EXISTENCE AND UNIQUENESS OF COLLOCATING ALGEBRAIC FUNCTION APPROXIMATIONS

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Abstract

The problem of approximating a real-valued function by an algebraic function, where the approximation is determined by collocation at a sufficient number of distinct nodes, is considered. Results are obtained for the existence, uniqueness and order of approximation for both 'normal' and 'non-normal' cases. Some illustrative examples are given.

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§1. Introduction.

This paper considers the problem of approximating a real-valued function by an algebraic function where the approximation is determined by collocation at a sufficient number of distinct nodes. The objective is to obtain analogous results to those obtained in [**13,17**] for the case of collocation at a single multiple node. The main results are the existence and uniqueness of an approximating algebraic form which determines the polynomial coefficients of the implicit equation for the algebraic function, and the existence and uniqueness of a distinguished branch of the algebraic function. In addition the order of approximation of the algebraic function is quantified in a variety of circumstances and it is shown how the set of "unattainable points" may be identified.

The generalization of the Taylor polynomial approximation to rational functions gives the Padé rational approximation. Such approximations have been found useful in problems in mathematical physics [**2**]. A variety of generalizations of this approach have been suggested in mathematical physics [**3**], and these are often collected in the class of what are called Hermite–Padé approximants [**4**]. A direct generalization of the Taylor polynomial approximation to algebraic functions was considered in [**13,17**].

A more recent and interesting application of the use of approximations by algebraic functions occurs in the field of computational geometry. This involves the use of algebraic curves and surfaces to approximate and model objects in computer geometry and computer modelling (e.g. [**1**]). While the approach used by these authors is generally quite different from that of the generalized Padé approximants, the problems are similar and deserving of further study.

While the approach of a generalized form of the Padé approximation requires that all the information about the given function be at a single node (usually taken to be the origin), such a requirement may be unrealistic in the case of computer modelling. An alternative formulation requires that the information about the given function be function values at a sufficient number of distinct nodes. This is the analogy of the 'Lagrange interpolation problem' for polynomial approximation, and the Newton–Padé [**9,10**] (sometimes the generalized or multipoint Padé approximation [**3,18**]) or 'rational interpolation problem' for rational function approximation. The general approach to this problem was first discussed by the author in colloquium papers [**11,12**].

The two separate aspects of this problem are the properties of the approximation and the computation of the approximation. An initial approach to the latter aspect, for the quadratic case, is given in [**15**] and further details are given in [**16**]. This paper is concerned with the properties of the approximation. The general approach in the case of the quadratic function collocating at a single multiple node was initiated in [**5**] and a more comprehensive analysis for the general algebraic function determined by collocation at a single multiple node is given in [**17**]. This paper is a revised and expanded version of [**14**], where many of these results were first obtained.

Although the results obtained in this paper are technically much easier than the corresponding results for the case of collocation at a single multiple node [**13,17**], the resolution of the conditions required to obtain analogous results was more complex. For example the two conditions in the definition of a normal branch (Definition 13) reduce to just a single condition (Definition 8 in [**17**]) in the case of a single multiple node. Furthermore the concept of "unattainable points" does not occur in the single multiple node case, although in retrospect, it might be argued that the "unattainable points" are the "higher repetitions" of the single multiple node. It is the objective of this paper to establish those concepts which seem to be most important. Consequently additional complications arising from possibilities such as the intersection of different branches of the algebraic function and confluence of the points in the collocation node set have been deliberately excluded in order to clearly establish the basic ground rules.

Following the approach in [**13,17**], a careful distinction is made between the approximation of the *algebraic form* which determines the polynomial coefficients of the implicit equation for the algebraic function, and the approximating properties of the algebraic function itself. The distinction between these two concepts leads to the separation of degenerate cases arising from different sources, and consequently a clear treatment of the effects of each type. As in [**13,17**], this leads to the definition of the surplus in the case of the algebraic form, and the definition of the deficiency in the case of the approximation by the algebraic function. Furthermore the general concept of the "unattainable points" (used by some authors [**9**] in the Newton–Padé approximation problem) arises naturally with this approach.

In Section 2 the problem of the approximation of the algebraic form which determines the coefficient polynomials is considered. The concept of the surplus in the approximation of the algebraic form is introduced. In Section 3 the approximating properties of the algebraic function are investigated. Attention is restricted to algebraic functions whose branches do not intersect in the interval containing the collocation nodes, so as to identify the basic behaviour of the approximation. The concept of the deficiency in the approximation is introduced and the order of approximation by the algebraic function is quantified. The set of "unattainable points" is also identified. Section 4 contains a series of illustrative examples to demonstrate the results of the previous sections in particular cases.

§2. The Algebraic Form Determined by Collocation.

Let $A_M = \{x_k, k = 0(1)M-1\}$ be the set of M *distinct* node points corresponding to the given set $\{L_k, k = 0(1)M-1\}$ of point evaluation functionals defined by $L_k g(x) = g(x_k)$.

DEFINITION 1 (ALGEBRAIC FORM).

Let $f(x)$ be a function whose values are given on the set $A_N \subset A_M$. Let $\underset{\sim}{n} = (n_0, n_1, \ldots, n_p)$, where n_i, $i = 0(1)p$, $n_i \geq -1$, are integers and $p \geq 1$.

The function

$$P_{\underset{\sim}{n},p}(f,x) \equiv \sum_{i=0}^{p} a_i(x) f(x)^i = O(\omega_N(x)) \tag{1}$$

will be called an $\underset{\sim}{n}$ algebraic form of degree p, where the $a_i(x)$ are algebraic polynomials with degree $a_i(x) \leq n_i$ for $i = 0(1)p$, and $a_i(x) \not\equiv 0$ for at least one value of i for $i = 1(1)p$, $N+1 = \sum_{i=0}^{p}(n_i+1)$, and $\omega_N(x) = \prod_{k=0}^{N-1}(x - x_k)$. The order notation is to be interpreted as meaning bounded in the neighborhood of the set, A_N, of collocation nodes. ∎

By convention, a polynomial of degree -1 is identically zero. *Generally the subscripts $\underset{\sim}{n}, p$ on P will be dropped when the context makes them obvious.*

Note that $P(f,x)$ may also be written as a function $r(x)$. This function $r(x)$ may be approximated by a polynomial of degree $< N$, determined by collocation on the points of A_N. As is shown in elementary numerical analysis texts, the error of the approximation may be written as $\omega_N(x)e(x)$. This is the rationale for the notation in (1). Further, if $A_N = \{x_0, k = 0(1)N-1\}$ with $x_0 = 0$, corresponding to the set of linear functionals $L_k = D^k$, where $D^k r(x) = r^{(k)}(0)/k!$, then this definition is equivalent to the algebraic form used in [**13,17**]. Also, if $p = 2$, this definition becomes the quadratic form defined in [**15,16**], which generalizes the form used in the Newton–Padé or 'rational interpolation' problem ($p = 1$), and the form for the so-called 'polynomial interpolation' problem ($p = 1, n_1 = 0$).

It should be noted that this form explicitly excludes the case of confluent node points and hence collocation in the Hermite sense, which is the sense used by Stahl [**18**] for the generalized (multipoint) Padé approximants.

The existence of such a form is easily confirmed.

THEOREM 2 (EXISTENCE).

For the given values of $f(x)$ defined on the given collocation node set A_N, there always exists an $\underset{\sim}{n}$ algebraic form of degree p.

PROOF: The algebraic form is defined by the coefficient polynomials $a_i(x)$. The existence of the $a_i(x)$ follows, since the application of the linear functionals, $L_k, k = 0(1)N-1$, to (1) leads to a system of N homogeneous linear equations for the $N+1$ unknown coefficients of the $a_i(x)$. Hence a non-trivial solution, with $a_i(x)$ not all zero, exists. Further if $a_i(x) \equiv 0$ for $i = 1(1)p$, then $a_0(x) \equiv 0$, and hence $\underset{\sim}{\overline{a}}(x) = (a_1(x), \ldots, a_p(x)) \not\equiv 0$. ∎

However the question of whether this algebraic form can be determined uniquely requires a more careful argument. The matrix form of the system of linear equations represented by applying the functionals $L_k, k = 0(1)N-1$, to equation (1) has the coefficient matrix:

$$F = \left[F_{n_0} : F_{n_1} : \cdots : F_{n_p}\right],$$

$$\text{where} \quad F_{n_k} = \begin{bmatrix} f_0^k & x_0 f_0^k & \cdots & x_0^{n_k} f_0^k \\ f_1^k & x_1 f_1^k & \cdots & x_1^{n_k} f_1^k \\ \cdot & \cdot & \cdots & \cdot \\ f_{N-1}^k & x_{N-1} f_{N-1}^k & \cdots & x_{N-1}^{n_k} f_{N-1}^k \end{bmatrix}$$

The matrix F has dimensions $N \times (N+1)$ and hence has a solution space of dimension at least 1. If the rank of the matrix F is N (and hence the solution space has dimension exactly 1), then any constant multiple of the coefficient polynomials will also be a solution. These solutions may be called *essentially unique* [4]. A unique representative of this class of solutions may be defined by choosing any convenient suitable normalization of the coefficients. Thus if $\underset{\sim}{a}$ represents the vector of coefficients of $\underset{\sim}{a}(x)$ then $\|\underset{\sim}{a}\| = 1$ for some convenient norm, could be used. However a normalization such as $a_1(0) = 1$, which is used by some authors [**2**] in the rational case ($p = 1$), is not a suitable normalization since there is no requirement that $a_1(0) \neq 0$.

A more serious problem however, is the case when the solution space is multi–dimensional. This occurs when the matrix F has rank $< N$. If the rank of F is $N+1-k$, then the solution space of the algebraic form has dimension $k > 1$. This situation, in the case that all the node points are coincident and the functionals $L_k = D^k$, was discussed in [**13,17**].

