A KOROVKIN THEOREM FOR AL-SPACES

by

P.F. Renaud

No. 27 July 1982

1. INTRODUCTION

In [4] Wulbert obtained the following Korovkin-type theorem for linear operators on the space $L^1[0,1]$.

THEOREM. Let T_n be a sequence of contraction operators on $L^1[0,1]$. If

(i) $T_n 1 \xrightarrow{s} 1$ and

(ii) $T_n f \xrightarrow{w} f$ for the two functions $f(x) = x$ and $f(x) = x^2$,

then $T_n f \xrightarrow{s} f$ for all f in $L^1[0,1]$.

Probably the most interesting aspect of this result is that the usual positivity assumption of the operators T_n is absent. Meir [2] obtained a similar result where T_n is now assumed positive and condition (ii) is weakened by assuming only that $T_n f \xrightarrow{w} f$ for the function $f(x) = x$.

The purpose of this paper is to establish a result which generalizes both Wulbert's and Meir's theorems. Moreover to allow our result to apply to ℓ^1 spaces as well as L^1 spaces, we replace the space $L^1[0,1]$ by an Abstract Lebesgue (AL) space and the constant function 1 by a generalized weak unit. Specifically we will prove the following.

THEOREM. Let E be an AL-space with generalized weak unit $\{e_\alpha\}$ and let T_n be a sequence of contraction linear operators on E such that $T_n e_\alpha \xrightarrow{s} e_\alpha$ for all α.

Let $N = \{f : T_n f \xrightarrow{w} f\}$.

Then N is a (closed) sublattice of E and $T_n f \xrightarrow{s} f$ for all $f \in N$.

Meir's theorem follows at once by noting that the smallest closed sublattice of $L^1[0,1]$ containing 1 and the function $f(x) = x$ is all of $L^1[0,1]$.

2. NOTATION

DEFINITION. A (real) Banach lattice E is called an AL space if

$\|x + y\| = \|x\| + \|y\|$ whenever $x, y \geqslant 0$.

For E an AL space define $E^+ = \{x : x \geqslant 0\}$ and E_1 the unit ball in E with similar definitions for E^* - the dual space of E.

A generalized weak unit $\{e_\alpha\}$ for E is a maximal orthogonal family in E^+. Such families clearly exist via Zorn's lemma.

The following properties of AL-spaces will be needed. A subset $A \subset E$ is weakly sequentially precompact (w.s.p.) if every sequence in A has a weakly Cauchy subsequence. Since E is weakly sequentially complete ([3], p.119) "Cauchy" can be replaced by "convergent". If A is norm bounded then ([3], p.152) A is w.s.p. iff for all disjoint majorized sequences $\{\psi_n\}$ in E^{*+},

$$\lim_n \sup_{x \in A} \langle |x|, \psi_n \rangle = 0 .$$

The map from $E^+ \to \mathbf{R}$ given by $x \to \|x\|$ extends to define a linear functional $\psi_0 \in E^*$ - the evaluating functional.

Finally if T is a linear operator on E, the linear modulus $|T|$ can be defined by

$$|T|x = \sup_{|y| \leqslant x} |Ty| , \quad x \in E^+$$

and extends to a linear operator on E satisfying

$$-|T| \leqslant T \leqslant |T| \text{ and } \| |T| \| = \|T\|$$

(see [3], Ch IV. §1, especially Corollary 2).

For notational simplicity we write τ for $|T|$.

3. THE KOROVKIN THEOREM

Throughout E denotes an AL space with generalized weak unit $\{e_\alpha\}$. T_n is a sequence of contraction linear operators on E with linear moduli τ_n. Let $N = \{f : T_n f \xrightarrow{w} f\}$. Clearly N is a (closed) subspace of E. To prove the Korovkin theorem stated above we first need the following

PROPOSITION. Let $T_n e_\alpha \xrightarrow{w} e_\alpha$ for all α. Then N is a sublattice of E.

The proof requires the following

LEMMA. Let $u \geqslant 0$ and suppose that $T_n u \xrightarrow{w} u$. Then

(i) $|T_n u| \xrightarrow{w} u$ and

(ii) $\tau_n u \xrightarrow{w} u$.

