Automorphisms of Baer-Levi Semigroups

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AUTOMORPHISMS OF BAER-LEVI SEMIGROUPS

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Let $T_X$ be the semigroup, under composition, of all transformations of the set $X$ to itself, and $G_X$ be the group, inside $T_X$, of all bijections of $X$. An automorphism $\phi$ of a semigroup $S$ in $T_X$ is said to have the inner automorphism property (i.a.p.) if the automorphisms of $S$ are precisely those of the form $\phi(f) = hfh^{-1}$, for all $f$ in $S$, where $h$ is an element of $G_X$. There is a readily stated unsolved problem concerning $T_X$: determine all subsemigroups $S$ of $T_X$ which have the inner automorphism property.

Amidst research on this problem there are two landmarks, the works of Schreier [5] and Fitzpatrick and Symons [2]. Schreier showed that if $S$ contains the semigroup of all constant maps $I_X$, then $S$ has the i.a.p., while Fitzpatrick and Symons showed that for semigroups $S$ containing $G_X$ the i.a.p. holds.

A large family of semigroups (first considered in [1]) which are disjoint from $I_X$ and $G_X$ are the Baer-Levi semigroups. In this paper we show that such semigroups
also possess the i.a.p.

The family of Baer-Levi semigroups are defined in the following way. Let $|X|$, the cardinality of the set $X$, be $p$, and $q$ be an infinite cardinal less than or equal to $p$. Then

$$BL(p,q) = \{ f \in \mathcal{T}_X : f \text{ is one-to-one and } |X \setminus R(f)| = q \}$$

is the Baer-Levi semigroup of type $(p,q)$. All congruences on $BL(p,p)$ were found by Sutov [6], while those on $BL(p,q)$ were found by Lindsey and Madison [3].

Let $B(p,q) = \{ A \subseteq X : |A| = p \text{ and } |X \setminus A| = q \}$. We call $B(p,q)$ the family of all Baer-Levi sets of type $(p,q)$. We view $B(p,q)$ as a partially ordered set, the order being given by set inclusion. A central part of our proof is a result of independent interest concerning $B(p,q)$, so we present it in the form of a lemma. For this we need a definition: a bijection $H$ of $B(p,q)$ is said to be induced by the bijection $h$ of $X$ if $H(A) = h(A) = \{ h(x) : x \in A \}$ for every $A$ in $B(p,q)$.

**Lemma:** Let $H$ be a bijection of $B(p,q)$. Then $H$ is induced if and only if $H$ and $H^{-1}$ are order-preserving.

**Proof:** If $H$ is induced it is clear that $H$ and $H^{-1}$ are order-preserving. We show the converse in four steps.
1. Let \( A, B \in B(p,q) \) with \( A \subseteq B \) and \( |B \setminus A| \) finite.

Then \( |B \setminus A| = |H(B) \setminus H(A)| \).

If \( |B \setminus A| = n \), then \( |\{S: A \subseteq S \subseteq B\}| = 2^n \).

But \( H \) and \( H^{-1} \) are order preserving bijections so
\[ |\{H(S): H(A) \subseteq H(S) \subseteq H(B)\}| = 2^n \] also. Thus
\[ |\{T: H(A) \subseteq T \subseteq H(B)\}| = 2^n \] and hence \( |H(B) \setminus H(A)| = n \).

2. Given \( x \in X \) there exists a \( y \in X \) such that
\( H(B \cup \{x\}) = H(B) \cup \{y\} \) for every \( B \in B(p,q) \) with \( x \notin B \).

For brevity we write \( B \cup \{x\} \) as \( B \cup x \) in future. Take \( A \in B(p,q) \) with the given \( x \) not in \( A \). Step 1 and the fact that \( H \) is order preserving together imply
\( H(A \cup x) = H(A) \cup y \) for some \( y \) in \( X \).

We show \( H(B \cup x) = H(B) \cup y \) for every other \( B \in B \) with \( x \notin B \). This is done in three stages.

i) Case \( B \subseteq A \): Let \( H(B \cup x) = H(B) \cup z \). Now \( B \cup x \subseteq A \cup x \) so \( H(B \cup x) \subseteq H(A \cup x) \) or \( H(B) \cup z \subseteq H(A) \cup y \) so \( z \in H(A) \cup y \).