DEFINITION 3 (ORDER).

The order of the $\underset{\sim}{n}$ *algebraic form associated with the subset* $\{L_k, k = 0(1)N-1\}$ *of the given set of functionals* $\{L_k, k = 0(1)M\}$ *is defined to be*

$$R = \mathrm{Ord}\,(P(f,x)) \text{ if } P(f,x) = O(\omega_R(x)), \neq O(\omega_{R+1}(x))$$

for x in the neighborhood of the collocation node set $A_{R+1} \subset A_M$. If the solution space of equation (1) is multidimensional, then the order of the algebraic form is to be interpreted as the maximum of the orders of the multiple solutions. ∎

DEFINITION 4 (ALGEBRAIC FUNCTION).

An $\underset{\sim}{n}$ *algebraic function of degree p is the function $Q(x)$ which satisfies*

$$P_{\underset{\sim}{n},p}(Q,x) \equiv \sum_{i=0}^{p} a_i(x) Q(x)^i = 0 \tag{2}$$

where the $a_i(x)$ are algebraic polynomials with degree $a_i(x) \leq n_i$ for $i = 0(1)p$. ∎

Note that if $P(f,x) = 0$ in Definition 3, then f is in fact an algebraic function and we may regard the order $R = \infty$ in this case. In general we seek the order R as large as possible since this will give a better approximation.

If the dimension of the solution space is $k > 1$, then to obtain a unique representative we seek a one dimensional subspace whose elements satisfy $P(f,x) = O(\omega_R(x))$ where the order R is maximal over the space of $\underset{\sim}{n}$ algebraic forms of degree p. For example, if $P^{(i)}(f,x)$, $i = 1(1)k$, represent the k linearly independent solutions to (1) then $\mathrm{Ord}(P^{(i)}(f,x)) = R^{(i)}$. Thus for an algebraic form of maximal order, the solution $P^{(i)}(f,x)$ is chosen so that $R = R^{(i)} = \max_{1 \le i \le k}(R^{(i)})$. If $R^{(1)} = R^{(2)} = R$, and $\underset{\sim}{a}^{(i)}(x) = (a_0^{(i)}(x), \cdots, a_p^{(i)}(x)), i = 1,2$, are two linearly independent solutions to (1), then by taking a suitable linear combination, $c_1\underset{\sim}{a}^{(1)}(x) + c_2\underset{\sim}{a}^{(2)}(x)$, of these solutions, the term $(c_1 r_R^{(1)} + c_2 r_R^{(2)})\omega_R(x)$ may be eliminated and an $\underset{\sim}{n}$ algebraic form of order at least $R+1$ is obtained, with the dimension of the solution space decreased by 1.

THEOREM 5 (UNIQUENESS).

There always exists an essentially unique $\underset{\sim}{n}$ algebraic form of degree p corresponding to the given set of linear functionals $\{L_k, k = 0(1)M\}$, which is of maximal order $R \ge N$, and which may be chosen uniquely by a suitable normalization of the coefficients of the coefficient polynomials. This unique representative will be denoted by $P^*(f,x)$.

PROOF: If the coefficient matrix, F, of equation (1) has rank N then the solution space has dimension 1 and the result is trivial.

If the matrix F has rank $N+1-k$ for $k > 1$, then the solution space has dimension k. If $P^{(i)}(f,x)$, $i = 1(1)k$, represent the k linearly independent solutions in this space and $\mathrm{Ord}(P^{(i)}(f,x)) = R^{(i)}$, then let $I = \{i : R^{(i)} = \max_{1 \le i \le k}(R^{(i)}) = R\}$ represent the set of indices for which these solutions have maximal order.

(i) If the set I consists of a single element then the algebraic form $P^{(i)}(f,x)$ corresponding to this value of i represents the essentially unique algebraic form of maximal order.

(ii) If the set I consists of k_1 elements, where $2 \le k_1 \le k$, then the solution space has a subspace of dimension k_1 in which the algebraic forms all have order R such that $N \le R < M$. Let $\underset{\sim}{a}^{(i)}(x)$, $i = 1(1)k_1$, represent a basis for this solution subspace of algebraic forms of order R. Then $\sum_{i=1}^{k_1} c_i \underset{\sim}{a}^{(i)}(x)$ represents an $\underset{\sim}{n}$ algebraic form of order $R + k_1 - 1$ where the constants c_i are defined by the linear system

$$\sum_{i=1}^{k_1} c_i r_{R+j}^{(i)} = 0, \quad j = 0(1)k_1 - 2,$$

and where the r_{R+j} are the coefficients of the formal Newton series associated with $P(f,x) = r(x)$, i.e., $P(f,x) = r(x) = \sum_{m=0}^{M} r_m \omega_m(x) + \omega_{M+1}(x)e(x)$.

If the matrix of this system of linear equations has rank $k_1 - 1$, then there exists an essentially unique solution. However, if this matrix has rank $< k_1 - 1$, then we must iterate this process since there are still linearly independent solutions. Since the rank reduces by at least one at each step, the iterations are finite, noting of course, that if we obtain a zero matrix then $R = \infty$ and the exact algebraic form, and hence the exact algebraic function, has been obtained. ∎

If $f(x)$ is an even function then $f(x)^k$ is also even. In the case of collocation at one multiple point at the origin, it was noted in **[13,17]** that for $p = 2$ with $f(x)$ even and the n_k all even, the matrix F has rank at most $N-1$, and hence the solution space has dimension of at least 2. The analogous result for the case of distinct node points is that for $f(x)$ even and the n_k all even *and* the collocation node points symmetric about the origin, then the matrix F has rank at most $N-1$, and hence the solution space has dimension of at least 2.

In fact the situation may be stated more generally for $\underset{\sim}{n}$ algebraic forms of arbitrary degree p. To avoid unnecessary complications the possibility of "null" basis functions (i.e., $n_k = -1$) is excluded in the following theorem which illustrates the possibility of multiple solutions.

THEOREM 6 (MULTIPLE SOLUTIONS).

If $f(x)$ is an even function, the n_k are all even and the degree p is even, then the $\underset{\sim}{n}$ algebraic form of degree p with a full basis (and hence $n_k \geq 0$ for all k), corresponding to the collocation node set A_N which is symmetric about the origin, has the coefficient matrix F of rank at most $N-p/2$ and hence the solution space has dimension at least $p/2+1$.

PROOF: Under the hypotheses of the theorem N is even, and $N+1$ is odd.

The coefficient matrix F has its $N+1$ columns partitioned into $p+1$ blocks as given above. Let the columns be rearranged into blocks of up to $p+1$ columns, in the following way. The first block consists of the first columns from each of the blocks $F_{n_k}, k = 0(1)p$. The second block of up to $p+1$ columns consists of the third columns from each of the blocks $F_{n_k}, k = 0(1)p$. If any block F_{n_k} does not have a third column then this null column is ignored and this block in the rearranged matrix will have less than $p+1$ columns. Continuing in this manner, when all the odd numbered columns of the blocks F_{n_k} have been dealt with, the procedure is repeated for the even numbered columns of the blocks F_{n_k}.

Now since $f(x)$ is even and A_N is symmetric about the origin, then $x_j = -x_{N-1-j}$ and $f_j^k = f_{N-1-j}^k$ for $j = 0(1)(N/2-1)$ and $k = 0(1)p$. Hence by elementary row operations the matrix F is transformed to the equivalent form:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A, B both have $N/2$ rows, except in the case $n_k = 0, k = 0(1)p$, when the matrix B is null, $N = p$, and A has $p/2$ rows of zeros, which implies that the matrix F has rank at most $p/2 = N - p/2$.

If $n_k > 0$ for at least one value of k then the corresponding block F_{n_k} in the original matrix F has one more odd numbered column than even numbered column, since n_k is even and F_{n_k} has $n_k + 1$ columns. Hence A has $p+1$ more columns than B. By considering a Laplacian expansion of the determinant of an arbitrary $N \times N$ submatrix of the above equivalent matrix, in terms of the first $N/2$ rows, it is clear that each cofactor has at least $p/2$ columns of zeros. Hence its rank (and hence that of F) is at most $N - p/2$.

The rank of the matrix F may in fact be even less than $N-p/2$ if either the rank of A or the rank of B is not full. ∎

§2.1. **Degeneracies in the Algebraic Form.**

In the algebraic form $P(f,x)$ defined by equation (1), it is possible that some of the coefficient polynomials are not of full degree, i.e., degree $(a_i(x)) < n_i$ for some i. This deficiency is of no particular importance except in the case when *all* the coefficient polynomials are not of full degree – a situation that will be considered later. The case when just some of the coefficient polynomials are not of full degree may be compared to the Lagrange polynomial approximation of degree $1, p_1(x)$, to the function $f(x) = x^2 - 3x + 3$ on the node set $\{1,2\}$, which is $p_1(x) = 1$, i.e., there is a zero coefficient of the linear term.

Of greater significance is the fact that the particular set of coefficient polynomials which solves (1) may in fact eliminate more of the coefficients of the formal Newton series form of $r(x)$ than just the first N. A trivial example in the polynomial case is that of zero degree Lagrange polynomial approximation to $f(x) = x^2 - 3x + 3$ on the collocation node set $A_M = \{1,2,3\}$. In this case $N = 1$, but the form

$$-1 + 1f(x) = (x-1)(x-2) = O(\omega_2(x)).$$

Although the right side of (1) is expected to be $O(\omega_1(x)) = O((x-1))$ in this case, we have in fact obtained $O(\omega_2(x))$. In [**13,17**] the term "surplus" was used to describe this effect in case of collocation at one multiple point.

DEFINITION 7 (SURPLUS).

