SPACE-LIKE INFINITY AND ASYMPTOTIC PHOTON FIELDS IN Q.E.D.

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ABSTRACT

Infra-red effects in Q.E.D. are discussed from the point of view of space-like infinity. Using the properties of spin and conformally weighted functions defined on a null cone, an elementary proof of the infra-red coherence condition is given and it is shown that Lorentz symmetry is spontaneously broken on all charged superselection sectors. The implications of these results for quantum gravity are discussed.

1. INTRODUCTION

Many important physical theories such as electromagnetism, non-linear gauge theories and general relativity possess long range coulomb-like fields which dominate at space-like infinity (S.L.I.). It has long been realised that it is precisely these fields that cause the so-called infra-red effects in the quantum domain. In quantum electrodynamics (Q.E.D.), for example, these long range fields necessitate the use of non-Fock representations of the asymptotic photon fields (Kulish and Faddeev, 1970) and cause a spontaneous break down of Lorentz symmetry on all charged sectors (Fröhlich et.al, 1979). Such effects do not occur in theories possessing only short range fields whose asymptotic components vanish at S.L.I.

In linear theory, it is always possible to choose a short range field which completely agrees with any given long range field over any given finite region of space-time. It is therefore impossible to distinguish between such fields by means of an experiment confined, necessarily, to a finite region of space-time. Arguments based on this fact are often evoked to show that infra-red effects are, in principle, unobservable and therefore devoid of any physical content. To some extent this is true for theories such as Q.E.D. containing only linear massless fields, and for such theories it is possible to use some regularization scheme which does not appreciably alter the physical content of the theory but rids it of all infra-red effects. However, for theories containing non-linear massless fields, such an argument does not apply. This is particularly evident for a pure gravitational field described by a singularity free solution of Einstein's equations, $R_{ab} = 0$. According to recently proved positive mass theorems (see, for example, Witten, 1981), any such field which is non-trivial over a finite region, necessarily possesses a non-zero asymptotic component at S.L.I.; non-trivial short range gravitational fields simply do not exist. Thus, at least as far as gravity is concerned, and possibly even for other non-linear fields, there is a real possibility that infra-red effects may be of physical importance and not devoid of physical content as in Q.E.D. Such effects therefore warrant consideration.

Of course, quantized, non-linear massless fields are notoriously difficult beasts to deal with. Nevertheless, the infra-red properties of any theory, independent of whether it contains linear or non-linear massless fields, appear to have the following general features in common:

1. they are asymptotic in nature and independent of the detailed local structure of the theory,
2. they are essentially determined by fields at S.L.I. and,
3. they are independent of the detailed dynamics of the theory.

In this paper we shall concentrate on these general features in the relatively well understood context of Q.E.D. in the hope that this may provide some insight into the nature of infra-red effects in more complex theories such as quantum gravity, where these effects may be of physical importance.

By using the properties of spin and conformally weighted functions defined on a null cone, we shall give an elementary but non-rigorous proof of the infra-red coherence condition (Zwanziger, 1976) and show that Lorentz symmetry is spontaneously broken on all charged superselection sectors. Throughout this paper we shall emphasise the important rôle played by long range coulomb-like fields, a rôle which presumably carries over into more complex situations. Furthermore, even though we deal exclusively with Maxwell fields on flat space-time, much of our analysis is based on asymptotic quantities at S.L.I. which have direct analogues in general relativity and gauge theories.

In section 2 we investigate the asymptotic behaviour at S.L.I. of Maxwell fields associated with classical, charged particle scattering systems. In particular, we show that the leading asymptotic component of such a field is completely determined by a single function defined on a sphere. In section 3, we extend this analysis to the quantum domain and show, given certain conditions on the asymptotic behaviour of the quantized current distribution, that this function has a well defined quantum analogue which acts as a superselection operator. We then use this result to show that Lorentz symmetry is spontaneously broken in all but the vacuum sector , and to demonstrate the necessity for non-Fock asymptotic photon fields. Finally, in section 4, we discuss the possible relevance of these results to quantum gravity.

A similar analysis of these issues, but from the point of view of null infinity, has been conducted by Ashtekar (1980) and Ashtekar and Narain (1980) .

Much of our analysis is based on the wellknown properties of spin and conformally weighted functions and their associated 'edth' operators $\eth$ and $\bar{\eth}$ (Newman and Penrose, 1966). Our metric has signature $(+,-,-,-,)$. When no confusion can arise, we shall omit tensor indices.

