EXISTENCE AND UNIQUENESS OF ALGEBRAIC FUNCTION APPROXIMATIONS

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Abstract

The problem of approximating a real-valued, locally analytic function, $f(x)$, by an algebraic function, $Q(x)$ is considered. Existence and uniqueness theorems are obtained under fairly general conditions, including those of "non-normality".

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§1. Introduction.

The problem of approximating a real-valued function by an algebraic function has been receiving increased attention recently, particularly in the field of computational geometry. There are two separate aspects of this problem that merit attention. Firstly, the properties of such an approximation, including existence, uniqueness and convergence of the approximation should be investigated. Secondly, the difficulties with the computation of such approximations need to be considered.

It is the objective of this paper to consider the first aspect; specifically the existence and uniqueness of approximations by a general algebraic function, where the approximations are determined by sufficient derivative information about the given function at the origin. Such a generalization of the Taylor polynomial approximation and Padé rational approximation was considered by Padé [7], but he gave little attention to the questions of existence. However the approximations determined in this way are now generally known in the literature as algebraic Hermite-Padé approximations [2,6,8]. The recent paper by Baker and Lubinsky [2] considered the question of existence, uniqueness and convergence for a *normal* function. However, the approach taken in this paper is quite different and more general. It includes the case of the *non-normal* function.

In this paper a careful distinction is made between the approximation of the *algebraic form* which determines the polynomial coefficients of the implicit equation for the algebraic function, and the approximating properties of the algebraic function itself. Some of the difficulties in Padé approximation may be due to the fact that this clear distinction was not made as these concepts tend to overlap in this case (eg. [9]).

The distinction between these two concepts leads to the separation of degenerate cases arising from different sources, and consequently a clear treatment of the effects of each type. This has led to the definition of the "surplus" in the case of the algebraic form, and the definition of the "deficiency" in the case of the approximation of the algebraic function.

The result of this distinction is the reversion from the Baker definition of the Padé rational approximation to the classical Padé-Frobenius definition (see [1, p. 20]). The advantage of the change is that it provides a more general concept of the approximation and, more importantly, carries over to the more general algebraic approximation. Hence under this formulation, Padé approximants, and more general algebraic approximants always exist and may be always uniquely defined.

The significant concepts introduced in this paper are the importance of the algebraic form and the notation introduced for this, and the introduction of the concepts of the "surplus" and the "deficiency". In Section 2 the problem of the approximation of the algebraic form which determines the coefficient polynomials is considered. The concept of the "surplus" in the approximation of the algebraic form is introduced. In Section 3 the approximating properties of the algebraic function are investigated. The concept of the "deficiency" in the approximation is introduced, and the order of the approximation by the algebraic function is quantified. Section 4 contains a series of illustative examples to demonstrate the results of the previous sections in particular cases.

§2. The Algebraic Form.

Consider the problem of approximating a real-valued, locally analytic function $f(x)$, by an algebraic function, $Q(x)$. We may suppose the $f(x)$ is analytic in the neighborhood of the origin.

DEFINITION 1.

Let $f(x)$ be a formal power series. Let $p \geq 1$, and n_i, $i = 0(1)p$, $n_i \geq -1$, be integers and $\underset{\sim}{n} = (n_0, n_1, ..., n_p)$.

The function

$$P_{\underset{\sim}{n},p}(f, x) \equiv \sum_{i=0}^{p} a_i(x) f(x)^i = O(x^N) \tag{1}$$

will be called an $\underset{\sim}{n}$ algebraic form of degree p where $a_i(x)$ is a polynomial in x with degree $(a_i(x)) \leq n_i$ for $i = 0(1)p$ and $N + 1 = \sum_{i=0}^{p}(n_i + 1)$. (By convention the polynomial of degee -1 is identically zero). Generally the subscripts $\underset{\sim}{n}, p$ on P will be dropped where the context makes them obvious.

Note that $P(f, x)$ may also be written as a formal power series, $r(x) = \sum_{i=0}^{\infty} r_i x^i$, and that the conditions (1) are equivalent to the requirement that the linear functionals $D^k(P(f, x)) = D^k r(x) = r^{(k)}(0)/k! = 0$ for $k = 0(1)N - 1$.

If $p = 2$, the problem reduces to quadratic Hermite-Padé approximation [4] and if $p = 1$, it reduces to ordinary Padé approximation. If $p = 1$ and $n_1 = 0$ the problem reduces to the familiar Taylor polynomial approximation.

The question of the existence of such a form is easily disposed of.

THEOREM 2 (EXISTENCE).

There always exists an $\underset{\sim}{n}$ algebraic form of degree p for a given formal power series $f(x)$.

PROOF:

The algebraic form is defined by the coefficient polynomials $a_i(x)$. The existence of the $a_i(x)$ follows since these polynomials contain $N + 1$ coefficients, while (1) leads to N homogeneous linear equations for these coefficients. Hence a non-trivial solution, with $a_i(x)$ not all zero, exists. Further, if $a_i(x) \equiv 0$ for $i = 1(1)p$, then $a_0(x) \equiv 0$ and hence $A_p(x) = (a_1(x), ..., a_p(x)) \not\equiv 0$.

The question of the uniqueness of such a form requires rather more care. The matrix form of the system of linear equations represented by (1) has the coefficient matrix

$$F = [F_{n_0} : F_{n_1} : ... : F_{n_p}]$$

$$\text{where} \quad F_{n_0} = \begin{bmatrix} I_{n_0+1} \\ 0 \end{bmatrix}$$

$$\text{and} \quad F_{n_k} = \begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ \cdot & \cdot & & \\ g_{n_k} & g_{n_k-1} & \cdots & g_0 \\ \cdot & \cdot & & \cdot \\ g_{N-1} & \cdots & & g_{N-1-n_k} \end{bmatrix}$$

where the g_j in this submatrix are defined by $f(x)^k = \sum_{j=0}^{\infty} g_j x^j$.

The matrix F has dimensions $N\times(N+1)$ and hence has a solution space of dimension at least 1. If the solution space dimension is exactly 1 then any constant multiple of the coefficient polynomials $\{a_i(x)\}_{i=0}^{p}$ will also be a solution. These solutions may be called [2] *essentially unique*. A unique representative of the class may be defined by choosing any convenient suitable normalization of the coefficients. For example in [2], the normalization is chosen so that all the coefficients in $A_p(x)$ have absolute value at most one, with equality for at least one coefficient. However the traditional normalization used in the rational (Padé) approximation case ($p=1$) in [1] is not a suitable normalization since there is no requirement that $a_1(0)\neq 0$.

Of more serious concern however, is the case when the solution space is multi-dimensional. This may occur if the matrix F has rank $< N$. If the rank of F is $N+1-k$, then the solution space of the algebraic forms has dimension $k>1$. This situation, for the quadratic case, $p=2$, was discussed in [5].