The surplus, $S(\underset{\sim}{n})$, of the $\underset{\sim}{n}$ algebraic form of degree p corresponding to the given set of linear functionals $\{L_k, k = 0(1)M\}, P^*(f,x)$, is defined by

$$S(\underset{\sim}{n}) = \operatorname{Ord}(P^*(f,x)) - N$$

where the order of the algebraic form is defined in Definition 3. ∎

The surplus, $S(\underset{\sim}{n}) = S \geq 0$, is the amount of extra matching obtained from $P^*(f,x)$. This may be achieved by serendipity for a particular function or by the process of obtaining a unique solution as outlined in the proof of Theorem 5. It is clear that in general we would like the surplus to be as large as possible since if $S = \infty$ then the form represents an algebraic function exactly.

This concept of surplus is more general than Stahl's concept of "overinterpolation" [**18**] for rational functions. Using the present notation, Stahl considers a node set in which the nodes are not necessarily distinct. He then implies that "overinterpolation" occurs if the multiplicity of a node satisfying the algebraic form is greater than that which occurs in the set A_N. This concept is equivalent to the concept of the surplus in the case of a single multiple node as defined in [**13,17**]. It should be noted in passing that it was shown in [**17**] that a positive surplus may or may not translate into an "over-approximation" of the corresponding algebraic function. However Stahl's concept of "overinterpolation" does not include the concept of surplus defined above for *distinct* node points. For distinct node points the surplus implies that the algebraic form is satisfied by the node set $A_{N+S} \supset A_N$. Compare Examples 11,12,23,25,26.

In [**13,17**] the surplus was used to define an S–table of the algebraic forms collocating at a single multiple point. This table was shown in [**8**], in the case $p = 1$, to give the block structure of the Padé table in a somewhat easier fashion than the more traditional

C–table [**2**]. In the case of algebraic forms of degree p, the S–table of the algebraic forms $P^*(f,x)$ would be a $(p+1)$–dimensional table. A basic theorem for the structure of the S–table of algebraic forms collocating at a single multiple point was given in [**13,17**]. Although the situation is complicated by overlapping structures (as also noted in [**13,17**]), an analogous basic theorem can also be given in this case.

THEOREM 8 (STRUCTURE).

If the $\underset{\sim}{n}$ algebraic form of degree p corresponding to the given set of linear functionals $\{L_k, k = 0(1)M\}, P^*_{\underset{\sim}{n}}(f,x)$, has a surplus $S(\underset{\sim}{n}) = S > 0$, and $R = N + S$, then

$$\omega_{r,R}(x)P^*_{\underset{\sim}{n}}(f,x), \quad r = 0(1)(S/p)$$

is an $\underset{\sim}{m}$ algebraic form with a surplus of $S(\underset{\sim}{m}) = S(\underset{\sim}{n}) - pr - \sum_{k=0}^{p} i_k$, where

(i)
$$\omega_{r,R}(x) = \prod_{k=0}^{r-1}(x - x_{R+k})$$

(ii)
$$\underset{\sim}{m} = (m_0, m_1, \cdots, m_p) \text{ with}$$
$$m_k = n_k + r + i_k, i_k \geq 0 \text{ for } k = 0(1)p,$$
$$\text{satisfies} \sum_{k=0}^{p} i_k \leq S - pr.$$

PROOF: Since $\operatorname{Ord}(P^*_{\underset{\sim}{n}}(f,x)) = N + S = R$, then

$$P^*_{\underset{\sim}{n}}(f,x) = O(\omega_R(x))$$

and hence

$$\omega_{r,R}(x)P^*_{\underset{\sim}{n}}(f,x) = O(\omega_{R+r}(x)),$$

so that $\operatorname{Ord}(\omega_{r,R}(x)P^*_{\underset{\sim}{n}}(f,x)) = R + r$.

Hence $\omega_{r,R}(x)P^*_{\underset{\sim}{n}}(f,x)$ will be an $\underset{\sim}{n} + r\underset{\sim}{o}$ (where $\underset{\sim}{o} = (1,1,\cdots,1)$) algebraic form of degree p provided $pr \leq S$. The surplus of this algebraic form is $S - pr$. Further, this algebraic form is also an algebraic form of the type $\underset{\sim}{m}$ where $\underset{\sim}{m} = (m_0, m_1, \ldots, m_p)$ and $m_k = n_k + r + i_k$, $i_k \geq 0$, for $k = 0(1)p$, satisfies $\sum_{k=0}^{p} i_k \leq S - pr$. This follows since $P_{\underset{\sim}{m}}(f,x) = O(x^{M+S(\underset{\sim}{m})})$ with $M + 1 = \sum_{k=0}^{p}(m_k + 1)$. Hence $\operatorname{Ord}(P_{\underset{\sim}{m}}(f,x)) = M + S(\underset{\sim}{m}) = N + S + r$ (from above). Substituting for M and N, and using the relation between m_k and n_k gives

$$\sum_{k=0}^{p} m_k + p + S(\underset{\sim}{m}) = \sum_{k=0}^{p} n_k + p + S + r.$$

Hence $\sum_{k=0}^{p} i_k + pr + S(\underset{\sim}{m}) = S$, and $S(\underset{\sim}{m}) \geq 0$ gives the required relation.

An initial investigation [8] of the case $p = 2$ for algebraic forms collocating at a single multiple point shows that some complexity can arise from the overlapping of this basic block structure. It is apparent that this same overlapping occurs in the present case. Further details on the block structure will be the subject of a future report.

Some simple examples can be given.

EXAMPLE 9.

Let $p = 1$ and $f(x) = x^2 - 3x + 3$ with $A_M = \{1, 2, 3\}$.

The $(0,0)$ algebraic form for this function has the polynomial coefficients $a_0(x) = -1, a_1(x) = 1$ and surplus $S = 1$. Hence this is also an algebraic form of types $(1,0)$ and $(0,1)$. The $(1,1)$ algebraic form has polynomial coefficients $(x-3)a_0(x) = -(x-3)$ and $(x-3)a_1(x) = x - 3$, with surplus $S = 0$. ∎

The following example is adapted from [3].

EXAMPLE 10. [3].

Let $p = 1$ and $f(x) = 1 + x + x^2$ with $A_M = \{-1, 0, 1\}$.

The $(0,0)$ algebraic form for this function is

$$-1 + 1f(x) = O((x+1)x)$$

with $S = 1$ since $N = 1$. Hence this is also the algebraic form of types $(1,0)$ and $(0,1)$. The $(1,1)$ algebraic form has polynomial coefficients $(x-1)a_2(x) = -(x-1)$ and $(x-1)a_1(x) = x - 1$, with surplus $S = 0$. ∎

Note that at this time we are simply asserting the existence of an algebraic form. The properties of an approximation generated by this form will be considered in the next section. A further example is adapted from [10].

EXAMPLE 11. [10].

Let $p = 1$ and $f(x) = \sum_{k=0}^{\infty} a_k \omega_k(x)$ where

$$a_0 = a_1 = a_3 = a_5 = a_6 = a_7 = 1;$$
$$a_2 = 0; a_4 = -1; \text{ and } a_k = 0 \text{ for } k \geq 8.$$

The collocation node set is $A_M = \{k : k \geq 1\}$.

The $(1,0)$ algebraic form for this function satisfies

$$-x + 1f(x) = O(\omega_3(x))$$

with $S = 1$ since $N = 2$. Hence this is also the algebraic form of types $(2,0)$ and $(1,1)$. The $(2,1)$ algebraic form has polynomial coefficients $(x-4)a_0(x) = -x(x-4)$ and $(x-4)a_1(x) = x-4$ with surplus $S = 0$ according to Theorem 8. However, in this case it happens that

$$-x(x-4) + (x-4)f(x) = O(\omega_5(x))$$

and hence this algebraic form has $S = 1$.

This is an example of the overlapping block structure mentioned above. A further application of Theorem 8 implies that the form is also an algebraic form of types $(3,1)$

and $(2,2)$*. The* $(3,2)$ *algebraic form has polynomial coefficients* $-x(x-4)(x-6)$ *and* $(x-4)(x-6)$ *with surplus* $S=0$. ∎

A final example illustrates the multiple solutions of Theorem 6. The set of collocation nodes in this example was chosen only to give a relatively simple solution to the problem. The use of non-rational function values generally produces a relatively unwieldy expression.

EXAMPLE 12.
Let $p=2$ *and* $f(x)=\cos(\pi x/6)$ *with the symmetric set of collocation nodes* $A_M=A_{14}=\{-2,2,-3,3,-4,4,-6,6,-8,8,-9,9,-10,10\}$*. Since* $f(x)$ *is an even function, the* $(2,2,2)$ *algebraic form on the symmetric collocation node set* A_8 *satisfies the conditions of Theorem 6 and two solutions are expected.*

$$P^{(1)}(f,x)=(14\,x^2-126)+(11\,x^2+58)f(x)+76\,f(x)^2=O(\omega_8(x)),$$
$$P^{(2)}(f,x)=(-68\,x^2+612)+(9\,x^2-792)f(x)+38\,x^2\,f(x)^2=O(\omega_8(x)).$$

An essentially unique representative for the $(2,2,2)$ *algebraic form may be obtained by the process of Theorem 5. In this case* $(126/19)\,P^{(1)}(f,x)+(17/19)\,P^{(2)}(f,x)$ *gives*

$$P^*(f,x)=(32\,x^2-288)+(81\,x^2-324)f(x)+(34\,x^2+504)f(x)^2=O(\omega_{10}(x)).$$

with $S=2$ *since* $N=8$.

