Multiquadric B-splines

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Abstract – Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular combinations of multiquadrics, called ψ–splines, will be defined that are analogues of polynomial B-splines. The paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ–splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines.

There are also results concerning the relationship between certain semi-infinite and bi-infinite combinations of ψ–splines. These results enable us to obtain error estimates for quasi-interpolation schemes involving generalized multiquadrics based on a finite number of centers.

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1 Introduction

There has recently been a great deal of interest in approximation by radial basis functions. That is by sums of translates of a single radially symmetric function. From this point of view univariate polynomial splines of odd degree are formed from polynomials plus sums of translates of the modulus raised to a fixed odd power. The generalized multiquadrics can then be viewed as obtained by smoothing out the derivative discontinuity of the function $|\cdot|^{2k-1}$. More precisely, let $k \in \mathbb{N}$ and $c > 0$. Then the basic generalized multiquadric of order $2k$ is defined by

$$\phi(x;2k) = \left(x^2+c^2\right)^{(2k-1)/2}, \tag{1.1}$$

and the ϕ function centered at t_j by

$$\phi_{j,2k}(x) = \phi(x-t_j;2k). \tag{1.2}$$

A multiquadric spline based on a finite number of centers is then a linear combination of appropriate $\phi_{j,2k}$'s supplemented by a polynomial of degree $2k-1$.

Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular combinations of multiquadrics, called ψ–splines, will be defined that are analogues of polynomial B–splines. The paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ-splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines. Furthermore, it is shown that if a polynomial is expressed as a bi-infinite series of ψ-splines then corresponding semi-infinite series sum to half the polynomial plus a few generalized multiquadrics. This result allows us to obtain error estimates for quasi-interpolation by generalized multiquadrics based on a finite number of centers, from the results for quasi-interpolation by ψ-splines on non-uniform bi-infinite meshes.

2 Preliminaries

In this section we define the ψ–splines and obtain some identities involving them.

Consider a mesh $t = \ldots < t_{j-1} < t_j < t_{j+1} < \ldots$, with $t_{\pm j} \to \pm\infty$ as $j \to \pm\infty$. Define the ψ–spline $\psi_{j,2k}$ $(= \psi_{j,2k,t})$ as the weighted divided difference

$$\psi_{j,2k}(x) = \frac{t_{j+2k} - t_j}{2}[x - t_j, x - t_{j+1}, \ldots, x - t_{j+2k}]\phi(x; 2k). \tag{2.1}$$

Hence making use of the definition of a divided difference as the leading coefficient in the power basis representation of an interpolating polynomial

$$\psi_{j,2k}(x) = \frac{t_{j+2k} - t_j}{2}[t_j, t_{j+1}, \ldots, t_{j+2k}]_u \phi(x - u; 2k), \tag{2.2}$$

where the subscript u, which will often be omitted, indicates that the divided difference is taken with respect to the u variable. This combination of $\phi_{j,2k}, \ldots, \phi_{j+2k,2k}$ will turn out to have some critical properties in common with the B-spline $N_{j,2k,t}$. (As is usual $N_{j,2k,t}$ denotes the B-spline of order $2k$ supported on $[t_j, t_{j+2k}]$ and normalized so that the sum of all the B-splines of a fixed order is 1.)

Before proceeding we need to derive some properties of the functions $\phi(x; 2k)$ and their derivatives and integrals. Firstly

$$D^{2k}\phi(x; 2k) = [(2k-1)!!]^2 \frac{c^{2k}}{(x^2 + c^2)^{(2k+1)/2}}, \qquad k \in \mathbb{N}, \tag{2.3}$$

where as usual

$$m!! = \prod_{\{j : j = m (\mathrm{mod}\ 2) \text{ and } 0 < j \leq m\}} j\ ,$$

Equation (2.3) can be shown by induction, the induction step following from applying Leibnitz's rule to

$$D^{2k+2}\{(x^2 + c^2)(x^2 + c^2)^{(2k-1)/2}\}.$$

Proceeding from (2.3) a similar induction argument shows that

$$D^{2k-1}\phi(x;2k) = \frac{p(x;2k)}{(x^2+c^2)^{(2k-1)/2}}, \qquad k \in \mathbf{N} \tag{2.4}$$

where $p(x;2k)$ is an odd polynomial in x defined by the recurrence

$$p(x;2k+2) = \begin{cases} x, & k=0, \\ (2k+1)[(2k-1)!!]^2 c^{2k} x & \\ \quad + (2k+1)2k(x^2+c^2)p(x;2k), & k \in \mathbf{N}. \end{cases} \tag{2.5}$$

It follows immediately from this recurrence that the power expansion of $p(x;2k)$ has positive coefficients and that

$$p(x;2k) = (2k-1)!x^{2k-1} + \sum_{j=1}^{k-1} a_{j,2k} c^{2k-2j} x^{2j-1}, \tag{2.6}$$

for some constants $a_{j,2k}$ not depending on c. Hence

$$D^{2k-1}\phi(x;2k) = \pm(2k-1)! + \mathcal{O}(|x|^{-2}), \qquad \text{as } x \to \pm\infty. \tag{2.7}$$

Defining

$$A(k) = \frac{(2k-1)!!}{(2k)!!} = \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{(2k-1)}{2k} \tag{2.8}$$

we have that

$$\int_{-\infty}^{\infty} \left(x^2+c^2\right)^{-(2k+1)/2} dx = \frac{1}{kA(k)c^{2k}}, \qquad k \in \mathbf{N}. \tag{2.9}$$

This can be proven by induction, the induction step following from the easily verified identity

$$\begin{aligned} \int \left(x^2+c^2\right)^{-(2k+1)/2} dx &= \frac{x}{(2k-1)c^2}\left(x^2+c^2\right)^{-(2k-1)/2} \\ &\quad + \frac{2(k-1)}{(2k-1)c^2}\int \left(x^2+c^2\right)^{-(2k-1)/2} dx. \end{aligned} \tag{2.10}$$

Also trivially

$$\int x\left(x^2+c^2\right)^{-(2k+1)/2} dx = -\frac{1}{2k-1}\left(x^2+c^2\right)^{-(2k-1)/2} + M, \tag{2.11}$$

so that

$$\int_{-\infty}^{\infty} |x| \left(x^2 + c^2\right)^{-(2k+1)/2} dx = \frac{2}{(2k-1)c^{2k-1}} \,. \tag{2.12}$$

Recall now that a multiple of the polynomial B-spline is the Peano kernel of the divided difference, so that in particular

$$[t_j, \ldots, t_{j+2k}]g = \frac{2k}{(t_{j+2k} - t_j)} \frac{1}{(2k)!} \int_{-\infty}^{\infty} N_{j,2k,t}(u) g^{(2k)}(u) du. \tag{2.13}$$

