mathbb{A}_\rho \) together with the singularity \( z = re^{i\theta} \) about the origin through an angle \( \sigma \) to obtain

\[
(6.4) G_{re^{i(\theta + \sigma)}}(Re^{i\phi}) = G_{re^{i\theta}}(Re^{i(\phi + \sigma)}),
\]

from which the definition of partial differentiation implies

\[
\frac{\partial}{\partial \theta} G_z(w) = \lim_{\sigma \to 0} \frac{1}{\sigma} \left( G_{re^{i(\theta + \sigma)}}(Re^{i\phi}) - G_{re^{i\theta}}(Re^{i\phi}) \right)
\]

\[
= \lim_{\sigma \to 0} \frac{1}{\sigma} \left( G_{re^{i\theta}}(Re^{i(\phi + \sigma)}) - G_{re^{i\theta}}(Re^{i\phi}) \right) \quad \text{by (6.4)}
\]

\[
= \lim_{\sigma \to 0} \frac{1}{-\sigma} \left( G_{re^{i\theta}}(Re^{i(\phi + \sigma)}) - G_{re^{i\theta}}(Re^{i\phi}) \right)
\]

\[
= -\frac{\partial}{\partial \phi} G_z(w).
\]

Hence,

\[
(6.5) \frac{\partial}{\partial \theta} \frac{\partial}{\partial n} G_z(w) = \frac{\partial}{\partial n} \frac{\partial}{\partial \theta} G_z(w) = -\frac{\partial}{\partial n} \frac{\partial}{\partial \phi} G_z(w) = -\frac{\partial}{\partial \phi} \frac{\partial}{\partial n} G_z(w).
\]
Let \( T = \{ t \in \mathbb{R} : (\partial/\partial t)\Psi_1(t) \text{ and } (\partial/\partial t)\Psi_\rho(t) \text{ both exist} \}, \) where
\[
\Psi_1(\theta) = \arg \frac{\partial}{\partial \theta} h(e^{i\theta}), \quad \Psi_\rho(\theta) = \arg \frac{\partial}{\partial \theta} h(\rho e^{i\theta}).
\]
Since the boundary components of \( h(A_\rho) \) are convex Jordan curves by Corollary 4.2, it follows that \( \mathbb{R} \setminus T \) is countable. Recall that the harmonic function \( \psi - \theta \) has the integral representation (see, e.g., [17, Theorem I.21])
\[
2\pi(\psi(z) - \theta) = \int_0^{2\pi} [\Psi_1(\phi) - \phi] \frac{\partial}{\partial n} G_z(e^{i\phi}) d\phi + \int_0^{2\pi} [\Psi_\rho(\phi) - \phi] \frac{\partial}{\partial n} G_z(\rho e^{i\phi}) \rho d\phi,
\]
where the integrals are taken over \( [0, 2\pi] \cap T \). Partial differentiation with respect to \( \theta \) followed by an application of (6.5) yields
\[
2\pi \left( \frac{\partial}{\partial \theta} \psi(z) - 1 \right) = \int_0^{2\pi} [\Psi_1(\phi) - \phi] \frac{\partial}{\partial \theta} \frac{\partial G_z}{\partial n} d\phi + \int_0^{2\pi} [\Psi_\rho(\phi) - \phi] \frac{\partial}{\partial \theta} \frac{\partial G_z}{\partial n} \rho d\phi
\]
\[
= - \int_0^{2\pi} [\Psi_1(\phi) - \phi] \frac{\partial}{\partial \phi} \frac{\partial G_z}{\partial n} d\phi + \int_0^{2\pi} [\Psi_\rho(\phi) - \phi] \frac{\partial}{\partial \phi} \frac{\partial G_z}{\partial n} \rho d\phi \quad \text{by (6.5)}
\]
\[
= \int_0^{2\pi} \frac{\partial G_z}{\partial n} d[\Psi_1(\phi) - \phi] + \int_0^{2\pi} \frac{\partial G_z}{\partial n} \rho d[\Psi_\rho(\phi) - \phi].
\]
Hence (see, e.g., [17, Theorem I.20]),
\[
(6.6) \quad 2\pi \frac{\partial}{\partial \theta} \psi(z) = \int_0^{2\pi} \frac{\partial G_z}{\partial n} d\Psi_1(\phi) + \int_0^{2\pi} \frac{\partial G_z}{\partial n} \rho d\Psi_\rho(\phi).
\]
It follows from (5.1) that \( \Psi_1(\phi) \) and \( \Psi_\rho(\phi) \) are non-decreasing functions of \( \phi \). Since \( \partial G / \partial n \) is positive on \( \partial A_\rho \), it follows from (6.6) that \( \partial \psi / \partial \theta \) is also positive on \( A_\rho \). Hence, \( h(T_r) \) is strictly convex for \( \rho < r < 1 \), which concludes our proof of Theorem 2.2.

**Remark.** The proof would have been much simpler if \( h \in C^2(\overline{A_\rho}) \), for it follows from (6.3) that (6.1) is then equivalent to
\[
1 + \frac{\partial}{\partial \theta} \left\{ \arg h_z + \arg \left( 1 - \frac{\bar{z} h_z}{z h_z} \right) \right\}
\]
\[
= 1 + \text{Re} \left( \frac{z h_{zz} + \bar{z} (z h_z h_{zz} + 2 h_z h_{zz} + \bar{z} h_z h_{zzz})}{h_z (z h_z - \bar{z} h_z)} \right) \geq 0.
\]
Since this holds on \( \partial A_\rho \), the maximum principle yields the same inequality on \( A_\rho \). The desired conclusion then follows from the observation that \( \partial \psi / \partial \theta \) cannot be identically zero on \( A_\rho \), as \( h(T_r) \) is a Jordan curve for \( \rho < r < 1 \).
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