This is also the algebraic form of types $(3,2,2)$, $(2,3,2)$, $(2,2,3)$ *with* $S=1$*; the algebraic form of types* $(3,3,2)$, $(3,2,3)$, $(2,3,3)$ *with* $S=0$*, and the* $(3,3,3)$ *algebraic form has polynomial coefficients* $(x+9)a_i(x)$, $i=0,1,2$ *(where the* $a_i(x)$ *are the polynomial coefficients of* $P^*(f,x)$ *above), with* $S=0$*, by Theorem 8.*

Further, these polynomial coefficients $a_i(x)$*, are also the coefficients of an algebraic form of types* $(4,2,2)$, $(2,4,2)$, $(2,2,4)$ *with* $S=0$*. But as noted above, the solution space for an even function with* n_k *all even has dimension 2 and hence there will be an additional linearly independent solution in these cases. Applying the uniqueness theorem (Theorem 5) gives an essentially unique algebraic form of nominal order 11. However, since the polynomial coefficients are all even (see [**13,17**]), the algebraic form is even and hence has order 12. Thus these forms also have surplus* $S=2$*. For example the* $(4,2,2)$ *algebraic form is*

$$P^*(f,x)=$$
$$(-32\,x^4+2880\,x^2-23328)+(685\,x^2+20300)f(x)+(90\,x^2+2840)f(x)^2=O(\omega_{12}(x)).$$

*This is an example of the overlapping block structure noted above and which is illustrated by the tables in [**8**].* ∎

§3. The Algebraic Approximation.

Once the unique $\underset{\sim}{n}$ algebraic form of degree p, $P^*(f,x)$, satisfying equation (1) has been obtained, it is clear that we can define an $\underset{\sim}{n}$ algebraic function of degree $p, Q(x)$, which satisfies the equation

$$P^*(Q,x) \equiv \sum_{i=0}^{p} a_i^*(x) Q(x)^i = 0. \tag{3}$$

Since the coefficients, $a_i^*(x)$, are determined by the given function $f(x)$ and the collocation node set A_M, this function, $Q(x)$, represents an approximation to $f(x)$. Recall that $f(x)$ is assumed given in the sense that its value at a sufficient number of distinct node points are known.

In this section we determine in what sense $Q(x)$ is an approximation to $f(x)$. In this investigation of the approximating properties of $Q(x)$ it will be shown that $Q(x)$ collocates $f(x)$ on some subset of the given collocation node set A_M. But the situation here is more complex than that considered in [**13**,**17**].

From the general theory of algebraic functions, it is known that the solution of equation (3) *normally* has p distinct analytic branches at each of the collocation points. This "normal" behaviour is true if the function $P^*(Q,x)$, as a polynomial in Q, has distinct roots at the collocation nodes. The implicit function theorem implies that we require the condition

$$\partial P^*(Q,x)/\partial Q|_{x \in A_R} \neq 0, \quad R = \mathrm{Ord}\,(P^*(f,x)),$$

for normal behaviour.

However, in contrast to the case of confluent nodes where A_R is a single, repeated point, in this case it is possible that $\partial P^*(Q,x)/\partial Q$ may also be zero at a point between the nodes. It was found in the case where the collocation nodes coincide at a single point [**6**,**7**], that it may be necessary to change branches at a point where they coincide, in order to obtain a single function with the desired approximating properties. Hence, in order to avoid any additional complications in this preliminary paper, it is necessary to require that $\partial P^*(Q,x)/\partial Q$ also has no zeros between the collocation nodes in order to attain normal or unexceptional behaviour. Thus we require the condition

$$\partial P^*(Q,x)/\partial Q|_{x \in \overline{A}_R} \neq 0,$$

where $R = \mathrm{Ord}\,(P^*(f,x))$ and $\overline{A}_R$ is the closure of A_R.

Further, for a particular $x_k \in A_R$, only one of the distinct branches of $Q(x)$ at $x = x_k$ will have $Q(x_k) = f(x_k)$ since $P^*(Q(x_k),x_k) = 0 = P^*(f(x_k),x_k)$. Hence there is a unique branch of the function $Q(x)$ – call it $Q^*(x)$ – which passes through $(x_k, f(x_k))$. For this branch we have

$$\partial P^*(f,x)/\partial f|_{x=x_k} = \partial P^*(Q^*,x)/\partial Q^*|_{x=x_k} \neq 0.$$

The difficulty in this case is that for each x_k the distinguished branch $Q^*(x)$ may in fact be a different branch of $Q(x)$. For normal behaviour we require that a single branch $Q^*(x)$ collocates at all the nodes $x_k \in A_R$. Thus for normal behaviour we may assume that the distinguished branch $Q^*(x)$ is determined by the node $x = x_0$ and that

$$\left.\frac{P^*(f,x) - P^*(Q^*,x)}{f - Q^*}\right|_{x \in A_R} \neq 0,$$

since this ratio will be $O((x - x_k))$ if $f(x_k)$ lies on another branch of $Q(x)$.

DEFINITION 13 (NORMAL BRANCH).

Given the function $f(x)$ with the set of point evaluation functionals $\{L_k, k = 0(1)R\}$, and the corresponding $\underset{\sim}{n}$ algebraic form of degree p, $P^*(f,x)$, then $Q^*(x)$ is the unique distinguished branch of the $\underset{\sim}{n}$ algebraic function of degree p, $Q(x)$, which is defined by $P^*(Q,x) = 0$, and which satisfies $Q^*(x_k) = f(x_k)$ for some $x_k \in A_R$.

The branch $Q^*(x)$ is called normal if

(i) $\partial P^*(Q,x)/\partial Q|_{x\in\overline{A}_R} \neq 0$, *where $\overline{A}_R$ is the closure of A_R and $R = \text{Ord}(P^*(f,x))$,*

(ii) $\dfrac{P^*(f,x) - P^*(Q^*,x)}{f - Q^*}|_{x\in A_R} \neq 0$. ∎

It is important to note that this definition of normality does not conflict with the previous definition introduced in [**13,17**] for the $\underset{\sim}{n}$ algebraic form which collocates at a single multiple point. In this latter case, $\overline{A}_R = \{0\}$, and for the unique branch of $Q(x)$ which satisfies $Q^*(0) = f(0)$, we have $\partial P^*(f,x)/\partial f|_{x=0} = \partial P^*(Q^*,x)/\partial Q^*|_{x=0} \neq 0$.

If $p = 1$, then $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = a_1(x)$. Thus this definition includes the usual conditions $(a_1(x)|_{x\in A_R} \neq 0)$ required for the existence of a non-degenerate multi-point Padé approximation [**3**]. However it does not correspond to the definition of normality for the Newton-Padé approximant used in [**10**]. The definition of normality in [**10**] appears to correspond more closely to the concept of $S = 0$, but as noted in [**13,17**], the notions of surplus and non-normality tend to overlap in the rational case.

THEOREM 14 (ORDER OF APPROXIMATION).

Let $P^*(f,x)$ be the $\underset{\sim}{n}$ algebraic form of degree p defined by equation (1), and satisfying $P^*(f,x) = O(\omega_R(x))$, where $R = N + S$ and $S \geq 0$ is the surplus and suppose that the distinguished branch is normal. Then the unique normal branch of the $\underset{\sim}{n}$ algebraic function of degree $p, Q^*(x)$, defined by the equation

$$P^*(Q^*,x) = 0$$

subject to the condition

$$Q^*(x_0) = f(x_0)$$

is an approximation to $f(x)$ satisfying

$$Q^*(x) = f(x) + O(\omega_R(x)).$$

That is, $Q^*(x)$ collocates $f(x)$ on the collocation node set $A_R = \{x_k, k = 0(1)R-1\}$.

PROOF: We have

$$P^*(f,x) = O(\omega_R(x)), \quad P^*(Q^*,x) = 0. \tag{4}$$

Since the branch $Q^*(x)$ is normal

$$[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*]|_{x\in A_R} \neq 0. \tag{5}$$

Applying the linear functionals L_k to (4) gives

$$L_k P^*(f,x) = 0 = L_k P^*(Q^*,x), \quad k = 0(1)R-1. \tag{6}$$

For $k = 0$ in (6): $P^*(f(x_0), x_0) = P^*(Q^*(x_0), x_0)$ since $Q^*(x_0) = f(x_0)$. Otherwise (6) gives

$$P^*(f(x_k), x_k) = P^*(Q^*(x_k), x_k), \quad k = 1(1)R-1.$$

Hence

$$\frac{P^*(f(x_k), x_k) - P^*(Q^*(x_k), x_k)}{f(x_k) - Q^*(x_k)} \cdot [f(x_k) - Q^*(x_k)] = 0,$$

and hence, using (5)

$$Q^*(x_k) = f(x_k), \quad k = 0(1)R-1.$$

Thus

$$Q^*(x) = f(x) + O(\omega_R(x)).$$

Hence, in the case of a normal branch, a unique branch of the $\underset{\sim}{n}$ algebraic function of degree p is obtained from the unique corresponding $\underset{\sim}{n}$ algebraic form, and this branch of the algebraic function collocates $f(x)$ on the same set of collocating nodes as the corresponding form. If the surplus is positive then "over–approximation" by the algebraic function will be obtained in this case.

§3.1. The Non-Normal Case.

The non–normal case for distinct nodes is rather more complicated than the analogous situation for a single multiple node treated in [**13,17**]. However, the general idea is that this situation leads to an approximation whose order is less than that expected in the normal case. Thus we might expect $Q^*(x) = f(x) + O(\omega_R(x))$ for $R < N + S$. This corresponds to the concept of "unattainable points" used by some authors (e.g. [**9**]) in the Newton–Padé approximation problem, and to the concept of "interpolation defect" introduced by Stahl [**18**] for generalized Padé approximation.

One additional complication is that the distinguished branch $Q^*(x)$ of the approximating function may shift to obtain the maximum approximation order as more of the collocating nodes are used. This has the consequence that the set of "unattainable points" may vary as N is increased.