It may happen that $r(x) = O(x^R),\ \neq O(x^{R+1})$ as $x\to 0$, for $R\geq N$. Following Paszkowski [8], we define the *order of the algebraic form* to be

$$R = Ord(P(f,x)) \text{ if } P(f,x) = O(x^R),\ \neq O(x^{R+1}) \text{ as } x\to 0.$$

Clearly if $R=\infty$, then $P(f,x)=0$ defines an algebraic function and so the general objective is to obtain an R as large as possible.

If the dimension of the solution space is $k>1$, then the choice of a unique representative form for the space is less clear. In this case the solution space is multi-dimensional and the order of the algebraic form $P(f,x)$ is to be interpreted as the minimum of the orders of the multiple solutions to equation (1). If $P(f,x)$ has order R, and $\underset{\sim}{a}^{(i)} = (a_0^{(i)},\cdots,a_p^{(i)}(x)),\ i=1,2$, are two linearly independent solutions to (1), then by taking a suitable linear combination, $c_1\underset{\sim}{a}^{(1)}(x)+c_2\underset{\sim}{a}^{(2)}(x)$, of these solutions, the term $(c_1r_R^{(1)}+c_2r_R^{(2)})x^R$ may be eliminated and an $\underset{\sim}{n}$ algebraic form of order at least $R+1$ is obtained, with the dimension of the solution space decreased by 1. Thus in general, beginning with a basis in the k-dimensional solution space, we seek a one dimensional subspace whose elements satisfy $P(f,x)=O(x^R)$ where the order R is maximal over the space of $\underset{\sim}{n}$ algebraic forms.

THEOREM 3 (UNIQUENESS).

There always exists an essentially unique $\underset{\sim}{n}$ algebraic form of order p, which is of maximal order $R\geq N$, and which may be chosen uniquely by a suitable normalization of the coefficients of the coefficient polynomials. This unique representative will be denoted by $P^*(f,x)$.

PROOF: As noted above, if the matrix F has rank N, the solution space is dimension 1 and the result is trivial.

If the matrix F has rank $N+1-k$ for $k>1$, then the solution space has dimension k. Let $\underset{\sim}{a}^{(i)}(x),\ i=1(1)k$, represent a basis for this solution space, and suppose $P(f,x)$ has order R (to be interpreted as the minimum order of the multiple solutions as noted above). Then $\sum_{i=1}^{k} c_i\underset{\sim}{a}^i(x)$ is an $\underset{\sim}{n}$ algebraic form of order $R+k-1$ where the constants c_i are defined by the linear system

$$\sum_{i=1}^{k} c_i\, r_{R+j}^{(i)} = 0,\ \ j=0(1)k-2.$$

If the matrix of this system of linear equations has rank $k-1$, then there exists an essentially unique solution. However, if this matrix has rank $< k-1$, then we must iterate this process since there are still linearly independent solutions. Since the rank reduces by at least one at each step, the iterations are finite, noting of course, that if we obtain a zero matrix then $R = \infty$ and the exact algebraic form, and hence the exact algebraic function, has been obtained.

It was noted in [5] that, in the case $p = 2$, if $f(x)$ is an even function, then $f(x)^2$ is also even. An examination of the matrix F in this case reveals that if the n_k are all even then the matrix has rank of at most $N-1$, and hence the solution space has dimension of at least 2. Some specific examples are given in [5].

Also if $p = 4$, then an examination of the matrix F for an even function in this case reveals that if the n_k are all even then the matrix has rank of at most $N-2$. Thus three linearly independent solutions may be obtained in this case. These may be represented by an essentially unique solution of order at least $R+2$.

§2.1 Degeneracies in the Algebraic Form.

In the algebraic form $P(f,x)$ defined by equation (1), it is possible that some of the coefficient polynomials are not of full degree, i.e. $deg(a_i(x)) < n_i$ for some i. This deficiency is of no particular consequence and may be compared to the fact that the Taylor polynomial approximation of degree one, $p_1(x)$, to the function $f(x) = 1 + x^2$ is $p_1(x) = 1$, i.e. there is a zero coefficient of the term in x.

Of greater importance is the fact that the particular set of coefficient polynomials which solve (1) may in fact eliminate more of the coefficients of $r(x)$ than just the first N. A trivial example in the polynomial case is that of the zero degree Taylor polynomial approximation to $f(x) = 1 + x^2$. In this case $N = 1$, but the form

$$-1 + 1f(x) = O(x^2).$$

Although the term "degeneracy" was used in this connection in [5], the term "surplus" seems more appropriate in that more matching than expected of the coefficients of $r(x)$ has occurred.

DEFINITION 4.

The surplus, $S(\underset{\sim}{n})$ *of the* $\underset{\sim}{n}$ *algebraic form* $P^*(f,x)$ *is defined by*

$$S(\underset{\sim}{n}) = Ord(P^*(f,x)) - N$$

where $Ord(r(x)) = R$ *if* $O(r(x)) = O(x^R), \neq O(x^{R+1})$ *as* $x \to 0$.

The surplus, $S(\underset{\sim}{n}) = S \geq 0$, is the amount of extra matching obtained from $P^*(f,x)$. Trefethen [9] introduced a similar concept for the Padé approximations rather than the Padé forms, when studying a related problem. However the approximation problem will be dealt with in the next section.

It is clear that in general we would like the surplus, S, to be as large as possible, since if $S = \infty$ then the form represents an algebraic function exactly.

In [5], the surplus (in the present notation) was used to define an S-table of algebraic forms (D-table in the notation of [5]) which was shown, in the case of $p = 1$, to give the block structure of the Padé table in a somewhat easier fashion than the more traditional

C-table [1]. In the case of algebraic forms of degree p, the S-table of algebraic forms $P^*(f,x)$ would be a $(p+1)$-dimensional table. The structure of this table could be expected to lead to the structure of the table for $\underset{\sim}{n}$ algebraic form of degree p in an analogous way. However an initial investigation of the case $p = 2$ in [5] shows that some complexity can arise from the overlapping of the basic block structure.

The results on the block structure of the S-table may be summarized in the following theorem which extends a theorem (for $p = 2$) in [5].

THEOREM 5.

If the $\underset{\sim}{n}$ algebraic form of degree p, $P^*_{\underset{\sim}{n}}(f,x)$, has a surplus $S(\underset{\sim}{n}) = S > 0$, then

$$x^r\, P^*_{\underset{\sim}{n}}(f,x),\ \ r = 0(1)(S/p)$$

is an $\underset{\sim}{m}$ algebraic form with a surplus of $S(\underset{\sim}{m}) = S(\underset{\sim}{n}) - pr - \sum_{k=0}^{p} i_k$, where

$$\underset{\sim}{m} = (m_0, m_1, \cdots, m_p) \text{ and } m_k = n_k + r + i_k,\ \ k = 0(1)p$$

and

$$i_k \geq 0,\ \ k = 0(1)p,\ \ \text{satisfies} \sum_{k=0}^{p} i_k \leq S - pr.$$

PROOF:

Since $Ord(P^*_{\underset{\sim}{n}}(f,x)) = N + S$, then

$$Ord(x^r\, P^*_{\underset{\sim}{n}}(f,x)) = N + S + r.$$

Hence $x^r P^*_{\underset{\sim}{n}}(f,x)$ will be an $\underset{\sim}{n} + r\underset{\sim}{o}$ (where $\underset{\sim}{o} = (1,1,\cdots 1)$) algebraic form of degree p provided $pr \lesssim S$. The surplus of this algebraic form is $S - pr$. Further, it follows that this algebraic form is also an algebraic form of the type $\underset{\sim}{m}$ where $\underset{\sim}{m} = (m_0, m_1, \cdots, m_p)$ and $m_k = n_k + r + i_k,\ \ k = 0(1)p$ with $i_k \geq 0, k = 0(1)p$ satisfies

$$\sum_{k=0}^{p} i_k \leq S - pr,$$

since $P_{\underset{\sim}{m}}(f,x) = O(x^M)$ with $M + 1 = \sum_{i=0}^{p}(m_i + 1)$.