2. CLASSICAL SCATTERING SYSTEMS

A classical, electromagnetic scattering system may be defined as a collection of interacting charged particles each of which attains a well-defined asymptotic velocity at time-like infinity. Thus, if $x^a = x_i^a(\tau)$ represents the world-line of the i^{th} particle, where $\sqrt{2}\tau$ is proper time†, the limits

$$\lim_{\tau\to\infty} \frac{dx_i}{d\tau} := v_i^{out} \tag{2-1}$$

and

$$\lim_{\tau\to-\infty} \frac{dx_i}{d\tau} := v_i^{in} \tag{2-2}$$

are well-defined. In terms of the total current distribution, J^a, generated by the particles, this implies that

$$\lim_{\lambda\to\infty} \lambda^3 J(\lambda x) = \sum_i e_i^{out} \int \delta(x - v_i^{out}\tau)\, v_i^{out}\, d\tau := J^{out}(x) \tag{2-3}$$

if x is future pointing, and

$$\lim_{\lambda\to\infty} \lambda^3 J(\lambda x) = \sum_i e_i^{in} \int \delta(x - v_i^{in}\tau)\, v_i^{in}\, d\tau := J^{in}(x) \tag{2-4}$$

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† For reasons connected with the standard Newman-Penrose spin coefficient notation, which we use later on in the paper, we normalize all four-velocity vectors such that $v^a v_a = 2$. Hence the factor $\sqrt{2}$.

if x is past pointing. Both J^{in} and J^{out} have support in the time-like region T inside the null cone N based at the origin, and satisfy

$$J^{in(out)}(\alpha x) = \alpha^{-3} J^{in(out)}(x) \tag{2-5}$$

for any given number α. They are therefore essentially functions on the space of all time-like lines through the origin. For non-trivial scattering, $J^{in} \neq J^{out}$. From equations (2-3) and (2-4) it is clear that J^{in} and J^{out} have the form

$$J_a^{in(out)} = \rho^{in(out)} x_a \tag{2-6}$$

where ρ^{in} and ρ^{out} are scalar quantities describing the velocity distribution of the 'in' and 'out' scattered particles.

In this section we shall investigate the asymptotic behaviour at S.L.I. of Maxwell fields associated with such current distributions, that is fields which satisfy

$$\begin{aligned} &\nabla_b F^{ab} = J^a \\ \text{and} \quad &\nabla_b {}^*F^{ab} = 0 \end{aligned} \tag{2-7}$$

where ${}^*F^{ab}$ is the dual of F^{ab}. However, for future convenience, we shall replace these two equations by the single complex equation

$$\nabla_b W^{ab} = J^a \tag{2-8}$$

where $W^{ab} = F^{ab} + i{}^*F^{ab}$.

If W_{ret} is the retarded field generated by J, one can show that

$$W^{as}_{ret} := \lim_{|\lambda| \to \infty} \lambda^2 W_{ret}(\lambda x) \tag{2-9}$$

is well-defined in the semi-space-like region S outside the null cone N, and is simply the retarded field generated by J^{in}. Similarly, W^{as}_{adv} is well defined in S and is the retarded field generated by J^{out}. If the scattering is non-trivial, we have $W^{as}_{ret} \neq W^{as}_{adv}$ and hence the asymptotic component,

$$W^{as}_{rad} := W^{as}_{ret} - W^{as}_{adv} ,$$

of the radiation field is non-zero.

In general, we shall refer to a field whose asymptotic component

$$W^{as}(x) := \lim_{|\lambda| \to \infty} \lambda^2 W(\lambda x) \qquad (2\text{-}10)$$

is well-defined over the whole of S as being asymptotically flat at S.L.I., or A.F. for short. If W^{as} vanishes over the whole of S we shall refer to the field as being empty at S.L.I. According to equation (2-10), W^{as} satisfies

$$W^{as}(\alpha x) = \alpha^{-2} W^{as}(x) \qquad (2\text{-}11)$$

for any given number α. It is therefore essentially a function on the space of all semi-space-like lines through the origin.