Using this with $g(u) = \phi(x-u; 2k)$ we find that

$$\psi_{j,2k}(x) = \frac{k}{(2k)!} \int_{-\infty}^{\infty} N_{j,2k,t}(u) \phi^{(2k)}(x-u) du. \tag{2.14}$$

Applying (2.3)

$$\psi_{j,2k}(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} N_{j,2k,t}(u) \left((x-u)^2 + c^2\right)^{-(2k+1)/2} du. \tag{2.15}$$

It follows from this and the formula

$$\int_{-\infty}^{\infty} N_{j,\ell,t}(x)\, dx = \frac{t_{j+\ell} - t_j}{\ell} \,, \tag{2.16}$$

for the integral of a B-spline that $\psi_{j,2k}$ is nonnegative and decays like $[t_{j+2k} - t_j]d(x, [t_j, t_{j+2k}])^{-(2k+1)}$ as $x \to \pm\infty$, where $d(.,.)$ denotes the usual distance for $\mathbf{R}$. Also

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \psi_{j,2k}(x) &= kA(k)c^{2k} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} N_{j,2k,t}(u) \left((x-u)^2 + c^2\right)^{-(2k+1)/2} du, \\
&= kA(k)c^{2k} \int_{-\infty}^{\infty} \left((x-u)^2 + c^2\right)^{-(2k+1)/2} du, \\
&= 1,
\end{aligned} \tag{2.17}$$

where in the last step we have used (2.9). Let

$$S_{2k}(u) = kA(k)c^{2k}(u^2 + c^2)^{-(2k+1)/2}, \qquad k \in N.$$

Jones [6, p.178] gives the formula

$$\int_{-\infty}^{\infty} e^{-itx}(1+x^2)^{-(k+\frac{1}{2})} dx = \frac{\pi^{1/2}|t|^k K_k(|t|)}{(k-\frac{1}{2})! 2^{k-1}},$$

where K_k is a modified Bessel function. Hence the Fourier transform of S_{2k} is

$$\widehat{S_{2k}}(t) = kA(k)c^{2k}\frac{2\pi^{1/2}}{(k-\frac{1}{2})!}\Big|\frac{t}{2c}\Big|^k K_k(|ct|)\,. \tag{2.18}$$

Now from Abromowitz and Stegun [1, p.375]

$$\begin{aligned} K_k(z) &= \frac{1}{2}(\frac{1}{2}z)^{-k}\sum_{j=0}^{k-1}\frac{(k-j-1)!}{j!}(-\frac{1}{4}z^2)^j \\ &\quad + (-)^{k+1}ln(\frac{1}{2}z)I_k(z) \\ &\quad + (-)^k\frac{1}{2}(\frac{1}{2}z)^k\sum_{j=0}^{\infty}(\psi_{(j+1)}+\psi_{(k+j+1)})\frac{(\frac{1}{4}z^2)^j}{j!(k+j)!} \end{aligned}$$

where

$$I_k(z) = \frac{(\frac{1}{2}z)^k}{\Gamma(k+1)} + O(z^{k+2}), \qquad \text{as } z \to 0.$$

Hence

$$\begin{aligned} \widehat{S_{2k}}(t) &= \sum_{j=0}^{k-1}\frac{(k-j-1)!}{(k-1)!j!}(\frac{-c^2}{4})^4 t^{2j} \\ &\quad + kA(k)c^{2k}\Big\{(-)^{k+1}\frac{2\pi^{1/2}}{(k-\frac{1}{2})!}(\frac{t}{2})^{2k}ln\Big(\frac{c|t|}{2}\Big)(1+O(c^2t^2)) \\ &\qquad + O(t^{2k})\Big\} \quad \text{as } t \to 0. \end{aligned} \tag{2.19}$$

Since $K_k(z)$ is infinitely differentiable at $z \neq 0$ it follows from (2.18) that $\widehat{S_{2k}} \in C^{2k-1}(\mathbf{R})$. Since $K_k(z)$ is positive for $z > 0$ and $k > -1$, $\widehat{S_{2k}}(t)$ is positive for $t \in R$. Also from (2.19)

$$\widehat{S_{2k}}^{(2j)}(0) = \frac{(k-j-1)!}{(k-1)!j!}\Big(\frac{-c^2}{4}\Big)^j(2j)!\,,\ 0 \le j \le k-1.$$

Hence from the formula $(\widehat{x^r f})(t) = \Big(\frac{id}{dt}\Big)^r \hat{f}(0)$

$$\int_{\mathbf{R}} x^{2j}S_{2k}(x)dx = \frac{(2j)!(k-j-1)!}{(k-1)!j!}\Big(\frac{c^2}{4}\Big)^j \tag{2.20}$$

for $0 \leq j \leq k-1$. Applying Cauchy-Schwarz we have

$$\begin{aligned} \int_{\mathbf{R}} |x|^{2j-1} S_{2k}(x)dx &\leq \left[\int_{\mathbf{R}} x^{2j} S_{2k}(x)dx \int_{\mathbf{R}} x^{2j-2} S_{2k}(x)dx\right]^{1/2} \\ &= \mathcal{O}(c^{2j-1}) \ , \ 0 \leq j \leq k-1. \end{aligned}$$

Also note that in the particular case $j = 1$ (2.20) gives

$$\int_{\mathbf{R}} x^2 S_{2k}(x)dx = \frac{c^2}{2(k-1)}, \qquad k \geq 2. \tag{2.21}$$

3 Basic properties of ψ–splines

In this section we derive some fundamental properties of the ψ–splines. These include global linear independence, polynomial reproduction properties, and expressions for ϕ and ψ splines as convolutions of a kernel with a power of the modulus and polynomial B–splines respectively.

It will be convenient to have the following notation. Given an infinite mesh $\boldsymbol{t} : \ldots < t_{j-1} < t_j < t_{j+1} < \ldots$ we define a coefficient sequence $\boldsymbol{d} = \{d_j\}_{j=-\infty}^{\infty}$ to be in the growth class $C(2k, \boldsymbol{t})$ if $d_j = \mathcal{O}(|t_j|^{2k-1})$ as $j \to \pm\infty$. We note that for meshes $\boldsymbol{t}$ of finite mesh size the condition is equivalent to the condition $\sum_{j=-\infty}^{\infty} d_j N_{j,2k}(x) = \mathcal{O}(|x|^{2k-1})$ as $x \to \pm\infty$. (See the proof of Lemma 3.)

We remind the reader of the following well known result.