If there is another structure of this type which overlaps this one, then for some of these values of $\underset{\sim}{m}$ there will be additional linearly independent solutions. Theorem 3 will need to be applied to find $P^*_{\underset{\sim}{m}}(f,x)$, which will come from the set of additional solutions since these have greater order. However, if there is no overlapping structure, then for those $\underset{\sim}{m}$ such that $i_k = 0$ for at least one value of k, $P_{\underset{\sim}{m}}(f,x)$ has maximal order. This follows since the polynomial coefficient corresponding to this value of k has full degree and hence no additional factors of x may be multiplied through in equation (1) to raise the nominal order. Thus, if there is no overlapping structure and $\underset{\sim}{m}$ is such that $i_k = 0$ for at least one value of k then $P_{\underset{\sim}{m}}(f,x) = P^*_{\underset{\sim}{m}}(f,x)$. Further details on the block stucture will be the subject of a future report.

A simple example of this type of structure is taken from [5].

EXAMPLE 6.

Let $p = 2$ and $f(x) = \cos(x)$. Since $f(x)$ is an even function, the $(2,2,2)$ algebraic form for this function has surplus $S = 2$ with polynomial coefficients

$$\begin{aligned} a_0(x) &= 578\,x^2 - 1425, \\ a_1(x) &= 344\,x^2 + 960, \\ a_2(x) &= 23\,x^2 + 465. \end{aligned}$$

This is also the algebraic form of types $(3,2,2)$, $(2,3,2)$, $(2,2,3)$ with $S = 1$; the algebraic form of types $(3,3,2)$, $(3,2,3)$, $(2,3,3)$ with $S = 0$, and the $(3,3,3)$ algebraic form has polynomial coefficients $xa_0(x)$, $xa_1(x)$, $xa_2(x)$, with $S = 0$. Further, these polynomial coefficients are also an algbraic form of types $(4,2,2)$, $(2,4,2)$, $(2,2,4)$ with $S = 0$. But as noted above, the solution space for an even function with n_k all even has dimension 2 and hence there will be an additional linearly independent solution in these cases. Applying the uniqueness theorem (Theorem 3) gives an essentially unique algebraic form of nominal order 11. However, since the polynomial coefficients are all even (see [3]), the algebraic form is even and hence has order 12. Thus these forms also have surplus $S = 2$. This is an example of the overlapping block structure noted above and which is illustrated by the tables in [5].

An even simpler example for the rational approximation case, $p = 1$, is adapted from [1].

EXAMPLE 7.

Let $p = 1$ and $f(x) = 1 + x^2$.

The $(0,0)$ algebraic form for this function has polynomial coefficients $a_0(x) = -1$, $a_1(x) = 1$ and surplus $S = 1$. Hence this is also an algebraic form of types $(0,1)$ and $(1,0)$ and the $(1,1)$ algebraic form has polynomial coefficients $xa_0(x) = -x$ and $xa_1(x) = x$, and surplus $S = 0$.

Similar examples have been verified in the higher degree cases.

§3. The Algebraic Approximation.

Once the unique $\underset{\sim}{n}$ algebraic form of degree p, $P^*(f,x)$, satisfying equation (1) has been obtained, it is clear that we can define an $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, which satisfies the equation

$$P^*(Q,x) = \sum_{i=0}^{p} a_i^*(x)Q(x)^i = 0. \tag{2}$$

Since the coefficients, $a_i^*(x)$, are determined by the given function, $f(x)$, this function, $Q(x)$, represents an approximation of $f(x)$. Recall that $f(x)$ is assumed given in the sense that a sufficient number of the derivative values at the origin are known.

From the general theory of algebraic functions, it is known that equation (2) *normally* has p distinct analytic branches at the origin. This "normal" behaviour is true if the function $P^*(Q,x)$, as a polynomial in Q, has distinct roots at the origin. Thus we require the condition

$$\partial P^*(Q,x)/\partial Q|_{x=0} \neq 0$$

for normal behaviour.

DEFINITION 8.

The function, $f(x)$, and its associated $\underset{\sim}{n}$ algebraic form $P^*(f,x)$, are called normal if

$$\partial P^*(f,x)/\partial f|_{x=0} \neq 0.$$

If $p = 1$, then $\partial P^*(f,x)/\partial f|_{x=0} = a_1(0)$. Thus this definition corresponds to the conventional definition used in Padé approximation. It also corresponds to the condition that Baker and Lubinsky [2] found to be "essential for the existence of a unique solution".

Only one of these distinct branches will have $Q(0) = f(0)$ since $P^*(Q(0),0) = 0 = P^*(f(0),0)$. Hence there is a unique branch of the function $Q(x)$ which passes through the point $(0, f(0))$, and we may consider the approximation properties of this branch as an approximation to $f(x)$. The following theorem corrects a similar theorem in [3] which did not consider the concept of the surplus.

THEOREM 9.

Let $P^*(f,x)$ be the normal $\underset{\sim}{n}$ algebraic form of degree p defined by equation (1), and satisfying $P^*(f,x) = O(x^{N+S})$, where $S \geq 0$ is the surplus. Then the $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, defined by the equation

$$P^*(Q,x) = 0,$$

subject to the condition

$$Q(0) = f(0),$$

is an approximation to $f(x)$ satisfying

$$Q(x) = f(x) + O(x^{N+S}).$$

PROOF:

Let $N + S = R$. Since we have

$$P^*(f,x) = O(x^R), \; P^*(Q,x) = 0$$

then

$$\frac{d^i}{dx^i} P^*(f,x)|_{x=0} = 0 = \frac{d^i}{dx^i} P^*(Q,x), \; i = 0(1)R-1. \tag{3}$$

For $i = 0$: $P^*(f(0),0) = P^*(Q(0),0)$ since $Q(0) = f(0)$.
For $i = 1$:

$$\left[\frac{\partial P^*(f,x)}{\partial f} \cdot \frac{df}{dx} + \frac{\partial P^*(f,x)}{\partial x}\right]|_{x=0} = \left[\frac{\partial P^*(Q,x)}{\partial Q} \cdot \frac{dQ}{dx} + \frac{\partial P^*(Q,x)}{\partial x}\right]|_{x=0}.$$

