A CHARACTERIZATION OF NEWTON MAPS

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Abstract

Conditions are given for a \( C^l \) map \( T \) to be a Newton map, that is, the map associated with a differentiable real-valued function via Newton's method. For finitely differentiable maps and functions, these conditions can only be necessary, but in the smooth case, i.e. for \( l = \infty \), they are also sufficient. The characterization rests upon the structure of the fixed point set of \( T \), and it is best possible as is demonstrated through examples.

Key words and phrases: Newton's method, Newton map, discrete dynamical system, fixed point set, attracting fixed point.


1 Introduction

Newton's method (NM) for computing successive approximations of zeros of functions is one of the most widely used methods in all of applied mathematics; variants and generalizations also play a prominent role in numerous other disciplines [2, 3, 8, 10, 11]. Conceptually, NM becomes especially transparent within a dynamical systems context. The purpose of this brief note is to characterize, in the simplest possible setting, the local properties of the dynamical systems thus encountered.

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Throughout, let $f : I \to \mathbb{R}$ be a differentiable function, defined on some open interval $I \subset \mathbb{R}$, and denote by $N_f$ its associated NM transformation, that is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \forall x \in I : f'(x) \neq 0; \quad (1)$$

for $N_f$ to be defined for every $x \in I$, set $N_f(x) := x$ whenever $f'(x) = 0$.

NM for finding roots (zeros) of $f$, i.e., real numbers $x^*$ with $f(x^*) = 0$, amounts to picking an initial point $x_0 \in I$ and iterating $N_f$, thus generating the sequence

$$x_n = N_f(x_{n-1}) = N^n_f(x_0), \quad \forall n \in \mathbb{N};$$

here and throughout, for any map $T : I \to \mathbb{R}$ and any $n \in \mathbb{N}, T^n(x) = T(T^{n-1}(x))$, provided that $T^{n-1}(x) \in I$, and $T^0(x) = x$. Note that $N_f(x) = x$ precisely if $f(x)f'(x) = 0$. Thus for $f(x_n)f'(x_n) = 0$, and only then, does NM terminate at $x_n$: If $f(x_n) = 0$, a root has been found, and otherwise (1) breaks down due to a horizontal tangent to the graph of $f$ at $x_n$ (see Figure 1).

![Figure 1](image)

Figure 1: Visualizing NM: The first few iterates $x_1, x_2, x_3$ are found graphically, both by means of tangents to the graph of $f$ (broken line) and via the graph of $N_f$ (solid line). Note how the point $x_2$ with $f'(x_2) = 0$ causes $N_f$ to have a discontinuity.

Clearly, if $(x_n)$ converges to $x^*$, say, and if $N_f$ is continuous at $x^*$, then $N_f(x^*) = x^*$, i.e., $x^*$ is a fixed point of $N_f$, and $f(x^*) = 0$. (The trivial alternative $f \equiv$ const. is tacitly excluded here, see Lemma 3 below.) It is this correspondence between the roots of $f$ and the fixed points of $N_f$ that suggests that NM be studied as a dynamical system. Under a mild assumption, each (isolated) fixed point $x^*$ is attracting, that is, $\lim_{n \to \infty} N^n_f(x_0) = x^*$ for all $x_0$ sufficiently close to $x^*$. (For $x_0$ further away from any root, the sequence $(x_n)$ may exhibit a considerably more complicated long-term behavior [2, 3, 11].) This aspect of NM is put into perspective by the main result of the present note, Theorem 10 below, which completely characterizes the local dynamical properties of $N_f$.}

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The definition of a Newton map given below entails a relationship between the analytic properties of a function \( f \) and its associated NM transformation \( N_f \), respectively. It is a simple fact, rarely alluded to in studies of NM, that generally these properties are quite independent. On the one hand, the \( C^1 \) function \( f(x) = |x|^{3/2} \), which is not \( C^2 \), has a \( C^\infty \) NM transformation, namely \( N_f(x) = \frac{1}{3} x \). On the other hand, \( N_f \) may lack decent analytic properties even if \( f \) is smooth.