For the distinguished branch $Q^*(x)$, and for $x_j \in A_R$, if $f(x_j) = Q^*(x_j)$ then

$$([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*])|_{x=x_j} = \partial P^*(Q^*,x)/\partial Q^*|_{x=x_j} \neq 0,$$

while if $f(x_j) \neq Q^*(x_j)$ then

$$([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*])|_{x=x_j} = 0 \text{ since } P^*(f,x)|_{x=x_j} = 0 = P^*(Q^*,x)|_{x=x_j}.$$

Conversely, if $([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*])|_{x=x_j} = 0$ then $f(x_j) \neq Q^*(x_j)$ and the point $(x_j, f(x_j))$ lies on another branch of the algebraic function $Q(x)$, unless the node x_j is a double node of $P^*(f,x)$ – a possibility that is excluded under the hypotheses of this paper. And if $([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*])|_{x=x_j} \neq 0$ then since the numerator is $O((x - x_j))$ the denominator must also be $O((x - x_j))$, which implies that $f(x_j) = Q^*(x_j)$.

Consequently it may be concluded, for the case of *distinct* node points and under the condition $\partial P^*(Q^*,x)/\partial Q^*|_{x \in \overline{A}_R} \neq 0$ (so that no branches intersect the distinguished branch in the closed interval containing the collocation node set), that $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = 0$ for $x = x_j$ if and only if the point $(x_j, f(x_j))$ lies on a branch of the algebraic function $Q(x)$ other than the distinguished branch. That is the node point x_j will be an *unattainable point* for this distinguished branch.

In [**13,17**] a deficiency index was defined to measure the amount by which the approximation falls short of the expected order. An analogous concept is appropriate in the case of distinct collocation nodes.

DEFINITION 15 (DEFICIENCY).

Let a distinguished branch $Q^*(x)$ of the $\underset{\sim}{n}$ algebraic function $Q(x)$, defined by $P^*(Q,x) = 0$, be determined by $Q^*(x_k) = f(x_k)$, and let the branch $Q^*(x)$ satisfy $\partial P^*(Q^*,x)/\partial Q^*|_{x\in\overline{A}_R} \neq 0$.

The deficiency, $D(\underset{\sim}{n})$, of the branch $Q^*(x)$ is defined by

$$D(\underset{\sim}{n}) = \mathrm{Ord}\,([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*])$$

where this quotient is $O(\omega_D(x))$, where

$$\omega_D(x) = \prod_{j=0}^{D-1}(x - x_{k_j})$$

and where $\{k_j, j = 0(1)D-1\} \subset \{k, k = 0(1)R-1\}$ constitute the set of subscripts for the set of unattainable points for the approximating function $Q^*(x)$. Note that $\omega_0(x)$ is to be taken as 1 and that $Q^*(x)$ with zero deficiency is a normal branch (Definition 13). ∎

The set of points $\{x_{k_j}, j = 0(1)D-1\}$ corresponds to the set of unattainable points in rational approximation. The deficiency, $D(\underset{\sim}{n}) = D \geq 0$, is the amount by which the order of approximation falls short of the expected order. The deficiency is similar to Stahl's concept of the "interpolation defect" [**18**].

We deal firstly with the case where the algebraic form has a factor $\omega_{r,R}(x), r > 0$, as defined in Theorem 8. This case corresponds most directly with the concept of unattainable points in rational approximation.

LEMMA 16 (COMMON FACTORS).

If $P^*_{\underset{\sim}{n}}(f,x) = \omega_{r,R}(x)P^*_{\underset{\sim}{m}}(f,x)$, where $P^*_{\underset{\sim}{m}}(f,x)$ has polynomial coefficients with no common factor $(x - x_k)$ for $x_k \in A_R$ (i.e. $[\sum_{i=0}^{p}|a_i(x)|]|_{x\in A_R} \neq 0$), then the deficiency of the distinguished branch $Q^*(x)$ is given by

$$D = D(\underset{\sim}{n}) = D(\underset{\sim}{m}) + r.$$

PROOF:

$$\mathrm{Ord}\,([P^*_{\underset{\sim}{n}}(f,x) - P^*_{\underset{\sim}{n}}(Q^*,x)]/[f - Q^*]) = D.$$

That is

$$\begin{aligned}&\mathrm{Ord}\,([\omega_{r,R}(x)\{P^*_{\underset{\sim}{m}}(f,x) - P^*_{\underset{\sim}{m}}(Q^*,x)\}]/[f - Q^*]) \\ &= r + \mathrm{Ord}\,([P^*_{\underset{\sim}{m}}(f,x) - P^*_{\underset{\sim}{m}}(Q^*,x)]/[f - Q^*]).\end{aligned}$$

The result follows from Definition 15.

Theorem 17 (Order of approximation).

Let

$$P^*_{\underset{\sim}{n}}(f,x)=\omega_{r,R}(x)P^*_{\underset{\sim}{m}}(f,x),$$

where the algebraic form $P^*_{\underset{\sim}{m}}(f,x)$ *has polynomial coefficients with no common factor* $(x-x_k)$ *for* $x_k \in A_R$, *and satisfies*

$$P^*_{\underset{\sim}{m}}(f,x)=O(\omega_{M+S(\underset{\sim}{m})}(x)),$$

where $S(\underset{\sim}{m})$ *is the surplus and* $M+1=\sum_{i=0}^{p}(m_i+1)$. *Suppose also that the unique branch of* $Q(x)$ *defined by* $P^*_{\underset{\sim}{m}}(Q,x)=0$ *which satisfies* $Q(x_0)=f(x_0)$ *is a normal branch.*

Then the unique branch of the $\underset{\sim}{n}$ *algebraic function of degree* p, $Q^*(x)$, *defined by the equation*

$$P^*_{\underset{\sim}{n}}(Q^*,x)=0,$$

subject to the condition

$$Q^*(x_0)=f(x_0),$$

is an approximation to $f(x)$ *satisfying*

$$Q^*(x)=f(x)+O(\omega_T(x)),$$

where $T=N+S-D$,

$D=D(\underset{\sim}{n})\geq 0$ *is the deficiency of* $P^*_{\underset{\sim}{n}}(f,x)$,

$S=S(\underset{\sim}{n})\geq 0$ *is the surplus of* $P^*_{\underset{\sim}{n}}(f,x)$,

and $\omega_T(x)=\omega_{N+S}(x)/\omega_D(x)$,

where $\omega_D(x)=\omega_{r,R}(x)$ *defines the set of unattainable points.*

Proof: Now $P^*_{\underset{\sim}{n}}(Q,x)=\omega_{r,R}(x)P^*_{\underset{\sim}{m}}(Q,x)=0$, and hence the $\underset{\sim}{n}$ algebraic function of degree p defined by $P^*_{\underset{\sim}{n}}(Q,x)=0$ is in fact the same as the $\underset{\sim}{m}$ algebraic function of degree p defined by $P^*_{\underset{\sim}{m}}(Q,x)=0$. By hypothesis, the distinguished branch determined by the condition $Q^*(x_0)=f(x_0)$ is a normal branch and hence by Theorem 14, we have

$$Q^*(x)=f(x)+O(\omega_{M+S(\underset{\sim}{m})}(x)).$$

By Lemma 16, $D=D(\underset{\sim}{n})=D(\underset{\sim}{m})+r$. Since the branch $Q^*(x)$ is normal, $D(\underset{\sim}{m})=0$ and hence $D=r$ and $\omega_D(x)=\omega_{r,R}(x)$.

Since $\omega_{r,R}(x)=\prod_{k=0}^{r-1}(x-x_{M+S(\underset{\sim}{m})+k})$, the set of nodes $\{x_{M+S(\underset{\sim}{m})+k}\}_{k=0}^{r-1}$ are the set of unattainable points for this approximation by Definition 15. Thus, in effect, we have the $\underset{\sim}{n}$ algebraic function approximation is in fact the $\underset{\sim}{m}$ algebraic function approximation and the additional nodes are unattainable points on the distinguished branch.

By Theorem 8, we have $N = M+(p+1)r$ and $S(\underset{\sim}{n}) = S(\underset{\sim}{m})-pr$. Hence $M+S(\underset{\sim}{m}) = N + S(\underset{\sim}{n}) - r = T$. ∎

Suppose now that $P^*_{\underset{\sim}{n}}(f,x)$ is an algebraic form whose polynomial coefficients have no common factors $(x - x_j)$ for $x_j \in A_R$, and that $[P^*_{\underset{\sim}{n}}(f,x) - P^*_{\underset{\sim}{n}}(Q^*,x)]/[f - Q^*] = 0$ for some $x = x_k \in A_R$ where $Q^*(x)$ is the branch determined by $Q^*(x_0) = f(x_0)$. (The subscript $\underset{\sim}{n}$ will be understood hereafter.)

If $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = 0$ for $x = x_k$, then the point $(x_k, f(x_k))$ lies on a branch of $Q(x)$ other than $Q^*(x)$. If, in addition, for $p > 1, \partial P(Q,x)/\partial Q|_{Q=Q^*,x=x_k} = 0$, then at least two of the branches of $Q(x)$ coincide at the node $x = x_k$, and x_k would be a repeated node in the collocation set. In particular, if $k = 0$, it will be necessary to impose additional conditions in order to distinguish a unique branch $Q^*(x)$. (Compare [**13,17**] for how this case was treated in the case of a single multiple node.) However, such additional complications will be left for a subsequent paper, and to avoid such problems but still illustrate the basic nature of the results, we will assume in this paper that

$$\partial P(Q^*,x)/\partial Q^*|_{x\in\overline{A}_R} \neq 0.$$

With this assumption, if $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = 0$ for $x = x_k$, then the point $(x_k, f(x_k))$ lying on a branch of $Q(x)$ other than $Q^*(x)$ becomes an unattainable point on this distinguished branch.