Since f is normal and $Q(0) = f(0)$, then

$$\partial P^*(f,x)/\partial f|_{x=0} = \partial P^*(Q,x)/\partial Q|_{x=0} \neq 0.$$

Also $Q(0) = f(0)$ implies

$$\partial P^*(f,x)/\partial x|_{x=0} = \partial P^*(Q,x)/\partial x|_{x=0}.$$

Hence equation (3) with $i = 1$ implies

$$Q'(0) = f'(0).$$

In general,

$$\left[\frac{\partial P^*(f,x)}{\partial f} \cdot \frac{d^i f}{dx^i} + F_i\right]|_{x=0} = 0 = \left[\frac{\partial P^*(Q,x)}{\partial Q} \cdot \frac{d^i Q}{dx^i} + G_i\right]|_{x=0}, \quad i = 1(1)R-1, \tag{4}$$

where

$$F_1 = \partial P^*(f,x)/\partial x,$$

$$F_i = F_i(x, f(x), f'(x), \cdots, f^{(i-1)}(x)) = \frac{d}{dx}\left(\frac{\partial P^*(f,x)}{\partial f}\right) \cdot \frac{d^{i-1}f}{dx^{i-1}} + \frac{dF_{i-1}}{dx},$$

$$G_1 = \partial P^*(Q,x)/\partial x,$$

$$G_i = F_i(x, Q(x), Q'(x), \cdots, Q^{(i-1)}(x)).$$

for $i = 2(1)R-1$.

The induction step implies that since

$$Q^{(j)}(0) = f^{(j)}(0), \quad j = 0(1)(i-1),$$

then $F_i = G_i$, and hence, since f is normal, equation (4) implies that $Q^{(i)}(0) = f^{(i)}(0)$. It follows that

$$Q^{(i)}(0) = f^{(i)}(0), \quad i = 0(1)R-1$$

and hence

$$Q(x) = f(x) + O(x^R).$$

Hence, in the case $f(x)$ is a normal function, a unique branch of the $\underset{\sim}{n}$ algebraic function of degree p is obtained from the unique corresponding $\underset{\sim}{n}$ algebraic form, and this branch of the algebraic function approximates $f(x)$ in the neighborhood of the origin with the same order of approximation as the corresponding form.

§3.1 The Non-Normal Case.

However as has been observed in [4], the case in which f is not normal is not uncommon. A simple example of when $\partial P^*(f,x)/\partial f|_{x=0} = 0$ occurs when the algebraic form $P^*(f,x)$ has a factor of x^r, $r > 0$, as was obtained in Theorem 5. But less obvious examples such as for $f(x) = \log(1+x)$, are also not uncommon as noted in [4]. This situation leads to an approximation whose order is less than that expected in the normal case.

As noted in [1], Padé defined a deficiency index to measure the shortcoming of the rational approximation. Since this concept will be shown to be a generalization of the Padé idea, the same terminology is appropriate.

Definition 10.

The deficiency, $D(\underset{\sim}{n}) = D \geq 0$, of the $\underset{\sim}{n}$ algebraic form $P^*(f,x)$ is defined by

$$D(\underset{\sim}{n}) = Ord(\partial P^*(f,x)/\partial f).$$

It will be shown that the deficiency, $D(\underset{\sim}{n}) = D \geq 0$, is the amount by which the order of the approximation falls short of the expected order.

A number of cases need to be dealt with in turn. Firstly we consider the case where the algebraic form has a factor of x^r, $r > 0$.

Lemma 11.

If $P^*_{\underset{\sim}{n}}(f,x) = x^r\, P^*_{\underset{\sim}{m}}(f,x)$ where $P^*_{\underset{\sim}{m}}(f,x)$ has polynomial coefficients with no common factor of x (i.e. $\sum_{i=0}^{p} |a_i(0)| \neq 0$ in the notation of [4]), then

$$D = D(\underset{\sim}{n}) = D(\underset{\sim}{m}) + r.$$

Proof:

$$\partial P^*_{\underset{\sim}{n}}(f,x)/\partial f|_{x=0} = [x^r\, \partial P^*_{\underset{\sim}{m}}(f,x)/\partial f]|_{x=0},$$

and the result follows from Definition 10.

Theorem 12.

Let

$$P^*_{\underset{\sim}{n}}(f,x) = x^r\, P^*_{\underset{\sim}{m}}(f,x),$$

where $P^*_{\underset{\sim}{m}}(f,x)$ has polynomial coefficients with no common factor of x, and $P^*_{\underset{\sim}{m}}(f,x)$ is the normal algebraic form satisfying $P^*_{\underset{\sim}{m}}(f,x) = O(x^{M+S(\underset{\sim}{m})})$, where $S(\underset{\sim}{m})$ is the surplus and $M + 1 = \sum_{i=0}^{p}(m_i + 1)$.

Then the $\underset{\sim}{n}$ algebraic function $Q(x)$ defined by the equation

$$P^*_{\underset{\sim}{n}}(Q,x) = 0,$$

subject to the condition

$$Q(0) = f(0),$$

is an approximation to $f(x)$ satisfying

$$Q(x) = f(x) + O(x^{N+S-D}),$$

where $D = D(\underset{\sim}{n}) \geq 0$ is the deficiency of $P^*_{\underset{\sim}{n}}(f,x)$,
and $S = S(\underset{\sim}{n}) \geq 0$ is the surplus of $P^*_{\underset{\sim}{n}}(f,x)$.

PROOF:

By Lemma 11, $D = D(\underset{\sim}{n}) = D(\underset{\sim}{m}) + r$. Since $P^*_{\underset{\sim}{m}}(f,x)$ is normal, $D(\underset{\sim}{m}) = 0$ and hence $D = r$.

$P^*_{\underset{\sim}{n}}(Q,x) = x^r P^*_{\underset{\sim}{m}}(Q,x) = 0$, and hence the $\underset{\sim}{n}$ algebraic function of degree p defined by $P^*_{\underset{\sim}{n}}(Q,x) = 0$ is in fact the same as the $\underset{\sim}{m}$ algebraic function of degree p defined by $P^*_{\underset{\sim}{m}}(Q,x) = 0$. Hence, by Theorem 9, we have

$$Q(x) = f(x) + O(x^{M+S(\underset{\sim}{m})}).$$

By Theorem 5, we have $N = M + (p+1)r$ and $S(\underset{\sim}{n}) = S(\underset{\sim}{m}) - pr$. Hence $M + S(\underset{\sim}{m}) = N + S(\underset{\sim}{n}) - r$.

Suppose now that $P^*_{\underset{\sim}{n}}(f,x)$ is an algebraic form whose polynomial coefficients have no common factor of x and that $\partial P^*_{\underset{\sim}{n}}(f,x)/\partial f|_{x=0} = 0$.

If $\partial P^*(f,x)/\partial f|_{x=0} = 0$ and $Q(0) = f(0)$ then $\partial P^*(Q,x)/\partial Q|_{x=0} = 0$. For $p > 1$, this means that at least two of the branches of $Q(x)$ coincide at the origin.