The 'in' and 'out' Maxwell fields of a scattering system may be defined by

$$W = W_{in} + W_{ret} \qquad (2\text{-}12)$$

and

$$W = W_{out} + W_{adv} \qquad (2\text{-}13)$$

where W is the total field. From these two relations we have, for non-trivial scattering,

$$W^{as}_{out} - W^{as}_{in} = W^{as}_{ret} - W^{as}_{adv} = W^{as}_{rad} \neq 0. \qquad (2\text{-}14)$$

Thus, if W_{out} is A.F., so is W_{in}. Furthermore, according to (2-14), W_{in} and W_{out} cannot <u>both</u> be empty at S.L.I. In the next section we shall show that this is related to the fact that the asymptotic 'in' and 'out' photon fields in Q.E.D. cannot belong to a Fock representation.

In order to consider the properties of the field W^{as} in greater detail, it is first necessary to develop a little mathematical machinery connected with weighted functions on the null cone N.

Given any covariantly constant, future pointing vector field v^a on N, normalized such that $v^a v_a = 2$, a null tetrad field $(n^a, l^a, m^a, \bar{m}^a)$ (m^a complex) can be defined on N by taking l^a to be the null vector which points along the null generators of N and which is normalized such that $v^a l_a = 1$, and by defining n^a and m^a by

$$\left.\begin{aligned} n^a &= v^a - l^a, \\ m^a m_a = m^a l_a &= m_a n_a = 0, \quad m^a \bar{m}_a = -1 \\ \text{and} \qquad l^a \nabla_a m^b &= 0. \end{aligned}\right\} \tag{2-15}$$

Under these conditions, l^a and n^a are determined uniquely and m^a is determined up to a phase factor:

$$m^{a\prime} = e^{i\lambda}\, m^a, \tag{2-16}$$

where $l^a \nabla_a \lambda = 0$. The vector field v^a also enables us to define a unique affine parameter, r, along the null generators of N according to

$$l^a \nabla_a r = 1 \quad \text{and } r(0) = 0. \tag{2-17}$$

If θ and ϕ are spherical coordinates labelling the generators of N, then (r,θ,ϕ) is a well-defined radial coordinate system on N.

In terms of this tetrad, the independent components of any Maxwell field W^{ab} on N are given by

$$\phi_0 = W_{ab} l^a m^b, \; \phi_1 = W_{ab} m^a \bar{m}^b \quad \text{and } \phi_2 = W_{ab} n^a \bar{m}^b \; . \tag{2-18}$$

Under the transformation (2-16) the component ϕ_0, for example, transforms according to

$$\phi_0{}' = e^{i\lambda}\phi_0$$

and is an example of a function with spin-weight (SW) unity. In general, a function η of which transforms according to

$$\eta' = e^{is\lambda}\eta \tag{2-19}$$

under (2-16), is said to have SW s. Two angular, differential operators associated with spin weighted functions are given by

$$\eth\eta = r\{m^a \nabla_a \eta - s(m^a \bar{m}^b \nabla_a m_b)\eta\} \tag{2-20}$$

and

$$\bar{\eth}\eta = r\{\bar{m}^a \nabla_a \eta - s(\bar{m}^a m^b \nabla_a \bar{m}_b)\eta\}$$

where η has SW s. It is easily checked that $\eth\eta$ and $\bar{\eth}\eta$ have spin weights s+1 and s-1, respectively. In terms of the coordinates θ and ϕ, these operators are given by

$$\eth\eta = -(\sin\theta)^{s}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\{(\sin\theta)^{-s}\eta\}$$

and

$$\bar{\eth}\eta = -(\sin\theta)^{-s}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\{(\sin\theta)^{s}\eta\} \quad , \tag{2-21}$$

provided Re m^a and Im m^a are tangent, respectively, to the curves ϕ = const. and θ = const. on each r = const. cross-section of N (Newman and Penrose, 1966).

