PRODUCTS OF IDEMPOTENTS IN REGULAR RINGS II

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Abstract

This paper characterizes products of idempotents in (von Neumann) regular rings which are unit-regular or right self-injective. For unit-regular rings, the minimum number of idempotents needed in such a product is determined, thereby generalizing a 1978 result of Ballantine in the case of a matrix with entries from a field.

Introduction

This paper continues the study (begun in [OM]) of the subsemigroup generated by the idempotents of a regular ring. We extend the results in [OM] by characterizing those elements which are a product of idempotents in any unit-regular ring or in any right self-injective regular ring. In the unit-regular case we can say precisely how many idempotents are needed in such a product, thereby generalizing Ballantine's 1978 result [B] which calculates this number in the case of a matrix with entries from a field.

Following Howie's study in 1966 of products of idempotents in a full transformation semigroup [H], several authors have characterized products of idempotents in semigroups which occur as the multiplicative semigroups of various rings. In 1967 Erdos [E] showed that the linear transformations of a finite-dimensional vector space which are products of proper ($\neq 1$) idempotents are precisely the singular transformations. Later, in 1978, Ballantine [B] showed that such a transformation a is a product of k idempotents if and only if the transformation $1 - a$ has rank at most $k.\nu(a)$, where $\nu(a)$ is the nullity of a. The problem for transformations on an arbitrary vector space was solved by Reynolds and Sullivan [RS] in 1985.

However none of these papers took advantage of the ring structure present. In [OM] it was shown that regular rings provide a natural setting for the above-mentioned results, and the characterization of products of idempotents in prime, right self-injective regular rings given there provides a ring-theoretical explanation for the results in [E] and [RS]. In this paper we extend the results in [OM] to a much larger class of regular rings including all unit-regular and all right self-injective regular rings. At the same time we sharpen the result in the unit-regular case so that Ballantine's result is also included in this more general context.

In Section 1 we consider the case of a unit-regular ring R. This includes Ballantine's matrix rings and the simple, directly finite, regular rings satisfying comparability considered in [OM], as well as many other rings. Surprisingly, Ballantine's result remains true in this case although, of course, it has to be translated so that it no longer uses the rank function, since an arbitrary unit-regular ring need not have a rank function. The appropriate version of Ballantine's result in this context is: an element $a \in R$ is a product of k idempotents if and only if $(1-a)R \lesssim k(r(a))$ (see below for the notation). In particular, in the cases studied by Ballantine [B] and O'Meara [OM] where R has a suitable rank function N, an element $a \in R$ is a product of k idempotents if and only if $N(1-a) \leq k(1-N(a))$.

In Section 2 we look at right self-injective regular rings R and their factor rings. The sharp characterization for unit-regular rings is no longer true in this situation, but we show instead that $a \in R$ is a product of idempotents if and only if a satisfies the condition

$$(*) \qquad Rr(a) = \ell(a)R = R(1-a)R.$$

In the prime case the ideals in this equation are characterized by the values of the dimension functions used by Reynolds and Sullivan [RS] and O'Meara [OM], and so we can obtain both their results as corollaries.

In Section 3 we consider the question of whether $(*)$ characterizes products of idempotents in arbitrary regular rings. We show that if the ring is simple and directly finite, then this problem is equivalent to Goodearl's open problem [G] as to whether such rings are unit-regular.

Preliminaries

All rings in this paper are associative with an identity element. The unqualified term *ideal* always means a two-sided ideal. For a subset X of a ring R we write $r(X)$ or $r_R(X)$ for the *right annihilator*, $\{r \in R | Xr = 0\}$, of X in R. Similarly $\ell(X)$ or $\ell_R(X)$ denotes the left annihilator. Modules are usually unital right modules. We say that a module A is *subisomorphic* to a module B if A is isomorphic to a submodule of B, and in this case we write $A \lesssim B$. If n is a positive integer we write nA for the direct sum of n copies of the module A.

A ring R is *(von Neumann) regular* if for any $a \in R$ there is some $x \in R$ such that $a = axa$. We refer the reader to Goodearl's book [G] for all notation, terminology and properties of regular rings. A ring R is *right self-injective* if the module R_R is injective. A module A is *directly finite* if it is not isomorphic to a proper direct summand of itself; otherwise A is *directly infinite.* A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ in R. This is equivalent to the module R_R being directly finite. A regular right self-injective ring R is called *purely infinite* if there are no nonzero central idempotents $e \in R$ such that eR is directly finite. Any regular right self-injective ring decomposes uniquely as a direct product of a directly finite ring and a purely infinite ring [G, Proposition 10.21]. A regular ring R satisfies the *comparability axiom* if for any $x, y \in R$ either $xR \lesssim yR$ or $yR \lesssim xR$, while it satisfies *general comparability* if for any $x, y \in R$ there is a central idempotent e such that $exR \lesssim eyR$ and $(1-e)yR \lesssim (1-e)xR$. By [G, Corollary 9.15] any right self-injective regular ring satisfies general comparability.

