Group divisible designs with two associate classes

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1 Introduction

The work in this research report was done while visiting The University of Canterbury, and was completed jointly in cooperation with H.L. Fu and D. Sarvate. These results follow upon previous efforts where we were investigating the existence of group divisible designs with first and second associates and with block size 3. Background information concerning this problem will be added to the final version which will incorporate all our work on the topic.

Graph theoretically, we are looking for a partition of the edges of a graph $H$ into copies of $K_3$ (each $K_3$ is also called a triple). In our case, $H$ is the multigraph with vertex set $V = V_0 \cup V_1 \cup \ldots \cup V_{m-1}$, $|V_i| = n$ for each $i \in \mathbb{Z}_m$, in which two vertices are joined by $\lambda_1$ edges if they both occur in $V_i$ for some $i$, and otherwise are joined by $\lambda_2$ edges. Edges joining vertices in the same or different groups are called pure or cross edges respectively. Such a decomposition of $H$ into copies of $K_3$ is called a group divisible design and is denoted by a GDD$(n, m)$ of index $(\lambda_1, \lambda_2)$. Formally, such a GDD is represented by the ordered triple $(V, \{V_0, \ldots, V_{m-1}\}, B)$, where $B$ is the collection of triples. If $m = 1$ then the GDD is simply a triple system, so a GDD$(n, 1)$ of index $(\lambda_1, \lambda_2)$ is denoted more simply by a TS$(n)$ of index $\lambda_1$.

We have already completely solved this problem in the case where $n, m \geq 3$, proving the following result.

**Theorem 1.1** Let $n, m \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a GDD$(n, m)$ of index $(\lambda_1, \lambda_2)$ if and only if

1. $2$ divides $\lambda_1(n-1) + \lambda_2(m-1)n$, and
2. $3$ divides $\lambda_1n(n-1) + \lambda_2m(m-1)n^2$.
In this report, the case where \( m = 2 \) is completely solved. At first sight, this would seem to be quite simple to handle compared to the myriad of cases that have to be considered to prove Theorem 1.1. However, it turns out to be a very interesting case, requiring different solution techniques and another necessary condition.

**Lemma 1.2** If there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\) then

(1) 2 divides \(\lambda_1(n - 1) + \lambda_2n\),

(2) 3 divides \(\lambda_1n(n - 1) + \lambda_2n^2\), and

(3) \(\lambda_1 \geq \lambda_2 n/2(n - 1)\).

**Proof:** (1) and (2) follow because each vertex must have even degree, and the number of edges must be divisible by 3. (3) follows since any cross edge must be contained in a triple that contains another cross edge and a pure edge, so the number of pure edges must be at least half the number of cross edges.

We will now proceed to show that these three conditions are also sufficient for the existence of a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\). The main result is finally stated as Theorem 3.7.

## 2 Preliminary Results

In this section we obtain several building blocks. In Section 3, these will be put together in various ways to obtain the main result.

**Lemma 2.1** Let \( n \geq 3 \). There exists a GDD\((n, 2)\) of index \((n, 2n - 2)\).

**Proof:** Define

\[
B = \{((a, 0), (b, 0), (c, 1)), ((a, 1), (b, 1), (c, 0)) \mid 0 \leq a < b \leq n - 1, c \in \mathbb{Z}_n\}.
\]

Then \((\mathbb{Z}_n \times \mathbb{Z}_2, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_2\}, B)\) is a GDD\((n, 2)\) of index \((n, 2n - 2)\).

The following is a result of Petersen.
**Theorem 2.2** [1] Let $H$ be a regular multigraph of even degree. Then there exists a 2-factorization of $H$.

Lemma 2.3 is a special case of a result of Rodger and Stubbs.

**Lemma 2.3** [3] Let $\lambda, n \geq 1$. Suppose that $0 \leq x \leq \lambda(n-1)$, $x$ is even, and 3 divides $xn$. Then there exists an $x$-regular multigraph of multiplicity at most $\lambda$ with $n$ vertices whose edges can be partitioned into triples.

These two results can be combined to obtain Corollary 2.4. Let $E(H)$ be the set of edges in $H$.

**Corollary 2.4** Suppose that $\lambda, n \geq 1$, $0 \leq x \leq \lambda(n-1)$, 3 divides $xn$, and $x$ are even. Then there exists an $x$-regular multigraph $H$ of multiplicity at most $\lambda$ with $n$ vertices whose edges can be partitioned into triples, such that $\lambda K_n - E(H)$ has a 2-factorization.

**Proof:** Choose $H$ using Lemma 2.3, then apply Theorem 2.2 to $\lambda K_n - E(H)$. \qed

We will need a companion result to Corollary 2.4 to cope with the situation where $\lambda(n-1)$ is odd. Obtaining this result will require the following results, the first by Stern and Lenz, the second by Rees, and the third by Simpson. For any $D \subseteq \mathbb{Z}_{[\lambda/2]}$, let $H[D]$ be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{(j, j + d) | d \in D, j \in \mathbb{Z}_n\}$, reducing the sum modulo $n$.

**Lemma 2.5** [5] There exists a 1-factorization of $H[D]$ if and only if there exists a $d \in D$ such that $d / \gcd(n, d)$ is even.

Notice that if $d = n/2 \in D$ then since $d / \gcd(n, d)$ is even, $H[D]$ has a 1-factorization.

