

A CHARACTERIZATION OF NEWTON MAPS

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Abstract

Conditions are given for a C^l map T to be a Newton map, that is, the map associated with a differentiable real-valued function via Newton's method. For finitely differentiable maps and functions, these conditions can only be necessary, but in the smooth case, i.e. for $l = \infty$, they are also sufficient. The characterization rests upon the structure of the fixed point set of T , and it is best possible as is demonstrated through examples.

Key words and phrases: Newton's method, Newton map, discrete dynamical system, fixed point set, attracting fixed point.

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1 Introduction

Newton's method (NM) for computing successive approximations of zeros of functions is one of the most widely used methods in all of applied mathematics; variants and generalizations also play a prominent role in numerous other disciplines [2, 3, 8, 10, 11]. Conceptually, NM becomes especially transparent within a dynamical systems context. The purpose of this brief note is to characterize, in the simplest possible setting, the local properties of the dynamical systems thus encountered.

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Throughout, let $f : I \rightarrow \mathbb{R}$ be a differentiable function, defined on some open interval $I \subset \mathbb{R}$, and denote by N_f its associated NM transformation, that is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \forall x \in I : f'(x) \neq 0; \quad (1)$$

for N_f to be defined for every $x \in I$, set $N_f(x) := x$ whenever $f'(x) = 0$.

NM for finding roots (zeros) of f , i.e., real numbers x^* with $f(x^*) = 0$, amounts to picking an initial point $x_0 \in I$ and iterating N_f , thus generating the sequence

$$x_n = N_f(x_{n-1}) = N_f^n(x_0), \quad \forall n \in \mathbb{N};$$

here and throughout, for any map $T : I \rightarrow \mathbb{R}$ and any $n \in \mathbb{N}$, $T^n(x) = T(T^{n-1}(x))$, provided that $T^{n-1}(x) \in I$, and $T^0(x) = x$. Note that $N_f(x) = x$ precisely if $f(x)f'(x) = 0$. Thus for $f(x_n)f'(x_n) = 0$, and only then, does NM terminate at x_n : If $f(x_n) = 0$, a root has been found, and otherwise (1) breaks down due to a horizontal tangent to the graph of f at x_n (see Figure 1).

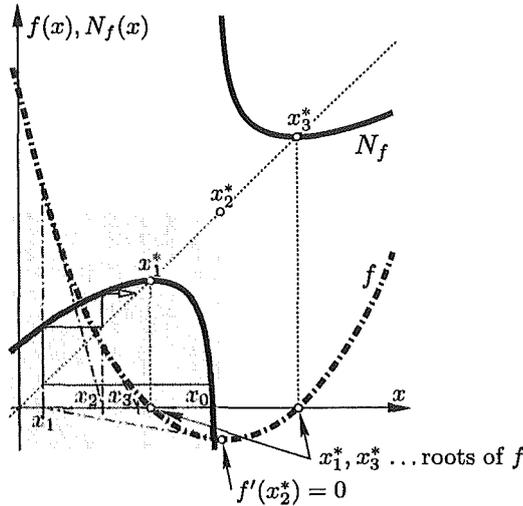


Figure 1: Visualizing NM: The first few iterates x_1, x_2, x_3 are found graphically, both by means of tangents to the graph of f (broken line) and via the graph of N_f (solid line). Note how the point x_2^* with $f'(x_2^*) = 0$ causes N_f to have a discontinuity.

Clearly, if (x_n) converges to x^* , say, and if N_f is continuous at x^* , then $N_f(x^*) = x^*$, i.e., x^* is a fixed point of N_f , and $f(x^*) = 0$. (The trivial alternative $f \equiv \text{const.}$ is tacitly excluded here, see Lemma 3 below.) It is this correspondence between the roots of f and the fixed points of N_f that suggests that NM be studied as a dynamical system. Under a mild assumption, each (isolated) fixed point x^* is *attracting*, that is, $\lim_{n \rightarrow \infty} N_f^n(x_0) = x^*$ for all x_0 sufficiently close to x^* . (For x_0 further away from any root, the sequence (x_n) may exhibit a considerably more complicated long-term behavior [2, 3, 11].) This aspect of NM is put into perspective by the main result of the present note, Theorem 10 below, which completely characterizes the local dynamical properties of N_f .