In order to avoid additional complications but still illustrate the nature of the results, we will assume that, for $p > 1$, only two of the branches of $Q(x)$ coincide at the origin, and hence $\partial^2 P^*(Q,x)/\partial Q^2|_{x=0} \neq 0$.

The two branches that coincide at the origin may be distinct branches (only coinciding at the origin), or they may coincide everywhere. If the two roots are in fact distinct branches, let the order of contact of these two roots at the origin be $t - 1 < \infty$, and hence $\partial P^*(Q,x)/\partial Q = O(x^t)$. In this case, in order to distinguish between these two branches it is necessary to impose an additional condition on $Q^{(t)}(0)$.

If $\partial P^*(f,x)/\partial f = O(x^t)$, then since

$$\partial P^*(f,x)/\partial f|_{x=0} = \partial P^*(Q,x)/\partial Q|_{x=0} \quad \text{if} \quad Q(0) = f(0),$$

we have $\partial P^*(Q,x)/\partial Q = O(x^t)$ and the algebraic function corresponding to the algbraic form $P^*(f,x)$ has two roots with order of contact $t-1$ at the origin. From the general theory of algebraic functions it is known that these branches are also locally analytic. By applying the argument of Theorem 9 to the function $\partial P^*(f,x)/\partial f$, it immediately follows that

$$Q^{(i)}(0) = f^{(i)}(0), \qquad i = 0(1)t-1,$$

since $\partial P^*(f,x)/\partial f = O(x^t) = \partial P^*(Q,x)/\partial Q$. Consequently, by imposing the additional condition

$$Q^{(t)}(0) = f^{(t)}(0)$$

we are able to distinguish a unique branch of the approximating function $Q(x)$.

LEMMA 13.

If the $\underset{\sim}{n}$ algebraic form of degree p, $P^*(f,x)$, with $\sum_{i=0}^{p}|a_i(0)| \neq 0$, satisfies

$$\partial P^*(f,x)/\partial f = O(x^t) \quad \text{and} \quad \partial^2 P^*(f,x)/\partial f^2|_{x=0} \neq 0,$$

then the deficiency $D = D(\underset{\sim}{n}) = t$.

PROOF:

This follows immediately from the Definition 10 since $P^*(f,x)$ has polynomial coefficients with no common factor of x.

THEOREM 14.

If S is the surplus and D is the deficiency of the $\underset{\sim}{n}$ algebraic form of degree $p > 1$, $P^*(f,x)$, satisfying

$$\begin{aligned} &(i) && \sum_{i=0}^{p}|a_i(0)| \neq 0, \\ &(ii) && \partial P^*(f,x)/\partial f = O(x^D), \quad D < (N+S)/2, \\ &(iii) && \partial^2 P^*(f,x)/\partial f^2|_{x=0} \neq 0, \end{aligned}$$

then the associated $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, defined by the equation

$$P^*(Q,x) = 0,$$

subject to the initial conditions

$$Q^{(i)}(0) = f^{(i)}(0), \qquad i = 0(1)D,$$

is an approximation to $f(x)$ satisfying

$$Q(x) = f(x) + O(x^{N+S-D}).$$

PROOF:

Let $N + S = R$. Following a similar argument to the proof of Theorem 9, we have

$$\text{(5)} \qquad \begin{aligned} &P^*(f,x) = O(x^R), \quad P^*(Q,x) = 0, \\ &\frac{\partial P^*(f,x)}{\partial f} = O(x^D), \quad \frac{\partial P^*(Q,x)}{\partial Q} = O(x^D). \end{aligned}$$

From equations (5)

$$\text{(6)} \qquad \frac{d^i}{dx^i}P^*(f,x)|_{x=0} = 0 = \frac{d^i}{dx^i}P^*(Q,x)|_{x=0}, \quad i = 0(1)R-1,$$

$$\text{(7)} \qquad \frac{d^j}{dx^j}\left(\frac{\partial P^*(f,x)}{\partial f}\right)|_{x=0} = 0 = \frac{d^j}{dx^j}\left(\frac{\partial P^*(Q,x)}{\partial Q}\right)|_{x=0}, \quad j = 0(1)D-1,$$

and

$$(8)\qquad \frac{d^D}{dx^D}\left(\frac{\partial P^*(f,x)}{\partial f}\right)|_{x=0} = \frac{d^D}{dx^D}\left(\frac{\partial P^*(Q,x)}{\partial Q}\right)|_{x=0} \neq 0,$$

since $Q^{(i)}(0) = f^{(i)}(0),\ i = 0(1)D$.

(a) Consider the case $D = 1$. For $i = 3$ in equation (6)

$$\left[\frac{\partial P^*}{\partial f}\cdot\frac{d^3 f}{dx^3} + 2\frac{d}{dx}\left(\frac{\partial P^*}{\partial f}\right)\frac{d^2 f}{dx^2} + \frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial f}\right)\cdot\frac{df}{dx} + \frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial x}\right)\right]|_{x=0} = 0,$$

and

$$(9)\qquad \left[\frac{\partial P^*}{\partial Q}\cdot\frac{d^3 Q}{dx^3} + 2\frac{d}{dx}\left(\frac{\partial P^*}{\partial Q}\right)\frac{d^2 Q}{dx^2} + \frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial Q}\right)\cdot\frac{dQ}{dx} + \frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial x}\right)\right]|_{x=0} = 0.$$

For each of these equations (9), the first term vanishes using (7), and the last two terms may be rewritten in the form

$$\frac{d}{dx}\left(\frac{\partial P^*}{\partial f}\right)\frac{d^2 f}{dx^2} + F_3(x,f,f') \quad \text{and} \quad \frac{d}{dx}\left(\frac{\partial P^*}{\partial Q}\right)\frac{d^2 Q}{dx^2} + F_3(x,Q,Q')$$

respectively. Hence from (9)

$$(10)\quad \left[3\frac{d}{dx}\left(\frac{\partial P^*}{\partial f}\right)\cdot\frac{d^2 f}{dx^2} + F_3(x,f,f')\right]|_{x=0} = \left[3\frac{d}{dx}\left(\frac{\partial P^*}{\partial Q}\right)\frac{d^2 Q}{dx^2} + F_3(x,Q,Q')\right]|_{x=0}.$$

Equation (10), using the initial conditions and (8), implies

$$Q^{(2)}(0) = f^{(2)}(0).$$

Proceeding now as in the proof of Theorem 9, we obtain in general

$$\left[\frac{\partial P^*}{\partial f}\cdot\frac{d^k f}{dx^k} + \binom{k}{1}\frac{d}{dx}\left(\frac{\partial P^*}{\partial f}\right)\cdot\frac{d^{k-1} f}{dx^{k-1}} + \binom{k}{2}\frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial f}\right)\frac{d^{k-2} f}{dx^{k-2}} + \frac{d}{dx}F_{k-1}\right]|_{x=0} = 0$$

and

$$\left[\frac{\partial P^*}{\partial Q}\frac{d^k Q}{dx^k} + \binom{k}{1}\frac{d}{dx}\left(\frac{\partial P^*}{\partial Q}\right)\cdot\frac{d^{k-1} Q}{dx^{k-1}} + \binom{k}{2}\frac{d^2}{dx^2}\left(\frac{\partial P^*}{\partial Q}\right)\frac{d^{k-2} Q}{dx^{k-2}} + \frac{d}{dx}G_{k-1}\right]|_{x=0} = 0$$

$$(11)\qquad \text{for } k = 4(1)R-1,$$

where

F_3 is the function obtained in (10),

$F_k = F_k(x, f(x), f'(x)), \ldots, f^{(k-2)}(x)),$

$G_k = F_k(x, Q(x), Q'(x), \ldots, Q^{(k-2)}(x)),$

for $k = 4(1)R-1$.