However, on the assumption that we seek a 'best' approximation $Q^*(x)$ with maximum order, in the sense that $Q^*(x)$ collocates the largest number of nodes from A_R, it may well be necessary to choose $Q^*(x)$ determined by $x_k \in A_R$ with $k \neq 0$. Thus we choose $Q^*(x)$ from the finite number of possible branches so that $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = O(\omega_D(x))$ as in Definition 15, with $D \geq 0$ as small as possible.

These comments may be summarized in the following lemma.

LEMMA 18. *Let $P^*(f,x)$ be the $\underset{\sim}{n}$ algebraic form of degree p, with $(\sum_{i=0}^{p}|a_i(x)|)|_{x\in A_R} \neq 0$. Then a distinguished branch $Q^*(x)$ is chosen which minimizes the value of the deficiency $D = D(\underset{\sim}{n})$ where*

$$\operatorname{Ord}([P^*(f,x) - P^*(Q^*,x)]/[f - Q^*]) = D,$$

and

$$\partial P^*(Q^*,x)/\partial Q^*|_{x\in\overline{A}_R} \neq 0.$$

If $\omega_D(x) = \prod_{j=0}^{D-1}(x - x_{k_j})$ then the subset $A_D = \{x_{k_j}, j = 0(1)D-1\} \subset A_R$ is the set of unattainable points for this approximation. ∎

If a sequence of approximations is being calculated for increasing values of N, it should be noted that both $Q^*(x)$ and the set of unattainable points may vary with increasing N. Thus, although it has sometimes been suggested for rational approximation that the set of collocation nodes should be re–ordered so that the unattainable points correspond to the final nodes in the set A_R, this process does not seem practical in general if a sequence of approximations is to be calculated.

THEOREM 19 (ORDER OF APPROXIMATION).

Let S be the surplus of the $\underset{\sim}{n}$ algebraic form of degree $p > 1, P^*(f, x)$, corresponding to the set A_M of distinct collocation nodes.

Let D be the minimum deficiency of the distinguished branch $Q^*(x)$ of the corresponding $\underset{\sim}{n}$ algebraic function defined by $P^*(Q^*, x) = 0$, and satisfying $Q^*(x_k) = f(x_k)$ for some $x_k \in A_R$.

If the unique branch of the $\underset{\sim}{n}$ algebraic function of degree $p, Q^*(x)$, defined by the equation

$$P^*(Q^*, x) = 0$$

subject to the initial condition

$$Q^*(x_k) = f(x_k) \text{ for a suitable } x_k \in A_R,$$

satisfies

(i)

$$\left(\sum_{i=0}^{p} |a_i(x)| \right) |_{x \in A_R} \neq 0, \text{ where } R = N + S,$$

(ii)

$$\operatorname{Ord}\left([P^*(f, x) - P^*(Q^*, x)]/[f - Q^*]\right) = D,$$

(iii)

$$\partial P^*(Q^*, x)/\partial Q^*|_{x \in \overline{A}_R} \neq 0,$$

then $Q^*(x)$ is an approximation to $f(x)$ satisfying

$$Q^*(x) = f(x) + O(\omega_T(x))$$

where $\omega_T(x) = \omega_R(x)/\omega_D(x)$, and $T = N + S - D$.

PROOF: Following a similar argument to Theorem 14, we have

$$P^*(f, x) = O(\omega_R(x)), \quad P^*(Q^*, x) = 0, \tag{7}$$
$$[P^*(f, x) - P^*(Q^*, x)]/[f - Q^*] = O(\omega_D(x)), \tag{8}$$
$$\partial P^*(Q^*, x)/\partial Q^*|_{x \in \overline{A}_R} \neq 0. \tag{9}$$

Applying the linear functionals L_k to (7) gives

$$L_k P^*(f, x) = 0 = L_k P^*(Q^*, x), \qquad k = 0(1)R - 1. \tag{10}$$

Hence $\dfrac{P^*(f(x_k), x_k) - P^*(Q^*(x_k), x_k)}{f(x_k) - Q^*(x_k)} \cdot [f(x_k) - Q^*(x_k)] = 0.$

Using (8), for $x_k \in A_R \backslash A_D$ this gives

$$Q^*(x_k) = f(x_k),$$

and hence

$$Q^*(x) = f(x) + O(\omega_T(x)).$$

Note that for $x_k \in A_R \backslash A_D$

$$\frac{P^*(f(x_k), x_k) - P^*(Q^*(x_k), x_k)}{f(x_k) - Q^*(x_k)} = \sum_{i=1}^{p} a_i(x_k) \left[\sum_{j=0}^{i-1} f(x_k)^{i-1-j} Q^*(x_k)^j \right]$$
$$= \partial P^*(Q^*, x)/\partial Q^*|_{x=x_k} \neq 0, \text{ since } Q^*(x_k) = f(x_k).$$

The remaining values of x in (9) are necessary to ensure that no other branches of $Q(x)$ intersect $Q^*(x)$ in this range, and hence all the points $(x_k, f(x_k))$ lie on this branch $Q^*(x)$.

For $x_{k_j} \in A_D$, equation (8) implies that the point $(x_{k_j}, f(x_{k_j}))$ must lie on another branch of the algebraic function $Q(x)$ defined by $P^*(Q, x) = 0$. Since, by (9), no other branches of $Q(x)$ intersect $Q^*(x)$ in the closed interval containing the set of collocation nodes, these points are unattainable points for this approximation.

COROLLARY 20.

For an algebraic form satisfying the conditions of Theorem 19, the order of approximation by an algebraic function will always be at least R/p. That is, at worst, $Q^*(x)$ satisfies

$$Q^*(x) = f(x) + O(\omega_T(x)) \text{ with } T = R/p.$$

PROOF: Since the algebraic form of degree p has at most p distinct branches, the number of points collocating each branch must be R/p in the worst case in which R/p points lie on each branch. In this case, from the point of view of order of approximation, it makes no difference which branch is chosen as the distinguished branch $Q^*(x)$ in Theorem 19. ∎

The fundamental process for the order of approximation of a non–normal branch of the algebraic function may be summarized by the following theorem.

THEOREM 21 (ORDER OF APPROXIMATION).

Let S be the surplus of the $\underset{\sim}{n}$ algebraic form of degree p, $P^*_{\underset{\sim}{n}}(f, x)$, corresponding to the set A_M of distinct collocation nodes.

Let D be the minimum deficiency of a distinguished branch $Q^*(x)$ of the corresponding $\underset{\sim}{n}$ algebraic function defined by $P^*_{\underset{\sim}{n}}(Q^*, x) = 0$ and satisfying $Q^*(x_k) = f(x_k)$ for some $x_k \in A_R$, which is subject to the condition $\partial P^*_{\underset{\sim}{n}}(Q^*, x)/\partial Q^*|_{x \in \overline{A}_R} \neq 0$.

Let $P^*_{\underset{\sim}{n}}(f, x) = \omega_{r,R}(x) P^*_{\underset{\sim}{m}}(f, x)$, where $P^*_{\underset{\sim}{m}}(f, x)$ has polynomial coefficients which satisfy $(\sum_{i=0}^{p} |a_i(x)|)|_{x \in A_R} \neq 0$.

Then the distinguished branch of the associated $\underset{\sim}{n}$ algebraic function of degree p, $Q^*(x)$, defined by the equation

$$P^*_{\underset{\sim}{n}}(Q^*, x) = 0$$

subject to the initial condition

$$Q^*(x_k) = f(x_k) \textit{ for a suitable } x_k \in A_R,$$

is an approximation to $f(x)$ *satisfying*

$$Q^*(x) = f(x) + O(\omega_T(x))$$

where

$$\omega_T(x) = \omega_R(x)/\omega_D(x), \textit{ and } T = N + S - D.$$

PROOF: This result follows by combining the results of Theorems 17 and 19. Note that in this case we have $\omega_D(x) = \omega_{r,R}(x).\ \omega_{D(\underset{\sim}{m})}(x)$ and hence $D = D(\underset{\sim}{m}) + r$.

The approximation by the $\underset{\sim}{n}$ algebraic function defined by $P^*_{\underset{\sim}{n}}(Q^*, x) = 0$ is in fact the same as the $\underset{\sim}{m}$ algebraic function of degree p defined by $P^*_{\underset{\sim}{m}}(Q^*, x) = 0$. This latter approximation is obtained by Theorem 19 (instead of Theorem 14 as was the case in Theorem 17). The set of unattainable points for the branch $Q^*(x)$ is given by the subset A_D, as in the previous theorems.

This section has developed theorems for the basic behaviour of the order of approximation by an $\underset{\sim}{n}$ algebraic function determined by a given set of distinct point evaluation functionals. It is interesting to observe the close analogy with the case of a collocation set consisting of a single multiple node which was discussed in [**13,17**]. The basic result is that the 'order of approximation' is determined by $N + S - D$ in both cases. It is suggested that this basic order will also hold true for the cases of more complex coincidences between the branches of $Q(x)$. These results will be detailed in a further report.

§4. Examples.

This section contains some illustrative examples of the results of the previous sections. The first examples consider the rational case with $p = 1$.

EXAMPLE 22.

This is the same example as Example 10 in Section 2. The $(1,1)$ *algebraic form on* $A_M = \{-1, 0, 1\}$ *is*

$$-(x-1) + (x-1)f(x) = O((x+1)x(x-1)) = O(\omega_3(x))$$

with $n_0 = n_1 = 1, N = 3$ *and* $S = 0$. *We have*

$$\begin{aligned} Q^*(x) = (x-1)/(x-1) = 1 &= 1 + x + x^2 + O((x+1)x) \\ &= f(x) + O(\omega_2(x)). \end{aligned}$$

In this case $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = (x-1) = O(\omega_D(x))$, *and so* $D = 1$. *Thus this result illustrates Theorem 17 with* $2 = T = N + S - D$. *The node* $x = 1$ *is an unattainable point for this approximation.* ∎

EXAMPLE 23. *This is the same example as Example 11 in Section 2.*

(i) $n_0 = 1, n_1 = 0$ *gives* $N = 2$. *The* $(1,0)$ *algebraic form is*

$$-x + 1f(x) = O(\omega_3(x)).$$

Hence $S = 1$. *We have*

$$[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = 1 \neq 0$$

and $\partial P^*(Q,x)/\partial Q = 1 \neq 0$. *Hence this branch is normal (with* $D = 0$*) and by Theorem 14 we obtain*

$$Q^*(x) = x = f(x) + O(\omega_3(x))$$

with $3 = R = N + S = N + S - D$.