2 Newton Maps

The definition of a Newton map given below entails a relationship between the analytic properties of a function f and its associated NM transformation N_f , respectively. It is a simple fact, rarely alluded to in studies of NM, that generally these properties are quite independent. On the one hand, the C^1 function $f(x) = |x|^{3/2}$, which is not C^2 , has a C^∞ NM transformation, namely $N_f(x) = \frac{1}{3}x$. On the other hand, N_f may lack decent analytic properties even if f is smooth.

Example 1. It is easily seen that the function

$$f(x) = \begin{cases} \exp(-x^{-2} + |x| + \cos(x^{-2})) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is C^∞ , and both f and f' vanish only at $x^* = 0$. Nevertheless

$$-1 = \liminf_{x \rightarrow 0} N_f(x) < \limsup_{x \rightarrow 0} N_f(x) = 1,$$

hence N_f is not even *continuous* at x^* .

Since N_f may fail to be continuous even if f is C^∞ , in order to ensure the applicability of NM, some explicit assumption on the smoothness of N_f has to be imposed. To neatly formulate such conditions, let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and stipulate that $\infty^{-1} := 0$ as well as $\infty \pm j = \infty$ and $j < \infty$ for all $j \in \mathbb{N}$. In view of (1), for N_f to be C^l for some $l \in \mathbb{N}_\infty$, one might demand that f be at least C^{l+1} , but this assumption has just proved neither necessary nor sufficient. Simply imposing further conditions on N_f also seems problematic as long as it is not clear whether any such condition can be met for a reasonably large class of functions. Thus it is inevitable to address in some generality the following inverse problem: Given a C^l map T , does there exist a function f such that $T = N_f$?

Definition 2. Let $I \subset \mathbb{R}$ be an open interval, and $l \in \mathbb{N}_\infty$. A map $T \in C^l(I)$ is called a *Newton map* (associated with f), if $T = N_f$ for some differentiable function $f : I \rightarrow \mathbb{R}$.

Evidently, not every $T \in C^l(I)$ is a Newton map, not even if $l = \infty$, as the trivial example $T(x) = -x$ shows, for which every f with $N_f = T$ lacks differentiability at $x^* = 0$. As will become clear shortly, the question raised above does not have a satisfactory answer for finitely differentiable maps. However, in the smooth case, i.e. for $l = \infty$, there is a simple characterization of Newton maps, as provided by Theorem 10. Statement and proof of this main result are preceded by a few simple preliminaries. For any map T , denote by $\text{Fix}[T]$ the set of fixed points of T , that is, $\text{Fix}[T] := \{x \in I : T(x) = x\}$, and say that $\text{Fix}[T]$ is *attracting* if $\lim_{n \rightarrow \infty} T^n(x_0) \in \text{Fix}[T]$ for all x_0 sufficiently close to $\text{Fix}[T]$.

Lemma 3. Let $f : I \rightarrow \mathbb{R}$ be differentiable, and assume that N_f is continuous. Then $\text{Fix}[N_f]$ is either empty or a (possibly one-point) interval; in the latter case,

$$\limsup_{x \rightarrow x^*} \frac{N_f(x) - x^*}{x - x^*} = \delta \quad \text{for some } \delta \in [0, 1] \quad (2)$$

holds for every $x^* \in \text{Fix}[N_f]$.

Proof. It will first be proved that both sets $Z_0 := \{x \in I : f(x) = 0\}$ and $Z_1 := \{x \in I : f'(x) = 0\}$ of zeros of f and f' , respectively, are (possibly empty or one-point) subintervals of I . Moreover, if $Z_1 \neq I$, that is, if f is not constant, then $Z_1 \subset Z_0$; in fact, the two

sets coincide unless Z_0 contains exactly one point, in which case Z_1 may be empty. Since $\text{Fix}[N_f] = Z_0 \cup Z_1$ the first part of the lemma follows immediately from this.

