A KOROVKIN THEOREM FOR AL-SPACES

by

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1. INTRODUCTION

In [4] Wulbert obtained the following Korovkin-type theorem for linear operators on the space $L^1[0,1]$.

THEOREM. Let $T_n$ be a sequence of contraction operators on $L^1[0,1]$. If

(i) $T_n 1 \xrightarrow{S} 1$ and

(ii) $T_n f \xrightarrow{W} f$ for the two functions $f(x) = x$ and $f(x) = x^2$,

then $T_n f \xrightarrow{S} f$ for all $f$ in $L^1[0,1]$.

Probably the most interesting aspect of this result is that the usual positivity assumption of the operators $T_n$ is absent. Meir [2] obtained a similar result where $T_n$ is now assumed positive and condition (ii) is weakened by assuming only that $T_n f \xrightarrow{W} f$ for the function $f(x) = x$.

The purpose of this paper is to establish a result which generalizes both Wulbert's and Meir's theorems. Moreover to allow our result to apply to $\ell^1$ spaces as well as $L^1$ spaces, we replace the space $L^1[0,1]$ by an Abstract Lebesgue (AL) space and the constant function 1 by a generalized weak unit. Specifically we will prove the following.

THEOREM. Let $E$ be an AL-space with generalized weak unit $\{e_\alpha\}$ and let $T_n$ be a sequence of contraction linear operators on $E$ such that $T_n e_\alpha \xrightarrow{S} e_\alpha$ for all $\alpha$.

Let $N = \{f : T_n f \xrightarrow{W} f\}$.
Then \( N \) is a (closed) sublattice of \( E \) and \( T_n f \xrightarrow{E} f \) for all \( f \in N \).

Meir's theorem follows at once by noting that the smallest closed sublattice of \( L^1[0,1] \) containing 1 and the function \( f(x) = x \) is all of \( L^1[0,1] \).

2. NOTATION

**DEFINITION.** A (real) Banach lattice \( E \) is called an AL space if

\[
\|x + y\| = \|x\| + \|y\| \quad \text{whenever} \quad x, y \geq 0.
\]

For \( E \) an AL space define \( E^+ = \{x : x \geq 0\} \) and \( E_1 \) the unit ball in \( E \) with similar definitions for \( E^* \) - the dual space of \( E \).

A generalized weak unit \( \{e^*_\alpha\} \) for \( E \) is a maximal orthogonal family in \( E^+ \). Such families clearly exist via Zorn's lemma.

The following properties of AL-spaces will be needed. A subset \( A \subset E \) is weakly sequentially precompact (w.s.p.) if every sequence in \( A \) has a weakly Cauchy subsequence. Since \( E \) is weakly sequentially complete ([3], p.119) "Cauchy" can be replaced by "convergent". If \( A \) is norm bounded then ([3], p.152) \( A \) is w.s.p. iff for all disjoint majorized sequences \( \{\psi_n\} \) in \( E^{**} \),

\[
\lim \sup_{n \in A} \langle |x|, \psi_n \rangle = 0.
\]

The map from \( E^+ \to \mathbb{R} \) given by \( x \to \|x\| \) extends to define a linear functional \( \psi_0 \in E^* \) - the evaluating functional.

Finally if \( T \) is a linear operator on \( E \), the linear modulus \( |T| \) can be defined by
\[ |T|x = \sup_{|y| \leq x} |Ty|, \ x \in \mathbb{E}^+ \]

and extends to a linear operator on \( \mathbb{E} \) satisfying

\[ -|T| \leq T \leq |T| \text{ and } \|T\| = |T| \]

(see [3], Ch IV. §1, especially Corollary 2).

For notational simplicity we write \( \tau \) for \( |T| \).

3. THE KOROVKIN THEOREM

Throughout \( \mathbb{E} \) denotes an AL space with generalized weak unit \( \{e_\alpha\} \). \( T_n \) is a sequence of contraction linear operators on \( \mathbb{E} \) with linear moduli \( \tau_n \). Let \( N = \{f : T_n f \overset{w}{\to} f\} \). Clearly \( N \) is a (closed) subspace of \( \mathbb{E} \). To prove the Korovkin theorem stated above we first need the following

PROPOSITION. Let \( T_n e_\alpha \overset{w}{\to} e_\alpha \) for all \( \alpha \). Then \( N \) is a sublattice of \( \mathbb{E} \).