**Example 1.** It is easily seen that the function
\[
  f(x) = \begin{cases} 
    \exp(-x^{-2} + |x| + \cos(x^{-2})) & \text{if } x \neq 0, \\
    0 & \text{if } x = 0,
  \end{cases}
\]
is \( C^\infty \), and both \( f \) and \( f' \) vanish only at \( x^* = 0 \). Nevertheless
\[
  -1 = \liminf_{x \to 0} N_f(x) < \limsup_{x \to 0} N_f(x) = 1,
\]
hence \( N_f \) is not even continuous at \( x^* \).

Since \( N_f \) may fail to be continuous even if \( f \) is \( C^\infty \), in order to ensure the applicability of NM, some explicit assumption on the smoothness of \( N_f \) has to be imposed. To neatly formulate such conditions, let \( N_\infty = \mathbb{N} \cup \{\infty\} \) and stipulate that \( \infty^{-1} := 0 \) as well as \( \infty \pm j = \infty \) and \( j < \infty \) for all \( j \in \mathbb{N} \). In view of (1), for \( N_f \) to be \( C^l \) for some \( l \in N_\infty \), one might demand that \( f \) be at least \( C^{l+1} \), but this assumption has just proved neither necessary nor sufficient. Simply imposing further conditions on \( N_f \) also seems problematic as long as it is not clear whether any such condition can be met for a reasonably large class of functions. Thus it is inevitable to address in some generality the following inverse problem: Given a \( C^1 \) map \( T \), does there exist a function \( f \) such that \( T = N_f \)?

**Definition 2.** Let \( I \subset \mathbb{R} \) be an open interval, and \( l \in N_\infty \). A map \( T \in C^l(I) \) is called a *Newton map* (associated with \( f \)), if \( T = N_f \) for some differentiable function \( f : I \to \mathbb{R} \).

Evidently, not every \( T \in C^l(I) \) is a Newton map, not even if \( l = \infty \), as the trivial example \( T(x) = -x \) shows, for which every \( f \) with \( N_f = T \) lacks differentiability at \( x^* = 0 \).

As will become clear shortly, the question raised above does not have a satisfactory answer for finitely differentiable maps. However, in the smooth case, i.e. for \( l = \infty \), there is a simple characterization of Newton maps, as provided by Theorem 10. Statement and proof of this main result are preceded by a few simple preliminaries. For any map \( T \), denote by \( \text{Fix}(T) \) the set of fixed points of \( T \), that is, \( \text{Fix}(T) := \{x \in I : T(x) = x\} \), and say that \( \text{Fix}(T) \) is *attracting* if \( \lim_{n \to \infty} T^n(x_0) \in \text{Fix}(T) \) for all \( x_0 \) sufficiently close to \( \text{Fix}(T) \).

**Lemma 3.** Let \( f : I \to \mathbb{R} \) be differentiable, and assume that \( N_f \) is continuous. Then \( \text{Fix}(N_f) \) is either empty or a (possibly one-point) interval; in the latter case,
\[
  \limsup_{x \to x^*} \frac{N_f(x) - x^*}{x - x^*} = \delta \quad \text{for some } \delta \in [0, 1]
\]
holds for every \( x^* \in \text{Fix}(N_f) \).

**Proof.** It will first be proved that both sets \( Z_0 := \{x \in I : f(x) = 0\} \) and \( Z_1 := \{x \in I : f'(x) = 0\} \) of zeros of \( f \) and \( f' \), respectively, are (possibly empty or one-point) subintervals of \( I \). Moreover, if \( Z_1 \neq I \), that is, if \( f \) is not constant, then \( Z_1 \subset Z_0 \); in fact, the two
sets coincide unless \( Z_0 \) contains exactly one point, in which case \( Z_1 \) may be empty. Since \( \text{Fix}[N_f] = Z_0 \cup Z_1 \), the first part of the lemma follows immediately from this.

If \( Z_1 = I \) then \( \text{Fix}[N_f] = I \), so let \( Z_1 \neq \emptyset \) be different from \( I \). Pick \( a \in Z_1 \), assume \( f(a) \neq 0 \) and, without loss of generality, that \( b \) := \( \sup \{ x \geq a : f(y) = f(a) \text{ for all } y \in [a,x] \} \) belongs to \( I \). Clearly, \( f(b) = f(a) \) and \( f'(b) = 0 \), hence \( N_f(b) = b \). By the Mean Value Theorem there exists a sequence \( b_n \searrow b \) such that \( 0 < |f'(b_n)| \leq 1 \) for all \( n \). But then