If $Z_1 = I$ then $\text{Fix}[N_f] = I$, so let $Z_1 \neq \emptyset$ be different from I . Pick $a \in Z_1$, assume $f(a) \neq 0$ and, without loss of generality, that $b := \sup\{x \geq a : f(y) = f(a) \text{ for all } y \in [a, x]\}$ belongs to I . Clearly, $f(b) = f(a)$ and $f'(b) = 0$, hence $N_f(b) = b$. By the Mean Value Theorem there exists a sequence $b_n \searrow b$ such that $0 < |f'(b_n)| \leq 1$ for all n . But then

$$\liminf_{n \rightarrow \infty} |N_f(b_n) - b| \geq \lim_{n \rightarrow \infty} |f(b_n)| = |f(b)| = |f(a)| > 0,$$

clearly contradicting the continuity of N_f . Therefore $f(a) = 0$, hence $Z_1 \subset Z_0$. Since the set Z_0 is closed, it contains, with any two points, the whole segment joining these points. Thus Z_0 is an interval. If Z_0 is not a singleton then $Z_0 \subset Z_1$ and therefore $Z_0 = Z_1$. The latter equality also holds if Z_0 is one-point because $Z_1 \neq \emptyset$. Finally, if Z_1 is empty then clearly Z_0 cannot contain more than one point.

Assertion (2) is trivially true if x^* is an interior point of $\text{Fix}[N_f]$. Without loss of generality therefore assume that x^* is a, say, *right* boundary point of $\text{Fix}[N_f] = Z_0$. Choose $\delta > 0$ so small that $J :=]x^*, x^* + \delta] \subset I$ and, for $0 < t \leq \delta$, let

$$h(t) := \frac{N_f(x^* + t) - x^*}{t}; \quad (3)$$

the function h is continuous on $]0, \delta]$, and $h(t) \neq 1$ for all $t > 0$. Since $x \neq N_f(x)$ for $x \in J$,

$$\frac{f'(x)}{f(x)} = \frac{1}{x - N_f(x)}, \quad \forall x \in J,$$

which after integrating both sides from x to $x^* + \delta$, and using the auxiliary function h defined in (3), can be written as

$$f(x) = f(x^* + \delta) \exp\left(-\int_{x-x^*}^{\delta} \frac{1}{1-h(t)} \frac{dt}{t}\right), \quad \forall x \in J. \quad (4)$$

Assume $f(x^* + \delta) > 0$ without loss of generality. If $h(t) > 1$ for all $t > 0$ then (4) implies that $f(x^*) \neq 0$, contradicting $x^* \in Z_0$. Thus $h(t) < 1$ for all $t > 0$, and in particular

$$\limsup_{t \searrow 0} h(t) = \limsup_{x \searrow x^*} \frac{N_f(x) - x^*}{x - x^*} \leq 1.$$

Fix $j \in \mathbb{N}$. Dividing (4) by $(x - x^*)^j = \delta^j \exp\left(-j \int_{x-x^*}^{\delta} t^{-1} dt\right)$ yields

$$(x - x^*)^{-j} f(x) = f(x^* + \delta) \delta^{-j} \exp\left(\int_{x-x^*}^{\delta} \frac{j-1-jh(t)}{1-h(t)} \frac{dt}{t}\right), \quad \forall x \in J. \quad (5)$$

To bound $\limsup_{t \searrow 0} h(t)$ from below, pick $\varepsilon > 0$ and assume that $h(t) < -\varepsilon$ for all sufficiently small $t > 0$. In this case, (5) with $j = 1$ shows that

$$(x - x^*)^{-1} f(x) \geq f(x^* + \delta) \delta^{-\frac{\varepsilon}{1+\varepsilon}} (x - x^*)^{\frac{\varepsilon}{1+\varepsilon}} \rightarrow \infty, \quad \text{as } x \searrow x^*,$$

which clearly contradicts the differentiability of f at x^* . Since $\varepsilon > 0$ was arbitrary, $\limsup_{t \searrow 0} h(t) \geq 0$, and the proof is complete. \square

Remark 4. (i) Lemma 3 should be contrasted with the simple fact that for *every* closed set $A \subset \mathbb{R}$ there exists a C^∞ map T with $T(I) \subset I$ and $\text{Fix}[T] = A \cap I$.

(ii) Under the conditions of Lemma 3 there is no analogue to (2) for the corresponding \liminf which, as simple examples show, can be any number between, and including, the trivial bounds $-\infty$ and δ .

As pointed out earlier, the applicability of NM rests on the correspondence between the roots of f and the the fixed points of N_f — *and* the attractiveness of the latter. Mere continuity of N_f does not guarantee that $\text{Fix}[N_f]$ is attracting.