Since our tetrad is linked to an arbitrary time-like vector v^a, it is important to determine how it transforms under a change in v^a. A new tetrad field $(n^{a'}, l^{a'}, m^{a'}\ \bar{m}^{a'})$ corresponding to a new time-like vector $v^{a'}$, where $v^{a'}v'_a = 2$, may be obtained by means of the pure Lorentz boost L (i.e. a linear, isometric, generator preserving mapping of N onto itself) which transforms v^a into $v^{a'}$. Since l^a and $l^{a'}$ are obviously proportional and $v^{a'}l_a{}' = 1$, we have

$$l^{a'} = \frac{l^a}{V} \tag{2-22}$$

where

$$V = v^{a'}l_a \quad . \tag{2-23}$$

Similarly,

$$\left.\begin{aligned} n^{a'} &= v^{a'} - l^{a'} \\ m^{a'} &= m^{a'} - \left(\frac{m^b v'_b}{V}\right) l^a \\ \text{and} \qquad r' &= Vr \end{aligned}\right\} \tag{2-24}$$

In general, we say that a quantity η which transforms according according to

$$\eta' = V^{-w}\eta \tag{2-25}$$

under this change of tetrad, has conformal weight (CW) w. Thus, for example, r has CW -1 and $l^{[a}m^{b]}$ and ϕ_0 each have CW 1. Another example of a conformally weighted function is the 'unit' spherical surface element given by

$$d\Omega_0 := r^{-2}d\Omega \tag{2-26}$$

where $d\Omega$ is the surface element of the r = const. cross sections of N. This transforms according to

$$d\Omega_0{}' = \frac{d\Omega_0}{V^2} , \tag{2-27}$$

and therefore has CW 2. A function which has both SW s and CW w will be said to have weight (s,w). ϕ_0, for example, has weight (1,1).

In general, the operators $\eth$ and $\bar{\eth}$ do not have a well defined conformal weight. However, it can be shown (see, for example, Newman and Penrose, 1966) that if $\pi(\theta,\phi)$ has weight (-1,-1) then

$$\phi := \eth\pi \tag{2-28}$$

has weight (-2,0). Conversely, if $\phi(\theta,\phi)$ has weight (-2,0), and can be written in the form (2-28), then π is determined uniquely and has weight (-1,-1). This result will prove useful later.

When expressed in terms of the components ϕ_i (i = 0,1,2), Maxwell's equations on N yield the following well-known radial spin-coefficient equations

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\phi_1 + \frac{1}{r}\bar{\eth}\phi_0 &= 0 \\ \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\phi_2 + \frac{1}{r}\bar{\eth}\phi_1 &= 0 \end{aligned} \tag{2-29}$$

(Newman and Penrose, 1968). In particular, the components, ϕ_i^{as}, of W^{as} on N satisfy equations (2-29). Furthermore, by equation (2-11), these components also satisfy

$$\frac{d}{dr}(r^2\phi_i^{as}) = 0, \tag{2-30}$$

and when this equation is substituted into (2-29) we obtain the following result:

$$\phi_0^{as} = 0, \quad \phi_1^{as} = \frac{\phi}{r^2} \text{ and } \phi_2^{as} = \frac{\bar{\eth}\phi}{r^2} \tag{2-31}$$

where $\phi(\theta,\phi)$ is a function of weight (-2,0). The boundary values of W^{as} on N are therefore completely determined by specifying the single function ϕ. Moreover, since any Maxwell field which is free on the semi-space-like region S is completely determined by its boundary values on N, we have the important result that W^{as} in S is determined by a single function $\phi(\theta,\phi)$ of weight (-2,0).

By Gauss' theorem together with equations (2-18), (2-26) and (2-31) we see that the total charge of the field is given by

$$e = \lim_{r \to \infty} \oint W_{ab}\, m^a \bar{m}^b \, d\Omega$$
$$= \oint \phi \, d\Omega_0 \ . \qquad (2\text{-}32)$$

Since ϕ and $d\Omega_0$ have weights (-2,0) and (2,0), respectively, the expresssion on the right hand side of (2-32) has zero weight. It is thus invariant in the sense that it is independent of the choice of tetrad.

If W and W* are two A.F. Maxwell fields which are related by the Lorentz boost L, i.e. L(W) = W*, then it is an easy matter to check that their corresponding weighted functions ϕ and ϕ^* at S.L.I. are related by

$$\phi^*(\theta,\phi) = V^{-2}\phi(\theta,\phi) . \qquad (2\text{-}33)$$

In particular, $\phi^* \neq \phi$ provided L is non-trivial. In the next section we shall use this fact to show that Lorentz symmetry is spontaneously broken in all but the vacuum sector in Q.E.D.