**Theorem 2.6** [2] For all $\equiv 0 \pmod{6}$ and for all even $x$ with $0 \leq x < n$ except for $(n, x) \in \{(12, 10), (6, 4)\}$, there exists an $x$-regular simple graph $H$ on $n$ vertices whose edges can be resolvably partitioned into triples, such that $K_n - E(H)$ has a 1-factorization.

**Theorem 2.7** [4] For any $y \geq 1$ and for some $s \in \{3y, 3y + 1\}$, the integers in $\{y + 1, y + 2, \ldots, 3y + 1\} \setminus \{s\}$ can be partitioned into pairs $(a_i, b_i)$ with $b_i > a_i$ such that $\{b_i - a_i | 1 \leq i \leq y\} = \{1, 2, \ldots, y\}$. 

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We can now present the companion to Corollary 2.4. It is probably a result that is of interest in its own right.

**Theorem 2.8** Suppose that $\lambda \geq 1, n \geq 3, 0 \leq x \leq \lambda(n - 1)$, 3 divides $xn$ and 2 divides $n$ and $x$. Then

(i) there exists an $x$-regular graph $H$ on $n$ vertices and of multiplicity at most $\lambda$ whose edges can be partitioned into triples, and

(ii) such that $\lambda K_n - E(H)$ has a 1-factorization.

**Proof:** For each $\lambda \geq 1$ and each even $n \geq 3$, let $S(n, \lambda)$ be the set of integers $x$ for which (i) and (ii) are true. Let $\ell = 2$ if $n \equiv 0$ or 4 (mod 6) and let $\ell = 6$ if $n \equiv 2$ (mod 6).

Since there exists a 1-factorization of $K_n$, if $x \in S(n, \lambda)$ then $x \in S(n, \lambda')$ for all $\lambda' \geq \lambda$. Also, since there exists a $TS(n)$ of index $\ell$, if $x = y \ell(n - 1) + x'$ with $0 \leq x' < \lambda(n - 1)$ and $\lambda \leq \ell$, and if $x' \in S(n, \lambda)$, then $x \in S(n, \lambda + y\ell)$. Therefore we need only consider the cases where $x < \ell(n - 1)$.

Suppose that $n \equiv 0$ (mod 6). We need only consider the cases where $x < 2(n - 1)$. If $x < n$ then the result follows from Theorem 2.6 unless $(n, x) \in \{(12, 10), (6, 4)\}$. Fortunately, since we do not require the set of triples to be resolvable, we can obtain solutions in these cases too: for each $m \in \{3, 6\}$ the complement of the edges in the triples of a GDD(2, $m$) of index $(0, 1)$ is a 1-factor. If $n \leq x \leq 2n - 4$ then we can simply combine a solution where $x' = n - 2$ and $\lambda' = 1$ with a solution where $x'' = x - (n - 2)$ and $\lambda = 1$.

If $n \equiv 2$ or 4 (mod 6) then since $x$ is even and 3 divides $xn$, we have that $x \equiv 0$ (mod 6), so let $x = 6y$. If $x = n - 2$ then $n \equiv 2$ (mod 6); since there exists a GDD(2, $3y + 1$) of index $(0, 1)$ we have that $n - 2 \not\in S(n, 1)$. If $x < n - 2$ then define $s, a_i$ and $b_i$ as in Theorem 2.7, and let $T = \{(j, a_i + j, b_i + j)| j \in \mathbb{Z}_n\}$, reducing sums modulo $n$. Then $T$ is a set of triples that partition $H = H[D']$ where $D' = \{1, 2, \ldots, 3y + 1\}\{s\}$, and $K_n - E(H) = H[D]$ where $d = \{1, 2, \ldots, n/2\}\{s\}$. Since $x < n - 2$, $n/2 \in D$, so $K_n - E(H)$ has a 1-factorization by Lemma 2.5. So it remains to consider $x \geq n$.

If $n \equiv 4$ (mod 6) then $\ell = 2$ so we can assume that $x < 2(n - 1)$; so $n + 2 \leq x \leq 2n - 8$ (since $x \equiv 0$ (mod 6)). We can combine a solution where $x' = n - 4$ and $\lambda' = 1$ with a solution where $x'' = x - (n - 4) \leq n - 8$ and $\lambda'' = 1$.

If $n \equiv 2$ (mod 6) then $\ell = 6$, so we can assume that $x < 6(n - 1)$; so $n + 4 \leq x \leq 6n - 12$ (since $x \equiv 0$ (mod 6)). Let $\ell'$ be such that $\ell'(n - 2) < x \leq (\ell' + 1)(n - 2)$. Combine
solutions where $x' = n - 2$ and $\lambda' = 1$ with a solution where $x'' = x - \ell'(n - 2) \leq n - 2$ and $\lambda'' = 1$.

It will be useful to define $[x, y, z]$ to denote the graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_2$ in which two vertices $(u, i)$ and $(v, j)$ are joined by $x$ edges if $i = j = 0$, by $y$ edges if $i \neq j$, and by $z$ edges if $i = j = 1$.

The next four results are crucial building blocks in the construction of the GDD's in Section 3.

**Lemma 2.9** For each $i \in \mathbb{Z}_2$, let $T_i$ be an $x\times$-regular multigraph on the vertex set $\mathbb{Z}_n \times \{i\}$ that has a 1-factorization. Then there exists a set of triples whose edges partition the edges of $[0, x, 0] + T_i$.