(ii) It was noted in Example 11 that the $(3,2)$ *algebraic form with* $N = 6$ *is*

$$(x-4)(x-6)P^*_{(1,0)}(f,x) = O(\omega_6(x))$$

with surplus $S = 0$. *That is*

$$-x(x-4)(x-6) + (x-4)(x-6)f(x) = O(\omega_6(x)) = O(\omega_R(x))$$

where $\omega_6(x) = \prod_{k=1}^{6}(x-k)$.

Thus $[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] = (x-4)(x-6) = O(\omega_D(x))$ *and hence* $D = 2$. *By part (i), the branch* $Q^*(x)$ *of the form* $P^*_{(1,0)}(f,x)$ *is normal and hence, by Theorem 17, the* $(3,2)$ *algebraic (rational) approximation is given by*

$$Q^*(x) = x = f(x) + O(\omega_T(x))$$

where $\omega_T(x) = \omega_R(x)/\omega_D(x) = (x-1)(x-2)(x-3)(x-5)$, *and* $4 = T = N + S - D$.

That is the function $Q^*(x) = x$ *collocates the function* $f(x)$ *at the nodes* $\{1, 2, 3, 5\}$ *and the nodes* $\{4, 6\}$ *correspond to unattainable points for this approximation.* ∎

The next three examples consider how these results extend to the case of quadratic function approximation, i.e. $p = 2$.

EXAMPLE 24.

Let $f(x) = e^x$ *and* $p = 2$ *with* $A_M = \{0,1,2,3,4\}$. *From [16], the* $(1,1,1)$ *algebraic form for this function is*

$$\begin{aligned}&[2e^5 + 4e^4 + e^4(e^2-1)x]\\ &+ [e(e^2-1)(e+1)^2(x-2)]f(x)\\ &+ [-2(e+2)+(e^2-1)(x-4)]f(x)^2 = O(\omega_5(x)).\end{aligned}$$

For this example $N = 5, S = 0$. *Also*

$$[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] \neq 0 \text{ for } x \in A_R$$

where $Q^*(x)$ *is the branch distinguished by* $Q^*(0) = f(0) = 1$ *and* $\partial P^*(Q^*,x)/\partial Q^* \neq 0$ *for* $x \in [0,4]$.

Hence this branch is normal (with $D = 0$*) and*

$$Q^*(x) = (-a_1(x) - \sqrt{a_1(x)^2 - 4a_2(x)a_0(x)})/2a_2(x),$$

where the coefficients $a_i(x)$ *are given above, is an approximation satisfying*

$$Q^*(x) = e^x + O(\omega_5(x)).$$

Hence this branch collocates the function $f(x) = e^x$ *on the collocation set* $A_R = \{0,1,2,3,4\}$ *in accordance with Theorem 14.*

EXAMPLE 25.

Let $p = 2$ *and* $A_M = \{0,1,2,3,4\}$ *with given function values*

$$f(0) = 2,\ f(1) = 1,\ f(2) = 2,\ f(3) = 0,\ f(4) = -\frac{1}{2}.$$

(i) With $n_0 = n_1 = n_2 = 0, N = 2$ *and the* $(0,0,0)$ *algebraic form is*

$$2 - 3f(x) + f^2(x) = O(x(x-1)(x-2)) = O(\omega_3(x)).$$

Hence $S = 1$ *and Theorem 8 implies that this is also the algebraic form of types* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ *with* $S = 0$.

$P^*(Q,x) = 0$ *with* $Q(0) = f(0) = 2$ *implies that the distinguished branch is* $Q^*(x) = 2$. *Thus*

$$\begin{aligned}[P^*(f,x) - P^*(Q^*,x)]/[f - Q^*] &= -1 + f = O((x-1))\\ &= O(\omega_D(x)) \text{ with } D = 1,\\ \text{and } \partial P^*(Q^*,x)/\partial Q^* &= -3 + 2Q^* \neq 0.\end{aligned}$$

Hence

$$Q^*(x) = 2 = f(x) + O(\omega_T(x))$$

where

$$\omega_T(x) = \omega_R(x)/\omega_D(x) = x(x-2), \text{ and } 2 = T = N + S - D.$$

That is the function $Q^*(x) = 2$ collocates the function $f(x)$ at the nodes $\{0, 2\}$ and the node $\{1\}$ is an unattainable point for this approximation.

The same function is also the $(1,0,0), (0,1,0), (0,0,1)$ type approximation, with T unchanged since N increases by 1 and S decreases by 1 in these cases.

(ii) With $n_0 = n_1 = 1$, and $n_2 = 0, N = 4$ and the $(1,1,0)$ algebraic form is

$$6 - 2x - (7 - x)f(x) + 2f(x)^2 = O(\omega_5(x)).$$

Hence $S = 1$ and this will also be the algebraic form for other values of $\underset{\sim}{n}$. For $(1,1,0)$ we have

$$P^*(Q, x) = 0 \text{ gives } ((3 - x) - 2Q)(2 - Q) = 0.$$

If the condition $Q(0) = f(0) = 2$ is used to obtain the distinguished branch $Q^*(x)$, then $Q^*(x) = 2$. Thus

$$\begin{aligned}[P^*(f, x) - P^*(Q^*, x)]/[f - Q^*] &= -3 + x + 2f \\ &= O((x - 1)(x - 3)(x - 4)) \\ &= O(\omega_D(x)) \text{ with } D = 3,\end{aligned}$$

and $\partial P^*(Q^*, x)/\partial Q^* = x + 1 \neq 0$ for $x \in \overline{A}_R = [0, 4]$. Thus for this branch

$$Q^*(x) = 2 = f(x) + O(\omega_T(x))$$

where

$$\omega_T(x) = \omega_R(x)/\omega_D(x) = x(x - 2) \text{ and } 2 = T = N + S - D.$$

That is the algebraic function $Q^*(x) = 2$ collocates the function $f(x)$ at the nodes $\{0, 2\}$ and the set of nodes $\{1, 3, 4\}$ are unattainable points for this approximation.

However, a better approximation, in the sense that it will collocate more points, is given by the alternate branch which is distinguished by $Q(1) = f(1) = 1$. In this case $Q^*(x) = (3 - x)/2$. Thus

$$\begin{aligned}[P^*(f, x) - P^*(Q^*, x)]/[f - Q^*] &= -2(2 - f) \\ &= O(x(x - 2)) \\ &= O(\omega_D(x)) \text{ with } D = 2,\end{aligned}$$

and $\partial P^*(Q^*, x)/\partial Q^* = -(x + 1) \neq 0$ for $x \in \overline{A}_R = [0, 4]$. Thus for this branch

$$Q^*(x) = (3 - x)/2 = f(x) + O(\omega_T(x))$$

where

$$\omega_T(x) = \omega_R(x)/\omega_D(x) = (x - 1)(x - 3)(x - 4) \text{ and } 3 = T = N + S - D.$$

That is the function $Q^*(x) = (3-x)/2$ is a $(1,1,0)$ algebraic function which collocates the function $f(x)$ at the nodes $\{1, 3, 4\}$ and the nodes $\{0, 2\}$ are unattainable points for this branch.

These results are in accordance with Theorem 21 where D is chosen as the minimum deficiency of the possible branches. ∎

EXAMPLE 26.

Let $p = 2$ and $f(x) = \cos(\pi x/6)$ with the symmetric set of collocation nodes $A_M = A_{14} = \{-2, 2, -3, 3, -4, 4, -6, 6, -8, 8, -9, 9, -10, 10\}$.

Using the results from Example 12, for $n_0 = 4,\ n_1 = n_2 = 2$ and hence $N = 10,\ S = 2$, the $(4, 2, 2)$ algebraic form is

$P^*(f, x) =$

$$(-32\,x^4 + 2880\,x^2 - 23328) + (685\,x^2 + 20300)f(x) + (90\,x^2 + 2840)f(x)^2 = O(\omega_{12}(x)).$$

The branch distinguished by $Q^*(2) = f(2) = 1/2$ may be written explicitly as

$$Q^*(x) = (-a_1(x) + \sqrt{a_1(x)^2 - 4a_2(x)a_0(x)})/2a_2(x),$$

where the coefficients $a_i(x)$ are the coefficients given above.

For this branch

$$([P^*(f, x) - P^*(Q^*, x)]/[f - Q^*])|_{x \in A_{12}} =$$
$$((685\,x^2 + 20300) + (90\,x^2 + 2840)(f(x) + Q^*(x)))|_{x \in A_{12}} \neq 0,$$

and

$$\partial P^*(Q^*, x)/\partial Q^* = (685\,x^2 + 20300) + 2(90\,x^2 + 2840)Q^*(x).$$

Substituting for $Q^*(x)$ from above

$$\partial P^*(Q^*, x)/\partial Q^* = \sqrt{5(2304\,x^6 - 40811\,x^4 + 698456\,x^2 + 135419216)} \neq 0$$

for $x \in \overline{A}_{12} = [-9, 9]$. So this branch is normal ($D = 0$), and

$$Q^*(x) = f(x) + O(\omega_{12}(x)) = f(x) + O(\omega_R(x)),$$

where $12 = R = N + S = N + S - D$.

That is this branch collocates the function $f(x) = \cos(\pi x/6)$ on the collocation node set A_{12} as indicated by Theorem 14.