\[
\lim \inf_{n \to \infty} |N_f(b_n) - b| \geq \lim \inf_{n \to \infty} |f(b_n)| = |f(b)| = |f(a)| > 0,
\]
clearly contradicting the continuity of \( N_f \). Therefore \( f(a) = 0 \), hence \( Z_1 \subseteq Z_0 \). Since the set \( Z_0 \) is closed, it contains, with any two points, the whole segment joining these points. Thus \( Z_0 \) is an interval. If \( Z_0 \) is not a singleton then \( Z_0 \subset Z_1 \) and therefore \( Z_0 = Z_1 \). The latter equality also holds if \( Z_0 \) is one-point because \( Z_1 \neq \emptyset \). Finally, if \( Z_1 \) is empty then clearly \( Z_0 \) cannot contain more than one point.

Assertion (2) is trivially true if \( x^* \) is an interior point of \( \text{Fix}[N_f] \). Without loss of generality therefore assume that \( x^* \) is a, say, right boundary point of \( \text{Fix}[N_f] = Z_0 \). Choose \( \delta > 0 \) so small that \( J := [x^*, x^* + \delta] \subseteq I \) and, for \( 0 < t \leq \delta \), let

\[
h(t) := \frac{N_f(x^* + t) - x^*}{t} ; \tag{3}
\]
the function \( h \) is continuous on \([0, \delta]\), and \( h(t) \neq 1 \) for all \( t > 0 \). Since \( x \neq N_f(x) \) for \( x \in J \),

\[
\frac{f'(x)}{f(x)} = \frac{1}{x - N_f(x)}, \quad \forall x \in J,
\]
which after integrating both sides from \( x \) to \( x^* + \delta \), and using the auxiliary function \( h \) defined in (3), can be written as

\[
f(x) = f(x^* + \delta) \exp \left( -\int_{x-x^*}^{\delta} \frac{1}{1-h(t)} \frac{dt}{t} \right), \quad \forall x \in J . \tag{4}
\]
Assume \( f(x^* + \delta) > 0 \) without loss of generality. If \( h(t) > 1 \) for all \( t > 0 \) then (4) implies that \( f(x^*) \neq 0 \), contradicting \( x^* \in Z_0 \). Thus \( h(t) < 1 \) for all \( t > 0 \), and in particular

\[
\lim \sup_{x \to x^*} h(t) = \lim \sup_{x \to x^*} \frac{N_f(x) - x^*}{x - x^*} \leq 1 .
\]
Fix \( j \in \mathbb{N} \). Dividing (4) by \((x - x^*)^j = \delta^j \exp \left( -j \int_{x-x^*}^{\delta} t^{-1} dt \right) \) yields

\[
(x - x^*)^{-j} f(x) = f(x^* + \delta) \delta^{-j} \exp \left( \int_{x-x^*}^{\delta} \frac{j - 1 - jh(t)}{1-h(t)} \frac{dt}{t} \right), \quad \forall x \in J . \tag{5}
\]
To bound \( \lim \sup_{x \to x^*} h(t) \) from below, pick \( \varepsilon > 0 \) and assume that \( h(t) < -\varepsilon \) for all sufficiently small \( t > 0 \). In this case, (5) with \( j = 1 \) shows that

\[
(x - x^*)^{-1} f(x) \geq f(x^* + \delta) \delta^{-1} \frac{\varepsilon}{1+\varepsilon} (x - x^*) \frac{\varepsilon}{1+\varepsilon} \to \infty \text{, as } x \searrow x^* ,
\]
which clearly contradicts the differentiability of \( f \) at \( x^* \). Since \( \varepsilon > 0 \) was arbitrary, \( \lim \sup_{x \to x^*} h(t) \geq 0 \), and the proof is complete. \( \square \)

**Remark 4.** (i) Lemma 3 should be contrasted with the simple fact that for every closed set \( A \subseteq \mathbb{R} \) there exists a \( C^\infty \) map \( T \) with \( T(I) \subset I \) and \( \text{Fix}[T] = A \cap I \).

(ii) Under the conditions of Lemma 3 there is no analogue to (2) for the corresponding \( \text{liminf} \) which, as simple examples show, can be any number between, and including, the trivial bounds \(-\infty \) and \( \delta \).
As pointed out earlier, the applicability of NM rests on the correspondence between the roots of f and the fixed points of NJ and the attractiveness of the latter. Mere continuity of NJ does not guarantee that Fix[NJ] is attracting.