*Proof:* Partition the $x\times$-factors in a 1-factorization of $T_i$ into $x$ sets $S_0, S_1, \ldots, S_{n-1}$, each of size $x$. For each $a \in \mathbb{Z}_n$ and for each edge $\{(u, i), (v, i)\}$ in a 1-factor in $S_a$, let $B$ contain the triple $\{(u, a), (v, a), (a, i + 1)\}$, reducing the sum modulo 2. \hfill \square

**Lemma 2.10** Let $n$ be odd, and let $F$ be any 1-factor of $[0, 1, 0]$. Then there exists an edge-disjoint decomposition of $[1, 1, 0] - F$ and of $[0, 1, 1] - F$ into copies of $K_3$.

*Proof:* Let $(\mathbb{Z}_n, \circ)$ be a symmetric idempotent quasigroup of order $n$. Let $i \in \mathbb{Z}_n$ and let $F' = \{(a, 0), (a, 1)\} | a \in \mathbb{Z}_n$. Let $B'_i = \{(a, i), (b, i), (a \circ b, i + 1)\} | 0 \leq a < b \leq n - 1$, reducing $i + 1$ modulo 2. Then clearly the triples in $B'_i$ partition the edges in $[1, 1, 0] - F'$ or $[0, 1, 1] - F'$ if $i = 0$ or 1 respectively. The first coordinate of the symbols in the triples in $B'_i$ whose second coordinate is $i + 1$ can easily be renamed to produce a set of triples $B_i$ that partition the edges of $[1, 1, 0] - F$ or $[0, 1, 1] - F$ as required. \hfill \square

**Lemma 2.11** Let $i \in \mathbb{Z}_2$, and let $H_i$ be a $2\times$-regular graph on the vertex set $\mathbb{Z}_n \times \{i\}$. Then there exists a $2\times$-regular multigraph $T$ consisting of 2-factor 1-factors, each being in $[0, 1, 0]$, such that there exists an edge-disjoint decomposition of $H_i + T$ into copies of $K_3$.

*Proof:* By Theorem 2.2, $H_i$ has a 2-factorization into $x$ 2-factors $T_0, T_1, \ldots, T_{x-1}$. For each $j \in \mathbb{Z}_x, T_x$ consists of vertex disjoint cycles which we can arbitrarily orient to form directed cycles; call the resulting directed graph $T'_j$. Let $H'_i$ be the corresponding directed graph. For each directed edge $(a, b)$ in $T'_j$, let $\{(a, i), (b, i + 1)\} \in F_{2j}$ and...
{(a, i), (a, i + 1)} ∈ F_{2j+1}. Let T be the 2x-regular multigraph formed by the sum of $F_0, \ldots, F_{2x-1}$. Then $B = \{(a, i), (b, i), (b, i + 1)\} | (a, b) \in E(H_i)\}$ is a set of triples whose edges partition the edges of $H_i + T$. □

**Lemma 2.12** Let $n \geq 4$ be even. Let $\epsilon = 0$ if $n \equiv 0 \pmod{4}$, $\epsilon = 1$ if $n \equiv 6 \pmod{12}$, and $\epsilon = 3$ if $n \equiv 2$ or $10 \pmod{12}$. For each $i \in \mathbb{Z}_2$ there exists a simple graph $H_i$ on the vertex set $\mathbb{Z}_n \times \{i\}$ such that:

(i) $H_0$ is $(n/2 + \epsilon)$-regular and $H_1$ is $(n/2 - \epsilon)$-regular,

(ii) the edges of $[0, 1, 0] + H_0 + H_1$, can be partitioned into triples, and

(iii) there exists a 1-factorization of $K_n - E(H_i), i \in \mathbb{Z}_2$.

**Proof:** Let $D = \{2k - 1 | 1 \leq k \leq n/4\}$. Define

$$D_0 = \begin{cases} D & \text{if } \epsilon = 0, \\ D \cup \{2\} & \text{if } \epsilon = 1, \\ D \cup \{2, 4\} & \text{if } \epsilon = 3, \end{cases}$$

and define

$$D_1 = \begin{cases} D & \text{if } \epsilon = 0, \\ (D \cup \{2\}) \backslash \{n/2 - 2\} & \text{if } \epsilon = 1, \\ D \cup \{n/2 - 4\} & \text{if } \epsilon = 3. \end{cases}$$

In any case, define $H_i = H[D_i]$ on the vertex set $\mathbb{Z}_n \times \{i\}$, for each $i \in \mathbb{Z}_2$. Then clearly $H_i$ satisfies (i), and since $n/2 \in D$, it follows from Lemma 2.5 that (iii) is satisfied.

If $\epsilon = 0$ then let $B = \{((j, 0), (j + 2k - 1, 0), (j + k + n/4, 1)) \mid j \in \mathbb{Z}_n, 1 \leq k \leq n/4\}$.

If $\epsilon = 1$ then let $B = \{((j, 0), (j + 2k - 1, 0), (j + k + (n + 2)/4, 1)), ((j, 0), (j + 2, 0), (j + 1, 1)) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 2)/4 \} \cup \{((j, 1), (j + 2k - 1, 1), (j + k + (n + 2)/4, 0)) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 6)/4\}$.

If $\epsilon = 3$ then let $B = \{((j, 0), (j + 2k - 1, 0), (j + k + (n + 6)/4, 1)) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 2)/4 \} \cup \{((j, 0), (j + 2, 0), (j + 1, 1)), ((j, 0), (j + 4, 0), (j + 2, 1)), ((j, 1), (j + (n - 4)/2, 1), (j + (n - 4)/2, 0)) \mid j \in \mathbb{Z}_n \} \cup \{((j, 1), (j + 2k - 1, 1), (j + k + (n - 2)/4, 0)) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 10)/4\}$.