Lemma 1 *Local linear independence of the B–spline basis.*
Let $k \in \mathbf{N}$ and consider an infinite mesh $\boldsymbol{t} : \ldots < t_{j-1} < t_j < t_{j+1} < \ldots$. Let $s = \sum_{j=-\infty}^{\infty} a_j N_{j,k}$ and $r = \sum_{j=-\infty}^{\infty} b_j N_{j,k}$. Then $s(x) = r(x)$ for all $x \in (t_j, t_{j+1})$ if and only if $a_\ell = b_\ell$ for all $j - k < \ell \leq j$.

Proof: Let p be any polynomial of degree $k-1$. Then p can be expressed as a linear combination of B–splines of order k. Because of the support properties of the B–splines this implies that $E := \{N_{\ell,k} : j - k < \ell \leq k\}$ is a spanning set for the polynomials of degree $k-1$ considered as a vector space of functions from (t_j, t_{j+1}) to $\mathbf{R}$. From the cardinality of E it follows that E is not merely a spanning set but also a basis for this vector space. This is the required result.

Lemma 2 *ψ–splines as convolutions.*
Let $k \in \mathbf{N}$ and consider an infinite mesh

$$\boldsymbol{t} : \ldots < t_{j-1} < t_j < t_{j+1} < \ldots, t_{\pm j} \to \pm\infty \, , \qquad \text{as } j \to \infty,$$

and $h = \sup_j (t_{j+1} - t_j) < \infty$. Suppose $\boldsymbol{\alpha} = \{\alpha_j\}_{j=-\infty}^{\infty} \in C(2k, \boldsymbol{t})$. Then

$$s(x) = \sum_{j=-\infty}^{\infty} \alpha_j \psi_{j,2k,t}(x)$$

is absolutely convergent for each x and is given by

$$s(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j N_{j,2k,t}(s)((x-u)^2 + c^2)^{-(2k+1)/2} du$$

in which the integral is absolutely convergent.

Proof: Define g as the B–spline series

$$g(x) = \sum_j \alpha_j N_{j,2k,t}(x)$$

and

$$M(x) = \sum_j |\alpha_j| N_{j,2k,t}(x).$$

As is familiar there is no convergence problem with these series as only a finite number of the B–splines are non-zero at any x. Indeed on $[t_i, t_{i+1}]$ only $N_{i-2k+1,2k,t}, \ldots, N_{i,2k,t}$ are non-zero. From this, the growth condition on $\boldsymbol{\alpha}$, the partition of unity property of the B–splines, and the finiteness of the mesh size, it follows that

$$|g(x)| \leq M(x) = \mathcal{O}(|x|^{2k-1}) \quad \text{as } x \to \pm\infty.$$

Hence, for each fixed, x

$$0 \leq kA(k)c^{2k} M(u)((x-u)^2 + c^2)^{-(2k+1)/2} = \mathcal{O}(u^{-2}), \qquad \text{as } u \to \pm\infty.$$

Thus the middle quantity above is integrable with respect to u on $\mathbf{R}$. It now follows from (2.15) and the Lebesgue dominated convergence theorem that

$$\sum_{j=-\infty}^{\infty} \alpha_j \psi_{j,2k,t}(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_j \alpha_j N_{j,2k,t}(u)((x-u)^2 + c^2)^{-(2k+1)/2} du$$

with the series on the left converging absolutely for each x.

Lemma 3 *Polynomials in the space spanned by the ψ-splines .*
Let $\boldsymbol{t}$ satisfy the conditions of Lemma 2. Suppose that $p \in \pi_{2k-1}$ has B–spline series expansion

$$p(x) = \sum_j d_j N_{j,2k}(x) .$$

Then

$$s(x) = \sum_j d_j \psi_{j,2k} ,$$

is a polynomial of the same degree as p and with the same leading coefficient. Moreover, if $p \in \pi_1$ then s and p are identical and

$$d_j = p(t_j^*), \qquad j = 0, \pm 1, \pm 2, \dots ,$$

where the points

$$t_j^* = \frac{t_{j+1} + \cdots + t_{j+2k-1}}{2k-1} ,$$

are the special points occurring in the definition of Schoenberg's variation diminishing spline.

Proof: Recall the remarkable condition property of the B–spline basis (see for example de Boor [5, p.155]

$$|d_i| \leq D_{2k} \| \sum_j d_j N_{j,2k} \|_{L^\infty[t_{i+1}, t_{i+2k-1}]}$$

where D_{2k} is independent of the mesh. Hence

$$|d_i| \leq D_{2k} \|p\|_{L^\infty[t_{i+1}, t_{i+2k-1}]} .$$

This together with the finiteness of the mesh size, implies that the coefficients belong to the growth class $C(2k, \boldsymbol{t})$. Hence, from Lemma 2,

$$\begin{aligned} s(x) &= kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j N_{j,2k}(u)((x-u)^2 + c^2)^{-(2k+1)/2} du \\ &= kA(k)c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2 + c^2)^{-(2k+1)/2} du . \qquad (3.1) \end{aligned}$$

Supposing now p is of exact degree $m, 0 \leq m \leq 2k-1$, so that

$$p(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0 \, ,$$

with $a_m \neq 0$, (3.1) implies

$$\begin{aligned} s(x) &= kA(k)c^{2k} a_m x^m \int_{-\infty}^{\infty} (u^2 + c^2)^{-(2k+1)/2} du \\ &\quad + \sum_{i=0}^{m-1} b_i x^i \int_{-\infty}^{\infty} p_i(u)(u^2 + c^2)^{-(2k+1)/2} du \, , \end{aligned}$$

where $p_i(u) \in \pi_m$. The first part of the lemma follows.

For $p \in \pi_1$ Schoenberg's variation diminishing spline with coefficients $d_j = p(t_j^*)$ satisfies

$$p(x) = \sum_j p(t_j^*) N_{j,2k}(x), \quad \forall x.$$

From Lemma 1, that is the local linear independence of the B–splines, the coefficients $p(t_j^*)$ are the only coefficients with this property. An application of (3.1) now gives

$$\begin{aligned} s(x) &= \sum_j p(t_j^*) \psi_{j,2k}(x) \\ &= kA(k)c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2 + c^2)^{-(2k+1)/2} du \\ &= kA(k)c^{2k} p(x) \int_{-\infty}^{\infty} (u^2 + c^2)^{-(2k+1)/2} du \\ &= p(x), \end{aligned}$$

where in the second to last step we have used that if g is odd and integrable $\int_{-\infty}^{\infty} g = 0$. ∎

Unfortunately when p has degree greater than 1 the coefficients used to express p in terms of the B–splines do not suffice to express it in terms of the ψ–splines. For example if $k > 1$ and

$$p(x) = x^2 = \sum_j d_j N_{j,2k}(x);,$$

then by (2.21)

$$s(x) = \sum_j d_j \psi_{j,2k}(x)$$

$$
\begin{aligned}
&= x^2 + kA(k)c^{2k}\int_{-\infty}^{\infty} u^2(u^2+c^2)^{-(2k+1)/2}du \\
&= x^2 + Mc^2
\end{aligned}
$$

where the non-zero constant M, depends on k.