1 The unit-regular case

Let R be a ring. An element $a \in R$ is called *unit-regular* if $a = aua$ for some unit $u \in R$. In a regular ring this is equivalent to the element being "balanced" in the sense that

$$r(a) \cong R/aR.$$

The ring R is called *unit-regular* if all its elements are unit-regular. By [G, Theorem 4.1] this is equivalent to R being a regular ring in which

$$eR \cong fR \text{ implies } (1-e)R \cong (1-f)R$$

for all idempotents $e, f \in R$. Unit-regular rings form a large class of directly finite regular rings, including all regular rings whose primitive factors are artinian, all regular rings with bounded index of nilpotence, all directly finite regular rings with general comparability, and all $\aleph_0$-continuous regular rings [G, 5.2,6.10,7.11,8.12,14.24].

In this section we generalize Ballantine's result for $n \times n$ matrices over a field (mentioned in the introduction) to elements of an arbitrary unit-regular ring, by characterizing when an element of a unit-regular ring is a product of idempotents and determining precisely how many idempotents are needed in such a product. This is the content of Theorem 1.2, the principal result of this section. We begin by presenting that part of the theorem which holds in a general regular ring.

Proposition 1.1 *Let R be a regular ring and let k be a positive integer. If $a \in R$ is a product of k idempotents, then*

$$(1-a)R \lesssim k(r(a)).$$

Proof. We use induction on k. Suppose R is a regular ring and $a \in R$ is a product of k idempotents. If $k = 1$, certainly $(1-a)R = r(a) \lesssim r(a)$. Now suppose $k > 1$. Write $a = a_1 f$ where f is idempotent and a_1 is a product of $k-1$ idempotents. By induction $(1-a_1)R \lesssim (k-1)r(a_1)$. Notice that $r(a) = (fR \cap r(a_1)) \oplus (1-f)R$. Write $r(a_1) = (fR \cap r(a_1)) \oplus hR$ for some $h \in R$. Then $hR \cap fR = 0$ implies $hR \lesssim (1-f)R$. Hence $r(a_1) \lesssim r(a)$. Now

$$(1-a_1)fR \subseteq (1-a_1)R \lesssim (k-1)r(a_1) \lesssim (k-1)r(a)$$

whence

$$(1-a)R = (1-a_1)fR + (1-f)R \lesssim (k-1)r(a) \oplus r(a) = k(r(a)),$$

giving $(1-a)R \lesssim k(r(a))$.

Theorem 1.2 *Let R be a unit-regular ring and k any positive integer. Then $a \in R$ is a product of k idempotents if and only if*

$$(1-a)R \lesssim k(r(a)).$$

Proof. Suppose $(1-a)R \lesssim k(r(a))$. We proceed by induction on k to show that a is a product of k idempotents. When $k = 1, (1-a)R \lesssim r(a) \subseteq (1-a)R$ and so by direct finiteness of R we have $r(a) = (1-a)R$, and hence $a = a^2$ is a product of 1 idempotent. Now assume $k \geq 2$ and that the result holds for $k-1$.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} &
\begin{pmatrix}
0 & 0 & * & * & 0 \\
0 & 1 & * & * & 0 \\
0 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 1: *The (two-sided) Peirce decomposition of a relative to* $1 = e_1 + e_2 + e_3 + e_4 + e_5$.

Since R is a regular ring, we can write

$$
\begin{aligned}
aR &= (r(a) \cap aR) \oplus r(1-a) \oplus bR, \\
R &= (r(a) + aR) \oplus cR, \\
r(a) &= (r(a) \cap aR) \oplus dR
\end{aligned}
$$

for some $b, c, d \in R$. Then

$$R = (r(a) \cap aR) \oplus r(1-a) \oplus bR \oplus cR \oplus dR.$$

Let e_1, e_2, e_3, e_4, e_5 be the orthogonal idempotents of R associated with this decomposition, that is, $1 = e_1+e_2+e_3+e_4+e_5$ with $e_1R = r(a) \cap aR$, $e_2R = r(1-a)$, $e_3R = bR$, $e_4R = cR$ and $e_5R = dR$. Let $e = e_1 + e_2 + e_3$ and $f = e_2 + e_3 + e_4$. Then $aR = eR$ and, because $r(a) = (1-f)R$, $Ra = Rf$. The form of a relative to the e_i is shown in Fig.1.

