RESEARCH REPORT

ALGEBRAS for MATRIX LIMITATION

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1. Let $A = (a_{m,n})$ be a regular matrix; by $\mathcal{A}$ is denoted the set of sequences limited by A and by $\mathcal{A}_0$ those sequences limited to zero by A. If $\xi = \{\xi_n\}$ is a bounded sequence and $\{\xi_n s_n\} \in \mathcal{A}_0$ whenever $\{s_n\} \in \mathcal{A}_0$ then ξ is called an f-sequence for the matrix $A = (a_{m,n})$. The set of f-sequences form an algebra denoted by $\mathcal{A}^0$. By $\mathbf{A}(\mathbf{A}_0)$ are denoted those bounded sequences limited (to zero) by A and by $\mathbf{A}^0$ the algebra of bounded sequences ξ such that $\xi x \in \mathbf{A}_0$ whenever $x \in \mathbf{A}_0$. It is easy to prove that $\mathbf{A}^0$ is a Banach Algebra.

The sequences of $\mathbf{A}^0$ and $\mathcal{A}^0$ have been used by many authors, notably by Agnew, [1], Brudno [3], Bosanquet [6], Erdös and Piranian [7] and Zeller [20]. Goes [8] took up the study of Banach Algebras of bounded sequences in general.

2. In 1933 Mazur and Orlicz [10] stated some important theorems on the sequences limited by a regular matrix. In 1955 the authors gave their Functional Analytic proofs in [11]. In the intervening years and subsequently different proofs of an element any character were given. Notably, the paper by Burdno [3] goes over the whole ground and then goes on to new theorems. The first and most important theorem may be stated

<u>Theorem 1.</u> If $\mathbf{B} \supset \mathbf{A}$ then $\mathbf{B}_0 \supset \mathbf{A}_0$.

The proof requires the construction of $\xi \in \mathbf{A}^0$ such that if $x \in \mathbf{A}_0 \cap \mathbf{B} \setminus \mathbf{B}_0$, $\xi x \in \mathbf{A}_0$ but $\xi x \notin \mathbf{B}$. The proof is staight-forward though perhaps obscured by manipulation.

A sequence x is called <u>thin</u> with respect to a matrix $A = (a_{m,n})$ if $x_n = 0$, $n \notin E$ where

$$\sum_{n \in E} |a_{m,n}| = 0. \qquad (1)$$

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There is a basic theorem concerning the structure of A^0 and $\mathbf{A}^0$ for positive matrices.

Theorem 2. If $A = (a_{m,n})$ and $x \in A^0 \cap A_0$ or $\mathbf{A}^0 \cap \mathbf{A}_0$ then $x = r + t$ where t is a thin sequence and r converges to zero.

The proof, see [13] follows almost immediately from the fact that $x^2 \in A_0$ (or $\mathbf{A}_0$) so that for any $\varepsilon > o$, $x_n^2 > \varepsilon$ if and only if $n \in E$ where E satisfies (1).

This second theorem enables us to use A^0 to establish Tauberian conditions for A. For example, let A be the Cesàro mean so that the f-sequences ξ will be bounded sequences such that the limit of the arithmetic mean of ξx is zero whenever that of x is zero. Performing an Abel transformation on the mean of ξx,

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=1}^{n} k(\xi_k - \xi_{k+1}) \frac{1}{k} \sum_{1}^{k} x_r + \frac{\xi_{n+1}}{n} \sum_{1}^{n} x_r.\right] \tag{2}$$

Since ξ is bounded the second term has limit zero. By a suitable choice of $\{x_r\}$, $\{\gamma_k\}$, $\gamma_k = \frac{1}{k} \sum_{1}^{k} x_r$ may be any sequence converging to zero so that a necessary and sufficient condition for ξ to be an f-sequence is that the matrix $B = (b_{n,k})$ defined by

$$b_{n,k} = \frac{k(\xi_k - \xi_{k+1})}{n}, \quad k \leqslant n$$

$$b_{n,k} = 0 \text{ elsewhere}$$

map sequences convergent to zero into sequences convergent to zero. This will be the case if

$$(\xi_k - \xi_{k+1}) = 0(\tfrac{1}{k}) \tag{4}$$

and since this implies

$$(\xi_k - \xi_{k+f(k)}) \to 0$$

unless $f(k) = 0(k)$ and $0(k)$ is not a counting function for the Cesàro mean (4) is a Tauberian condition for the Cesàro mean.

A careful analysis would give the same result for (3). This analysis can give Tauberian conditions for quite general matrices and I do not think its full effects have been explored.