The proof requires the following

LEMMA. Let \( u \geq 0 \) and suppose that \( T_n u \overset{w}{\to} u \). Then

(i) \( |T_n u| \overset{w}{\to} u \) and

(ii) \( \tau_n u \overset{w}{\to} u \).

PROOF.

(i) Since \( T_n u \overset{w}{\to} u \), \( \{T_n u\} \) is w.s.p. and hence so is \( \{|T_n u|\} \).

So we can choose a subsequence \( T_n(j) \) such that \( |T_n(j) u| \overset{w}{\to} v \) say. Clearly \( v \geq u \).
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Using the evaluating functional we have $\|T_n(j)u\| \to \|v\|$. i.e. $\|v\| = \lim \|T_n(j)u\| \leq \|u\|$. 

This combined with $v \geq u$ shows that $v = u$. Applying this argument to any subsequence of $T_n$ we have $|T_n u| \overset{w}{\to} u$.

(ii) For $\psi \in E_i^*$ we have

$$|\langle \tau_n u - u, \psi \rangle| \leq |\langle \tau_n u - |T_n u|, \psi \rangle| + |\langle |T_n u| - u, \psi \rangle|.$$

The second term $\to 0$ by (i) and the first term is bounded by $\|\tau_n u - T_n u\| = \|\tau_n u\| - \|T_n u\|$

$$\leq \|u\| - \|T_n u\| \to 0 \text{ by (i)}.$$

So $\tau_n u \overset{w}{\to} u$.

PROOF OF PROPOSITION. It suffices to show that if $f \in N$, so does $|f|$. Fix $f \in N$. We firstly reduce the problem to the case where a weak unit for $E$ exists.

Let $\text{spt } f = \{\alpha : |f| \land e_{\alpha} > 0\}$. Since the $e_{\alpha}$'s are disjoint and positive we have for any $\alpha_1, \ldots, \alpha_n$,

$$\|f\| \geq \sum_{i=1}^{n} \|f\| \land e_{\alpha_i}$$

so that $\text{spt } f$ is countable.

Define $e = \sum \frac{e_n}{2^n\|e_n\|}$ with summation over $\text{spt } f$.

For $A \subseteq E$ let $\downarrow A = \{x : |x| \land |y| \text{ for all } y \in A\}$. Then ([1], p.309) $\downarrow(e)$ is a sub $\text{AL}$-space of $E$ for which $e$ is a weak unit and which contains $f$. Hence we can assume that $E$ has a weak unit $e \geq 0$. Clearly $T_n e \overset{w}{\to} e$ and so by the lemma $\tau_n e \overset{w}{\to} e$.

We now show that $T_n |f| \overset{w}{\to} |f|$.
For \( m = 1,2, \ldots \) we have \( 0 \leq |f| \leq \text{me} \leq \text{me} \) so that
\[
\tau_n(|f| \text{me}) \leq m \tau_n \text{e}.
\]
Since \( \tau_n \text{e} \) is weakly convergent, \( \tau_n(|f| \text{me}) \) is w.s.p. for each fixed \( m \).

Choose a subsequence \( n(1,j) \) such that
\[
\tau_{n(1,j)}(|f| \text{e}) \xrightarrow{w} g_1.
\]
Now choose a subsequence \( n(2,j) \) of \( n(1,j) \) such that
\[
\tau_{n(2,j)}(|f| \text{e}) \xrightarrow{w} g_2 \quad \text{etc.}
\]
By diagonalization we have a subsequence \( n(j,j) \) such that
\[
\tau_{n(j,j)}(|f| \text{me}) \xrightarrow{w} g_m \quad \text{for each } m.
\]
Clearly \( \{g_m\} \) is increasing and via the evaluating functional we see that
\[
\|g_m\| = \lim_j \|\tau_{n(j,j)}(|f| \text{me})\| \leq \|f\|.
\]
So \( \{g_m\} \) converges (order and strongly) to \( g \) say and \( \|g\| \leq \|f\| \).