The same function $Q^*(x)$ is also the $(5, 2, 2)$, $(4, 3, 2)$, $(4, 2, 3)$ type algebraic approximation (with N increased by 1 and S decreased by 1), as well as the algebraic approximation of types $(5, 3, 2)$, $(4, 3, 3)$, $(5, 2, 3)$ (with $N = 12$ and $S = 0$). $Q^*(x)$ is also the $(5, 3, 3)$ algebraic approximation since by Theorem 8,

$$P^*_{(5,3,3)}(f, x) = (x + 10)P^*_{(4,2,2)}(f, x),$$

and by Theorem 17, $x = -10$ is an unattainable point for the branch $Q^*(x)$. ∎

This section concludes with a graphical presentation of the results of Example 26. Figure 1 is a plot of the distinguished branch $Q^*(x)$ and the given function $f(x) = \cos(\pi x/6)$. This branch is effectively obtained by collocation on 11 nodes, since the original collocation on 10 nodes produces two solutions by Theorem 6, and the unique representative obtained by Theorem 5 effectively requires collocation at an additional node point. There is little apparent difference between the the functions except for a slight difference at $x = 0$ and more apparent differences at $x = \pm 10$ outside the domain of the collocation nodes.

Figure 2 plots the alternative branch of the algebraic function $Q(x)$ which satisfies $P^*(Q, x) = 0$. In this example this branch has no points of agreement with $f(x)$.

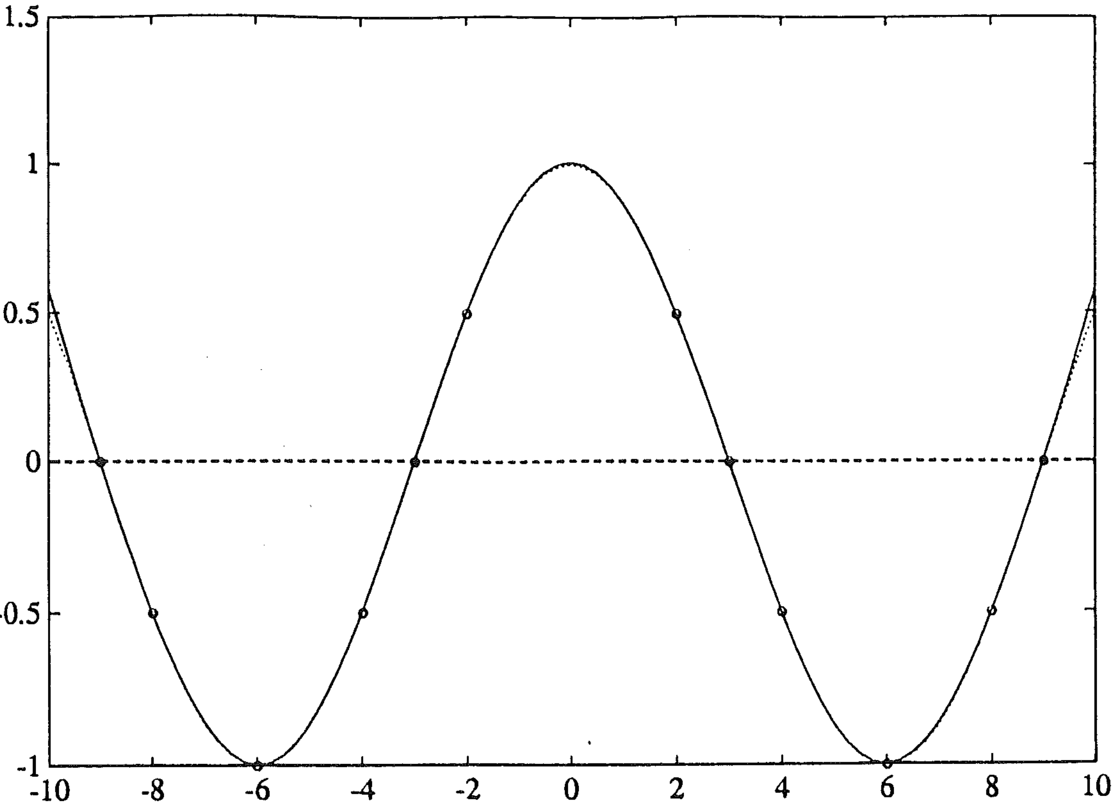

Figure 1. Plot of branch $Q^*(x)$ and $f(x) = \cos(\pi x/6)$, showing collocation points.

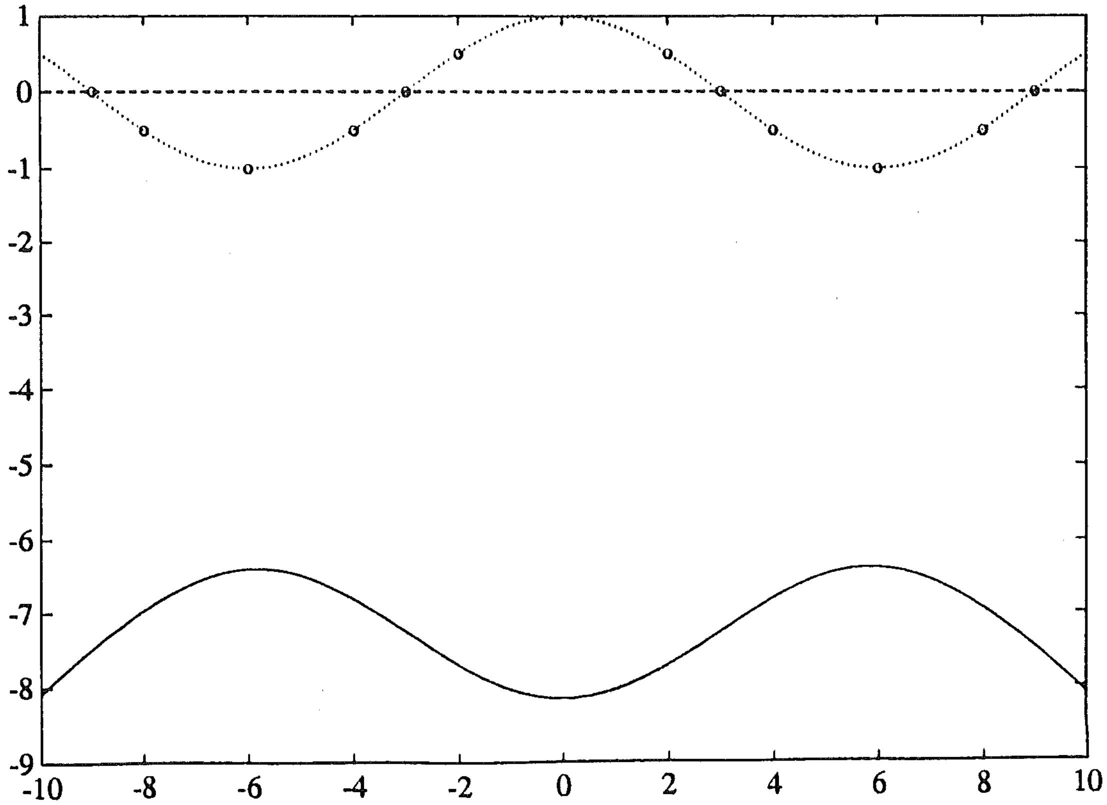

Figure 2. Alternate branch $Q(x)$ and $f(x) = \cos(\pi x/6)$, showing collocation points.

§5. Conclusion.

This paper has considered the problem of approximating a real–valued function, $f(x)$, whose values on a sufficiently large set of nodes A_M are assumed to be known, by an $\underset{\sim}{n} = (n_0, n_1, \ldots, n_p)$ algebraic function of degree p, $Q(x)$, with N defined by $N + 1 = \sum_{i=0}^{p}(n_i + 1)$. Following the analogous procedure in [**13,17**], a careful distinction was made between the approximating properties of the algebraic form, by which the polynomial coefficients of the algebraic function are defined, and the approximating properties of the algebraic function itself.

By defining a surplus, S, for the algebraic form a unique algebraic form could be defined. It was shown how a basic structure within a table of algebraic forms could be constructed, but the global nature of such a table is complicated by the possibility of overlapping structures.

A concept of the deficiency, D, of the distinguished branch $Q^*(x)$ of the algebraic function was also defined. This is analogous to the concept of deficiency introduced in [**13,17**] for collocation at a single multiple node. A normal branch of the algebraic function approximation has deficiency $D = 0$, and in this case there is a unique branch of the algebraic function with specified approximating properties. If a non–normal branch of the algebraic function (with a non–zero deficiency) is obtained then the basic behaviour of the order of approximation of the algebraic function was shown to be $N + S - D$ as in the case of collocation at a single multiple node. The analysis also identified the set of "unattainable points" for this algebraic approximation. However the branch of the algebraic function may not be unique in this case as it is possible that two (or more) different branches have the same order of approximation for collocation on different subsets of the set of collocating nodes A_M. A unique branch could be defined by choosing from this finite set of branches with the same order of approximation, that branch for which $Q^*(x_k) = f(x_k)$, $x_k \in A_M$, for the minimum value of k. This paper avoided cases where the branches of the algebraic function intersect and dealt with the order of approximation in the basic case.

Note that the final theorem allows for both "over–approximation" (as in Example 26) and "under–approximation" (as in Example 25) to be obtained, depending on the nature of the algebraic form and the branch of the algebraic function. This should be compared to the "interpolation defect" and "overinterpolation" concepts noted by Stahl [**18**] in the context of generalized (or multipoint) Padé approximation. This paper both identifies and quantifies the concepts of both "over–approximation" and "under–approximation" for the general algebraic approximation determined on a distinct set of collocation nodes.

The results obtained have been illustrated by some simple examples in Section 4.

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