3. A different type of problem is to find the matrices corresponding to a given algebra. For example, when does A^0 (or $\mathbf{A}^0$) consist of all the bounded sequences $\mathbf{C}$. For submatrices of the identity matrix, i.e. matrices defined by transformations of the form:

$$t_k = s_{n_k}$$

both algebras are $\mathbf{C}$. The complete set of matrices for which $\mathbf{A}^0 = \mathbf{C}$ is known [12] - it is clear that for such matrices all members of $\mathbf{A}_0$ are of the form r + t where r is convergent and t is thin. A by-product of this investigation was the solution of the problem of finding all matrices for which $x\ y \in \mathbf{A}_0$ implies $x\ y \in \mathbf{A}_0$, [12]. The solution of the corresponding problem for A_0 is already known. It also turned out that the matrices for which $\mathbf{A}^0 = \mathbf{C}$ are also those for which $A^0 = \mathbf{C}$.

Another problem is given a matrix A, to find all of the matrices B such that $B^0 = A^0$ or $\mathbf{B}^0 = \mathbf{A}^0$. For positive matrices, this can be very difficult, for non-positive matrices a solution can often be found.

4. Let $A = (a_{m,n})$ by the matrix defined by the transformations:

$$t_n = \tfrac{1}{2}(s_n + s_{n+1}). \tag{6}$$

For example, this matrix limits the sequence $\{(-1)^n\}$ to zero so that it is clear that if $\xi \in \mathbf{A}^0$ then

$$\xi_n - \xi_{n+1} = o(1). \tag{7}$$

On the other hand if ξ satisfies (7) and x is bounded and limited to zero by A, then

$$\tfrac{1}{2}(\xi_n x_n + \xi_{n+1} x_{n+1})$$

$$= \tfrac{1}{2}\xi_n(x_n + x_{n+1}) + \tfrac{1}{2}x_{n+1}(\xi_{n+1} - \xi_n). \qquad (8)$$

The second term has limit zero since x is bounded and hence A^0 consists of those bounded sequences satisfying (7).

We turn now to A^0. Since

$$s_{n+1} = 2[t_n - t_{n-1} + \ldots + (-1)^{n-1} t_1] + (-1)^n s_1$$

and $\{t_n\}$ has limit zero, we see that $s \in A_0$ implies that $s_n = o(n)$. If $x \in A_0$ then the first term in (8) has limit zero as before. However, now unless $(\xi_n - \xi_{n+1}) = 0(\frac{1}{n})$ we can choose $x \in A_0$ such that the second term is unbounded. We could investigate this matrix further by defining $A_0\ (\rho_n)$ as those sequences x limited to zero by A and satisfying the condition $x = 0\ (\rho_n)$, $\rho_n \nearrow \infty$. The algebra $A^0\ (\rho_n)$ could be defined in the obvious way. We would find that $\xi \in A^0\ (\rho_n)$, if and only if

$$\xi_n - \xi_{n+1} = 0(\frac{1}{\rho_n}).$$

We also observe that for regular matrices in general

$$A^0 \supset A^0\ (\rho_n)$$

and hence since A^0 is closed

$$A^0 \supset [\bigcup_{\rho_n} A^0\ (\rho_n)]^{-}. \qquad (9)$$

For the particular matrix defined by (6)

$$A^0 = [\bigcup_{\rho_n} A^0\ (\rho_n)]^{-} \qquad (10)$$

It would be most useful to know if (10) were true for all regular matrices.

5. A triangular matrix $A = (a_{m,n})$ is called an M-matrix if for some $k > 0$,

$$\left|\sum_{k=1}^{n} a_{m,k}\, s_k\right| \leqslant k \left|\sum_{k=1}^{n} a_{n',k}\, s_k\right|$$

for some n', $n' = n'(n)\,(0 \leqslant n' \leqslant n)$ and for all m. The number n' depends on n and $\{s_n\}$ but is independent of m. The concept of an M-matrix extends to non-regular matrices. It may turn out that not only is A an M-matrix but that $C = (c_{m,n})$ is an M-matrix, where

$$c_{m,n} = f(m)\, a_{m,n} \text{ and } f(m) \uparrow \infty.$$

Such a function we shall call a regulating function, see [9] and [16], [17]. Using the ideas of the previous paragraph it is shown in [16] that if

$$\frac{1}{f(m)} \sum_{n=1}^{m} f(n)\ |s_{n+1} - s_n| \leqslant M$$

and $s \in A_0$ then $s = r + t$ where r converges to zero and t is thin.

Many well known matrices, such as that of the Riesz mean and some cases of the Nørlund mean are M-matrices and the above analysis is effective.

Examples can be concocted however where this technique gives very poor results.

6. Suppose $A = (a_{m,n})$ is a regular matrix and in general the elements may be positive or negative. Let

$$N(A) = \max\,[\lim_{m\to\infty} \sum a_{m,n}\, s_n]$$

where the maximum is taken over all sequences s in the unit ball, i.e. $|s_n| \leqslant 1$, $(n = 1, 2, \ldots)$. This maximum exists and is attained by a sequence in the unit ball, see [2] and [18]. Sequences for which the maximum is attained are called extreme points.