Example 5. Consider the C^1 function

$$f(x) = \begin{cases} |x|^{\frac{3}{2}} \exp\left(-\int_0^{|x|^{-1}} t^{-1} \sin t \, dt\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for which the associated NM transformation

$$N_f(x) = \begin{cases} x \frac{1 + 2 \sin(|x|^{-1})}{3 + 2 \sin(|x|^{-1})} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous yet obviously not C^1 . The only fixed point of N_f , and correspondingly the only root of f and f' , is $x^* = 0$. Since, for every $j \in \mathbb{N}$, the points $\pm \frac{2}{\pi}(4j - 1)^{-1}$ are 2-periodic, $\text{Fix}[N_f] = \{0\}$ is *not* attracting.

Thus while $\text{Fix}[N_f]$ is topologically simple whenever N_f is continuous, to make NM practical for approximating zeros, more smoothness is required. Only the case of N_f being at least C^1 will therefore be considered from now on. (For the same reason, the legitimate case $l = 0$ has been excluded from Definition 2.) Also, the properties of N'_f , albeit not completely determined by the smoothness of f , do depend on the latter. To describe this dependence, for every $k \in \mathbb{N}_\infty$, define the set

$$\Delta_k := \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, 1 - k^{-1}\right\} \cup]1 - k^{-1}, 1], \quad (6)$$

and note that $[0, 1] = \Delta_1 \supset \Delta_2 \supset \dots \supset \Delta_\infty = \{1 - j^{-1} : j \in \mathbb{N}_\infty\}$.

Lemma 6. *Let $f : I \rightarrow \mathbb{R}$ be differentiable, and assume that $N_f \in C^1(I)$. Then $\text{Fix}[N_f]$ is either empty or an attracting (possibly one-point) interval. Moreover, if $\text{Fix}[N_f] \neq \emptyset$ and $f \in C^k(I)$ with $k \in \mathbb{N}_\infty$ then*

$$N'_f(\text{Fix}[N_f]) = \{\delta\} \quad \text{for some } \delta \in \Delta_k. \quad (7)$$

Proof. The assertions are trivially true if f is constant or $\text{Fix}[N_f] = \emptyset$. Therefore assume that f is not constant and $\text{Fix}[N_f]$ is not empty, hence a subinterval of I , by Lemma 3. If x^* is an interior point of $\text{Fix}[N_f]$ then $N'_f \equiv 1$ in a neighborhood of x^* , and the assertion is again true. Thus assume without loss of generality that x^* is a *right* boundary point of $\text{Fix}[N_f]$. According to Lemma 3, $N'_f(x^*) \in \Delta_1$, so x^* obviously is attracting from the right, unless perhaps for $N'_f(x^*) = 1$. In the latter case, with the notations introduced in the proof of Lemma 3, the function h according to (3), supplemented by $h(0) := N'_f(x^*) = 1$ is continuous on $[0, \delta]$ and can be written as $h(t) = 1 - H(t)$ where H is also continuous on $[0, \delta]$, and $H(t) \neq 0$ unless $t = 0$. With this, (4) takes the form

$$f(x) = f(x^* + \delta) \exp\left(-\int_{x-x^*}^{\delta} \frac{dt}{tH(t)}\right), \quad \forall x \in J.$$

Since $f(x^*) = 0$ and $f(x^* + \delta) \neq 0$, the integral $\int_0^{\delta} \frac{dt}{tH(t)}$ must diverge to $+\infty$. As H is continuous and, except possibly at $t = 0$, does not change sign, $H(t) > 0$ and so $h(t) < 1$ whenever $0 < t \leq \delta$. From $N_f(x^* + t) - x^* = th(t) < t$ and $h(0) = 0$ it follows that

$x^* < N_f(x_0) < x_0$ and therefore $N_f^n(x_0) \searrow x^*$ provided that $x_0 \in J$. In other words, x^* is attracting from the right.

It remains to verify (7) for $f \in C^k(I)$. To this end, assume first that $k < \infty$ and $f(x^*) = f'(x^*) = \dots = f^{(k)}(x^*) = 0$. In this case, since f is C^k , the left-hand side in (5) with $j = k$ tends to a finite limit as $x \searrow x^*$. Consequently,

$$\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\delta} \frac{k-1-kh(t)}{1-h(t)} \frac{dt}{t} < +\infty. \quad (8)$$

If $h(0) < 1 - k^{-1}$, then the integrand in (8) would eventually be positive near $t = 0$, which clearly is impossible. Therefore $h(0) \geq 1 - k^{-1}$. Since $h(0) \leq 1$ by the same argument,

$$N'_f(x^*) = h(0) \in [1 - k^{-1}, 1] \subset \Delta_k.$$

If $k = \infty$ and $f^{(j)}(x^*) = 0$ for all $j \in \mathbb{N}$ then similar reasoning shows that $N'_f(x^*) \in \bigcap_{j \in \mathbb{N}} [1 - j^{-1}, 1] = \{1\} \subset \Delta_{\infty}$.