Observe that since $fR \cong eR$ (with left multiplication by a providing an isomorphism) and R is unit-regular, we have $(1-f)R \cong (1-e)R$. Hence

$$r(a) \cong e_4R \oplus e_5R. \tag{1}$$

Also $r(1-a) = e_2R$ implies $(1-e_2)R \cong (1-a)R$ whence, from $(1-a)R \lesssim k(r(a))$, we conclude that $e_1R \oplus e_3R \oplus e_4R \oplus e_5R \lesssim k(r(a))$ and, from (1), that

$$e_1R \oplus e_3R \oplus r(a) \lesssim (k-2)r(a) \oplus r(a) \oplus e_1R \oplus e_5R.$$

Unit-regularity of R entitles us by [G, Corollary 4.6] to cancel common terms, whereby we obtain

$$e_3R \lesssim (k-2)r(a) \oplus e_5R.$$

$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_{31} & e_{32} & e_4 & e_5 \\
\hline
e_1 & 1 & 0 & * & 0 & * & * \\
e_2 & 0 & 1 & * & 0 & * & * \\
e_{31} & 0 & 0 & * & 0 & * & * \\
e_{32} & 0 & 0 & * & 1 & * & *-1 \\
e_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_5 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Figure 2: *The Peirce decomposition of a_1 relative to $1 = e_1 + e_2 + e_{31} + e_{32} + e_4 + e_5$. The e_2, e_{31} and e_4 columns of a_1 are the same as those of a, while its e_5 column is obtained from the e_{32} column of a by removing e_{32} and shifting the remainder under right multiplication by y.*

By [G, Corollary 2.9] we can write $e_3 = e_{31} + e_{32}$ where e_{31} and e_{32} are orthogonal idempotents of R with

$$e_{31}R \lesssim (k-2)r(a) \tag{2}$$

and

$$e_{32}R \lesssim e_5 R. \tag{3}$$

From (3) we can find $y \in e_{32}Re_5$ and $z \in e_5Re_{32}$ with $yz = e_{32}$. The wedge for induction is now provided by the factorization

$$\begin{aligned} a &= (e_1 + e_{32} + a(1-e_{32}) + (a-1)y)(f+z) \\ &= a_1(f+z) \end{aligned} \tag{4}$$

where $a_1 = e_1 + e_{32} + a(1-e_{32}) + (a-1)y$, because

We claim: $(1-a_1)R \lesssim (k-1)r(a_1)$.

The motivation for the choice of a_1 comes from Fig.2 (a_1 is chosen such that $r(1-a_1)$ is "larger" than $r(1-a)$). To verify the claim, we have by unit-regularity that $R(e_4+e_5) \subseteq \ell(a_1)$ implies $(e_4+e_5)R \lesssim r(a_1)$. Consequently by (1)

$$r(a) \lesssim r(a_1). \tag{5}$$

Also $e_1R \oplus e_2R \oplus e_{32}R \subseteq r(1-a_1)$, whence $e_{31}R \oplus e_4R \oplus e_5R$ contains a complement right ideal of $r(1-a_1)$. Hence

$$\begin{aligned}(1-a_1)R &\lesssim e_{31}R \oplus e_4R \oplus e_5R \\ &\lesssim (k-2)r(a) \oplus e_4R \oplus e_5R && \text{by}(2) \\ &= (k-2)r(a) \oplus r(a) = (k-1)r(a) && \text{by}(1) \\ &\lesssim (k-1)r(a_1) && \text{by}(5).\end{aligned}$$

This proves the claim.

By induction, $a_1 = f_1f_2\cdots f_{k-1}$ for some idempotents $f_1, \cdots, f_{k-1} \in R$. Let $f_k = f + z$. Then f_k is idempotent, and by (4) we have

$$a = a_1f_k = f_1f_2\cdots f_{k-1}f_k$$

is a product of k idempotents, as desired.

The converse is given in Proposition 1.1.

Remark 1.3 *In the proof of Theorem 1.2, since $\ell(a) = \ell(a_1)$, we actually have $aR = a_1R$ and $f_kR \cong fR \cong aR$. Hence, by induction, we can arrange the idempotents f_i in the product $a = f_1f_2\cdots f_k$ so that*

$$f_1R \cong f_2R \cong \cdots \cong f_kR \cong aR.$$

At the expense of not knowing exactly how many idempotents may be involved, the following corollary gives a very simple necessary and sufficient condition, in terms of two ideals being equal, for an element of a unit-regular ring to be a product of idempotents. In practice the condition is not hard to check because the first ideal is always contained in the second.

Corollary 1.4 *Let R be a unit-regular ring. Then $a \in R$ is a product of idempotents of R if and only if*

$$Rr(a) = R(1-a)R.$$

Proof. We always have $Rr(a) \subseteq R(1-a)R$ because $r(a) \subseteq (1-a)R$. On the other hand, by [G, Corollary 2.23], $R(1-a)R \subseteq Rr(a)$ if and only if $(1-a)R \lesssim k(r(a))$ for some positive integer k. Thus the Corollary now follows from Theorem 1.2.