We also consider $\|A\|$ (norm A) defined by

$$\|A\| = \inf (\sup_m \sum |a'_{m,n}|)$$

where the inf is taken over all matrices $A' = (a'_{m,n})$ for which $A' = A$. The following important theorems are due to Brudno [4], see [5].

<u>Theorem 3</u>. $N(A) = \|A\|$.

<u>Theorem 4</u>. If $B \supset A$ then $\|B\| \geqslant \|A\|$.

There may be a regular matrix $A' = (a_{m,n})$ such that

$$\sup_m \sum |a'_{m,n}| = \|A\| .$$

If such is the case, we say the norm of A is <u>attained</u>. Otherwise the norm is <u>not attained</u>.

If the norm is attained, there is a sequence of 1's and -1's among the extreme points and all other extreme points differ from this sequence by a thin sequence. If the norm is attained $A^0 \cap A_0$ consists of sequences of the form $r + t$ where r converges to zero and t is thin, see [2] and [18].

If the norm is not attained, the above statement may not be true. Let $A = (a_{m,n})$ be the matrix defined by

$$t_{2n} = \tfrac{1}{2}(s_{4n-3} + s_{4n-2})$$

$$t_{2n+1} = \tfrac{1}{2}(s_{4n-3} + s_{4n-2}) + s_{4n-1} - s_{4n}$$

Since the matrix defined by the even numbered rows is stronger than A, it is clear that $\|A\| = 1$. On the other hand the sequence x

$$x_{4n-3} = x_{4n-2} = 0, \; x_{4n-1} = x_{4n} = 1$$

is a memeber of $A^0 \cap A_0$ and is not thin so that the norm of A is not attained.

As yet no means of distinguishing the summability fields of those matrices for which the norm is attained from those for which it is not attained is known. Perhaps some combination of facts about extreme points and algebras will serve the purpose.

Let $t \in \mathbf{A}^0$ then $t^k \in \mathbf{A}^0$, $(k = 1, 2, \ldots)$. From the fact that $\mathbf{A}^0$ is closed we find that for complex sequences $e^{2\pi i t_n} \in \mathbf{A}^0$. Also $e^{2\pi i h t_n} \in \mathbf{A}^0$, $(h = 1, 2, \ldots)$, hereafter we write $e(x_n)$ for $e^{2\pi i x_n}$. The sequence s is said to be uniformly distributed with respect to A if A limits the sequences $e(h s_n)$ to zero, $(h = 1, 2, \ldots)$. Clearly if $t \in \mathbf{A}^0$, $e(h[s_n + t_n])$ is limited to zero so the sequence $s + t$ is uniformly distributed. Let $\mathbf{A}^*$ be such that $t \in \mathbf{A}^*$ if and only if $s + t$ is uniformly distributed with respect to A whenever s is uniformly distributed. We have seen

$$\mathbf{A}^* \supset \mathbf{A}^0 .$$

Unfortunately, $\mathbf{A}^*$ is not an algebra, but it can be proved that $\mathbf{A}^*$ is closed and that if

$$x, y \in \mathbf{A}^*, \ \|x\| \leqslant 1, \ \|y\| \leqslant 1$$

then $x y \in \mathbf{A}^*$. I intend to submit a paper proving this statement shortly.

For sequences <u>almost convergent</u> to zero, i.e. such that

$$\frac{1}{n} \sum_{k=p}^{n+p} s_k$$

has limit zero uniformly in p $\mathbf{A}^*$ and $\mathbf{A}^0$ are known, see [19] and in fact $\mathbf{A}^* = \mathbf{A}^0$ for sequences in the unit ball. The algebra A^0 is not relevant here.

8. The algebra A^0 can also be used upon occasion to prove Tauberian theorems in a gap form. Examples of matrices with the same A^0 can be found by choosing ξ so $\xi_n > \alpha > 0$ $(n = 1, 2, \ldots)$,

$$\liminf_{m\to\infty} \sum a_{m,n} \xi_n > \beta > 0$$

and then defining $B = (b_{m,n})$ by

$$b_{m,n} = a_{m,n} \xi_n \Big/ \sum_{n=1}^{\infty} a_{m,n} \xi_n .$$

It is clear from the structure of B and the simple mapping from A_0 to B_0 that $A^0 = B^0$.

There do not appear to be any simple relationships such as $\mathbf{A}_0 \supset \mathbf{B}_0$ implies $\mathbf{A}^0 \supset \mathbf{B}^0$ or $\mathbf{A}^0 \subset \mathbf{B}^0$ etc.

I conclude then that there are many questions associated with the algebras of a regular matrix whose answers would much advance our knowledge of the sequences it limits.

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