Finally assume that $f(x^*) = f'(x^*) = \dots = f^{(j)}(x^*) = 0$ yet $f^{(j+1)}(x^*) \neq 0$ for some j with $0 \leq j < k$. The same argument as before with k replaced by j shows that $N'_f(x^*) \in [1 - (j+1)^{-1}, 1]$. If $h(0) > 1 - (j+1)^{-1}$ then (5) with j replaced by $j+1$ would imply that $\lim_{x \searrow x^*} (x - x^*)^{-(j+1)} f(x) = 0$, which contradicts $f^{(j+1)}(x^*) \neq 0$. Thus $N'_f(x^*) = h(0) = 1 - (j+1)^{-1} \in \Delta_{\infty} \subset \Delta_k$. \square

Remark 7. Lemma 6 is best possible in the following sense: For every $k \in \mathbb{N}_{\infty}$ and $\delta \in \Delta_k$ there exists a C^k function f with $N_f \in C^1$ having a single fixed point x^* such that $N'_f(x^*) = \delta$. For $k \in \mathbb{N}$ and $\delta \in \Delta_k \setminus \{1\}$ let $\gamma = (1 - \delta)^{-1}$ and consider the function

$$f(x) = \begin{cases} x^{\gamma} \left(1 + \frac{1}{2k+4} x^{(1+\gamma)(1+k)} \sin(x^{-\gamma}) \right) & \text{if } 0 < |x| < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

where, for non-integer γ , each argument x has to be replaced by $|x|$. Taking $I =]-1, 1[$, it is readily checked that $f \in C^k(I)$ and $N_f \in C^1(I)$. Moreover, $x^* = 0$ is the only fixed point of N_f in I , and $N'_f(x^*) = 1 - \gamma^{-1} = \delta$. For $\delta = 1$, an example is provided by the C^k function $f(x) = \exp(-|x|^{-1}) + \frac{1}{2} \exp(-(k+4)|x|^{-1}) \sin(\exp(|x|^{-1}))$ for which N_f is C^1 , has $x^* = 0$ as its only fixed point, and $N'_f(x^*) = 1$. Simple examples in the case $k = \infty$ are $f(x) = x^{\gamma}$ for $\delta < 1$, and $f(x) = \exp(-|x|^{-1})$ for $\delta = 1$, respectively.

An important special case for which Lemma 6 can be strengthened is the case of a root of finite multiplicity. Recall that $x^* \in I$ is a root of $f \in C^k(I)$ of multiplicity $j \in \mathbb{N}$ if $f(x) = (x - x^*)^j g(x)$ for all $x \in I$, where $g \in C^k(I)$ and $g(x^*) \neq 0$.

Corollary 8. *Let x^* be a root of $f \in C^k(I)$ of finite multiplicity j . Then, for some open interval $J \subset I$ containing x^* , $N_f \in C^{k-1}(J)$, and $N'_f(x^*) = 1 - j^{-1}$; in particular, $\text{Fix}[N_f] \cap J = \{x^*\}$ is attracting.*

Proof. Since $f(x) = (x - x^*)^j g(x)$ for some $g \in C^k$ with $g(x^*) \neq 0$,

$$N_f(x) - x^* = (x - x^*) \frac{(j-1)g(x) + (x - x^*)g'(x)}{jg(x) + (x - x^*)g'(x)} = (x - x^*)h(x), \quad (9)$$

where h is C^{k-1} on some open interval $J \subset I$ containing x^* , and $N'_f(x^*) = h(x^*) = 1 - j^{-1}$. Thus, for J chosen sufficiently small, $\text{Fix}[N_f] \cap J = \{x^*\}$, and the fixed point x^* clearly is attracting. \square

Lemma 6 contains necessary conditions for a map to be Newton. In general it would be too optimistic to expect that every $T \in C^1(I)$ whose fixed point set is attracting and satisfies (7) were a Newton map associated with some $f \in C^k(I)$.

