ON THE NON-EXISTENCE OF
5-(24,12,6) AND 4-(23,11,6) DESIGNS

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No. 10
November 1977

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Abstract

It is shown that 5-\((24,12,6)\) and 4-\((23,11,6)\) designs cannot exist and that consequently a non-trivial 2-\((2n+1,n,n-1)\) design can be extended to a 4-\((2n+3,n+2,n-1)\) design if and only if \(n = 4\).
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DEFINITIONS AND NOTATION

A $t-(v,k,\lambda)$ design is formed from a set of $v$ symbols (or points, or varieties) by taking $b$ subsets, called blocks, each of size $k$, subject to the conditions that every unordered $t$-tuple must appear just $\lambda$ times and no two blocks are identical. Let $\lambda_i$, $(0 \leq i \leq t)$, be the number of times each unordered $i$-tuple appears in the design. Then $\lambda_0 = b$ the number of blocks; $\lambda_i = r$, the number of replications of each symbol; and $\lambda_t = \lambda$. The complete set of the $\lambda_i$'s is given by the standard equations

$$\lambda_i = \lambda \frac{(v-i)(v-i-1)(v-i-2)\ldots(v-t+1)}{(k-i)(k-i-1)(k-i-2)\ldots(k-t+1)}; \quad 0 \leq i \leq t-1. \quad (1)$$

Let $n_i$, $(0 \leq i \leq k)$, be the number of blocks intersecting a given block in exactly $i$ points. Note that $n_k = 1$. Another standard set of equations connects the $\lambda_i$'s and the $n_i$'s through

$$\sum_{i=0}^{k} n_i = b = \lambda_0;$$

and

$$\sum_{i=0}^{k} i(i-1)\ldots(i-j)n_i = [k(k-1)\ldots(k-j+1)]\lambda_j; \quad 1 \leq j \leq t. \quad (2)$$

Given a $t$-design a $(t-1)$-design can be formed by rejecting all blocks not containing a given symbol which is then deleted from the remaining blocks. The smaller design is called a restriction or contraction of the larger. The reverse process is sometimes possible, in which case the same new symbol
is added to all the blocks of (t-1)-design and the further blocks not containing the new symbol are adjoined to make a t-design. The new design is called an extension of the original design.

The complement of a t-design is also a t-design and is formed by replacing each block by the set of symbols not contained in it.

BACKGROUND

It is well-known that any 2-(2n+1,n,λ) design can be extended to a 3-(2n+2,n+1,λ) design by complementation; that is to say by adding the same new symbol to each original block and then adjoining the complement of each block with respect to the 2n+2 symbols then available. For the Hadamard 2-designs, in which $\lambda = \frac{1}{2}$ and n is odd, it is known that this is the only way of extending to a 3-design. Furthermore, if a Hadamard design has a repeated extension to a 4-(2n+3,2n+2,\(\frac{n}{2} - \frac{1}{2}\)) design then from (1), $(n+2)|6$. Since n has to be odd the only possibility is $n = 1$ corresponding to a vacuous situation. Therefore a Hadamard design never extends to a 4-design.

For the family of designs with the larger value $\lambda = n-1$ however, it is sometimes possible to extend to a 3-(2n+2,n+1,n-1) design other than by complementation. In particular it has been shown (Breach, [1]) that among the eleven 2-(9,4,3) designs there are two which have extensions other than by complementation and one of these extends to a 4-(11,6,3) design. Are there any other values of n for which an extension to a 4-design is possible? If so the
condition \((n+2)|12\) must be satisfied. Thus \(n = 1, 2, 4\) or 10. The case \(n = 1\) is vacuous; the case \(n = 2\) is trivial since a 4-(7,4,1) design contains all possible combinations of 4 symbols from 7 symbols; the case \(n = 4\) is already decided; thus the only further possibility is that there exists a 2-(21,10,9) design which extends to a 4-(23,12,9) design. This note shows that such a 4-design cannot exist. This is achieved by showing that the complementary design 4-(23,11,6), and its possible extension to a 5-(24,12,6) design cannot exist.

**THE PROOF OF NON-EXISTENCE**

(a) *Any* 4-(23,11,6) *design can be extended to a 5-(24,12,6) *design* by *complementation.*

For if \(N_5\) is the number of blocks of the 4-design which contain all of 5 given symbols and \(N_0\) is the number containing none of them, then by the principle of inclusion and exclusion,

\[
N_0 = \lambda_0 - \binom{5}{1}\lambda_1 + \binom{5}{2}\lambda_2 - \binom{5}{3}\lambda_3 + \binom{5}{4}\lambda_4 - N_5. 
\]  

(3)

But for a 4-(23,11,6) design, from (1),

\[
\lambda_0 = 161, \quad \lambda_1 = 77, \quad \lambda_2 = 35, \quad \lambda_3 = 15, \quad \lambda_4 = 6. 
\]  

(4)

Therefore \(N_0 = 6 - N_5\) and the result follows.

(b) *A self-complementary* 5-(24,12,6) *design cannot exist.*

Consider two blocks A and B. Then if A intersects B in \(i\) points, A intersects the complement of B in 12-\(i\) points. Thus the block intersection numbers for A are such that \(n_1 = n_{12-i}^\prime\). In particular \(n_0 = n_{12} = 1\).
Let \((\lambda_1)\) stand for the block intersection equation of (2) in which \(\lambda_1\) appears. Then the linear combination 
\[ (\lambda_1) - 2(\lambda_4) - 360(\lambda_2) + 1800(\lambda_1) \]
reduces to 
\[ 21,600 n_1 + 8,640 n_2 + 2,592 n_3 + 432 n_4 = -12^2\cdot 10^9. \]
This has no solution in non-negative integers. Therefore a self-complementary 5-(24,12,6) design cannot exist.

(c) There are no 5-(24,12,6) or 4-(23,11,6) designs.
This follows by combining (a) and (b) since any such 5-design would have a restriction to a 4-design which in turn would have an extension to a self-complementary 5-design.

REMARKS.

(i) Since a 4-(23,12,9) design cannot exist, being the complement of a 4-(23,11,6) design, it follows that a non-trivial 2-(2n+1, n, n-1) design can be extended to a 4-design if and only if \(n=4\). The resulting 4-(11,6,3) design is unique.
A construction and proof of uniqueness is given in Breach [1].

(ii) Although a 4-(23,12,9) design cannot exist there are 3-(22,11,9) designs. The writer has constructed some by ad hoc methods.

(iii) A chain of restrictions from 5-(24,12,6) leads to a 2-(21,9,6) design. There is an example of such a design listed in Hall [3]. This raises the question, is it ever possible to extend a 2-(21,9,6) design to a 3-(22,10,6) design?
(iv) It has been shown that a 5-(24,12,\(\lambda\)) design cannot exist if \(\lambda = 6\) but there does exist a 5-design with a larger value of \(\lambda\), namely 48. This is described by Conway [2] who shows that the Mathieu group \(M_{24}\) is quintuply transitive on a set of 2576 dodecads. Is \(\lambda = 48\) the smallest value for which a 5-(24,12,\(\lambda\)) exists?

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