Lemma 4 *Global linear independence of the ψ–spline basis.*
Let $\boldsymbol{t}$ be as in Lemma 2 and suppose $\boldsymbol{d} \in C(2k, \boldsymbol{t})$. Then

$$
s(x) = \sum_j d_j \psi_{j,2k}(x) = 0, \quad \text{for all } x,
$$

implies $\boldsymbol{d}$ is the zero sequence.

Proof: The assumptions on $\boldsymbol{t}$ and $\boldsymbol{d}$ ensure

$$
g = \sum_j d_j N_{j,2k},
$$

satisfies $g(x) = \mathcal{O}(|x|^{2k-1})$ as $x \to \pm\infty$ hence is a tempered distribution, having a generalized Fourier Transform well defined except possibly at zero. (The properties of tempered distributions we use can be found in Rudin [7, particularly pp.173–178].) Now

$$
0 = s(x) = \sum_j d_j \psi_{j,2k}(x), \quad \text{for all } x,
$$

implies by Lemma 2

$$
0 = kA(k)c^{2k}\{\sum_j d_j N_{j,2k}(\cdot)\} * (\cdot^2 + c^2)^{-(2k+1/2}\}, \quad \text{for all } x. \tag{3.2}
$$

Taking the generalized Fourier transform we obtain

$$
0 = \widehat{g}(\xi)\widehat{S_{2k}}(\xi), \quad \text{for all } \xi \neq 0 \ , \tag{3.3}
$$

where $\widehat{S_{2k}} \in C^{2k-1}(\mathbf{R})$ is the everywhere positive transform of S_{2k}, previously discussed (see (2.18) and (2.19)) . Hence (3.3) implies the support of $\widehat{g}(\xi)$ is $\xi = 0$. Thus g must be a polynomial. Since from above $g(x) = O(|x|^{2k-1})$, this polynomial has exact degree not exceeding $2k-1$. Then, from Lemma 3, s is a polynomial of the same exact degree. But from the hypotheses s being identically zero has exact degree -1. Hence $\sum d_j N_{j,2k}$ is identically zero. The result follows from Lemma 1 .

The following corollary goes in the opposite direction to Lemma 3.

Corollary 5 *Polynomial reproduction and a dual representation.*
Let $k \in \mathrm{N}$, and $\boldsymbol{t}$ be as in Lemma 2. Let

$$s(x) = \sum_j d_j \psi_{j,2k}(x) \quad \text{and} \quad g(x) = \sum_j d_j N_{j,2k}(x). \tag{3.4}$$

where the first sum may be divergent. Then:

(a) Given any polynomial $q \in \pi_{2k-1}$ there is a unique choice of coefficients $\boldsymbol{d}$ such that both $s = q$ and the growth condition $\boldsymbol{d} \in C(2k, \boldsymbol{t})$ hold.

(b) If $\boldsymbol{d} \in C(2k, \boldsymbol{t})$ and $s \in \pi_{2k-1}$ then s and g have the same exact degree and leading coefficient.

(c) If $\boldsymbol{d} \in C(2k, \boldsymbol{t})$ and $s \in \pi_1$ then s and g are identical.

Proof: (a) Let $q \in \pi_{2k-1}$ be fixed and of exact degree m. From Lemma 3 choosing $p(x)$ there as x^m, $x^{m-1}, \ldots, 1$ in turn, and then taking linear combinations, we can find coefficients $\boldsymbol{d}$ satisfying the growth condition $\boldsymbol{d} \in C(2k, \boldsymbol{t})$ and $s = q$. From Lemma 4 these coefficients are unique. Note that in this construction $\sum_j d_j N_{j,2k}$ is a polynomial with the same exact degree and leading coefficient as q.

(b) Let $\boldsymbol{d} \in C(2k, \boldsymbol{t})$ and $s \in \pi_{2k-1}$. Then from the uniqueness part of part (a) the coefficients $\boldsymbol{d}$ must be those of the construction of part (a). The conclusion follows from the remark at the end of the proof of part (a).

(c) From (b) if $s \in \pi_1$ then $g \in \pi_1$. The conclusion then follows from Lemma 3.

Lemma 6 Let $k \in \mathrm{N}$ and $c > 0$. Then

$$I_k = \int \frac{u^{2k-1}}{((x-u)^2 + c^2)^{(2k+1)/2}} du = \left\{ \frac{p(x,u)}{((x-u)^2 + c^2)^{(2k-1)/2}} \right\} + C, \tag{3.5}$$

where $p(x,u)$, considered as a polynomial in u, has degree $2k-1$ and constant part

$$-\frac{(x^2 + c^2)^{2k-1}}{2kA(k)c^{2k}}. \tag{3.6}$$

Proof: By differentiation one easily establishes the recurrences

$$\begin{aligned}\int \frac{u^m}{((x-u)^2+c^2)^{(2k+1)/2}}du &= \frac{u^{m-1}}{(m-2k)((x-u)^2+c^2)^{(2k-1)/2}}\\ &\quad - \left(\frac{2k-2m+1}{m-2k}\right) x \int \frac{u^{m-1}}{((x-u)^2+c^2)^{(2k+1)/2}}du\\ &\quad - \left(\frac{m-1}{m-2k}\right)(x^2+c^2)\int \frac{u^{m-2}}{((x-u)^2+c^2)^{(2k+1)/2}}du\end{aligned}$$

and

$$\begin{aligned}\int ((x-u)^2+c^2)^{-(2k+1)/2}du &= \frac{(u-x)}{(2k-1)c^2}((x-u)^2+c^2)^{-(2k-1)/2}\\ &\quad + \frac{2(k-1)}{(2k-1)c^2}\int ((x-u)^2+c^2)^{-(2k-1)/2}du.\end{aligned}$$

Since

$$\int ((x-u)^2+c^2)^{-\frac{3}{2}}du = \frac{(u-x)}{c^2}((x-u)^2+c^2)^{\frac{-1}{2}} + D,$$

an easy induction shows that there is an indefinite integral I_k of the form stated in (3.5). It remains to show that the constant part of the polynomial in u, $p(x,u)$, is given by (3.6).