□

The next corollary can be viewed as an extension of Erdos' result [E] that an $n \times n$ matrix over a field is a product of proper idempotent matrices exactly when it is singular. It also further extends [OM, Theorem 3], which was the analogous result for simple, directly finite regular rings satisfying the comparability axiom.

Corollary 1.5 *Let R be a simple unit-regular ring. Then $a \in R$ is a product of proper $(\neq 1)$ idempotents of R if and only if a is not a unit.*

Proof. This is immediate from Corollary 1.4.

□

If a simple unit-regular ring R also satisfies the comparability axiom, then R has a unique rank function $N : R \to [0,1]$ and N determines subisomorphism of principal right ideals, that is $xR \lesssim yR$ if and only if $N(x) \leq N(y)$ [G, Corollary 16.15]. This enables us to replace the subisomorphism condition, $(1-a)R \lesssim k(r(a))$, in Theorem 1.2 by a simple inequality involving the ranks of the elements a and $1-a$, as in the following corollary. Ballantine's result can then be deduced immediately from this.

Corollary 1.6 *Let R be a simple, directly finite regular ring satisfying the comparability axiom, and let $N : R \longrightarrow [0,1]$ be its unique rank function. Let k be an arbitrary positive integer. Then $a \in R$ is a product of k idempotents of R if and only if*

$$N(1-a) \leq k(1-N(a)).$$

Proof. By [G, Theorem 8.12], R is unit-regular. Observe that for principal right ideals xR and yR of R, we have by [G, Corollary 16.15 and Proposition 8.2] that

$$xR \lesssim k(yR) \quad \text{iff} \quad N(x) \leq kN(y).$$

Let $a \in R$ and let $r(a) = yR$. Note that $N(y) = 1 - N(a)$ because $R \cong r(a) \oplus aR$. By applying the above observation to the principal right ideals $(1-a)R$ and yR, we have by Theorem 1.2 that:

$$\begin{array}{llcl} & \multicolumn{3}{l}{a \text{ is a product of } k \text{ idempotents}} \\ \text{iff} & (1-a)R & \lesssim & k(r(a)) \\ \text{iff} & (1-a)R & \lesssim & k(yR) \\ \text{iff} & N(1-a) & \leq & kN(y) \\ \text{iff} & N(1-a) & \leq & k(1-N(a)). \end{array}$$

By Remark 1.3, the k idempotents in Corollary 1.6 can be chosen to have the same rank as a.

Corollary 1.7 (Ballantine [B]). *Let D be a division ring, k and n arbitrary positive integers, and A an $n \times n$ matrix over D. Then A is a product of k idempotent matrices over D if and only if*

$$rank(I-A) \leq k.nullity(A).$$

Proof. The ring $R = M_n(D)$ of $n \times n$ matrices over D is a simple, directly finite regular ring satisfying the comparability axiom. Its unique rank function N is given by

$N(x) = rank(x)/n$, where $rank(x)$ is the usual matrix rank. The corollary now follows immediately from Corollary 1.6.

2 Right self-injective regular rings

Our main aim in this section is to characterize products of idempotents in right self-injective regular rings, by a modification of the condition in Corollary 1.4 which characterized such products in unit regular rings. This class of rings overlaps to some extent with the class of unit-regular rings (the intersection being precisely the class of directly finite, right self-injective, regular rings [G, Theorem 9.17]). However there are enough new rings for the condition $(1-a)R \lesssim k(r(a))$ in Theorem 1.2 to no longer characterize products of k idempotents. Indeed, using the next lemma (based on [D, Lemma 4.2]), we shall see that the connexion can break down even for $k = 1$ and $k = 2$.

Lemma 2.1 *Let R be any ring. If $a \in R$ is a product of 2 idempotents and $r(a) \subseteq aR$, then $a^2 = a^3$.*

Proof. Suppose $a = ef$ where e, f are idempotents in R. Then $1 - f \in r(a) \subseteq aR \subseteq eR$ and so $e(1-f) = 1 - f$. Hence $fe(1-f) = 0$ so that $fe = fef$. Thus $a^3 = efefef = efef = a^2$.

Example 2.2 *Let R be any ring such that $R_R \cong 2R_R$. Then there is a nilpotent element $a \in R$ of index 3 such that*

- *(i)* $r(a) \subseteq aR$ and $a^2 \neq a^3$,
- *(ii)* a is a product of three idempotents but no fewer,
- *(iii)* $R = (1-a)R \cong r(a) \cong 2r(a)$,
- *(iv)* $R = R(1-a) \cong \ell(a) \cong 2\ell(a)$.