But \( z \notin H(A) \) (for then \( H(B \cup x) \subseteq H(A) \) whence \( B \cup x \subseteq A \), since \( H^{-1} \) is order preserving, a contradiction), so \( z = y \).

ii) Case \( A \cap B \in B(p,q) \): For this case we need the following small result: if \( H(C \cup x) = H(C) \cup y \), then
\( H(C \cup D \cup x) = X(C \cup D) \cup y \), where \( C \in B(p,q) \), \( C \cup D \in B(p,q) \), \( C \cap D = \phi \) and \( x \notin C \cup D \). The proof is as follows.
Suppose $y \in H(C \cup D)$. Certainly $H(C) \subseteq H(C \cup D)$ so $H(C \cup x) \subseteq H(C \cup D)$, hence $C \cup x \subseteq C \cup D$, a contradiction, so $y \not\in H(C \cup D)$. Now let $H(C \cup D \cup x) = H(C \cup D) \cup z$. We have $C \cup x \subseteq C \cup D \cup x$ so $H(C \cup x) \subseteq H(C \cup D \cup x)$ or $y \in H(C \cup D \cup x) = H(C \cup D) \cup z$. Since $y \not\in H(C \cup D)$, $y = z$.

Returning to the case $A \cap B \in B(p, q)$, we know $H((A \cap B) \cup x) = H(A \cap B) \cup y$ from i). Letting $C = A \cap B$ and $D = B \setminus A$ in the small result now gives $H(B \cup x) = H(B) \cup y$ as required.

iii) Case $A \cap B \not\in B(p, q)$: Suppose $q < p$. Then either $|A \cap B| = p$ and $|(A \cap B)'| \neq q$, whence $|A \cap B'| = |A' \cup B'| \neq q$, contradicting $|A'| = |B'| = q$, or $|A \cap B| < p$, whence $|(A \cap B)'| = |A' \cup B'| = p$, so one of $|A'|, |B'|$ is $p$, again a contradiction. Thus in this case we have $q = p$.

Then we can find a $C \in B(p, p)$ with $x \not\in C$ such that $A \cap C$ and $B \cap C \in B(p, p)$. Then $H(A \cup x) = H(A) \cup y$ implies $H(C \cup x) = H(C) \cup y$, using ii) and $A \cap C \in B(p, p)$. But in turn this implies $H(B \cup x) = H(B) \cup y$, using ii) and $B \cap C \in B(p, p)$.

We are now able to produce the required bijection of $X$:

Definition: Given $x \in X$, define a mapping $h : X \to X$ by $h(x) = y$, where $H(B \cup x) = H(B) \cup y$ for some $B \in B(p, q)$ with $x \in B$. 
3. \( h \) is a well-defined bijection of \( X \).

Step 2 ensures \( h \) is well-defined. Now suppose \( h(x) = h(x') = y \), say and take \( B \in \mathcal{B}(p,q) \) with \( x, x' \in B \). Then

\[
H((B \cup x) \cup x') = H(B \cup x) \cup y = H(B) \cup y = H(B \cup x).
\]

Since \( H \) is one-to-one we must have \( x = x' \), so \( h \) also is one-to-one.

Finally, take \( y \in X \) and \( C \in \mathcal{B}(p,q) \) with \( y \in C \).
Consider \( H^{-1}(C \cup y) = H^{-1}(C) \cup x \), for some \( x \). Then

\[
H(H^{-1}(C) \cup x) = C \cup y \hspace{1cm} \text{so} \hspace{1cm} h(x) = y, \hspace{1cm} \text{or} \hspace{1cm} h \text{ is onto.}
\]

4. \( H \) is induced by \( h \).

We must show \( H(A) = h(A) \), for each \( A \in \mathcal{B}(p,q) \), where \( h(A) = \{h(x) : x \in A\} \). From the definition of \( h \) we at once have \( h(A) \subseteq h(A) \). Take \( y \in H(A) \).
Then \( H^{-1}(H(A) \setminus y) = A \setminus x \), for some \( x \in A \), so

\[
H(A \setminus x) = H(A) \setminus y \hspace{1cm} \text{or} \hspace{1cm} h(x) = y. \hspace{1cm} \text{Thus} \hspace{1cm} H(A) \subseteq h(A), \hspace{1cm} \text{so} \hspace{1cm} \text{equality follows.}
\]

**THEOREM:** \( BL(p,q) \) has the inner automorphism property.

**PROOF:** \( BL(p,q) \) is certainly \( G_x \)-normal, so it suffices to show that every automorphism \( \phi \) of \( BL(p,q) \) has the form \( \phi(f) = hfh^{-1} \), for all \( f \in BL(p,q) \), and some fixed bijection \( h \) of \( X \). This is carried out in four steps. Throughout, \( f, g \) and \( k \) are elements of \( BL(p,q) \).
1. \( R(f) \subseteq R(g) \) if and only if for each \( k \) such that \( kg = g \) we have \( kf = f \).