To this end make the substitution

$$\cos\theta = \frac{c}{\sqrt{(x-u)^2+c^2}} \qquad \text{and} \qquad \sin\theta = \frac{-(x-u)}{\sqrt{(x-u)^2+c^2}},$$

implying $u = x + c\tan\theta$. Note in particular that the expression defining $\cos\theta$ is defined everywhere and always positive. Then

$$\begin{aligned}I_k &= \int \frac{(x+c\tan\theta)^{2k-1}c\sec^2\theta}{(c\sec\theta)^{2k+1}}d\theta\\ &= \frac{1}{c^{2k}}\int (x\cos\theta + c\sin\theta)^{2k-1}\,d\theta\\ &= \frac{(x^2+c^2)^{(2k-1)/2}}{c^{2k}}\int \cos^{2k-1}t\,dt\end{aligned}$$

where

$$t = \theta - \gamma, \qquad \cos\gamma = \frac{x}{\sqrt{x^2+c^2}} \qquad \text{and} \qquad \sin\gamma = \frac{c}{\sqrt{x^2+c^2}}.$$

Then with $v = \sin t$

$$\int \cos^{2k-1} t dt = \int (1-v^2)^{k-1} dv = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j v^{2j+1}}{2j+1} + E.$$

Therefore we may choose

$$p(x,u) = \left(\frac{c}{\cos\theta}\right)^{2k-1} \frac{(x^2+c^2)^{(2k-1)/2}}{c^{2k}} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j \sin^{2j+1} t}{2j+1}.$$

When $u = 0$, $\cos\theta = c/\sqrt{x^2+c^2} = \sin\gamma$, $\sin\theta = -x/\sqrt{x^2+c^2} = -\cos\gamma$, and $\sin t = \sin(\theta-\gamma) = \cos\gamma\sin\theta - \cos\theta\sin\gamma = -1$. Therefore

$$p(x,0) = -\frac{(x^2+c^2)^{2k-1}}{c^{2k}} \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{2j+1}.$$

Finally

$$\frac{1}{2kA(k)} = \frac{(2k-2)!!}{(2k-1)!!} = \int_0^{\pi/2} \sin^{2k-1} x \, dx = \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{2j+1},$$

which completes the proof.

Lemma 7 *ϕ and ψ splines as convolutions.*
Let $k \in \mathbf{N}$. Then

$$\phi(x;2k) = kA(k)c^{2k} |\cdot|^{2k-1} * (\cdot^2 + c^2)^{-(2k+1)/2}, \tag{3.7}$$

and

$$\psi_{j,2k}(x) = kA(k)c^{2k} N_{j,2k,t} * (\cdot^2 + c^2)^{-(2k+1)/2}. \tag{3.8}$$

Proof: For $f \in C^{2k}(\mathbf{R})$ of compact support a straight forward integration by parts argument shows

$$f(x) = \frac{1}{2\cdot(2k-1)!} (|\cdot|^{2k-1} * f^{(2k)})(x).$$

That this also holds for the function ϕ, which grows at infinity, follows from the following more direct argument.

Let $g(x,u)$ be the indefinite integral

$$\int \frac{u^{2k-1}}{((x-u)^2+c^2)^{(2k+1)/2}}du = \frac{p(x,u)}{((x-u)^2+c^2)^{(2k-1)/2}} + C,$$

discussed in Lemma 6, with C chosen to be 0. Then

$$\int_{-\infty}^{\infty} \frac{|u|^{2k-1}}{((x-u)^2+c^2)^{(2k+1)/2}}du = \lim_{u\to\infty} g(x,u) + \lim_{u\to-\infty} g(x,u) - 2g(x,0).$$

But from the previous lemma the first two terms on the right above cancel and the last term equals

$$\frac{(x^2+c^2)^{(2k-1)/2}}{kA(k)c^{2k}},$$

which establishes (3.7).

The second part of the lemma is already contained in (2.15) and Lemma 2.

4 Polynomials as semi-infinite sums of ψ-splines.

Fundamental to the work of Beatson and Powell [2], is that not only are linear polynomials in the space of all bi-infinite combinations of $\psi_{j,2}$'s but also in the space of semi-infinite combinations (modulo a few *edge* ϕ's). In this section we will obtain an analogous result for ψ–splines of general order.

The proof used in [2] was a direct integration. An alternative argument collapsing sum argument is as follows. Let

$$\beta(x) = \sum_{j=0}^{\infty} \psi_{j,2k}(x).$$

Then

$$\begin{aligned}
\beta(x) &= \lim_{m\to\infty} \sum_{j=0}^{m} \psi_{j,2k}(x) \\
&= \frac{1}{2} \lim_{m\to\infty} \sum_{j=0}^{m} \{[t_{j+1},\ldots,t_{j+2k}]\phi(x-u;2k) - [t_j,\ldots,t_{j+2k-1}]\phi(x-u;2k)\} \\
&= \frac{1}{2}\{\lim_{m\to\infty}[t_{m+1},\ldots,t_{m+2k}]\phi(x-u;2k) - [t_0,\ldots,t_{2k-1}]\phi(x-u;2k)\}
\end{aligned}$$

where all the divided differences are with respect to the u variable. Using the asymptotic expression for $\phi^{(2k-1)}$ (2.7) to express the first term on the right, and the familiar formula

$$[t_\ell, t_{\ell+1}, \ldots, t_{\ell+n}]f = \sum_{j=\ell}^{\ell+n} \frac{f(t_j)}{\Pi_{i=\ell, i\neq j}^{\ell+n}(t_j - t_i)}$$

for a divided difference to express the last, it follows that

$$\beta(x) = \frac{1}{2} - \frac{1}{2}\sum_{j=0}^{2k-1} \left[\Pi_{i=0,i\neq j}^{2k-1}(t_j - t_i)\right]^{-1} \phi_{j,2k}(x).$$

More generally we have

Theorem 8 *Polynomials as semi-infinite sums of ψ–splines.*
Suppose $\boldsymbol{t}$ satisfies the conditions of Lemma 2 and $\boldsymbol{d} \in C(2k, \boldsymbol{t})$. Further suppose $p = \sum_{j=-\infty}^{\infty} d_j \psi_{j,2k}$ is in π_{2k-1}. Then $q = \sum_{j=-\infty}^{\infty} d_j N_{j,2k}$ is also in π_{2k-1}. Furthermore the function, s, defined by the semi-infinite sum

$$s(x) = \sum_{j=0}^{\infty} d_j \psi_{j,2k}(x) ,$$

can be rewritten as

$$s(x) = \frac{p(x)}{2} + \frac{1}{2}\sum_{\ell=0}^{2k-1} \lambda_\ell \phi_{\ell,2k}(x) ,$$

where the vector λ is the unique solution of

$$\sum_{\ell=0}^{2k-1} \lambda_\ell (\cdot - t_\ell)^{2k-1} = q .$$

The theorem also holds in the polynomial spline case $c = 0$.