PROOF.

(i) Since $T_n u \xrightarrow{w} u$, $\{T_n u\}$ is w.s.p. and hence so is $\{|T_n u|\}$. So we can choose a subsequence $T_{n(j)}$ such that $|T_{n(j)} u| \xrightarrow{w} v$ say. Clearly $v \geqslant u$.

Using the evaluating functional we have $\|T_{n(j)}u\| \to \|v\|$. i.e. $\|v\| = \lim \|T_{n(j)}u\| \leq \|u\|$.

This combined with $v \geq u$ shows that $v = u$. Applying this argument to any subsequence of T_n we have $|T_n u| \xrightarrow{w} u$.

(ii) For $\psi \in E_1^*$ we have

$$|< \tau_n u - u, \psi >| \leq | < \tau_n u - |T_n u|, \psi > | + | < |T_n u| - u, \psi > | .$$

The second term $\to 0$ by (i) and the first term is bounded by $\|\tau_n u - T_n u\| = \|\tau_n u\| - \||T_n u|\|$

$$\leq \|u\| - \||T_n u|\|$$

$$\to 0 \text{ by (i)} .$$

So $\tau_n u \overset{w}{—} u$.

PROOF OF PROPOSITION. It suffices to show that if $f \in N$, so does $|f|$. Fix $f \in N$. We firstly reduce the problem to the case where a weak unit for E exists.

Let $\operatorname{spt} f = \{\alpha : |f| \wedge e_\alpha > 0\}$. Since the e_α's are disjoint and positive we have for any $\alpha_1, \ldots, \alpha_n$,

$$\|f\| \geq \sum_{i=1}^{n} \||f| \wedge e_{\alpha_i}\|$$

so that $\operatorname{spt} f$ is countable.

Define $e = \Sigma \dfrac{e_n}{2^n \|e_n\|}$ with summation over $\operatorname{spt} f$.

For $A \subset E$ let $\perp A = \{x : |x| \wedge |y| \text{ for all } y \in A\}$. Then ([1], p.309) $\perp\perp(e)$ is a sub AL-space of E for which e is a weak unit and which contains f. Hence we can assume that E has a weak unit $e \geq 0$. Clearly $T_n e \xrightarrow{w} e$ and so by the lemma $\tau_n e \xrightarrow{w} e$. We now show that $T_n|f| \xrightarrow{w} |f|$.

For $m = 1,2,\ldots$ we have $0 \leqslant |f| \wedge me \leqslant me$ so that

$$\tau_n(|f| \wedge me) \leqslant m\, \tau_n e .$$

Since $\tau_n e$ is weakly convergent, $\tau_n(|f| \wedge me)$ is w.s.p. for each fixed m.

Choose a subsequence $n(1,j)$ such that

$$\tau_{n(1,j)}(|f| \wedge e) \xrightarrow{w} g_1 .$$

Now choose a subsequence $n(2,j)$ of $n(1,j)$ such that

$$\tau_{n(2,j)}(|f| \wedge 2e) \xrightarrow{w} g_2 \quad \text{etc.}$$

By diagonalization we have a subsequence $n(j,j)$ such that

$$\tau_{n(j,j)}(|f| \wedge me) \xrightarrow{w} g_m \text{ for each } m.$$

Clearly $\{g_m\}$ is increasing and via the evaluating functional we see that

$$\|g_m\| = \lim_j \|\tau_{n(j,j)}(|f| \wedge me)\| \leqslant \|f\|.$$

So $\{g_m\}$ converges (order and strongly) to g say and $\|g\| \leqslant \|f\|$.