Example 9. Let $I =]-1, 1[$ and consider the map

$$T(x) = \begin{cases} \frac{x}{\log|x|} & \text{if } 0 < |x| < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

which has $x^* = 0$ as its only, attracting fixed point and, with $T'(x^*) := 0$, is C^1 on I . Obviously $T'(x^*) \in \Delta_k$ for all $k \in \mathbb{N}_\infty$. Suppose that $N_f = T$ for some $f \in C^k(I)$. Then, with some nonzero constant C ,

$$f(x) = Cx(1 - \log x), \quad \forall x : 0 < x < 1.$$

Evidently, this function cannot be extended to even a *differentiable* function on I . Thus $N_f \neq T$ for every $f \in C^k(I)$. The fact that in this example T is barely C^1 is not important, as it is easy to find similar examples with T showing any *finite* degree of differentiability: For every $l \in \mathbb{N}$ (and $k \in \mathbb{N}_\infty$) there exist maps $T \in C^l(I)$ such that $T'(\text{Fix}[T]) = \{\delta\}$ with $\delta \in \Delta_k$, yet $N_f \neq T$ for all $f \in C^k(I)$.

Example 9 shows that there is no hope to reverse Lemma 6, not even if N_f is assumed to be more regular than C^1 . However, the situation is much clearer for smooth maps, that is, for $l = \infty$. In this case, the converse of Lemma 6 does actually hold, i.e., the stated conditions are also sufficient.

Theorem 10. *Let $k \in \mathbb{N}_\infty$, and suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^k(I)$, if and only if $\text{Fix}[T]$ either is empty or an attracting (possibly one-point) interval, and*

$$T'(\text{Fix}[T]) = \{\delta\}, \quad \text{for some } \delta \in \Delta_k. \quad (10)$$

Moreover, the function f is uniquely determined up to a multiplicative constant if either $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \dots, 1 - k^{-1}\} \setminus \{1\}$ or the set $I \setminus \text{Fix}[T]$ is connected.

Proof. Only the sufficiency of the stated conditions, and the uniqueness assertion have yet to be proved. To this end, three cases will be distinguished; throughout let $g(x) := x - T(x)$.

Case 1: Assume that $\text{Fix}[T] = \emptyset$. Then g is nonvanishing and C^∞ on I , and so is

$$f(x) = \exp\left(\int_\xi^x \frac{dt}{g(t)}\right), \quad \forall x \in I,$$

where ξ is any point in I . Since g is C^∞ and does not vanish on I , the solution f of the first-order ODE $f'/f = 1/g$, or equivalently, $N_f = T$, is unique up to multiplication by a constant.

Case 2: Assume that $x^* \in \text{Fix}[T]$ and $T'(x^*) = \delta$ with $\delta \in \Delta_k \setminus \{1\}$. Clearly this implies that $\text{Fix}[T] = \{x^*\}$, and T can be written as

$$T(x) = x^* + \delta(x - x^*) + (1 - \delta)(x - x^*)^2 h(x),$$

with a uniquely determined $h \in C^\infty$. Note that $(x - x^*)h(x) \neq 1$ for all $x \in I$. Let $\gamma = (1 - \delta)^{-1}$, pick points $x^-, x^+ \in I$ with $x^- < x^* < x^+$, and define $f : I \rightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} c^+(x^+ - x^*)^\gamma \exp\left(-\int_{x^+}^x \frac{dt}{g(t)}\right) & \text{if } x > x^*, \\ 0 & \text{if } x = x^*, \\ c^-(x^* - x^-)^\gamma \exp\left(\int_{x^-}^x \frac{dt}{g(t)}\right) & \text{if } x < x^*; \end{cases} \quad (11)$$

here c^+, c^- are nonzero real constants. Since x^* is the only fixed point of T in I it follows that $f \in C^\infty(I \setminus \{x^*\})$, and $N_f = T$. By using the identity

$$(x - x^*)^\gamma = (x^+ - x^*)^\gamma \exp\left(-\gamma \int_x^{x^+} \frac{dt}{t - x^*}\right), \quad \forall x > x^*, \quad (12)$$

a short computation yields

$$(x - x^*)^{-\gamma} f(x) = c^+ \exp\left(-\gamma \int_x^{x^+} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right), \quad \forall x > x^*.$$