Proof. By hypothesis $R_R \cong 2R_R \cong 3R_R$ so there are orthogonal idempotents $e_1, e_2, e_3 \in R$ such that each $e_i R \cong R_R$ and $e_1 + e_2 + e_3 = 1$. From the isomorphisms $e_1 R \cong e_2 R \cong e_3 R$ we get elements $e_{ij} \in e_i R e_j$ such that $e_i = e_{ij} e_{ji}$ whenever $i \neq j$. Let $a = e_{21} + e_{32}$. Then a is nilpotent of index 3 and $r(a) = e_3 R \subseteq (e_2 + e_3)R = aR$, which proves (i). By Lemma 2.1, a cannot be a product of 2 idempotents, but it is a product of 3 idempotents since

$$\begin{aligned} a &= (e_2 + e_3 + e_{21})(e_1 + e_{32}) \\ &= (e_2 + e_3 + e_{21})(e_1 + e_3 + e_{32})(e_1 + e_2). \end{aligned}$$

Thus (ii) is proved. Since a is nilpotent, $(1 - a)R = R$ and so (iii) is true. Finally (iv) follows because $\ell(a) = Re_1$. Thus the element a satisfies $(1-a)R \precsim k(r(a))$ and its left-hand analogue for $k = 1$ and $k = 2$, but a is neither idempotent nor a product of 2 idempotents.

The simplest example of a ring satisfying the hypotheses of Example 2.2 is the ring of all linear transformations of an infinite dimensional (right) vector space over a division ring. However, by [G, Theorem 10.16 and Proposition 10.21], any right self-injective regular ring R which is not unit-regular has a direct factor which satisfies the hypotheses of the example and so R has an element satisfying conditions (ii), (iii) and (iv). Thus among the right self-injective regular rings only the unit-regular ones satisfy the conclusions in Theorem 1.2.

The example is only a minor set-back though. Proposition 1.1 shows that in any regular ring R we have $(1 - a)R \precsim k(r(a))$ for some k whenever a is a product of idempotents and, by symmetry, $R(1 - a) \precsim k(\ell(a))$. The example shows that in a general regular ring we need to relax the connexion between the number of idempotents in the product and the number of copies of $r(a)$ needed to cover $(1 - a)R$ (or the number of copies of $\ell(a)$ needed to cover $R(1 - a)$). As in Corollary 1.4 we can use [G, Corollary 2.23] to restate these conditions in terms of ideals, thus removing the explicit counting of the copies of $r(a)$ or $\ell(a)$. Thus by Proposition 1.1 we have:

Proposition 2.3 *In any regular ring R, if $a \in R$ is a product of idempotents then*

$$(*) \qquad Rr(a) = \ell(a)R = R(1-a)R.$$

For a unit-regular ring the converse also holds by Corollary 1.4. (We did not need to mention $\ell(a)R$ in the unit-regular case because there $\ell(a)R = Rr(a)$ for all $a \in R$.) Notice that the Proposition fails in a general ring, as can be seen by taking R to be the ring of 2×2 upper triangular matrices over a field and letting a be a nonzero strictly upper triangular matrix. In this section we shall show (Theorems 2.7 and 2.8) that $(*)$ also characterizes products of idempotents in right self-injective regular rings and their factor rings.

The condition $(*)$ is essentially of the same nature as the conditions found by Reynolds and Sullivan [RS] and O'Meara [OM] when they studied (respectively) the cases where R is a full linear ring or a prime right self-injective regular ring. This is because in both cases the ideals of R correspond precisely to the values of the dimension functions which they use (but which are no longer available in our case). In [OM, Theorem 6], the condition used to characterize an element a of a prime, regular, right self-injective ring R as a product of idempotents is

$$(**) \qquad \mu(r(a)) = \mu\text{-}codim(aR) = \mu((1-a)R)$$

where μ is the Goodearl-Boyle infinite dimension function (see [G, Chapter 12]). However, because $\mu(xR) = \mu(yR)$ if and only if $RxR = RyR$, statements involving principal right ideals having the same μ-dimension can be translated into ones involving equality of their two-sided principal ideals, and vice versa. Furthermore the use of a complement right ideal of aR corresponds to our use of $\ell(a)$, since if $aR = eR$ with $e = e^2$, then $\ell(a) = R(1-e)$ and $(1-e)R$ is a complement of aR, and so $\ell(a)$ and the complement generate the same ideal $R(1-e)R$. Thus the condition $(**)$ is equivalent to $(*)$ in this setting. It was shown in [OM, Corollary 12] how one could deduce the conditions of Reynolds and Sullivan [RS] from $(**)$, and hence $(*)$, for characterizing a linear transformation $a \in End_F V$ as a product of proper idempotent transformations, namely

$$\begin{array}{rlll} n(a) & = d(a) & = s(a) & \geq \aleph_0 \\ \text{or} \quad 0 < n(a) & = d(a) & \leq s(a) & < \aleph_0 \end{array}$$

where $n(a) = dim\ Ker(a), d(a) = codim\ Im(a), s(a) = codim\{u \in V | a(u) = u\}$.