Then in each case, $B$ is a set of triples which partition the edges of $[0, 1, 0] + H_0 + H_1$. □

The following structure will be needed in Section 3.
Let \( n \) be even, and let \( F \) be a partition of \( \mathbb{Z}_n \) into sets of size 2. A symmetric quasigroup \((\mathbb{Z}_n, \circ)\) with holes \( F \) and of order \( n \) is an \( n \times n \) array in which: cell \((a, b)\) contains exactly one symbol in \( \mathbb{Z}_n \) if \( \{a, b\} \notin F \) and no symbols if \( \{a, b\} \in F \); for each \( a \in \mathbb{Z}_n \) row and column \( a \) contain each symbol in \( \mathbb{Z}_n \) exactly once except for symbols \( a \) and \( b \), where \( \{a, b\} \in F \); and cells \((a, b)\) and \((b, a)\) either contain the same symbol or are both empty, for \( 0 \leq a < b \leq n - 1 \). The following is well known.

**Lemma 2.13** For all even \( n \geq 6 \), there exists a symmetric quasigroup with holes \( F \) and of order \( n \), where \( F \) is a partition of \( \mathbb{Z}_n \) into sets of size 2.

Since maximum packings and minimum coverings of triple systems have been completely determined, we have the following result.

**Lemma 2.14** Let \( n \equiv 2 \pmod{6} \), \( n \geq 8 \) and let \( L \) be a set of 2 independent edges in \( K_n \). Then there exists an edge-disjoint decomposition of \((6y + 2)K_n + 2L\) and of \((6y + 4)K_n - 2L\) into copies of \( K_3 \), for all \( y \geq 0 \).

Finally, it will probably help enormously to list the values of \( n \) that satisfy conditions (1) and (2) of Lemma 1.2 for all values of \( \lambda_1 \) and \( \lambda_2 \). This is done in Table 1.

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<th>( 0 )</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0,3</td>
<td>even</td>
<td>0,3</td>
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<td>3</td>
<td>-</td>
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</tr>
<tr>
<td>2</td>
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<td>0</td>
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<td>-</td>
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<td>-</td>
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<tr>
<td>5</td>
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<td>-</td>
<td>3,5</td>
<td>-</td>
<td>3</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1. The values of \( n \pmod{6} \) for each value of \( \lambda_1 \pmod{6} \) and \( \lambda_2 \pmod{6} \) that satisfy conditions (1) and (2) of Lemma 1.2.
3 The Main Results

We begin with a result that helps us deal with condition (3) of Lemma 1.2. It allows us to focus on large values of \( n \), so then this lower bound on \( \lambda_1 \) will no longer be a moving target (that is, a function of \( n \)).

**Proposition 3.1** If conditions (1-3) of Lemma 1.2 are sufficient for the existence of a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\) whenever \( \lambda_2 \leq 2(n - 1) \), then they are sufficient for all \( \lambda_2 \geq 1 \).

**Proof:** Suppose that \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1-3) of Lemma 1.2, that \( 2x(n-1) < \lambda_2 \leq (2x + 2)(n - 1) \), and that \( x \geq 1 \). Then by (3),

\[
\lambda_1 \geq \begin{cases} 
\lambda_2/2 + x + 1 & \text{if } \lambda_2 \text{ is odd and } \lambda_2 > (2x + 1)(n - 1), \\
\lambda_2/2 + x & \text{otherwise}.
\end{cases}
\]

Let \( \epsilon = 1 \) if \( \lambda_2 \) is odd and \( \lambda_2 > (2x+1)(n-1) \), and \( \epsilon = 0 \) otherwise. Since \( \lambda'_1 = \lambda_1 - xn \geq \lambda_2/2 + x + \epsilon - xn = (\lambda_2 - 2x(n-1))/2 + \epsilon = \lambda_2/2 + \epsilon \), so \( \lambda_1 \geq \lambda_2 n/2(n - 1) \), (3) is satisfied by \( n, \lambda_1 \) and \( \lambda_2 \), and (1) and (2) are easily seen to be satisfied too. Therefore, by our assumption there exists a GDD\((n, 2)\) of index \((\lambda_1 - xn, \lambda_2 - 2x(n-1))\). Also, by Lemma 2.1 there exists a GDD\((n, 2)\) of index \((xn, x(2n-2))\) for any \( x \geq 1 \). So together these two GDD's form a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

Therefore, it remains to consider the case where \( \lambda_2 \leq 2(n - 1) \); or \( n \geq \lambda_2/2 + 1 \). Under this condition, (3) simply becomes \( \lambda_1 \geq (\lambda_2 + 1)/2 \). So throughout the rest of this paper we will assume that \( n \) and \( \lambda_1 \) satisfy these lower bounds imposed by \( \lambda_2 \).