By using the Lienard-Wiechert potentials, one can show that the weighted functions ϕ_{ret} and ϕ_{adv}, corresponding to W_{ret} and W_{adv}, are given by

$$\phi_{ret} = \int \frac{\rho^{in}}{(x.l)^2}\, dV = \sum_i \frac{e_i^{in}}{(l.v_i^{in})^2}$$

and

$$\phi_{adv} = \int \frac{\rho^{out}}{(x.l)^2}\, dV = \sum_i \frac{e_i^{out}}{(l.v_i^{out})^2} \, , \qquad (2\text{-}34)$$

where dV is the volume element of the (future) time-like hyperboloid $x^a x_a = 2$.

In order to obtain a useful expression for the weighted function ϕ corresponding to any A.F. free field W (e.g. W^{in} or W^{out}), it is first necessary to expand W in terms of plane waves:

$$W_{ab} = \int \widetilde{W}_{ab}(k)\, e^{ik.x}\, dN \qquad (2\text{-}35)$$

where k^a is the null position vector on N and where dN is the invariant volume element of N. By writing $k^a = rl^a$ and using the tetrad orthogonality relations, it can be seen that equation (2-35) may be written in the form

$$W_{ab}(x) = \oint \int_{r=-\infty}^{\infty} l_{[a}m_{b]} \, \alpha e^{irl.x} \, r \, dr \, d\Omega_0 , \tag{2-36}$$

where α is a function on N which, in order for this expression to be invariant, has weight (-1,-1). There thus exists a one to one correspondence between free fields and functions of weight (-1,-1) defined on N.

In terms of these functions, an invariant expression for the Hermitian scalar product between two free Maxwell fields corresponding to α_1 and α_2 is given by

$$(\alpha_1, \alpha_2) = \oint \int \frac{\bar{\alpha}_1 \alpha_2}{|r|} \, d\Omega_0 dr. \tag{2-37}$$

This product enables us to define two important classes of smooth functions of weight (-1, -1):

$$C_0 := \{\alpha | (\alpha, \alpha) < \infty\} \tag{2-38}$$

and $$C := \{\alpha | \; |(\alpha \; \beta)| < \infty \text{ for all } \beta \in C_0\} .$$

Obviously $C_0 \subset C_1$. In the next section we shall show that fields belonging to C_0 and C are related to Fock and non-Fock, infra-red, photon fields, respectively.

From equation (2-37) and the defining relations (2-38) we see that

$$\lim_{r \to 0} \alpha := \pi(\theta, \phi) \tag{2-39}$$

exists if $\alpha \in C$ and vanishes if $\alpha \in C_0$. Furthermore, by means of a non-trivial limiting procedure similar to that given in Ludvigsen (1982), one can show that any field belonging to C is A.F. and that its corresponding weighted function ϕ at S.L.I. is given by

$$\phi(\theta, \phi) = \eth \pi . \tag{2-40}$$

The zero frequency mode of a free field therefore determines the field's asymptotic component at S.L.I. Equation (2-40) implies that ϕ has weight $(-2,0)$, and also determines a one-to-one correspondence between the functions π and ϕ. (c.f. equation (2-28)).

Finally, from the above results we see that the weighted function ϕ at S.L.I. corresponding to the total field of a scattering system is given by

$$\phi = \phi_{in} + \phi_{ret} = \phi_{out} + \phi_{adv} \tag{2-41}$$

where

$$\phi_{in(out)} = \eth\pi_{in(out)} \tag{2-42}$$

and

$$\pi_{in(out)} = \lim_{r\to 0} \alpha_{in(out)} , \tag{2-43}$$

3. QUANTIZED SCATTERING SYSTEMS

In this section we shall consider asymptotic 'in' and 'out' photon fields corresponding to a quantized scattering system. As in the previous section, we shall avoid all questions concerning the detailed dynamics of such a system by imposing, what appear to be, reasonable asymptotic conditions on the quantized current distribution and by concentrating on the properties of total quantized Maxwell field at S.L.I. By this means, we shall demonstrate the necessity for non-Fock asymptotic photon fields and show that Lorentz symmetry is spontaneously broken in all but the vacuum sector. We shall also give an elementary proof of the infra-red coherence condition. We start with free fields.

To obtain a quantum description of a free Maxwell field one first replaces its associated weighted function α by a weighted operator distribution $\hat{\alpha}$ which satisfies the canonical commutation relations (C.C.R's)

$$\begin{aligned} [\hat{\alpha}_f, \hat{\alpha}^{\dagger}_{f'}] &= (f,f') \\ [\hat{\alpha}_f, \hat{\alpha}_{f'}] &= 0 \end{aligned} \tag{3-1}$$

where f and f' belong to C_0 and $\hat{\alpha}_f := (f,\hat{\alpha})$.