Before presenting our principal result in this section, we require some preliminaries. The first is a lemma which was used in [OM] and is probably folklore (stemming from Litoff's Theorem).

Lemma 2.4 *Let J be an ideal of a regular ring R. For any $x_1, \cdots, x_n \in J$, there exists an idempotent $g \in J$ such that $x_i \in gRg$ for all $i = 1, \cdots, n$.*

Proof. There are idempotents $e, f \in J$ such that $\sum_1^n x_i R = eR$ and $\sum_1^n Rx_i = Rf$, because R is regular. Also there exist $u, v \in R$ such that $(1-f)R = (eR \cap (1-f)R) \oplus uR$ and $R = (eR + (1-f)R) \oplus vR$. Observe that $vR \cap (1-f)R = 0$ implies $vR \lesssim fR$, whence $vR \subseteq RfR \subseteq J$ and $v \in J$. Notice too that $R = eR \oplus uR \oplus vR$. Let $g = g^2 \in R$ be such that $gR = eR \oplus vR$ and $(1-g)R = uR$. Then $g \in J$ because $e, v \in J$. Also $eR \subseteq gR$ and $(1-g)R \subseteq (1-f)R$, hence $eRf \subseteq gRg$. Now $x_i \in eRf \subseteq gRg$ gives $x_i \in gRg$ for all i, as required.

Our next lemma is a reduction technique from [OM] that still works in our more general setting.

Lemma 2.5 *Suppose R is a regular ring and that $a \in R$ satisfies the condition (*) in Proposition 2.3. Then there is an idempotent $g \in R$ and an element $y \in A \equiv gRg$ such that*

$$a = y + (1-g)$$

and in the ring A

$$Ar(y) = \ell(y)A = A.$$

Proof. This is contained in the proof of [OM, Theorem 6] but for completeness we repeat the argument here. Let $x = 1 - a$ and denote the ideal $R(1-a)R$ by J. Since $x \in J$, Lemma 2.4 shows that there is an idempotent $g \in J$ such that $x \in gRg$. Clearly $RgR = J$. Let $y = g + x \in A$ so that $a = y + (1-g)$. Then, since $y \in gR$,

$$r_A(y) = r_R(a) \cap Rg = r_R(a)g$$

and so

$$\begin{aligned} Ar_A(y) &= (gRg)r_R(a)g \\ &= gRr_R(a)g \qquad\qquad (\text{since } r_R(a) \subseteq (1-a)R \subseteq gR) \\ &= gJg \\ &= gRgRg = gRg = A. \end{aligned}$$

By symmetry we must also have $\ell_A(y)A = A$ and so the proof is complete.

Notice that, in the notation of the lemma, if we can write y as a product of idempotents e_i in the ring A then a is the corresponding product of the idempotents $e_i + (1-g)$ in the ring R. Since the ring A is right self-injective and regular whenever R is (by [G, Corollary 9.3]) this means that we can reduce to the case where the ideals in $(*)$ are the whole ring. The next two lemmas will be used to construct products of idempotents in just this situation.

Lemma 2.6 *Suppose R is a regular ring with idempotents e, f, g such that*

$$eR \cap gR = 0, \;\; fR \cap gR = 0 \text{ and } fR \lesssim gR.$$

Then each $a \in eRf$ is a product $e_1e_2e_3$ of three idempotents where $e_1R = eR$, $e_2R \lesssim (1-e_2)R$ and $e_3 = f$.

Proof. (Based on [RS, Lemma 7]) We may assume that e and g are orthogonal. Since $fR \lesssim gR$ there are orthogonal idempotents g_1, g_2 such that $g = g_1 + g_2$ and $fR \cong g_1R$. Hence we can find $x \in fRg_1$ and $y \in g_1Rf$ such that $xy = f$. Since $fR \cap g_1R = 0$ there are orthogonal idempotents h_1, h_2 such that $g_1R = h_1R$ and $fR = h_2R$. Then

$$a = [e + ax][h_1 + yh_2][f]$$

is a product ($e_1e_2e_3$, say) of three idempotents. Clearly $e_1R = eR$. Also $e_2R = h_1R = g_1R \lesssim (1 - g_1)R$ since $g_1R \cong fR$ and $fR \cap g_1R = 0$. As $(1 - g_1)R \cong (1 - e_2)R$ it follows that $e_2R \lesssim (1 - e_2)R$, as required.