**Example 5.** Consider the $C^1$ function

$$f(x) = \begin{cases} |x|^{\frac{1}{3}} \exp(-\int_0^{|x|^{-1}} t^{-1} \sin t \, dt) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for which the associated NM transformation

$$N_f(x) = \begin{cases} \frac{1 + 2 \sin(|x|^{-1})}{3 + 2 \sin(|x|^{-1})} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous yet obviously not $C^1$. The only fixed point of $N_f$, and correspondingly the only root of $f$ and $f'$, is $x^* = 0$. Since, for every $j \in \mathbb{N}$, the points $\pm \frac{2}{3}(4j - 1)^{-1}$ are 2-periodic, Fix[$N_f$] = {0} is not attracting.

Thus while Fix[$N_f$] is topologically simple whenever $N_f$ is continuous, to make NM practical for approximating zeros, more smoothness is required. Only the case of $N_f$ being at least $C^1$ will therefore be considered from now on. (For the same reason, the legitimate case $l = 0$ has been excluded from Definition 2.) Also, the properties of $N_f$, albeit not completely determined by the smoothness of $f$, do depend on the latter. To describe this dependence, for every $k \in \mathbb{N}_\infty$, define the set

$$A_k := \{0, \frac{1}{2}, \frac{2}{3}, \ldots, 1 - k^{-1}\} \cup [1 - k^{-1}, 1],$$

and note that $[0, 1] = \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_\infty = \{1 - j^{-1} : j \in \mathbb{N}_\infty\}$.

**Lemma 6.** Let $f : I \to \mathbb{R}$ be differentiable, and assume that $N_f \in C^1(I)$. Then Fix[$N_f$] is either empty or an attracting (possibly one-point) interval. Moreover, if Fix[$N_f$] $\neq \emptyset$ and $f \in C^k(I)$ with $k \in \mathbb{N}_\infty$ then

$$N_f(x) \in \delta \text{ for some } \delta \in A_k.$$  

**Proof.** The assertions are trivially true if $f$ is constant or Fix[$N_f$] = $\emptyset$. Therefore assume that $f$ is not constant and Fix[$N_f$] is not empty, hence a subinterval of $I$, by Lemma 3. If $x^*$ is an interior point of Fix[$N_f$] then $N_f^\prime = 1$ in a neighborhood of $x^*$, and the assertion is again true. Thus assume without loss of generality that $x^*$ is a right boundary point of Fix[$N_f$]. According to Lemma 3, $N_f^\prime(x^*) \in \Delta_1$, so $x^*$ obviously is attracting from the right, unless perhaps for $N_f(x^*) = 1$. In the latter case, with the notations introduced in the proof of Lemma 3, the function $h$ according to (3), supplemented by $h(0) := N_f^\prime(x^*) = 1$ is continuous on $[0, \delta]$ and can be written as $h(t) = 1 - H(t)$ where $H$ is also continuous on $[0, \delta]$, and $H(t) \neq 0$ unless $t = 0$. With this, (4) takes the form

$$f(x) = f(x^* + \delta) \exp \left( -\int_{x^*}^{x^* + \delta} \frac{dt}{tH(t)} \right), \quad \forall \ x \in J.$$  

Since $f(x^*) = 0$ and $f(x^* + \delta) \neq 0$, the integral $\int_0^\delta \frac{dt}{tH(t)}$ must diverge to $+\infty$. As $H$ is continuous and, except possibly at $t = 0$, does not change sign, $H(t) > 0$ and so $h(t) < 1$ whenever $0 < t \leq \delta$. From $N_f(x^* + t) - x^* = th(t) < t$ and $h(0) = 0$ it follows that
If $x^* < N_f(x_0) < x_0$ and therefore $N_f^n(x_0) \setminus x^*$ provided that $x_0 \in J$. In other words, $x^*$ is attracting from the right.