**Proposition 3.2** Suppose that \( n \) is odd, \( \lambda_1 \geq \lambda_2/2 + 1 \) and \( n \geq \lambda_2/2 + 1 \). Let \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1) and (2) of Lemma 1.2. Then there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

**Proof:** Since \( n \) is odd, \( \lambda_2 \) is even (see Table 1). Let \( \lambda = \lambda_1 - \lambda_2/2 \). So \( \lambda \geq 1 \). The result will follow if we can find an integer \( t \) that satisfies the following conditions:

(i) \( 0 \leq 2t \leq \lambda(n - 1) \) and 3 divides \( \lambda(n - 1) - 2t \), and

(ii) \( \lambda_2 - \lambda(n - 1) \leq 2t \leq \lambda_2 \), and 3 divides \( \lambda(n - 1) - \lambda_2 + 2t \).
For, once these conditions are met, we can proceed as follows. Condition (i) ensures that the conditions of Corollary 2.4 are met when 
\[ x = \lambda(n - 1) - 2t, \]
so there exists a \((\lambda(n - 1) - 2t)\)-regular graph \( H_0 \) on the vertex set \( \mathbb{Z}_n \times \{0\} \) such that there exists a set \( B_0 \) of triples which partition the edges of \( H_0 \); so \( \lambda K_n - E(G_0) \) is a \( 2t \)-regular graph. Similarly, condition (ii) ensures that the conditions of Corollary 2.4 are met with 
\[ x = \lambda(n - 1) - \lambda_2 + 2t, \]
so there exists a \((\lambda(n - 1) - \lambda_2 + 2t)\)-regular graph \( H_1 \) on the vertex set \( \mathbb{Z}_n \times \{1\} \) such that there exists a set \( B_1 \) of triples which partition the edges of \( H_1 \); so \( \lambda K_n - E(H_1) \) is a \((-2t)\)-regular graph.

Since \( \lambda_2 \) is even, by Lemma 2.11 there exists a set \( F_0 \) of \( 2t \) 1-factors and a set \( F_1 \) of \( \lambda_2 - 2t \) 1-factors, each 1-factor being in \([0,1,0]\), such that for each \( i \in \mathbb{Z}_2 \) there exists a collection \( B_i \) of triples which partition the edges of \( \lambda K_n - E(H_i) \) and the edges in the 1-factors in \( F_i \).

Finally, if \( F \) is the \( \lambda \)-regular multigraph consisting of all the edges in \( F_0 \) and \( F_1 \), then by Lemma 2.10 there exists a collection \( B \) of triples that partition the edges of \([0,2,0]\) and the edges in \( F \).

If \( \lambda_2 = 6x + 2 \) and \( n \equiv 3 \pmod{6} \) then \( \lambda_1 \geq 3x + 2 \) (since \( \lambda_1 \geq \lambda_2/2 + 1 \) and \( n \geq 3x + 3 \) (since \( n \geq \lambda_2/2 + 1 \)). Choose \( t = [(3x + 1)/2] \). Then \( 2t \leq n - 1, 3 \) divides \( n \), and \( \lambda_2 - (n - 1) \leq 2t \).

If \( \lambda_2 = 6x + 2 \) and \( n \equiv 5 \pmod{6} \) then \( \lambda_1 \equiv 2 \pmod{3} \) (see Table 1), so \( \lambda \equiv 1 \pmod{3} \).

If \( x \) is odd then \( n \geq 3x + 2 \), so choose \( t = (3x + 1)/2 \).
If \( x \) is even then \( n \geq 3x + 5 \) (since \( n \equiv 5 \pmod{6} \)), so choose \( t = (3x + 4)/2 \).

If \( \lambda_2 = 6x + 4 \) and \( n \equiv 3 \pmod{6} \) then \( n \geq 3x + 3 \), so choose \( t = [(3x + 1)/2] \).

If \( \lambda_2 = 6x + 4 \) and \( n \equiv 5 \pmod{6} \) then \( \lambda_1 \equiv 1 \pmod{3} \) (see Table 1). If \( x \) is even then \( n \geq 3x + 5 \), so choose \( t = (3x + 2)/2 \). If \( x \) is odd then \( n \geq 3x + 8 \), so choose \( t = (3x + 5)/2 \).

If \( \lambda_2 = 6x \) and \( n \equiv 1 \pmod{6} \) then: if \( x \) is odd then \( n \geq 3x + 4 \), so choose \( t = (3x + 3)/2 \); if \( x \) is even then \( n \geq 3x + 1 \), so choose \( t = 3x/2 \).

If \( \lambda_2 = 6x \) and \( n \equiv 3 \pmod{6} \) then \( n \geq 3x + 3 \), so choose \( t = [3x/2] \).

If \( \lambda_2 = 6x \) and \( n \equiv 5 \pmod{6} \) then \( \lambda_1 \equiv 0 \pmod{3} \) (see Table 1) and so \( \lambda \geq 3 \), and \( n \geq 3x + 2 \). If \( x \) is even then choose \( t = 3x/2 \), and if \( x \) is odd then choose \( t = (3x + 3)/2 \).

It turns out that if \( \lambda_2 \) is odd then we need to consider the smallest value of \( \lambda_1 \) by itself.
Proposition 3.3 Suppose that $\lambda_2$ is odd and $\lambda_1 = (\lambda_2 + 1)/2$. Let $n, \lambda_1$ and $\lambda_2$ satisfy conditions (1-3) of Lemma 1.2. Then there exists a $GDD(n, 2)$ of index $(\lambda_1, \lambda_2)$.