Proof: We consider firstly the case when $c = 0$ so that $\psi_{j,2k}$ is the polynomial B–spline $N_{j,2k}$ and p and q are identical. Then

$$s(x) = \sum_{j=0}^{\infty} d_j N_{j,2k}(x)$$

and by the properties of B–splines

$$s(x) = \begin{cases} 0, & x \le t_0, \\ q(x), & x \ge t_{2k-1}, \end{cases}$$

and has possible jump discontinuities in its $(2k-1)^{st}$ derivative at $t_0, t_1, \ldots, t_{2k-1}$. We note that

$$x_+^{2k-1} = \frac{x^{2k-1} + |x|^{2k-1}}{2}$$

has a jump discontinuity of magnitude $(2k-1)!$ in its $(2k-1)^{st}$ derivative at $x = 0$ and none elsewhere. Hence,

$$s(x) = \sum_{\ell=0}^{2k-1} \frac{\lambda_\ell}{2}\{(x-t_\ell)^{2k-1} + |x-t_\ell|^{2k-1}\},$$

for some constants $\lambda_0, \ldots, \lambda_{2k-1}$. This can be rewritten as

$$s(x) = \left\{\sum_{\ell=0}^{2k-1} \frac{\lambda_\ell}{2}(x-t_\ell)^{2k-1}\right\} + \left\{\sum_{\ell=0}^{2k-1} \frac{\lambda_\ell}{2}|x-t_\ell|^{2k-1}\right\}.$$

But for $x > t_{2k-1}$ both terms in curly brackets reduce to the same thing and sum to $q(x)$. Hence the first term equals $q(x)/2$ for all $x > t_{2k-1}$, and being a polynomial for all $x \in \mathbf{R}$. Thus

$$s(x) = \frac{q(x)}{2} + \left\{\sum_{\ell=0}^{2k-1} \frac{\lambda_\ell}{2}|x-t_\ell|^{2k-1}\right\}.$$

Comparing these last two expressions for $s(x)$ we find

$$\sum_{\ell=0}^{2k-1} \lambda_\ell(\cdot - t_j)^{2k-1} = q.$$

Since $\{(\cdot - t_\ell)^{2k-1} : \ell = 0, \ldots, 2k-1\}$ forms a basis for π_{2k-1} it follows that the coefficients $\lambda_0, \ldots, \lambda_{2k-1}$ are uniquely determined by this last equation. This establishes the theorem when $c = 0$.

We now turn to the case $c > 0$. From Corollary 5, Lemma 1 and Lemma 2

$$q(x) = \sum_{j=-\infty}^{\infty} d_j N_{j,2k}(x) \tag{4.1}$$

is the unique polynomial in π_{2k-1} such that

$$p = \sum_{j=-\infty}^{\infty} d_j \psi_{j,2k} = kA(k)c^{2k}\Big\{ \sum_{j=-\infty}^{\infty} d_j N_{j,2k} * (\cdot + c^2)^{-(2k+1)/2} \Big\}, \qquad (4.2)$$

with the last equality holding term by term. Hence

$$\begin{aligned} s &= \sum_{j=0}^{\infty} d_j \psi_{j,2k} \\ &= kA(k)c^{2k} \sum_{j=0}^{\infty} d_j \Big\{ N_{j,2k} * (\cdot + c^2)^{-(2k+1)/2} \Big\} \\ &= kA(k)c^{2k} \Big\{ \frac{q(\cdot)}{2} + \frac{1}{2} \sum_{\ell=0}^{2k-1} \lambda_\ell | \cdot - t_\ell |^{2k-1} \Big\} * (\cdot + c^2)^{-(2k+1)/2} \\ &= \frac{p(\cdot)}{2} + \frac{1}{2} \sum_{\ell=0}^{2k-1} \lambda_\ell \phi_{\ell,2k}(\cdot), \end{aligned}$$

where in the second to last equality we have used the already proven result for $c = 0$. The last equality follows from (4.1), (4.2) and Lemma 7.

Note that in the special case

$$q = (\cdot - t_0)^{2k-1} = \sum_{j=-\infty}^{\infty} d_j N_{j,2k} \,,$$

Theorem 8 gives the especially simple expression

$$\sum_{j=0}^{\infty} d_j N_{j,2k} = \frac{1}{2}\{(\cdot - t_0)^{2k-1} + | \cdot - t_0 |^{2k-1}\} = (\cdot - t_0)_+^{2k-1} \,.$$

5 Approximation by ψ–splines

In this section we consider approximation properties of ψ–splines. We use quasi-interpolants to show Jackson-type error estimates for non-uniform meshes and continuous or continuously differentiable functions. The results are generalisations of some of the results of Buhmann [3, 4] for bi-infinite uniform meshes, and of results of Beatson and Powell [2]. for quasi-interpolation on a finite mesh with ordinary multiquadrics.

Theorem 9: Let $k \geq 1$, $c > 0$ and mesh $\boldsymbol{t} : \ldots < t_{j-1} < t_j < t_{j+1} < \ldots$, with $t_{\pm j} \to \pm\infty$ as $j \to \pm\infty$ be given. Suppose the mesh size $h = \sup_j (t_{j+1} - t_j)$ is finite. Then for each function f, uniformly continuous on $\mathbf{R}$, the quasi-interpolant $\mathcal{L}_B f = \sum_{j=-\infty}^{\infty} f(t_j^*)\psi_{j,2k}$ satisfies

$$\|f - \mathcal{L}_B f\|_\infty \leq (k + 1 + \frac{c}{h})\omega(f, h).$$

The same result holds when t_j^* is replaced by t_{j+k} in the definition of $\mathcal{L}_B$.

Proof: Firstly note that

$$|f(x)| \leq |f(0)| + (1 + |x|)\omega(f, 1), \quad x \in \mathbf{R}\,.$$

Hence, $|f(x)|$ grows at most linearly as $x \to \pm\infty$, and $\mathcal{L}_B f$ is well defined by Lemma 2.