Lemma 2.7 *Let R be a regular ring satisfying general comparability. If $e, f \in R$ are idempotents such that*

$$eR \lesssim (1 - e)R \quad \text{and} \quad fR \lesssim (1 - f)R$$

then each $a \in eRf$ is a product $e_1e_2e_3$ of three idempotents each of which satisfies $e_iR \lesssim (1 - e_i)R$.

Proof. We can write

$$eR = e_0R \oplus (eR \cap fR) \quad \text{and} \quad fR = f_0R \oplus (eR \cap fR)$$

where e_0, f_0 are orthogonal idempotents. By general comparability there is a central idempotent $u \in R$ such that

$$ue_0R \lesssim uf_0R \quad \text{and} \quad (1 - u)f_0R \lesssim (1 - u)e_0R.$$

By writing any $a \in eRf$ as $a = ua + (1-u)a$ we can concentrate on the rings uR and $(1-u)R$ separately: if we can write $ua = g_1g_2g_3$ in uR and $(1 - u)a = h_1h_2h_3$ in $(1 - u)R$, where each $g_iuR \lesssim (u - g_i)R$ and $h_i(1 - u)R \lesssim (1 - u - h_i)R$, then we can use the idempotents $e_i = g_i + h_i$ to get $a = e_1e_2e_3$ and have each $e_iR \lesssim (1 - e_i)R$. Hence it is enough to assume, in turn, that $e_0R \lesssim f_0R$ and that $f_0R \lesssim e_0R$.

Firstly let us consider the case $e_0R \lesssim f_0R$. We construct an idempotent $g \in R$ so that Lemma 2.6 can be used. Since $e_0R \lesssim f_0R$ there is some $x \in f_0Re_0$ with zero right annihilator in e_0R. Let $h = e_0 + x$ which is idempotent because e_0 and f_0 are orthogonal. Then $eR \cap hR = 0$ since if $z \in eR \cap hR$ we have $z = hz = e_0z + xe_0z$, giving $xe_0z \in eR \cap f_0R = 0$ and so $e_0z = 0$, which forces $z = 0$.

Similarly $fR \cap hR = 0$ since if $z \in fR \cap hR$ then $z = e_0z + xe_0z$ gives $e_0z \in fR \cap e_0R = 0$ and so $z = 0$.

Also $eR + fR = fR + hR$ since $e_0 = h - x \in hR + fR$. Now let h_1R be a complement of $eR + fR$ in R and let g be an idempotent of R such that $gR = hR + h_1R$. This is the g we want for Lemma 2.6 since clearly $eR \cap gR = 0$ and $fR \cap gR = 0$, while the decomposition

$$R = (fR \oplus hR) \oplus h_1R = fR \oplus gR$$

shows that $gR \cong (1 - f)R$ and so, by hypothesis, $fR \lesssim gR$. Thus in this case we are finished.

Similarly if we have $f_0R \lesssim e_0R$ the above construction gives an idempotent g such that $eR \cap gR = 0$ and $fR \cap gR = 0$ but $eR \lesssim gR$. However $f_0R \lesssim e_0R$ implies that $fR \lesssim eR$ and so $fR \lesssim gR$ in this case too.

We can now show that $(*)$ characterizes products of idempotents in right self-injective regular rings.

Theorem 2.8 *Let R be right self-injective and regular and let $a \in R$. Then a is a product of idempotents if and only if*

$$(*) \qquad Rr(a) = \ell(a)R = R(1 - a)R.$$

Proof. We have already seen in Proposition 2.3 that $(*)$ is a necessary condition, so suppose $a \in R$ satisfies $(*)$. By Lemma 2.5 and the remarks following it we can assume that $R(1-a)R = R$. By [G, Proposition 10.21] there is a central idempotent $u \in R$ such that $(1-u)R$ is directly finite and uR is purely infinite. By Corollary 1.4, $a(1-u)$ is a product of idempotents, so we just have to consider $au \in uR$. That is, we may assume that R is purely infinite. Let e, f be idempotents of R such that $eR = aR$ and $Rf = Ra$ so that our hypotheses become $R(1-f)R = R(1-e)R = R$. Since $a \in eRf$ it is enough, by Lemma 2.7, to show that $eR \lesssim (1-e)R$ and $fR \lesssim (1-f)R$. Hence it is enough to show that if $RgR = R$ then $R \lesssim gR$. But if $RgR = R$ then $R \lesssim n(gR)$ for some integer n, by [G, Corollary 2.23]. Since R is purely infinite we have $nR \cong R \lesssim n(gR)$, by [G, Theorem 10.16], and so $R \lesssim gR$ by [G, Theorem 10.34]. So the proof is complete.

We conclude this section by extending even further the class of rings for which we know that $(*)$ characterizes products of idempotents.