Proof: By (3) of Lemma 1.2, $n \geq \lambda_2 + 1$. Since $\lambda_2$ is odd, $n$ and $\lambda_1$ are even (see Table 1), so we can write $\lambda_1 = 6x + 2y$, $\lambda_2 = 12x + 4y - 1$, and $n \geq 12x + 4y$, where $y \in \mathbb{Z}_3$. So Table 1 shows that $\lambda_1, \lambda_2$ and $n$ are restricted even more: if $-\lambda_1 \equiv 0 \pmod{6}$ then $-\lambda_2 = 5 \pmod{6}$ so $n \equiv 0 \pmod{6}$; if $-\lambda_1 \equiv 2 \pmod{6}$ then $-\lambda_2 = 3 \pmod{6}$ so $n \equiv 0$ or $4 \pmod{6}$; and if $-\lambda_1 \equiv 4 \pmod{6}$ then $-\lambda_2 \equiv 1 \pmod{6}$ so $n \equiv 0$ or $2 \pmod{6}$. Notice that in every case

(a) either $n \equiv 0 \pmod{6}$ or $n/2 - \lambda_1 \equiv 0 \pmod{3}$.

It will also be useful later to notice that if $n \equiv 2$ or $10 \pmod{12}$ then $\lambda_1 \equiv 4$ or $2 \pmod{6}$ respectively, and so since $n/2 \geq (\lambda_2 + 1)/2 = \lambda_1$ we have:

(b) if $n \equiv 2$ or $10 \pmod{12}$ then $n/2 \geq \lambda_1 + 3$; and if $n \equiv 6 \pmod{12}$ then $n/2$ is odd, so we have:

(c) if $n \equiv 6 \pmod{12}$ then $n/2 \geq \lambda_1 + 1$.

Let $\epsilon$ be defined as in Lemma 2.12. By Lemma 2.12, for each $i \in \mathbb{Z}_2$, there exists a simple graph $H_i$ on the vertex set $\mathbb{Z}_n \times \{i\}$ satisfying (i-iii). Let $B_0$ be a set of triples that partitions the edges of $[0, 1, 0] + H_0 + H_1$ (see (ii)). By (iii), $K_n - E(H_i)$ can be partitioned into $n - 1 - (n/2 + (-1)^i\epsilon) = n/2 - 1 - (-1)^i\epsilon$ 1-factors.

We want to apply Theorem 2.8 with $x = n/2 - \lambda_1 - (-1)^i\epsilon$ and $\lambda = 1$, so we have some things to check. If $n \equiv 2$ or $4 \pmod{6}$ then $\epsilon \in \{0, 3\}$, so by (a) we have that 3 divides $x\lambda$. In each case $n/2 - (-1)^i\epsilon$ is even, so $x$ is even because $\lambda_1$ is even. Clearly $x \leq n - 1$, and by (b) and (c) we have that $x \geq 0$.

Therefore, by Theorem 2.8, for each $i \in \mathbb{Z}_2$ there exists a set of triples $B'_i$ and there exists an $(n/2 - \lambda_1 - (-1)^i\epsilon)$ - regular graph $H'_i$ with vertex set $\mathbb{Z}_n \times \{i\}$ whose edges are partitioned by the triples in $B'_i$ such that $K_n - E(H'_i)$ has a 1-factorization into $n - 1 - (n/2 - \lambda_1 - (-1)^i\epsilon) = n/2 + \lambda_1 - 1 + (-1)^i\epsilon$ 1-factors.

Finally, for each $i \in \mathbb{Z}_2$, since $\lambda_1 \geq 2$ we can take the $(\lambda_1 - 2)(n - 1)$ 1-factors in a 1-factorization of $(\lambda_1 - 2)K_n$ on the vertex set $\mathbb{Z}_n \times \{i\}$. So for each $i \in \mathbb{Z}_2$, altogether on the vertex set $\mathbb{Z}_n \times \{i\}$ we have defined $(n/2 - 1 - (-1)^i\epsilon) + (n/2 + \lambda_1 - 1 + (-1)^i\epsilon) + (\lambda_1 - 2)(n - 1) = n(\lambda_1 - 1) = n(\lambda_2 - 1)/2$ 1-factors. By Lemma 2.9, there exists a set $B_1$ of triples that partition the edges in these 1-factors together with the edges in $[0, \lambda_2 - 1, 0]$.

Then clearly the triples in $B_0, B_1, B'_0$ and $B'_1$ form a $GDD(n, 2)$ of index $(\lambda_1, \lambda_2)$. □
Before presenting our last proposition, we need to deal with two exceptional cases.

**Lemma 3.4** Let \( n \equiv 2 \text{ or } 4 \pmod{6} \), \( \lambda_1 = 6y + 6 \), \( \lambda_2 = 12y + 9 \) and \( n \geq 6y + 6 \). Then there exists a GDD\( (n, 2) \) of index \( (\lambda_1, \lambda_2) \).

**Proof:** If \( n \equiv 2 \pmod{6} \) then there exists a TS\( (2n) \) of index 2, and by Proposition 3.3 there exists a GDD\( (n, 2) \) of index \( (6y + 4, 12y + 7) \), which together produce a GDD\( (n, 2) \) of index \( (6y + 6, 12y + 9) \).