From the partition of unity property of the $\psi_{j,2k}$'s

$$f(x) - (\mathcal{L}_B f)(x) = \sum_{j=-\infty}^{\infty} \{f(x) - f(t_j^*)\}\psi_{j,2k}(x).$$

From the properties of the modulus of continuity

$$|f(x) - f(t_j^*)| \leq \omega(f, |x - t_j^*|) \leq \left(1 + \frac{|x - t_j^*|}{h}\right)\omega(f, h).$$

Hence using also Lemma 2

$$\begin{aligned} &|f(x) - \sum_{j=-\infty}^{\infty} f(t_j^*)\psi_{j,2k}(x)| \\ &\quad \leq \; kA(k)c^{2k} \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} |f(x) - f(t_j^*)| N_{j,2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du \\ &\quad \leq \; kA(k)c^{2k}\omega(f, h) \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} \left(1 + \frac{|x - t_j^*|}{h}\right) N_{j,2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du. \quad (5.1) \end{aligned}$$

Now recall that

$$t_j^* = \frac{t_{j+1} + \cdots + t_{j+2k-1}}{2k - 1}\,,$$

so that t_j^* is increasing in each of $t_{j+1}, \ldots, t_{j+2k-1}$. Hence $\max\{t_j^* - t_j, t_{j+2k} - t_j^*\}$ occurs when all the points are as far apart as possible and is kh. Since $\operatorname{supp}(N_{j,2k}) = [t_j, t_{j+2k}]$ it follows that $N_{j,2k}(u)$ is non-zero only when $|u - t_j^*| \le kh$. When this is the case $|x - t_j^*| \le |x - u| + kh$. Substituting into (5.1)

$$|f(x) - (\mathcal{L}_B f)(x)| \le kA(k)c^{2k}\omega(f,h) \int_{-\infty}^{\infty} \frac{(k+1) + \frac{|x-u|}{h}}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du.$$

Using the values for the integrals given in (2.9) and (2.12) we find

$$|f(x) - (\mathcal{L}_B f)(x)| \le (k + 1 + \frac{c}{h})\omega(f,h).$$

The argument when we replace t_j^* by t_{j+k} is almost identical.

Corollary 10: Let $k \in N$ and mesh $\boldsymbol{t} : t_0 < t_1 < \ldots < t_n$ be given. Let

$$\mathcal{B} = \operatorname{span}\{1, \phi_{0,2k}, \phi_{1,2k}, \ldots, \phi_{n,2k}\}.$$

Then

$$\operatorname{dist}(f, \mathcal{B}; L^\infty[t_0, t_n]) \le (2k + \frac{c}{h})\omega(f,h)$$

for all $f \in C[t_0, t_n]$, where $h = \max_{0 \le j \le n-1}(t_{j+1} - t_j)$ is the mesh size.

Proof: Case 1. $n \le 2k$. In this case approximate f by the constant function $s(x) = f(t_{[n/2]})$ and note

$$\|f - s\|_{L^\infty[t_0,t_n]} \le (n - [n/2])\omega(f,h) \le 2k\omega(f,h).$$

Case 2. $n > 2k$. In this case extend the mesh to $\pm\infty$ by requiring $t_{j+1} - t_j = h$ for all $j \in \mathbf{Z}\backslash[0,n)$. Then set

$$g(x) = \begin{cases} f(t_{k-1}), & x \le t_{k-1}, \\ f(x), & t_{k-1} \le x \le t_{n-k+1}, \\ f(t_{n-k+1}), & t_{n-k+1} \le x. \end{cases}$$

Note that $\max\{t_{k-1} - t_0, t_n - t_{n-k+1}\} \le (k-1)h$ implying that $\|f - g\|_{L^\infty[t_0,t_n]} \le (k-1)\omega(f,h)$, and also that g is uniformly continuous on $\mathbf{R}$ with $\omega(g,h) \le \omega(f,h)$. Hence by Theorem 9

$$\|g - \sum_{j=-\infty}^{\infty} g(t_{j+k})\psi_{j,2k}\|_{L^\infty(\mathbf{R})} \le (k + 1 + \frac{c}{h})\omega(f,h).$$

Thus

$$\|f - \sum_{j=-\infty}^{\infty} g(t_{j+k})\psi_{j,2k}\|_{L^\infty[t_0,t_n]} \leq (2k + \frac{c}{h})\omega(f,h)\,. \tag{5.2}$$

Let g_ℓ and g_r be constant functions with values $f(t_{k-1})$ and $f(t_{n-k+1})$ respectively. Then by Theorem 3 and Theorem 8

$$\sum_{j=-\infty}^{-1} g(t_{j+k})\psi_{j,2k} = \sum_{j=-\infty}^{-1} g_\ell(t_{j+k})\psi_{j,2k}$$

is in span$\{1, \phi_{0,2k}, \phi_{1,2k}, \ldots, \phi_{2k-1,2k}\}$ and hence is in the space $\mathcal{B}$. Similarly

$$\sum_{j=n-2k+1}^{\infty} g(t_{j+k})\psi_{j,2k} = \sum_{j=n-2k+1}^{\infty} g_r(t_{j+k})\psi_{j,2k}$$

also belongs to the space $\mathcal{B}$. Since $\psi_{j,2k} \in \mathcal{B}$ for $j = 0, \ldots, n-2k$ the corollary follows from (5.2).

Theorem 11 Let $k \in \mathrm{N}$ and $k \geq 2$. There exists a constant M, depending only on k, with the following property. Let $c > 0$ and mesh $\boldsymbol{t} : \ldots < t_{j-1} < t_j < t_{j+1} < \ldots$ with $t_{\pm j} \to \pm\infty$ as $j \to \infty$ be given. Suppose the mesh size $h = \sup_j (t_{j+1} - t_j)$ is finite. Then for each function f with f' uniformly continuous on $\mathbf{R}$, the quasi-interpolant,

$$\mathcal{L}_B f = \sum_{j=-\infty}^{\infty} f(t_j^*)\psi_{j,2k}\,,$$

satisfies

$$\|f - \mathcal{L}_B f\|_{L^\infty(\mathbf{R})} \leq M\Big\{\frac{c^2}{h} + c + h\Big\}\omega(f',h)\,.$$

Proof: Firstly note that

$$|f'(x)| \leq |f'(0)| + (1+|x|)\omega(f',1), \quad x \in \mathbf{R},$$

so that $|f(x)|$ grows at most quadratically as $x \to \pm\infty$. Hence $\mathcal{L}_B f$ is well defined by Lemma 2.