The problem of quantization now reduces to that of choosing a Hilbert space representation, $\{\hat{\alpha},H\}$, of these C.C.R's. We shall restrict our attention to the so-called infra-red representations (Roepstorff, 1970). These have the property that the function $< a \mid \hat{\alpha} \mid b >$ belongs to the class C for all normalizable states $|a>$ and $|b>$. Therefore, for such a representation, $\lim\limits_{r \to 0} < a|\hat{\alpha}|b >$ exists and determines an operator $\hat{\pi}(\theta,\phi)$ according to

$$\hat{\pi} = w - \lim_{r \to 0} \hat{\alpha} \quad , \tag{3-2}$$

where w - lim indicates weak limit, i.e. the limit of the matrix elements of the operator in question. The existence of this limit together with the results of the previous section imply that the quantized Maxwell field, given by

$$\hat{W}_{ab} = \oint\!\!\int l_{[a}m_{b]}\hat{\alpha}\ e^{irl.x} r dr\ d\Omega_0 \quad , \tag{3-3}$$

is A.F. at S.L.I. in the sense that $\hat{W}^{as}$ exists as a weak limit. Moreover, $\hat{W}^{as}$ is determined by the weighted operator

$$\hat{\phi} := \eth\hat{\pi} \quad . \tag{3-4}$$

From equations (3-1) we have

$$[\hat{\alpha},\hat{\alpha}^{\dagger}_f] = f$$

and

$$[\hat{\alpha},\hat{\alpha}_f] = 0 \ .$$

Therefore, since the limit (3-2) exists and $\lim\limits_{r \to 0} f = 0$, these two relations imply that

$$[\hat{\pi},\hat{\alpha}^{\dagger}_f] = \underset{r \to 0}{\text{w-lim}}[\alpha,\alpha^{\dagger}_f] = \lim_{r \to 0} f \quad = 0$$

and

$$[\hat{\pi},\hat{\alpha}_f] = \underset{r \to 0}{\text{w-lim}}[\alpha,\alpha_f] = 0 \quad .$$

The weighted operator, $\hat{\phi}$, given by equation (3-2) therefore commutes with $\hat{\alpha}_f$ and $\hat{\alpha}^{\dagger}_f$ for all functions f belonging to C_0, and thus defines superselection sectors on H. This is a reflection of the fact that $\hat{\phi}$ (or, equivalently, $\hat{W}^{as}$) is defined at S.L.I. and is therefore causally unrelated to any finite region of space-time. On these superselection sectors, which are usually referred to as infra-red sectors, $\hat{\phi}$ is a c-number function. It can be shown (see, for example, Roepstorff, 1970) that two sectors corresponding to c-number functions ϕ and ϕ' are unitarily equivalent if and only if $\phi = \phi'$, and unitarily equivalent to a Fock representation if and only if $\phi = 0$.

A curious, but wellknown, feature of sectors with non-vanishing ϕ is that they do not carry a unitary representation of the Lorentz group. To see this, we need only note that if $\hat{W}$ and $\hat{W}^*$ are related by a Lorentz boost L then, by equation (2-33), $\phi^* \neq \phi$. $\hat{W}$ and $\hat{W}^*$ therefore belong to inequivalent sectors.

Let us now move on to quantized scattering systems involving charged particles. We shall assume that such a system has the following properties:

1. The total field $\hat{W}$ and current $\hat{J}$ are operator distributions acting on some Hilbert space H and satisfy Maxwell's equations
$$\nabla_b \hat{W}^{ab} = \hat{J}^a . \tag{3-5}$$
2. $$\hat{W} = \hat{W}_{in} + \hat{W}_{ret} = \hat{W}_{out} + \hat{W}_{adv} \tag{3-6}$$
   where $\hat{W}_{in}$ and $\hat{W}_{out}$ are free fields belonging to a general infra-red representation.
3. For all time-like, past pointing position vectors
$$\hat{J}^{in}(x) := w - \lim_{\lambda \to \infty} \lambda^3 \hat{J}(\lambda x) \tag{3-7}$$
   is a well defined operator distribution and has the form $J^{in}_a = \rho^{in} x_a$.
4. If $$|\alpha, in> = |e_1^{in} v_1^{in}, \ldots, e_N^{in} v_N^{in}> \tag{3-8}$$
   is an in-state containing N particles of velocity v_i^{in} and charge $e_i^{in} (i = 1 \ldots N)$, then
$$\hat{J}^{in}|\alpha, in> = J^{in}|\alpha, in>, \tag{3-9}$$
   where J^{in} is given by equation (2-4) .