An analogous computation for $x < x^*$ yields

$$(x^* - x)^{-\gamma} f(x) = c^- \exp\left(\gamma \int_{x^-}^x \frac{h(t) dt}{1 - (t - x^*)h(t)}\right), \quad \forall x < x^*.$$

Since the integrand $\frac{h(t)}{1 - (t - x^*)h(t)}$ is C^∞ on I , both one-sided limits for $|x - x^*|^{-\gamma} f(x)$, as x approaches x^* , are finite and nonzero. If $\delta = 1 - j^{-1}$ for some $1 \leq j \leq k$ then, for f to be C^j on I , these two one-sided limits have to be equal or, equivalently,

$$c^- = (-1)^j c^+ \exp\left(-j \int_{x^-}^{x^+} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right)$$

must hold. In the latter case, for all $x \in I$,

$$f(x) = c^+ (x - x^*)^j \exp\left(-j \int_x^{x^+} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right),$$

which shows $f \in C^k(I)$. Since the two-parameter family defined in (11) contains all solutions of $N_f = T$ on $x < x^*$ and $x > x^*$ separately, the solution of $N_f = T$ is unique up to multiplication by a nonzero constant if $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \dots, 1 - k^{-1}\} \setminus \{1\}$.

If, on the other hand, $\delta > 1 - k^{-1}$, and correspondingly $\gamma > k$, then $f \in C^k(I)$ for any choice of the constants c^+, c^- , and $f(x^*) = f'(x^*) = \dots = f^{(k)}(x^*) = 0$.

Case 3: Assume that $T'(\text{Fix}[T]) = \{1\}$. If $\text{Fix}[T] = I$, then trivially T is the Newton map associated with $f \equiv 1$. Without loss of generality, therefore, assume that x^* is the right boundary point of $\text{Fix}[T]$. In this case

$$T(x) = x - (x - x^*)^2 h(x),$$

where $h \in C^\infty(I)$ and $h(x) > 0$ whenever $x > x^*$, and $h(x) = 0$ for all $x \in \text{Fix}[T]$; in particular, therefore, $h(x^*) = 0$. As before, pick $x^+ \in I$ with $x^+ > x^*$ and, analogously to (11), let

$$f^+(x) := \begin{cases} \exp\left(-\int_x^{x^+} \frac{dt}{g(t)}\right) & \text{if } x > x^*, \\ 0 & \text{if } x \leq x^*. \end{cases}$$

Using (12), with γ replaced by j , and recalling that $g(t) = (t - x^*)^2 h(t)$, it follows that $\lim_{x \searrow x^*} (x - x^*)^{-j} f^+(x) = 0$ for all $j \in \mathbb{N}$. Thus $f^+ \in C^\infty(I)$ and $N_{f^+}(x) = T(x)$ whenever $x > x^*$ or $x \in \text{Fix}[T]$. If $\text{Fix}[T]$ has a left boundary point in I as well, then define f^- in a “mirrored” manner and let $f = c^+ f^+ + c^- f^-$ with nonzero constants c^+, c^- . Clearly, $f \in C^\infty(I)$ and $N_f = T$ for any choice of c^+, c^- .

The assertion concerning uniqueness up to multiplication by a constant is now obvious from the three cases detailed above. \square

Corollary 11. *Suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^\infty(I)$, if and only if $\text{Fix}[T]$ is either empty or an attracting (possibly one-point) interval, and*

$$T'(\text{Fix}[T]) = \{1 - j^{-1}\}, \quad \text{for some } j \in \mathbb{N}_\infty. \quad (13)$$

Moreover, f is uniquely determined up to a multiplicative constant unless $j = \infty$ in (13) and the set $I \setminus \text{Fix}[T]$ is not connected.