Now from Lemma 3 $\mathcal{L}_{\mathcal{B}}$ reproduces linears. It is after all the analogue of the variation diminishing spline. Hence if p is the linear Taylor polynomial of f at x

$$\sum_{j=-\infty}^{\infty} p(t_j^*)\psi_{j,2k}(x) = p(x) = f(x).$$

Thus

$$|f(x) - (\mathcal{L}_B f)(x)| = |\sum_{j=-\infty}^{\infty} \{p(t_j^*) - f(t_j^*)\}\psi_{j,2k}(x)|.$$

Using the bound

$$|f(t_j^*) - p(t_j^*)| \le |t_j^* - x|\omega(f', |t_j^* - x|)$$

from Taylor's theorem this is bounded above by

$$\sum_{j=-\infty}^{\infty} |x-t_j^*|\omega(f', |x-t_j^*|)\psi_{j,2k}(x) \le \sum_{j=-\infty}^{\infty} |x-t_j^*|\Big(1+\frac{|x-t_j^*|}{h}\Big)\omega(f', h)\psi_{j,2k}(x) \quad (5.3)$$

Writing

$$S_{2k}(u) = kA(k)c^{2k}(u^2 + c^2)^{-(2k+1)/2},$$

as before, the right hand side of (5.3) becomes

$$\omega(f', h)\int_{s=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \Big\{\frac{(x-t_j^*)^2}{h} + |x - t_j^*|\Big\} N_{j,2k}(u) S_{2k}(x-u)du. \quad (5.4)$$

But if $N_{j,2k}(u)$ is non-zero then $|u - t_j^*| \le kh$ implying

$$|x - t_j^*| \le |x - u| + kh.$$

Hence (5.4) is bounded above by

$$\omega(f', h)\int_{u=-\infty}^{\infty} \Big\{\frac{1}{h}(x-u)^2 + (2k+1)|x-u| + k(k+1)h\Big\} S_{2k}(x-u)du.$$

Since $k \ge 2$, (2.9), (2.12) and (2.21) show there exists a constant M_1, depending only on k such that

$$\int_{-\infty}^{\infty} |u|^j S_{2k}(u)du \le M_1 c^j \quad , \ 0 \le j \le 2,$$

and the result follows. ∎

Corollary 12: Let $k \in \mathbf{N}$ and $k \ge 2$. There exists a constant M depending only on k with the following property. Let $c > 0$ and mesh $\boldsymbol{t} : t_0 < t_1 < \ldots < t_n$ be given. Let

$$\mathcal{C} = \operatorname{span}\{1, x, \phi_{0,2k}, \phi_{1,2k}, \ldots, \phi_{n,2k}\}.$$

Then

$$\operatorname{dist}(f,\mathcal{C};L^\infty[t_0,t_n]) \le M\Big\{\frac{c^2}{h}+c+h\Big\}\omega(f',h)$$

for all $f\in C^1[t_0,t_n]$ where $h=\max_j(t_{j+1}-t_j)$ is the mesh size.

Proof Case 1. $n\le 2k$. In this case approximate f by the linear function $s(x)=f(t_{[n/2]})+f'(t_{[n/2]})(x-t_{[n/2]})$ and note

$$\|f-s\|_{L^\infty[t_0,t_n]} \le (n-[n/2])h\omega(f',kh) \le k^2h\omega(f',h).$$

Case 2. $n>2k$. In this case extend the mesh to $\pm\infty$ by requiring $t_{j+1}-t_j=h$ for all $j\in\mathbf{Z}\backslash[0,n)$. Then set

$$g(x)=\begin{cases} f(t^*_{-1})+f'(t^*_{-1})(x-t^*_{-1}), & x\le t^*_{-1},\\ f(x), & t^*_{-1}\le x\le t^*_{n-2k+1},\\ f(t^*_{n-2k+1})+f'(t^*_{n-2k+1})(x-t^*_{n-2k+1}), & t^*_{n-2k+1}\le x.\end{cases}$$

Then $\|f-g\|_{L^\infty[t_0,t_n]} \le (k-1)h\omega(f',(k-1)h) \le (k-1)^2h\omega(f',h)$ and g' is uniformly continuous on $\mathbf{R}$ with $\omega(g',h)\le\omega(f',h)$. By an argument analogous that in the latter part of the proof of Corollary 10, excepting that the application of Theorem 9 is replaced by an application of Theorem 11, we find that $(\mathcal{L}_{\mathcal{C}}g):=\sum_{j=-\infty}^{\infty} g(t^*_j)\psi_{j,2k}\in\mathcal{C}$, and

$$\|f-\mathcal{L}_{\mathcal{C}}g\|_{L^\infty[t_0,t_n]} \le M_2\Big\{\frac{c^2}{h}+c+h\Big\}\omega(f',h)\ .$$

We now turn to the case $k=1$ discussed in Beatson and Powell [2]. Extend $f\in C^1[t_0,t_n]$ outside $[t_0,t_n]$ by appending first degree Taylor polynomials at t_0 and t_n. They show that the operator $\mathcal{L}_Bf$ of Theorem 9 applied to this extended f, becomes (in the notation of the current paper)

$$\begin{aligned}(\mathcal{L}_Bf)(x) &= \sum_{j=-\infty}^{\infty} f(t^*_j)\psi_{j,2}(x)=\sum_{j=-\infty}^{\infty} f(t_{j+1})\psi_{j,2}(x)\\ &= \frac{f'(t_0)}{2}\Big[(x-t_0)-\phi_0(x)\Big]+\frac{f(t_0)}{2}\Big[1+\frac{\phi_1(x)-\phi_0(x)}{t_1-t_0}\Big]\\ &\quad+\sum_{j=1}^{n-1} f(t_j)\psi_{j-1,2}(x) \qquad\qquad (5.5)\\ &\quad+\frac{f(t_n)}{2}\Big[1-\frac{\phi_n(x)-\phi_{n-1}(x)}{t_n-t_{n-1}}\Big]+\frac{f'(t_n)}{2}\Big[\phi_n(x)-(t_n-x)\Big].\end{aligned}$$

Note that in [2] ψ_j denotes a combination of $\phi_{j-1,2}, \phi_{j,2}$, and $\phi_{j+1,2}$ whereas here it denotes a combination of $\phi_{j,2}, \phi_{j+1,2}$ and $\phi_{j+2,2}$. They obtain an estimate for $\|f - \mathcal{L}_B f\|$ when f has a Lipschitz derivative. It is natural therefore to seek an estimate in terms of $\omega(f', h)$.

Theorem 13 Let $k = 1$. There exists a constant M with the following property. Let a mesh $\boldsymbol{t} : t_0 < t_1 < \ldots < t_n$ be given and $(\mathcal{L}_B f)$ be defined by (5.5). then

$$\|f - \mathcal{L}_B f\|_{L^\infty[t_0,t_n]} \leq M\Big\{c + h + \frac{c^2}{h} + \frac{c^2}{h}\log\Big(1 + \Big(\frac{t_n - t_0}{c}\Big)\Big)\Big\}\omega(f', h)$$

for all $f \in C^1[t_0, t_n]$ where h is the mesh size.

Proof: This proof is quite intricate but involves no essentially new ideas. It has therefore been omitted.

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