It remains to verify (7) for $f \in C^k(I)$. To this end, assume first that $k < \infty$ and $f(x^*) = f'(x^*) = \ldots = f^{(k)}(x^*) = 0$. In this case, since $f$ is $C^k$, the left-hand side in (5) with $j = k$ tends to a finite limit as $x \searrow x^*$. Consequently,

$$\lim_{x \searrow 0} \int_x^t \frac{k - 1 - kh(t)}{1 - h(t)} \frac{dt}{t} < +\infty.$$  

If $h(0) < 1 - k^{-1}$, then the integrand in (8) would eventually be positive near $t = 0$, which clearly is impossible. Therefore $h(0) \geq 1 - k^{-1}$. Since $h(0) \leq 1$ by the same argument,

$$N_f^n(x^*) = h(0) \in [1 - k^{-1}, 1] \subset \Delta_k.$$  

If $k = \infty$ and $f^{(j)}(x^*) = 0$ for all $j \in \mathbb{N}$ then similar reasoning shows that $N_f^n(x^*) \in \bigcap_{j \in \mathbb{N}} [1 - j^{-1}, 1] = \{1\} \subset \Delta_\infty$.

Finally assume that $f(x^*) = f'(x^*) = \ldots = f^{(j)}(x^*) = 0$ yet $f^{(j+1)}(x^*) \neq 0$ for some $j$ with $0 \leq j < k$. The same argument as before with $k$ replaced by $j$ shows that $N_f^n(x^*) \in [1 - (j + 1)^{-1}, 1]$. If $h(0) > 1 - (j + 1)^{-1}$ then (5) with $j$ replaced by $j + 1$ would imply that $\lim_{x \searrow x^*} (x - x^*)^{-(j+1)} f(x) = 0$, which contradicts $f^{(j+1)}(x^*) \neq 0$. Thus $N_f^n(x^*) = h(0) = 1 - (j + 1)^{-1} \in \Delta_\infty \subset \Delta_k$.  

**Remark 7.** Lemma 6 is best possible in the following sense: For every $k \in \mathbb{N}_\infty$ and $\delta \in \Delta_k$ there exists a $C^k$ function $f$ with $f \in C^1$ having a single fixed point $x^*$ such that $N_f^n(x^*) = \delta$. For $k \in \mathbb{N}$ and $\delta \in \Delta_k \setminus \{1\}$ let $\gamma = (1 - \delta)^{-1}$ and consider the function

$$f(x) = \begin{cases} x^\gamma \left(1 + \frac{1}{2k+4} x^{(1+\gamma)(1+k)} \sin(x^{-\gamma})\right) & \text{if } 0 < |x| < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

where, for non-integer $\gamma$, each argument $x$ has to be replaced by $|x|$. Taking $I = ]-1, 1[$, it is readily checked that $f \in C^k(I)$ and $N_f \in C^1(I)$. Moreover, $x^* = 0$ is the only fixed point of $N_f$ in $I$, and $N_f^n(x^*) = 1 - \gamma^{-1} = \delta$. For $\delta = 1$, an example is provided by the $C^k$ function $f(x) = \exp(-|x|^{-\gamma}) + \frac{1}{2} \exp(-(k + 4)|x|^{-\gamma}) \sin(\exp(|x|^{-\gamma}))$ for which $N_f \in C^1$, has $x^* = 0$ as its only fixed point, and $N_f^n(x^*) = 1$. Simple examples in the case $k = \infty$ are $f(x) = x^\gamma$ for $\delta < 1$, and $f(x) = \exp(-|x|^{-\gamma})$ for $\delta = 1$, respectively.

An important special case for which Lemma 6 can be strengthened is the case of a root of finite multiplicity. Recall that $x^* \in I$ is a root of $f \in C^k(I)$ of multiplicity $j \in \mathbb{N}$ if $f(x) = (x - x^*)^j g(x)$ for all $x \in I$, where $g \in C^k(I)$ and $g(x^*) \neq 0$.

**Corollary 8.** Let $x^*$ be a root of $f \in C^k(I)$ of finite multiplicity $j$. Then, for some open interval $J \subset I$ containing $x^*$, $N_f \in C^{k-1}(J)$, and $N_f^n(x^*) = 1 - j^{-1}$; in particular, Fix $[N_f] \cap J = \{x^*\}$ is attracting.