We also assume similar conditions for
$$J^{out} := w\text{-}\lim_{\lambda \to \infty} \lambda^3 J(\lambda x)$$
where the position vector x is now time-like and future pointing.

Under these conditions, the asymptotic component of $\hat{W}$ at S.L.I. exists as a weak limit and is determined by a single operator function $\hat{\phi}(\theta,\phi)$ given by

$$\hat{\phi} = \hat{\phi}_{ret} + \hat{\phi}_{in} = \hat{\phi}_{adv} + \hat{\phi}_{out} \quad . \tag{3-10}$$

By equation (2-32), the charge operator of the system is given by

$$\hat{e} = \oint \hat{\phi} \; d\Omega_0 . \tag{3-11}$$

Standard arguments (see, for example, Strocchi and Wightman, 1974), based on the fact that all observables are local and that $\hat{\phi}$ is defined at S.L.I., show that $\hat{e}$ commutes with all observables and hence defines charge superselection sectors on which $\hat{e}$ is a c-number. Moreover, as Zwanziger (1976) has pointed out, the same arguments also show that $\hat{\phi}$ is itself a c-number function on any irreducible charged sector. Therefore, on any such sector of charge e, we have

$$\phi = \hat{\phi}_{ret} + \hat{\phi}_{in} = \hat{\phi}_{adv} + \hat{\phi}_{out} \tag{3-12}$$

where ϕ is a c-number function such that

$$e = \oint \phi \, d\Omega_0 \quad . \tag{3-13}$$

It would be nice if we could choose $\hat{\phi}_{in}$ such that ϕ vanishes. However, as ϕ_{in} is charge free (i.e. $\oint \phi_{in} d\Omega_0 = \int \eth\pi_{in} d\Omega_0 = 0$), this can be done only on the vacuum sector. The best we can do on any charged sector is to choose ϕ_{in} such that $e = \phi$. In this case, equation (3-12) yields

$$\hat{\phi}_{in} = e - \hat{\phi}_{ret} \;, \tag{3-14}$$

and, therefore, on using equations (3-9) and (2- 4), we have

$$\hat{\phi}_{in}|\alpha, \text{in}> = \left\{ e - \sum_i \frac{e_i^{in}}{(l.v_i^{in})^2} \right\} |\alpha,\text{in}>, \tag{3-15}$$

provided $|\alpha,\text{in}>$ has total charge e. Equation (3-15) is equivalent to the wellknown infra-red coherence condition, obtained by different means by Kulish and Faddeev (1970), and discussed extensively by Zwanziger (1976). It shows that, when acting on states of the type $|\alpha,\text{in}>$, W_{in} must belong to the infra-red sector corresponding to

$$\phi_{in} = \left\{ e - \sum_i \frac{e_i^{in}}{(1.v_i^{in})^2} \right\} \tag{3-16}$$

In particular, it shows that W_{in} cannot belong to a Fock representation since this would imply

$$\phi_{in}|\alpha, in> = 0$$

for all states $|\alpha, in>$, in contradiction to equation (3-15). Similar remarks also apply to the 'out' fields.

Exactly as in the free field case, one can show that sectors on which ϕ is non-zero, and hence all charged sectors, do not carry a unitary representation of the Lorentz group. Lorentz symmetry is thus spontaneously broken in all but the vacuum sector. This is reflected in the non-invariant form of the infra-red coherence condition (3-15). On the vacuum sector it assumes the invariant and simpler form

$$\hat{\phi}_{in}|\alpha, in> = -\sum_i \frac{e_i^{in}}{(1.v_i^{in})^2} \quad |\alpha, in> . \tag{3-17}$$

4. DISCUSSION

From the above results, we see that if (unbroken) Lorentz symmetry is to be retained as a basic property of any quantum theory involving a Maxwell field - or any other massless, integer spin field for that matter - it is necessary to restrict attention to superselection sectors on which the Maxwell field is empty at S.L.I., i.e. $\phi = 0$. This obviously rules out all charged sectors. Nevertheless, this imposes no real physical restraint on the theory since, from an operational point of view, the creation of a charge e is always accompanied by the creation of a charge -e. It is thus always possible to obtain a Lorentz invariant quantum description of, for example, an electron, so long as we include within the same description the positively charged ion which the electron leaves behind. In this sense, the vacuum sector in Q.E.D. is, in principle, sufficiently rich in states to describe any realistic physical process involving charged particles.