If \( n \equiv 4 \pmod{6} \) then define \( \epsilon \) as in Lemma 2.12. By Lemma 2.12, for each \( i \in \mathbb{Z}_2 \) there exists a simple graph \( H_i \) on the vertex set \( \mathbb{Z}_n \times \{ i \} \) that is \( (n/2 + (-1)^i \epsilon) \)-regular, such that \( K_n - E(H_i) \) has a 1-factorization into a set \( F_1(i) \) of \( (n/2 - 1 - (-1)^i \epsilon) \) 1-factors. Since 6 divides \( x = 3n/2 - 6y - 6 - (-1)^i \epsilon \) and \( 0 \leq x \leq n - 1 \), by Theorem 2.8, for each \( i \in \mathbb{Z}_2 \) there exists a set \( B_i \) of triples and an \( x \)-regular graph \( H_i \) in \( (6y + 5)K_n \) defined on the vertex set \( \mathbb{Z}_n \times \{ i \} \) whose edges are partitioned by the triples in \( B_i \), such that \( (6y + 5)K_n - E(H) \) has a 1-factorization into a set \( F_2(i) \) of \( (6y + 5)(n - 1) - x \) 1-factors.

In \( F_1(i) \) and \( F_2(i) \), \( i \in \mathbb{Z}_2 \) there are a total of \( (6y + 4)n \) 1-factors, which altogether with the edges in \( [0, 12y + 8, 0] \) can be partitioned into a set \( B' \) of triples (by Lemma 2.9).

Clearly the triples in \( B, B', B_0 \) and \( B_1 \) together form a GDD\( (n, 2) \) of index \( (6y + 6, 12y + 9) \).

\[ \square \]

**Lemma 3.5** Let \( \lambda_1 \equiv 4 \pmod{6} \), \( \lambda_2 = 1 \) and \( n \equiv 2 \pmod{6} \). Let \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1-3) of Lemma 1.2. Then there exists a GDD\( (n, 2) \) of index \( (\lambda_1, \lambda_2) \).

**Proof:** Let \( \lambda_1 = 6y + 4 \). Let \( F = \{ \{2a, 2a + 1\} | a \in \mathbb{Z}_{n/2} \} \) and \( F_0 = \{ \{(a, 0), (b, 0)\} | \{a, b\} \in F \} \). Let \( L = \{ \{0, 1\}, \{2, 3\} \} \), and for each \( i \in \mathbb{Z}_2 \) let \( L_i = \{ \{(a, i), (b, i)\} | \{a, b\} \in L \} \).

Let \( (\mathbb{Z}_n, \circ) \) be a symmetric quasigroup with holes \( F \) and of order \( n \) (see Lemma 2.13). Define

\[
B = \{ \{(a, 0), (b, 0), (a \circ b, 1)\} | 0 \leq a < b \leq n - 1, \{a, b\} \notin F \} \cup \\
\{ \{(2a, 0), (2a + 1, 0), (2a, 1)\}, \{(2a, 0), (2a + 1, 0), (2a + 1, 1)\} | 2 \leq a \leq n/2 \} \cup \\
\{ \{(2a, 0), (2a, 1), (2a + 1, 1)\}, \{(2a + 1, 0), (2a, 1), (2a + 1, 1)\} | 0 \leq a \leq 1 \} .
\]

Then the triples in \( B \) contain: each edge \( \{(a, 0), (b, 0)\} \) exactly once if \( \{a, b\} \notin F \), exactly twice if \( \{a, b\} \in F \setminus L \), and not at all if \( \{a, b\} \in L \); each edge \( \{(a, 0), (b, 1)\} \) exactly once; and each edge \( \{(a, 1), (b, 1)\} \) exactly twice if \( \{a, b\} \in L \), and otherwise not at all.
Using Lemma 2.14, let \( B_0 \) be a collection of triples that partition the edges of \((6y + 2)K_n + 2L_0\) on the vertex set \( Z_n \times \{0\} \), and let \( B_1 \) be a collection of triples that partition the edges of \((6y + 4)K_n - 2L_1\) on the vertex set \( Z_n \times \{1\} \).

Finally, let \((Z_n \times \{0\}, F_0, B')\) be a GDD\((n, 2)\) of index \((0, 1)\).

Then the triples in \( B, B', B_0 \) and \( B_1 \) together form a GDD\((n, 2)\) of index \((6y + 4, 1)\).

\[ \square \]

**Proposition 3.6** Suppose that \( n \) is even, \( \lambda_1 \geq \lambda_2/2 + 1 \) and \( n \geq \lambda_2/2 + 1 \). Let \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1) and (2) of Lemma 1.2. Then there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

**Proof:** The result will follow if we can find an integer \( t \) that satisfies the following conditions:

(i) \( 0 \leq t, \ nt \leq \lambda_1(n - 1) \), and 3 divides \((\lambda_1(n - 1) - tn)n\), and

(ii) \( t \leq \lambda_2, \ (\lambda_2 - t)n \leq \lambda_1(n - 1) \), and 3 divides \((\lambda_1(n - 1) - (\lambda_2 - t)n)n\).

For, once these conditions are met, we proceed as follows.

Since \( n \) is even \( \lambda_1 \) is even, so \((\lambda_1(n - 1) - tn)\) is even. Therefore, by Theorem 2.8 and using (i), there exists a \((\lambda_1(n - 1) - tn)\)-regular graph \( H_0 \) on the vertex set \( Z_n \times \{0\} \) of multiplicity at most \( \lambda_1 \) and there exists a set \( B_0 \) of triples such that: these triples partition the edges of \( H_0 \); and \( T_0 = \lambda_1K_n - E(H_0) \) has a 1-factorization into \( tn \) 1-factors. Similarly, by Theorem 2.8 and (ii), there exists a \((\lambda_1(n - 1) - (\lambda_2 - t)n)\)-regular graph \( H_1 \) on the vertex set \( Z_n \times \{1\} \) and there exists a set \( B_1 \) of triples such that: these triples partition the edges of \( H_1 \); and \( T_1 = \lambda_1K_n - E(H_1) \) has a 1-factorization into \((\lambda_2 - t)n\) 1-factors. Finally, by Lemma 2.9, there exists a set \( B \) of triples which partition the edges of \([0, \lambda_2, 0] + T_0 + T_1\). Then clearly the triples in \( B_0, B_1 \) and \( B \) together form a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\). So it remains to find a suitable value of \( t \) in each case.