**Proof.** Since $f(x) = (x - x^*)^j g(x)$ for some $g \in C^k$ with $g(x^*) \neq 0$,

$$N_f(x) - x^* = (x - x^*) \frac{(j - 1)g(x) + (x - x^*)g'(x)}{jg(x) + (x - x^*)g'(x)} = (x - x^*) h(x),$$

where $h$ is $C^{k-1}$ on some open interval $J \subset I$ containing $x^*$, and $N_f^n(x^*) = h(x^*) = 1 - j^{-1}$. Thus, for $J$ chosen sufficiently small, Fix $[N_f] \cap J = \{x^*\}$, and the fixed point $x^*$ clearly is attracting.  

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Lemma 6 contains necessary conditions for a map to be Newton. In general it would be too optimistic to expect that every $T \in C^1(I)$ whose fixed point set is attracting and satisfies (7) were a Newton map associated with some $f \in C^k(I)$.

Example 9. Let $I = ]-1, 1[$ and consider the map

$$T(x) = \begin{cases} \frac{x}{\log |x|} & \text{if } 0 < |x| < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

which has $x^* = 0$ as its only, attracting fixed point and, with $T'(x^*) := 0$, is $C^1$ on $I$. Obviously $T'(x^*) \in \Delta_k$ for all $k \in \mathbb{N}_\infty$. Suppose that $N_f = T$ for some $f \in C^k(I)$. Then, with some nonzero constant $C$,

$$f(x) = Cx(1 - \log x), \quad \forall x : 0 < x < 1.$$  

Evidently, this function cannot be extended to even a differentiable function on $I$. Thus $N_f \neq T$ for every $f \in C^k(I)$. The fact that in this example $T$ is barely $C^1$ is not important, as it is easy to find similar examples with $T$ showing any finite degree of differentiability: For every $l \in \mathbb{N}$ (and $k \in \mathbb{N}_\infty$) there exist maps $T \in C^l(I)$ such that $T'(\text{Fix}[T]) = \{\delta\}$ with $\delta \in \Delta_k$, yet $N_f \neq T$ for all $f \in C^k(I)$.

Example 9 shows that there is no hope to reverse Lemma 6, not even if $N_f$ is assumed to be more regular than $C^1$. However, the situation is much clearer for smooth maps, that is, for $l = \infty$. In this case, the converse of Lemma 6 does actually hold, i.e., the stated conditions are also sufficient.

Theorem 10. Let $k \in \mathbb{N}_\infty$, and suppose $T \in C^\infty(I)$. Then $T$ is a Newton map, associated with $f \in C^k(I)$, if and only if $\text{Fix}[T]$ either is empty or an attracting (possibly one-point) interval, and

$$T'(\text{Fix}[T]) = \{\delta\}, \quad \text{for some } \delta \in \Delta_k. \quad (10)$$

Moreover, the function $f$ is uniquely determined up to a multiplicative constant if either $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\} \setminus \{1\}$ or the set $I \setminus \text{Fix}[T]$ is connected.

Proof. Only the sufficiency of the stated conditions, and the uniqueness assertion have yet to be proved. To this end, three cases will be distinguished; throughout let $g(x) := x - T(x)$.

Case 1: Assume that $\text{Fix}[T] = \emptyset$. Then $g$ is nonvanishing and $C^\infty$ on $I$, and so is

$$f(x) = \exp \left( \int_{\xi}^{x} \frac{dt}{g(t)} \right), \quad \forall x \in I,$$

where $\xi$ is any point in $I$. Since $g$ is $C^\infty$ and does not vanish on $I$, the solution $f$ of the first-order ODE $f'/f = 1/g$, or equivalently, $N_f = T$, is unique up to multiplication by a constant.

Case 2: Assume that $x^* \in \text{Fix}[T]$ and $T'(x^*) = \delta$ with $\delta \in \Delta_k \setminus \{1\}$. Clearly this implies that $\text{Fix}[T] = \{x^*\}$, and $T$ can be written as

$$T(x) = x^* + \delta(x - x^*) + (1 - \delta)(x - x^*)^2h(x),$$

with a uniquely determined $h \in C^\infty$. Note that $(x - x^*)h(x) \neq 1$ for all $x \in I$. Let $\gamma = (1 - \delta)^{-1}$, pick points $x^-, x^+ \in I$ with $x^- < x^* < x^+$, and define $f : I \to \mathbb{R}$ by