Theories involving linearized gravitational fields (i.e. spin-2, massless fields) have a very similar structure and, with little alteration, the methods of sections 2 and 3 can be extended to such theories. In particular, one can show that the asymptotic component at S.L.I. of an A.F. linearized gravitational field is completely determined by a single function $\psi(\theta,\phi)$ which differs from its Maxwell equivalent in having weight (-3,0) rather than weight (-2,0). As in Q.E.D., this function must vanish in the quantum domain if we wish to retain (unbroken) Lorentz symmetry. However, if we attempt to impose upon the theory the well attested fact that gravitational mass - the gravitational equivalent of electric charge - always has the same sign, we immediately run into difficulty: if massive particles are present, their Coulomb-like fields necessarily make ψ non-zero, and this in turn causes a spontaneous breakdown of Lorentz symmetry. The condition that gravitational mass always has the same sign is thus incompatible with unbroken Lorentz symmetry for linear quantum gravity.

Does this difficulty also occur for full non-linear quantum gravity? Of course, since an acceptable theory of quantum gravity has yet to be produced, we cannot give an unequivical answer to this question. There are, however, strong indications that this difficulty does not occur in the full theory in the sense that distinct superselection sectors on which ψ is a c-number cease to exist.

For the sake of simplicity, let us consider solutions of Einstein's equations which are matter and singularity free and which are asymptotically flat at both S.L.I. (Ashtekar and Hansen, 1978) and at null infinity (Penrose, 1968). Assuming certain regularity conditions at S.L.I., the leading asymptotic component of the Weyl tensor corresponding to such a solution is completely determined by a single function $\psi(\theta,\phi)$ of weight (-3,0), exactly as in the linear case. In terms of this function, the total (A.D.M.) momentum of the solution is given by

$$P_a = \oint \psi(\theta,\phi)\, l_a \, d\Omega_0 \qquad (4\text{-}1)$$

According to recently proved positive mass theorems (Witten, 1981), P_a is non-zero and future pointing for all non-trivial solutions, and is zero if and only if the solution is identically flat. Consequently, the function ψ corresponding to any non-trivial solution cannot be identically zero. Thus, as in the case of a linearized gravitational field produced by massive particles, a pure non-linear gravitational field cannot be empty at S.L.I.

If, as seems plausible, this function has a well defined quantum analogue, $\hat{\psi}$ say, we have the problem of interpreting $\hat{\psi}$. By analogy with linear theory, an obvious interpretation of $\hat{\psi}$ is that it acts as a superselection operator. If this were indeed the case, then the vacuum sector would contain a single state corresponding to flat space-time and (asymptotic) Lorentz symmetry would be spontaneously broken on all non-trivial sectors. We would thus encounter the same difficulty as in linear theory. However, as non-linear gravity is a highly non-local theory, one cannot envoke the principle of locality to argue that $\hat{\psi}$ acts as a superselection operator. Indeed, for any acceptable theory of quantum gravity, $\hat{\psi}$ cannot act in this way as this would imply, by equation 4-1, that the momentum operator $\hat{P}_a$ commutes with all 'local' observables.

A more plausible interpretation of $\hat{\psi}$ comes directly from asymptotic quantization schemes based on null infinity (Bramson, 1976 and Ashtekar, 1981). Under this interpretation, $\hat{\psi}$ acts as a generator of B.M.S. supertranslations in the sense that

$$[\oint \hat{\psi}\, \varepsilon\, d\Omega_0,\ \hat{N}(u,\theta,\phi)] = \hat{N}(u+\varepsilon,\theta,\phi) - \hat{N}(u,\theta,\phi) \qquad (4\text{-}2)$$

where $\hat{N}$ is the quantum analogue of the gravitational news function and $\varepsilon(\theta,\phi)$ is an infinitesimal supertranslation function of weight (1,0).

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