In the following, to check that \( nt \leq \lambda_1(n - 1) \) it is easiest to check that \( t \leq (\lambda_1 - t)(n - 1) \). Also, we will choose \( t \) so that \( t \geq \lambda_2/2 \), in which case \( nt \leq \lambda_1(n - 1) \) implies that \((\lambda_2 - t)n \leq \lambda_1(n - 1) \).

If \( \lambda_2 = 6x \) then \( \lambda_1 \geq 3x + 1 \) and \( n \geq 3x + 1 \). Choose \( t = 3x \). From Table 1, 3 divides \( \lambda_1, n \) or \( n - 1 \), and since 3 divides \( t \), the divisibility by 3 conditions in (i-ii) are met.

If \( \lambda_2 = 6x + 1 \) and \( n \equiv 0 \) (mod 6) then \( \lambda_1 \geq 3x + 2 \) and \( n \geq 3x + 2 \). Choose \( t = 3x + 1 \).
If \( \lambda_2 = 6x + 1 \) and \( n \equiv 2 \pmod{6} \), then \( \lambda_1 \equiv 4 \pmod{6} \) (see Table 1), so \( \lambda_1 \geq 3x + 4 \) and \( n \geq 3x + 2 \). Choose \( t = 3x + 2 \). Then all conditions in (i-ii) are met except that if \( x = 0 \) then \( \lambda_2 < t \); but then we seek a GDD\((n, 2)\) of index \((6y + 4, 1)\) which was constructed in Lemma 3.5.

If \( \lambda_2 = 6x + 2 \) then \( \lambda_1 \geq 3x + 2 \) and \( n \geq 3x + 2 \). Choose \( t = 3x + 1 \).

If \( \lambda_2 = 6x + 3 \) and \( n \equiv 0 \pmod{6} \) then \( \lambda_1 \geq 3x + 3 \) and \( n \geq 3x + 3 \). Choose \( t = 3x + 2 \).

If \( \lambda_2 = 6x + 3 \) and \( n \equiv 2 \pmod{6} \) then \( \lambda_1 \equiv 0 \pmod{6} \) (see Table 1), so \( \lambda_1 \geq 3x + 3 \) and \( n \geq 3x + 5 \). Choose \( t = 3x + 3 \). Then all conditions in (i-ii) are met except that if \( \lambda_1 = 3x + 3 \) then \( nt > \lambda_1(n - 1) \). However, if \( \lambda_1 = 3x + 3 \) then we can write \( \lambda_1 = 6y + 6, \lambda_2 = 12y + 9 \) and \( n \equiv 2 \pmod{6} \), so we can use Lemma 3.4.

If \( \lambda_2 = 6x + 3 \) and \( n \equiv 4 \pmod{6} \) then \( \lambda_1 \geq 3x + 3 \) and \( n \geq 3x + 4 \). Choose \( t = 3x + 3 \). Then all conditions in (i-ii) are satisfied unless \( \lambda_1 = 3x + 3 \), for then \( nt > \lambda_1(n - 1) \). If \( \lambda_1 = 3x + 3 \) then again the GDD can be obtained from Lemma 3.4.

If \( \lambda_2 = 6x + 4 \) then \( \lambda_1 \geq 3x + 3 \) and \( n \geq 3x + 3 \). Choose \( t = 3x + 2 \).

If \( \lambda_2 = 6x + 5 \) and \( n \equiv 0 \pmod{6} \) then \( \lambda_1 \geq 3x + 4 \) and \( n \geq 3x + 6 \). Choose \( t = 3x + 3 \).

If \( \lambda_2 = 6x + 5 \) and \( n \equiv 2 \pmod{6} \) then \( \lambda_1 \equiv 2 \pmod{6} \) (see Table 1), so \( \lambda_1 \geq 3x + 5 \) and \( n \geq 3x + 5 \). Choose \( t = 3x + 4 \).

Finally, we can present the main result.

**Theorem 3.7** Let \( n \geq 3 \) and \( \lambda_1, \lambda_2 \geq 1 \). There exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\) if and only if

1. \( 2 \) divides \( \lambda_1(n - 1) + \lambda_2n \),
2. \( 3 \) divides \( \lambda_1n(n - 1) + \lambda_2n^2 \), and
3. \( \lambda_1 \geq \lambda_2n/2(n - 1) \).

**Proof:** By Proposition 3.1, it suffices to consider the case where \( \lambda_2 \leq 2(n - 1), \) so \( n \geq \lambda_2/2 + 1 \) and therefore by (3) \( \lambda_1 \geq (\lambda_2 + 1)/2 \).

If \( n \) is odd (so \( \lambda_2 \) is even) the result follows from Proposition 3.2.
If $\lambda_1 = (\lambda_2 + 1)/2$ then the result follows from Proposition 3.3.

If $n$ is even and $\lambda_1 \geq \lambda_2/2 + 1$ then the result follows from Proposition 3.6. \hfill \Box

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References