As before the first term in each of the equations (11) vanishes by (7). The induction step implies that since $Q^{(j)}(0) = f^{(j)}(0),\ j = 0(1)(k-2)$, equations (11) imply

$$Q^{(k-1)}(0) = f^{(k-1)}(0).$$

It follows that

$$Q^{(j)}(0) = f^{(j)}(0),\ j = 0(1)R-2$$

and hence

$$Q(x) = f(x) + O(x^{R-1}) = O(x^{R-D}).$$

(b) The general case follows in a similar fashion. For $i = 2D+1$ in equation (6)

$$\left[\sum_{k=0}^{2D} \binom{2D}{k} \frac{d^k}{dx^k}\left(\frac{\partial P^*}{\partial f}\right)\frac{d^{2D+1-k} f}{dx^{2D+1-k}} + \frac{d^{2D}}{dx^{2D}}\left(\frac{\partial P^*}{\partial x}\right)\right] \Big|_{x=0} = 0,$$

and

$$\left[\sum_{k=0}^{2D} \binom{2D}{k} \frac{d^k}{dx^k}\left(\frac{\partial P^*}{\partial Q}\right)\frac{d^{2D+1-k} Q}{dx^{2D+1-k}} + \frac{d^{2D}}{dx^{2D}}\left(\frac{\partial P^*}{\partial x}\right)\right] \Big|_{x=0} = 0. \tag{12}$$

Now

$$\begin{aligned}&\sum_{k=D+1}^{2D} \binom{2D}{k} \frac{d^k}{dx^k}\left(\frac{\partial P^*}{\partial f}\right)\frac{d^{2D+1-k} f}{dx^{2D+1-k}} + \frac{d^{2D}}{dx^{2D}}\left(\frac{\partial P^*}{\partial x}\right) \\ &= \sum_{k=1}^{D} \binom{2D}{k-1} \frac{d^k}{dx^k}\left(\frac{\partial P^*}{\partial f}\right) \cdot \frac{d^{2D+1-k} f}{dx^{2D+1-k}} + F_{2D+1}(x, f(x), f'(x), \ldots, f^{(D)}(x)),\end{aligned}$$

and similarly for the form with $Q(x)$. Hence using equations (7), equations (12) imply

$$\begin{aligned}&\left[K \frac{d^D}{dx^D}\left(\frac{\partial P^*}{\partial f}\right) \frac{d^{D+1} f}{dx^{D+1}} + F_{2D+1}(x, \ldots, f^{(D)}(x))\right] \Big|_{x=0} \\ &= \left[K \frac{d^D}{dx^D}\left(\frac{\partial P^*}{\partial Q}\right) \frac{d^{D+1} Q}{dx^{D+1}} + F_{2D+1}(x, \ldots, Q^{(D)}(x))\right] \Big|_{x=0}\end{aligned}$$

where $K = \binom{2D}{D} + \binom{2D}{D-1}$.

With the initial conditions and (8) this equation implies

$$Q^{D+1}(0) = f^{D+1}(0).$$

Using equation (6) with $i = (2D+2)(1)(R-1)$, and proceeding by induction as before implies that

$$Q^{(j)}(0) = f^{(j)}(0), \quad j = 0(1)(R-1-D),$$

and it follows that

$$Q(x) = f(x) + O(x^{R-D}).$$

COROLLARY 15.

For an algebraic form satisfying the conditions of Theorem 14, the deficiency D satisfies $2D \leq R-1$ and the order of the approximation by an algebraic function will always be at least $R/2$ where $R = N + S$. That is, at worst, $Q(x)$ satisfies

$$Q(x) = f(x) + O(x^{R/2}).$$

PROOF:

Note that if we take $i = 2D$ in equation (6) this equation is automatically satisfied by the initial conditions. Hence if $2D > R-1$ then (6) is satisfied by the initial conditions. Hence $2D \leq R-1$. The order of the approximation, $R - D \geq R - (R-1)/2 \geq R/2$, taking account of the fact that R may be odd or even.

This result was also obtained for the case $p = 2$, $S = 0$ in [4], which also has some illustrative examples of how the approximation is degraded in the non-normal case.

The final case to be considered is that in which the two branches of $Q(x)$ coinciding at the origin in fact coincide everywhere. In this case the order of contact is infinite and $\partial P^*(Q,x)/\partial Q = 0$. The condition $Q(0) = f(0)$ distinguishes this coincident branch and we consider the order of approximation in this case.

LEMMA 16.

If the $\underset{\sim}{n}$ algebraic form of degree p, $P^*(f,x)$, with $R = N + S \geq N$, satisfies

$$\begin{aligned} &(i) && \sum_{i=0}^{p} |a_i(0)| \neq 0, \\ &(ii) && P^*(f,x) = O(x^R), \\ &(iii) && \partial P^*(f,x)/\partial f = O(x^{R/2}), \\ &(iv) && \partial^2 P^*(f,x)/\partial f^2|_{x=0} \neq 0, \end{aligned}$$

then the deficiency $D = D(\underset{\sim}{n}) = R/2$ and $\partial P^*(Q,x)/\partial Q = 0$.

PROOF:

The statement $D = R/2$ follows immediately from Definition 10, as in Lemma 13.

If $\partial P^*(f,x)/\partial f = O(x^{R/2})$, then two roots have order of contact $R/2 - 1$ at the origin. As noted previously, since $\partial^2 P^*(f,x)/\partial f^2|_{x=0} \neq 0$, the initial conditions, $Q^{(i)}(0) = f^{(i)}(0), i = 0(1)R/2 - 1$, immediately follow. Hence, as noted in the proof of Corollary 15, equation (6) is automatically satisfied by these initial conditions. Thus $P^*(Q,x) = O(x^R)$, which implies $P^*(Q,x) = 0$ since $R \geq N$ and $P^*(Q,x)$ is completely determined (up to a constant multiple) by these N conditions. Thus $P^*(Q,x) = 0$ has two roots which match up to $O(x^{R/2})$, which implies $P^*(Q,x) = O(x^R)$, which implies $P^*(Q,x) = 0$. That is, an additional condition to distinguish between these two roots would imply more conditions than are necessary to determine $P^*(Q,x)$. Hence $P^*(Q,x) = 0$ must have two identical roots and $\partial P^*(Q,x)/\partial Q = 0$.

THEOREM 17.