$$f(x) := \begin{cases} c^+(x^+ - x^*)^\gamma \exp \left( -j_{x^+}^{x} \frac{dt}{g(t)} \right) & \text{if } x > x^+, \\ 0 & \text{if } x = x^*, \\ c^-(x^* - x^-)^\gamma \exp \left( j_{x^*}^{x^-} \frac{dt}{g(t)} \right) & \text{if } x < x^*, \end{cases} \quad (11)$$
here $c^+, c^-$ are nonzero real constants. Since $x^*$ is the only fixed point of $T$ in $I$ it follows that $f \in C^\infty(I \setminus \{x^*\})$, and $N_f = T$. By using the identity
\[
(x - x^*)^{-\gamma} = \left(x^+ - x^*\right)^{-\gamma} \exp\left(-\gamma \int_{x^+}^{x} \frac{dt}{t - x^*}\right), \quad \forall x > x^*,
\]
a short computation yields
\[
(x - x^*)^{-\gamma}f(x) = c^+ \exp\left(-\gamma \int_{x}^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right), \quad \forall x > x^*.
\]
An analogous computation for $x < x^*$ yields
\[
(x^* - x)^{-\gamma}f(x) = c^- \exp\left(\gamma \int_{x^-}^{x} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right), \quad \forall x < x^*.
\]
Since the integrand $\frac{h(t)}{1 - (t - x^*)h(t)}$ is $C^\infty$ on $I$, both one-sided limits for $|x - x^*|^{-\gamma}f(x)$, as $x$ approaches $x^*$, are finite and nonzero. If $\delta = 1 - j^{-1}$ for some $1 \leq j \leq k$ then, for $f$ to be $C^j$ on $I$, these two one-sided limits have to be equal or, equivalently,
\[
c^- = (-1)^j c^+ \exp\left(-j \int_{x}^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right)
\]
must hold. In the latter case, for all $x \in I$,
\[
f(x) = c^+(x - x^*)^j \exp\left(-j \int_{x}^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right),
\]
which shows $f \in C^k(I)$. Since the two-parameter family defined in (11) contains all solutions of $N_f = T$ on $x < x^*$ and $x > x^*$ separately, the solution of $N_f = T$ is unique up to multiplication by a nonzero constant if $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\}\{1\}$.

If, on the other hand, $\delta > 1 - k^{-1}$, and correspondingly $\gamma > k$, then $f \in C^k(I)$ for any choice of the constants $c^+, c^-$, and $f(x^*) = f'(x^*) = \ldots = f^{(k)}(x^*) = 0$.

Case 3: Assume that $T'(\text{Fix} [T]) = \{1\}$. If $\text{Fix} [T] = I$, then trivially $T$ is the Newton map associated with $f \equiv 1$. Without loss of generality, therefore, assume that $x^*$ is the right boundary point of $\text{Fix} [T]$. In this case
\[
T(x) = x - (x - x^*)^2 h(x),
\]
where $h \in C^\infty(I)$ and $h(x) > 0$ whenever $x > x^*$, and $h(x) = 0$ for all $x \in \text{Fix} [T]$; in particular, therefore, $h(x^*) = 0$. As before, pick $x^+ \in I$ with $x^+ > x^*$ and, analogously to (11), let
\[
f^+(x) := \begin{cases} 
\exp\left(-\int_{x}^{x^+} \frac{dt}{g(t)}\right) & \text{if } x > x^*, \\
0 & \text{if } x \leq x^*.
\end{cases}
\]
Using (12), with $\gamma$ replaced by $j$, and recalling that $g(t) = (t - x^*)^2 h(t)$, it follows that $\lim_{x \to x^*}(x - x^*)^{-j} f^+(x) = 0$ for all $j \in \mathbb{N}$. Thus $f^+ \in C^\infty(I)$ and $N_{f^+}(x) = T(x)$ whenever $x > x^*$ or $x \in \text{Fix} [T]$. If $\text{Fix} [T]$ has a left boundary point in $I$ as well, then define $f^-$ in a "mirrored" manner and let $f = c^+ f^+ + c^- f^-$ with nonzero constants $c^+, c^-$. Clearly, $f \in C^\infty(I)$ and $N_f = T$ for any choice of $c^+, c^-$. The assertion concerning uniqueness up to multiplication by a constant is now obvious from the three cases detailed above. \qed
Corollary 11. Suppose $T \in C^\infty(I)$. Then $T$ is a Newton map, associated with $f \in C^\infty(I)$, if and only if $\text{Fix}[T]$ is either empty or an attracting (possibly one-point) interval, and

$$T'(\text{Fix}[T]) = \{1 - j^{-1}\}, \quad \text{for some } j \in \mathbb{N}_\infty. \quad (13)$$

Moreover, $f$ is uniquely determined up to a multiplicative constant unless $j = \infty$ in (13) and the set $I \setminus \text{Fix}[T]$ is not connected.