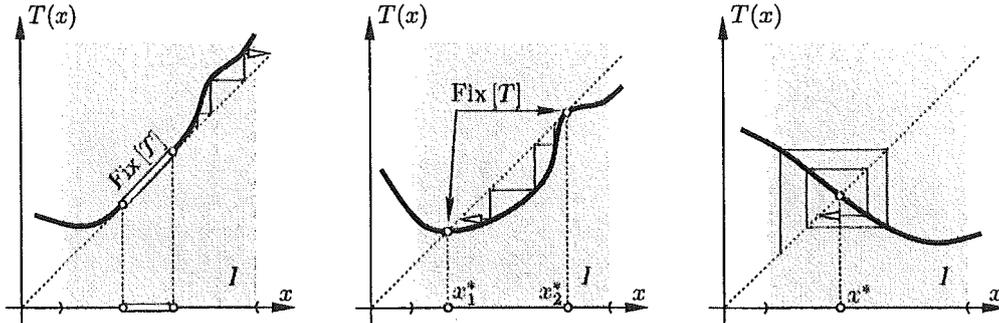


Figure 2: Three C^∞ maps T which are not Newton maps associated with any C^k function on the interval I because $\text{Fix}[T]$ is not attracting (left), $\text{Fix}[T]$ is not an interval (middle), and $T'(x^*) \notin \Delta_k$ for any $k \in \mathbb{N}_\infty$, respectively.

The next corollary requires T to be not only C^∞ but even real-analytic. Recall that a map is *real-analytic* if it can be represented by its Taylor's series in a neighborhood of every point in its domain. Real-analytic Newton maps are especially easy to characterize. Although analyticity is a strong assumption indeed, the class of real-analytic functions is of great historical [7, 11] and practical relevance, as it contains for example all rational and trigonometric functions and compositions thereof [1, 6]. If f is real-analytic then so is N_f , provided the latter map is continuous [1, 2].

Corollary 12. *Let T be real-analytic on I , and $T(x) \neq x$. Then T is a Newton map, associated with a real-analytic function f , if and only if T has at most one fixed point in I , and, in case a fixed point x^* exists, $T'(x^*) = 1 - j^{-1}$ for some $j \in \mathbb{N}$. Moreover, f is unique up to multiplication by a constant.*

Example 13. (i) For $f(x) = \exp(-x)$ and $f_j(x) = x^j$, $j \in \mathbb{N}$, clearly $N_f(x) = x + 1$ and $N_{f_j}(x) = (1 - j^{-1})x$, respectively. Thus all cases contained in Corollary 12 can occur.

(ii) The much-studied logistic map $F_\mu(x) = \mu x(1 - x)$ is a Newton map associated with a real-analytic function on $I =]0, 1[$ if and only if $\mu \in M$, with $M :=]-\infty, 1] \cup \{1 + j^{-1} : j \in \mathbb{N}\}$. Indeed, $F_\mu = N_{f_\mu}$ with functions

$$f_\mu(x) = \left(\frac{x}{\mu x + 1 - \mu} \right)^{(1-\mu)^{-1}} \quad \text{for } \mu \neq 1,$$

and $f_1(x) = \exp(-x^{-1})$. Note that while f_μ is real-analytic on I for all $\mu \in M$, it is only in the trivial case $\mu = 0$ that f_μ could be extended to a real-analytic function such that $N_{f_\mu}(x) = F_\mu(x)$ for all $x \in \mathbb{R}$. Consequently, F_μ is not a Newton map on \mathbb{R} unless $\mu = 0$.

Remark 14. (i) It must be emphasized that Theorem 10 and Corollaries 11, 12 do not force the set $\text{Fix}[T]$ of a Newton map $T \in C^\infty(I)$ to attract *all* points in I . In fact, the map T may at the same time exhibit some *stable* dynamical feature other than a fixed point. For a concrete example consider the (real-analytic) function

$$f(x) = x \frac{3 + x^2}{1 + x^2},$$

for which the associated Newton map

$$N_f(x) = -\frac{4x^3}{3 + x^4}$$

has the stable (in fact, super-attracting) 2-periodic orbit $\{\sqrt{3}, -\sqrt{3}\}$.

(ii) It is well known that if f is a *rational* function (i.e., a quotient of two polynomials) then N_f can be extended uniquely to (and studied appropriately as) a smooth function \overline{N}_f on \mathbb{R} , the one-point compactification of \mathbb{R} . Albeit finite, $\text{Fix}[\overline{N}_f]$ generally contains more than one point [2, 3]. Corollary 12, however, clearly still applies to $\text{Fix}[\overline{N}_f] \cap I$ for every interval I on which f is real-analytic.

The above results about Newton maps have an immediate bearing on the distribution of the floating-point fractions of the iterates $x_n = N_f^n(x_0)$, that is, on the numerical data generated by NM. (See [9] for an account on the relevance of fraction parts distributions for practical computations.) In particular, this distribution depends significantly on the analytic properties of N_f discussed in this note; the interested reader is referred to [6] for details.

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