Suppose \( R(f) \subseteq R(g) \). Then \( kg = g \) implies \( k \) is the identity on \( R(g) \), so also on \( R(f) \). Hence \( kf = f \).

Suppose now \( R(f) \nsubseteq R(g) \) (= \( A \) say). Let \( \{B_1, B_2, \ldots\} \) be a partition of \( A' \) such that \( |B_i| = q \), and \( k_i : B_i \to B_{i+1} \) an arbitrary bijection, for each \( i \geq 1 \).

Then \( k \), given by \( k(x) = x \) for \( x \in A \) and \( k(x) = k_i(x) \) for \( x \in B_i \), each \( i \), lies in \( BL(p,q) \) and has fixed points precisely \( R(g) \). Thus \( kg = g \), yet \( kf \neq f \).

2. \( R(f) = R(g) \) if and only if \( R(\phi(f)) = R(\phi(g)) \)

Using the result of step 1 we immediately have that \( R(f) \subseteq R(g) \) if and only if \( R(\phi(f)) \subseteq R(\phi(g)) \), from which step 2 follows.

Thus the automorphism \( \phi \) gives rise in a natural way to a mapping of \( BL(p,q) \):

Definition: Given \( A \in B(p,q) \), define \( H(A) = R(\phi(f)) \), where \( f \) in \( BL(p,q) \) is such that \( R(f) = A \).

3. \( H \) is a well-defined bijection of \( B(p,q) \), with \( H \) and \( H^{-1} \) order-preserving.

That \( H \) is well-defined is the content of step 2.

Suppose \( A \neq B, A, B \in B(p,q) \). Then if \( R(f) = A \) and \( R(g) = B \) we have \( R(\phi(f)) \neq R(\phi(g)) \), by step 2, so \( H(A) \neq H(B) \), or \( H \) is one-to-one. Now take \( B \in B(p,q) \).
and $f$ such that $R(\phi(f)) = B$. If $A = R(f)$ we must have $H(A) = R(\phi(f)) = B$, so $H$ is onto.

Finally, the definition of $H$, together with the fact that $R(f) \subseteq R(g)$ if and only if $R(\phi(f)) \subseteq R(\phi(g))$, ensures that $H$ and $H^{-1}$ are order-preserving.

4. $\phi$ is inner.

From the lemma we now have that $H$ is induced by a bijection $h$ of $X$. We show that $\phi(f) = hfh^{-1}$ for each $f$ in $BL(p,q)$.

Take such an $f$ and an $x \in X$ and suppose $f(x) = y$. Choose $A$ and $B$ in $B(p,q)$ such that $A \subseteq B$ and $B \setminus A = \{x\}$, together with $p$ and $q$ in $BL(p,q)$ such that $R(p) = A$ and $R(q) = B$.

Now $R(q) \setminus R(p) = B \setminus A = \{x\}$ so $R(\phi(q)) \setminus R(\phi(p)) = H(B) \setminus H(A) = \{h(x)\}$. On the other hand, $R(fq) \setminus R(fp) = \{y\}$ so $R(\phi(fq)) \setminus R(\phi(fp)) = \{h(y)\}$. But since $R(\phi(fq)) \setminus R(\phi(fp)) = R(\phi(f)\phi(q)) \setminus R(\phi(f)\phi(p))$ we must have $\phi(f)h(x) = h(y) = hf(x)$. Thus $\phi(f) = hfh^{-1}$.

On completion of this work the authors discovered that the result has been announced in [4]. Schein's quite different proof, yet to appear, involves showing that $\phi$ permutes the subsemigroups $S_x = \{f \in BL(p,q) : f(x) = x\}$, where $x \in X$. 
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