If S is the surplus and D is the deficiency of the $\underset{\sim}{n}$ algebraic form of degree $p > 1$, $P^*(f, x)$, satisfying

$$\begin{aligned}
&(i) && \sum_{i=0}^{p} |a_i(0)| \neq 0, \\
&(ii) && P^*(f, x) = O(x^{N+S}), \\
&(iii) && \partial P^*(f, x)/\partial f = O(x^D), \quad \text{with} \quad 2D = N + S, \\
&(iv) && \partial^2 P^*(f, x)/\partial f^2|_{x=0} \neq 0,
\end{aligned}$$

then the associated $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, defined by the equation

$$P^*(Q, x) = 0,$$

subject to the initial condition

$$Q(0) = f(0),$$

is an approximation to $f(x)$ satisfying

$$Q(x) = f(x) + O(x^{N+S-D}).$$

PROOF:

Let $N + S = R$. Using Lemma 16,

$$\begin{gathered}
P^*(f, x) = O(x^R), \quad P^*(Q, x) = 0, \\
\frac{\partial P^*(f, x)}{\partial f} = O(x^{R/2}), \quad \frac{\partial P^*(Q, x)}{\partial Q} = 0, \\
\partial^2 P^*(f, x)/\partial f^2|_{x=0} = \partial^2 P^*(Q, x)/\partial Q^2|_{x=0} \neq 0.
\end{gathered}$$

Note that the last equation follows from the condition $Q(0) = f(0)$.
Applying Theorem 9 to the normal function $\partial P^*(f, x)/\partial f$ gives

$$Q(x) = f(x) + O(x^{R/2}).$$

This may be written as

$$Q(x) = f(x) + O(x^{R-D})$$

since by Lemma 16 we have $D = R/2$.

A similar result was partially obtained for the case $p = 2$ in [4].
The preceding results may be summarised by the following theorem.

THEOREM 18.

Let S be the surplus and D be the deficiency of the $\underset{\sim}{n}$ algebraic form of degree p, $P^*_{\underset{\sim}{n}}(f, x)$.

Let $P^*_{\underset{\sim}{n}}(f, x) = x^r\, P^*_{\underset{\sim}{m}}(f, x)$ where $P^*_{\underset{\sim}{m}}(f, x)$ has polynomial coefficients which satisfy $\sum_{i=0}^{p} |a_i(0)| \neq 0$, and $\partial^2 P^*_{\underset{\sim}{m}}(f, x)/\partial f^2|_{x=0} \neq 0$.

Then the associated $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, defined by the equation

$$P^*_{\underset{\sim}{n}}(Q,x) = 0,$$

subject to the initial conditions

$$Q^{(i)}(0) = f^{(i)}(0), \qquad i = 0(1)t,$$

$$(\text{where, if} \quad \partial P^*_{\underset{\sim}{m}}(f,x)/\partial f = O(x^{t_1}), \quad \text{then}$$

$$\begin{array}{ll} t = t_1 \quad \text{if} \quad t_1 < (M + S(\underset{\sim}{m}))/2 \quad \text{and} & \\ t = 0 \quad \text{if} \quad t_1 = (M + S(\underset{\sim}{m}))/2 &), \end{array}$$

is an approximation to $f(x)$ satisfying

$$Q(x) = f(x) + O(x^{N+S-D}).$$

PROOF:
The result follows by Theorems 12,14,17 since by Lemma 11, $D = D(\underset{\sim}{m}) + r$, and by Lemmas 13,16, $D(\underset{\sim}{m}) = t_1$.

This theorem shows the fundamental process for the order of approximation of a non-normal function, i.e. a function which has at least one of its algebraic forms non-normal. The details for more complex coincidences between the roots will be the subject of a further report.

§4. Examples.

This section contains some illustrative examples of the results of the previous sections. The first two examples consider the rational case with $p = 1$.

EXAMPLE 19.
This is the same example as Example 7 in Section 2. The $(1,1)$ algebraic form is

$$-x + xf(x) = O(x^3),$$

with $n_0 = n_1 = 1$, $N = 3$ and $S = 0$. We have

$$Q(x) = x/x = 1 = 1 + x^2 + O(x^2) = f(x) + O(x^{N+S-D})$$

since

$$\partial P^*(f,x)/\partial f = x = O(x^1) \Rightarrow D = 1.$$

This example has also illustrated Lemma 11 and Theorem 12.

EXAMPLE 20 ([1, p.22]).

Let $f(x) = (1 + 2x + x^2 + x^3)/(1 + x + x^3)$ *and* $p = 1$. *Then*

$$f(x) = 1 + x - x^4 + x^5 - x^6 + 2x^7 - 3x^8 + 4x^9 - \ldots.$$

(i) $n_0 = 1\ \ n_1 = 0$ *implies* $N = 2$. *The* $(1,0)$ *algebraic form is*

$$-(1 + x) + f(x) + O(x^4).$$

Hence $S = 2$ *and* $\partial P^*(f,x)/\partial f = 1 = O(1) \Rightarrow D = 0$. *(Thus this is a normal form). The* $(1,0)$ *algebraic (Padé) approximation is given by*

$$Q(x) = 1 + x = f(x) + O(x^4) = f(x) + O(x^{N+S-D}),$$

which follows from Theorem 9.

(ii) Theorem 5 implies the $(3,2)$ *algebraic form, with* $N = 6$, *is* $x^2 P_{(1,0)}(f,x)$ *with* $S = 0$. *In this case* $\partial P^*(f,x)/\partial f = O(x^2)$ *and hence* $D = 2$. *Hence, by Theorem 12, the* $(3,2)$ *algebraic (Padé) approximation is given by*

$$Q(x) = 1 + x = f(x) + O(x^4) = f(x) + O(x^{N+S-D}).$$

(iii) It should be observed that the $(3,3)$ *algebraic form has* $P^*(f,x) = 0 \Rightarrow S = \infty$, *as is to be expected and as was noted in Section 2.1.*

The next two examples consider how these results extend to the case of quadratic function approximation, i.e. $p = 2$.

EXAMPLE 21.

Let $f(x) = \cos(x)$ *and* $p = 2$. *Using the results from Example 6 in Section 2, for* $n_0 = n_1 = n_2 = 3$ *and hence* $N = 11$, $S = 0$ *and* $D = 1$ *(since* $P^*(f,x)$ *has a factor of* x*), we obtain*

$$\begin{aligned} Q(x) &= (-x a_1(x) + x\sqrt{a_1(x)^2 - 4a_2(x)a_0(x)})/2x a_2(x) \\ &= \sum_{k=0}^{4} (-1)^k x^{2k}/2k! - \frac{653}{76204800} x^{10} + \ldots \\ &= \cos(x) + O(x^{10}) = f(x) + O(x^{N+S-D}). \end{aligned}$$

EXAMPLE 22.