The next corollary requires $T$ to be not only $C^\infty$ but even real-analytic. Recall that a map is real-analytic if it can be represented by its Taylor’s series in a neighborhood of every point in its domain. Real-analytic Newton maps are especially easy to characterize. Although analyticity is a strong assumption indeed, the class of real-analytic functions is of great historical [7, 11] and practical relevance, as it contains for example all rational and trigonometric functions and compositions thereof [1, 6]. If $f$ is real-analytic then so is $N_f$, provided the latter map is continuous [1, 2].

Corollary 12. Let $T$ be real-analytic on $I$, and $T(x) \neq x$. Then $T$ is a Newton map, associated with a real-analytic function $f$, if and only if $T$ has at most one fixed point in $I$, and, in case a fixed point $x^*$ exists, $T'(x^*) = 1 - j^{-1}$ for some $j \in \mathbb{N}$. Moreover, $f$ is unique up to multiplication by a constant.

Example 13. (i) For $f(x) = \exp(-x)$ and $f_j(x) = x^j$, $j \in \mathbb{N}$, clearly $N_f(x) = x + 1$ and $N_{f_j}(x) = (1 - j^{-1})x$, respectively. Thus all cases contained in Corollary 12 can occur.

(ii) The much-studied logistic map $F_\mu(x) = \mu x(1-x)$ is a Newton map associated with a real-analytic function on $I = [0, 1]$ if and only if $\mu \in M$, with $M := [\infty, 1] \cup \{1 + j^{-1} : j \in \mathbb{N}\}$. Indeed, $F_\mu = N_{f_\mu}$ with functions

$$f_\mu(x) = \left(\frac{x}{\mu x + 1 - \mu}\right)^{(1-\mu)^{-1}} \quad \text{for } \mu \neq 1,$$

and $f_1(x) = \exp(-x^{-1})$. Note that while $f_\mu$ is real-analytic on $I$ for all $\mu \in M$, it is only in the trivial case $\mu = 0$ that $f_\mu$ could be extended to a real-analytic function such that $N_{f_\mu}(x) = F_\mu(x)$ for all $x \in \mathbb{R}$. Consequently, $F_\mu$ is not a Newton map on $\mathbb{R}$ unless $\mu = 0$. 
Remark 14. (i) It must be emphasized that Theorem 10 and Corollaries 11, 12 do not force the set $\text{Fix}[T]$ of a Newton map $T \in C^\infty(I)$ to attract all points in $I$. In fact, the map $T$ may at the same time exhibit some stable dynamical feature other than a fixed point. For a concrete example consider the (real-analytic) function

$$f(x) = \frac{3 + x^2}{1 + x^2},$$

for which the associated Newton map

$$N_f(x) = \frac{-4x^3}{3 + x^4}$$

has the stable (in fact, super-attracting) 2-periodic orbit $\{\sqrt{3}, -\sqrt{3}\}$.

(ii) It is well known that if $f$ is a rational function (i.e., a quotient of two polynomials) then $N_f$ can be extended uniquely to (and studied appropriately as) a smooth function $\tilde{N}_f$ on $\mathbb{R}$, the one-point compactification of $\mathbb{R}$. Albeit finite, $\text{Fix}[\tilde{N}_f]$ generally contains more than one point [2, 3]. Corollary 12, however, clearly still applies to $\text{Fix}[\tilde{N}_f] \cap I$ for every interval $I$ on which $f$ is real-analytic.

The above results about Newton maps have an immediate bearing on the distribution of the floating-point fractions of the iterates $x_n = N_f^n(x_0)$, that is, on the numerical data generated by NM. (See [9] for an account on the relevance of fraction parts distributions for practical computations.) In particular, this distribution depends significantly on the analytic properties of $N_f$ discussed in this note; the interested reader is referred to [6] for details.

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