Further for $\psi \in E^*$ we have

$$\begin{aligned} |\langle \tau_{n(j,j)}|f| - g, \psi \rangle| &\leqslant |\langle \tau_{n(j,j)}(|f| - |f| \wedge me), \psi \rangle| \\ &\quad + |\langle \tau_{n(j,j)}(|f| \wedge me) - g_m, \psi \rangle| \\ &\quad + |\langle g_m - g, \psi \rangle| . \end{aligned}$$

For m sufficiently large, the first and third terms on the right are small and for fixed large m, the second term is small for large j. We deduce that

$$\tau_{n(j,j)}|f| \xrightarrow{w} g .$$

But then $\tau_{n(j,j)}|f| \geqslant |T_{n(j,j)}f| \geqslant T_{n(j,j)}f$ and in the limit we have $g \geqslant f$. Similarly $g \geqslant -f$ so that $g \geqslant |f|$. This together with $\|g\| \leq \|f\|$ shows that $g = f$. Now applying this reasoning to an arbitrary subsequence of $\{T_n f\}$ we have that $\tau_n|f| \xrightarrow{w} |f|$.

To show now that $T_n|f| \xrightarrow{w} |f|$, we first notice that since $|T_n(|f| \wedge me)| \leq \tau_n(|f| \wedge me)$ then for each fixed m, $\{T_n(|f| \wedge me)\}$ is w.s.p. By the argument above there exists a subsequence $T_{n(j,j)}$ and a sequence $\{h_m\}$ such that

$$T_{n(j,j)}(|f| \wedge me) \xrightarrow{w} h_m \text{ for all } m.$$

Fix $\psi \in E^{*+}$. Then

$$0 \leq <(\tau_{n(j,j)} - T_{n(j,j)})|f|,\psi> \leq <(\tau_{n(j,j)} - T_{n(j,j)})(|f| - |f| \wedge me),\psi> + <(\tau_{n(j,j)} - T_{n(j,j)})me,\psi>.$$

The first term on the right can be made small by choosing m large and for fixed large m the second term $\to 0$ as $j \to \infty$. We deduce that $(\tau_{n(j,j)} - T_{n(j,j)})|f| \xrightarrow{w} 0$ and hence that

$$T_{n(j,j)}|f| \xrightarrow{w} |f|.$$

Applying this to any subsequence of $(T_n|f|)$ we have

$$T_n|f| \xrightarrow{w} |f|.$$

PROOF OF THEOREM. Without loss of generality we may again assume that E has a weak unit e and that $T_n e \xrightarrow{s} e$. Fix $f \in N$. By the proposition above we have $\tau_n f \xrightarrow{w} f$. We first show that $\tau_n f \xrightarrow{w} f$.

In fact $\{\tau_n f\}$ is w.s.p. so that for some subsequence $n(j)$, $\tau_{n(j)} f \xrightarrow{w} g$ say. Again by the proposition we have

$$|f| \pm f \xleftarrow{w} T_n(|f| \pm f) \leq \tau_n(|f| \pm f) \xrightarrow{w} |f| \pm g$$

which shows that $g = f$. We now deduce that $\tau_n f \xrightarrow{w} f$.

Since E has a weak unit it may be represented as the L^1 space of a compact measure space ([3], p.114) where e becomes the constant function 1.

Meir's result ([2], Corollary) applies directly to show that

$$\tau_n f \xrightarrow{s} f \quad \text{for all } f \in N.$$

Let $f \in N$, $f \geqslant 0$. Then

$$0 \leqslant (\tau_n - T_n) f = (\tau_n - T_n)(f - f \wedge me) + (\tau_n - T_n)(f \wedge me) .$$

Choosing m large so that $\| f - f \wedge me \|$ is small and noting that $(\tau_n - T_n)(f \wedge me) \leqslant (\tau_n - T_n) me \xrightarrow{s} 0$ we have

$$\| \tau_n f - T_n f \| \to 0 .$$

Hence

$$T_n f \xrightarrow{s} f .$$

Applying this idea to $|f| \pm f$ we have $T_n f \xrightarrow{s} f$ for all $f \in N$ which proves the theorem.

REFERENCES

1. G. Birkhoff, "Lattice Theory", American Mathematical Society, Providence, RI, 1967.

2. A. Meir, A Korovkin type theorem on weak convergence, Proc. Amer. Math. Soc. 59, No. 1 (1976) 72 - 74.

3. H.H. Schaefer, "Banach Lattices and Positive Operators", Springer-Verlag, Berlin, 1974.

4. D.E. Wulbert, Convergence of operators and Korovkin's theorem, J. Approximation Theory 1 (1968), 381-390.