Let $f(x) = \cos(x)$ *and* $p = 2$. *Again using the results from Example 6 in Section 2, for* $n_0 = 4$, $n_1 = n_2 = 2$ *and hence* $N = 10$, $S = 2$ *and* $D = 0$, *we obtain*

$$\begin{aligned} a_0(x) &= -237x^4 + 5267x^2 - 11553, \\ a_1(x) &= 704x^2 + 11136, \\ a_2(x) &= 14x^2 + 417, \end{aligned}$$

and using Theorem 9 for this normal form, we obtain the branch

$$
\begin{aligned}
Q(x) &= (-a_1(x) + \sqrt{a_1(x)^2 - 4a_2(x)a_0(x)})/2a_2(x) \\
&= \sum_{k=0}^{5} (-1)^k x^{2k}/2k! + \frac{313}{9652608000} x^{12} + \dots \\
&= \cos(x) + O(x^{12}) = f(x) + O(x^{N+S-D}).
\end{aligned}
$$

These examples explain the remark in [5] that 'an algebraic form of maximal order does not necessarily yield an approximation of the same order'. This observation has now been quantified by the results in Theorem 18, which shows that, in the case of two roots of $P(f)$ coinciding at the origin, the approximating properties of the algebraic function will be degraded by an amount given by the order of the zero of the first derivative of P as a fuction of f at the origin. These results also hold for higher degree functions although the computations become more extensive.

EXAMPLE 23.

Let $f(x) = \log(1+x)$ with $p = 4$. Let $n_i = 2$, $i = 0(1)4$ and hence $N = 14$. We obtain the algebraic form

$$
\begin{aligned}
&5760\,x^2 + (-2295\,x^2 - 4590\,x)f(x) + (375\,x^2 - 1170\,x - 1170)f(x)^2 + \\
&(-30\,x^2 - 60\,x)f(x)^3 + (x^2 - 30\,x - 30)f(x)^4 = \frac{1}{113513400}\,x^{14} + \dots
\end{aligned}
$$

as expected. Hence $S = 0$. However, since

$$\partial P^*(f,x)/\partial f = O(x^1) \quad \text{and} \quad \partial^2 P^*(f,x)/\partial f^2|_{x=0} \neq 0,$$

$P^*(f,x)$ has a repeated root at the origin and $D = 1$. By Theorem 14, we require $Q(0) = f(0)$ and $Q^{(1)}(0) = f^{(1)}(0)$ to distinguish a unique branch of $Q(x)$, and for this branch

$$
\begin{aligned}
Q(x) &= \sum_{k=1}^{12} (-1)^{k-1} x^k/k + \frac{4654721077}{60511374000} x^{13} + \dots \\
&= \log(1+x) + O(x^{13}) = f(x) + O(x^{N+S-D}).
\end{aligned}
$$

The final example illustrates the more complex situation arising from the result of Theorem 5, and also coincident roots.

EXAMPLE 24.

Let $f(x) = \cos(x^2)$ with $p = 2$. The $(0,0,0)$ algebraic form is

$$P^*(f,x) \equiv\ 1 - 2f(x) + f(x)^2 = O(x^8).$$

Since $n_i = 0$, $i = 0(1)2$, and hence $N = 2$, this implies $S = 6$. Theorem 5 implies that $x\,P^*(f,x) = O(x^9)$ is the $(1,1,1)$ algebraic form with $S = 4$, $x^2\,P^*(f,x) = O(x^{10})$ is *the* $(2,2,2)$ *algebraic form with* $S = 2$ *and* $x^3\, P^*(f,x) = O(x^{11})$ *is the* $(3,3,3)$ *algebraic form with* $S = 0$.

(i) Consider the $(0,0,0)$ *algebraic form. For this case*

$$\partial P^*(f,x)/\partial f = -2 + 2f(x) = O(x^4) \quad \text{and} \quad \partial^2 P^*(f,x)/\partial f^2 = 2 \neq 0.$$

Hence the conditions for Theorem 17 apply and $Q(x) = 1$ *is a coincident root, with* $4 = D = (N+S)/2$. *Thus*

$$Q(x) = 1 = \cos(x^2) + O(x^4) = f(x) + O(x^{N+S-D}).$$

(ii) The $(2,2,2)$ *algebraic form,* $x^2\, P^*(f,x)$, *has* $N = 8$, $S = 2$. *The deficiency is given by*

$$\partial(x^2\, P^*(f,x))/\partial f = x^2\,(-2 + 2f(x)) = x^2\,(x^4 + \dots) = O(x^6) \Rightarrow D = 6.$$

Hence

$$Q(x) = 1 = \cos(x^2) + O(x^4) = f(x) + O(x^{N+S-D}).$$

Note that in this case $D = 2 + D(\underset{\sim}{m})$ *where* $\underset{\sim}{m} = (0,0,0)$ *by Lemma 11, and* $P^*_{\underset{\sim}{n}}(Q,x) = P^*_{\underset{\sim}{m}}(Q,x)$ *is given by part (i). Thus although* $D = 6$, *Theorem 18 implies we need only the initial condition for* $t = 0$.

(iii) The $(0,0,4)$ *algebraic form overlaps the structure based on* $(0,0,0)$ *in a similar way to that discussed in Example 6. This form (for* $\underset{\sim}{n} = (0,0,4)$*) is*

$$P^*_{\underset{\sim}{n}}(f,x) \equiv 5 - 16f(x) + (3\,x^4 + 11)f(x)^2 = O(x^{12}).$$

In this case $n_0 = n_1 = 0$ *and* $n_2 = 4$ *gives* $N = 6$ *and hence* $S = 6$. *Since* $\partial P^*_{\underset{\sim}{n}}(f,x)/\partial f|_{x=0} \neq 0$, *this form is normal and* $D = 0$. *Hence (as in Theorem 9), the branch satisfying the initial condition is*

$$\begin{aligned} Q(x) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{13}{144}\,x^{12} + \dots \\ &= \cos(x^2) + O(x^{12}) = f(x) + O(x^{N+S-D}). \end{aligned}$$

(iv) Theorem 5 implies that the $(2,2,6)$ *algebraic form is* $x^2\, P^*_{\underset{\sim}{n}}(f,x)$ *with* $N = 12$ *and* $S = 2$. *By Lemma 11,* $D = r = 2$. *Thus by Theorem 12, using the result (iii) gives*

$$\begin{aligned} Q(x) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{13}{144}\,x^{12} + \dots \\ &= \cos(x^2) + O(x^{12}) = f(x) + O(x^{N+S-D}). \end{aligned}$$

(v) The $(3,3,7)$ *algebraic form is* $x^3\, P^*_{\underset{\sim}{n}}(f,x)$ *with* $S = 0$ *and* $N = 15$. *By Lemma 11,* $D = r = 3$, *and the* $(3,3,7)$ *algebraic approximation is the same branch as in parts (iii) and (iv).*

§5. Conclusion.

This paper has considered the problem of approximating a real-valued, locally analytic function, $f(x)$, by an algebraic function $Q(x)$. A careful distinction was made between the approximating properties of the algebraic form, by which the polynomial coefficients of the algebraic function are defined, and the approximating properties of the algebraic function itself. It was shown that by defining a "surplus" for the algebraic form and a "deficiency" for the algebraic function, a unique algebraic form could be defined, and a unique branch of the algebraic function with specified approximating properties could be obtained. These results have been demonstrated in particular cases by a series of illustrative examples.

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