Further for \( \psi \in E^* \) we have
\[
| \langle \tau_{n(j,j)} |f| - g, \psi \rangle | \leq | \langle \tau_{n(j,j)} (|f| - |f| \text{me}), \psi \rangle | + | \langle \tau_{n(j,j)} (|f| \text{me}) - g_m, \psi \rangle | + | \langle g_m - g, \psi \rangle |.
\]
For \( m \) sufficiently large, the first and third terms on the right are small and for fixed large \( m \), the second term is small for large \( j \). We deduce that
\[
\tau_{n(j,j)} |f| \xrightarrow{w} g.
\]
But then \( \tau_{n(j,j)} |f| \geq |T_{n(j,j)} f| \geq T_{n(j,j)} f \) and in the limit we have \( g \geq f \). Similarly \( g \geq -f \) so that \( g \geq |f| \). This
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together with \( \|g\| \leq \|f\| \) shows that \( g = f \). Now applying this reasoning to an arbitrary subsequence of \( \{T_n f\} \) we have that \( \tau_n |f| \xrightarrow{w} |f| \).

To show now that \( T_n |f| \xrightarrow{w} |f| \), we first notice that since \( |T_n (|f| \wedge m e)| \leq \tau_n (|f| \wedge m e) \) then for each fixed \( m \), \( \{T_n (|f| \wedge m e)\} \) is w.s.p. By the argument above there exists a subsequence \( T_n(j,j) \) and a sequence \( \{h_m\} \) such that

\[
T_n(j,j) \left( |f| \wedge m e \right) \xrightarrow{w} h_m \text{ for all } m.
\]

Fix \( \psi \in E^{*+} \). Then

\[
0 \leq \langle (\tau_n(j,j) - T_n(j,j)) |f|, \psi \rangle \leq \langle (\tau_n(j,j) - T_n(j,j)) (|f| - |f| \wedge m e), \psi \rangle + \langle (\tau_n(j,j) - T_n(j,j)) m e, \psi \rangle.
\]

The first term on the right can be made small by choosing \( m \) large and for fixed large \( m \) the second term \( \to 0 \) as \( j \to \infty \). We deduce that \( (\tau_n(j,j) - T_n(j,j)) |f| \xrightarrow{w} 0 \) and hence that

\[
T_n(j,j) |f| \xrightarrow{w} |f|.
\]

Applying this to any subsequence of \( \{T_n |f|\} \) we have

\[
T_n |f| \xrightarrow{w} |f|.
\]

PROOF OF THEOREM. Without loss of generality we may again assume that \( E \) has a weak unit \( e \) and that \( T_n e \xrightarrow{s} e \). Fix \( f \in N \). By the proposition above we have \( \tau_n f \xrightarrow{w} f \). We first show that

\[
\tau_n f \xrightarrow{w} f.
\]

In fact \( \{\tau_n f\} \) is w.s.p. so that for some subsequence \( n(j) \), \( \tau_n(j) \xrightarrow{w} g \) say. Again by the proposition we have

\[
|f| \pm f \xleftarrow{w} T_n (|f| \pm f) \leq \tau_n (|f| \pm f) \xrightarrow{w} |f| \pm g
\]

which shows that \( g = f \). We now deduce that \( \tau_n f \xrightarrow{w} f \).
Since $E$ has a weak unit it may be represented as the $L^1$ space of a compact measure space ([3], p.114) where $e$ becomes the constant function $1$.

Meir's result ([2], Corollary) applies directly to show that $\tau_n f \overset{S}{\rightarrow} f$ for all $f \in N$.

Let $f \in N, f \geq 0$. Then

$$0 \leq (\tau_n - T_n)f = (\tau_n - T_n)(f - f \wedge me) + (\tau_n - T_n)(f \wedge me).$$

Choosing $m$ large so that $\| f - f \wedge me \|$ is small and noting that $(\tau_n - T_n)(f \wedge me) \leq (\tau_n - T_n)me \overset{S}{\rightarrow} 0$ we have

$$\| \tau_n f - T_n f \| \rightarrow 0.$$

Hence

$$T_n f \overset{S}{\rightarrow} f.$$

Applying this idea to $|f| \pm f$ we have $T_n f \overset{S}{\rightarrow} f$ for all $f \in N$ which proves the theorem.
REFERENCES