Theorem 2.9 *Let R be a regular ring. Then the property*

$$(*) \qquad\qquad Rr(a) = \ell(a)R = R(1-a)R$$

characterizes $a \in R$ as a product of idempotents whenever R is any of the following:

(i) *unit-regular*

(ii) *right continuous*

(iii) *a factor ring of a right self-injective ring.*

Proof. Corollary 1.4 looks after the unit-regular case. If R is right continuous then by [G, Theorem 3.17] R is a direct product of an abelian (and so unit-regular) ring and a right self-injective ring. Hence this case follows from Corollary 1.4 and Theorem 2.8. (Alternatively we could use the fact that R contains all the idempotents of its maximal right quotient ring and then use Theorem 2.8.) So we just have to consider rings R/I where I is an ideal of a right self-injective regular ring R. Suppose $a \in R/I$ satisfies $(*)$. By Theorem 2.8 it is enough to find some $b \in R$ satisfying $(*)$ in the ring R and such that

$\bar{b} = a$ (where $\bar{\ }$ denotes the image of the natural map $R \to R/I$). Choose any $b_1 \in R$ with $\bar{b}_1 = a$. Since a satisfies $(*)$ it is easy to see that

$$R(1-b_1)R + I = Rr(b_1) + I = \ell(b_1)R + I.$$

We begin by modifying the first equation. We have $(1-b_1)R \subseteq Rr(b_1) + yR$ for some $y \in I$. By general comparability there is some central idempotent u of R such that

$$ur(b_1) \lesssim uyR \quad \text{and} \quad (1-u)yR \lesssim (1-u)r(b_1).$$

Since $ur(b_1) \lesssim uyR \subseteq I$ we have $ur(b_1) \subseteq I$ and so $u(1-b_1) \in I$. Hence $\bar{u}\bar{b}_1 = \bar{u}$ in R/I. Let $b_2 = u + (1-u)b_1$ so that $\bar{b}_2 = \bar{b}_1$. Then we have $R(1-b_2)R = Rr(b_2)$ since this is clearly true on uR while on $(1-u)R$ we have

$$\begin{aligned}
(1-u)R(1-b_2)R &= (1-u)R(1-b_1)R \\
&\subseteq (1-u)Rr(b_1) + (1-u)yR \\
&\subseteq (1-u)Rr(b_1) \quad \text{since } (1-u)yR \lesssim (1-u)r(b_1) \\
&= (1-u)Rr(b_2).
\end{aligned}$$

By symmetry there is a central idempotent $v \in R$ such that $b_3 = v + (1-v)b_2$ satisfies $\bar{b}_3 = \bar{b}_2 = \bar{b}_1$ and $R(1-b_3)R = \ell(b_3)R$. But we still have $R(1-b_3)R = Rr(b_3)$ since this holds for b_2. Hence b_3 satisfies $(*)$ and $\bar{b}_3 = a$, and so the proof is complete.

3 General regular rings

In view of Theorem 2.9, we are prompted to ask the following question.

Question 3.1 *Does the property*

$$(*) \qquad Rr(a) = \ell(a)R = R(1-a)R$$

characterize products of idempotents in a general regular ring R?

As we saw in Proposition 2.3, the property is certainly necessary. It is also trivially sufficient if R is commutative. However if we don't require R to be regular, then $(*)$ is not sufficient to ensure $a \in R$ is a product of idempotents. For example, we could choose R to be a simple Noetherian ring (with identity) which has zero-divisors but no nontrivial idempotents (as in the Zalesskii and Neroslavskii example [ZN]). Notice that for a simple ring, if $(*)$ is to characterize products of idempotents, then elements which are left and right zero-divisors must be products of idempotents. It is also worth noting that $(*)$ is a "local" property, that is, an element satisfies $(*)$ in R if and only if it satisfies $(*)$ in some finitely generated subring of R. Consequently, $(*)$ characterizes products of idempotents when a ring is locally one of those in Theorem 2.9.

In this section we establish that any regular ring for which $(*)$ characterizes products of idempotents, must satisfy a certain "weak unit-regularity" property. For directly finite, simple regular rings, this is equivalent to unit-regularity. This, together with Theorem 2.9, shows that even for the class of directly finite, simple regular rings, Question 3.1 is equivalent to the open Problem 3 in [G]: is a directly finite, simple regular ring necessarily unit-regular?

We begin with a lemma, which may be known, although we have been unable to find a reference to it.

Lemma 3.2 *The unit-regular elements of a regular ring R form a multiplicative subsemigroup.*

Proof. Let $a, b \in R$ be unit-regular. Since R is a regular ring, there exist $c, d, e \in R$ with $bR = (r(a) \cap bR) \oplus cR, r(a) = (r(a) \cap bR) \oplus dR$, and $R = (r(a) + bR) \oplus eR$. Then

$$R = bR \oplus dR \oplus eR. \tag{1}$$

Also by regularity, there exist $f, g \in R$ such that $R = r(b) \oplus fR \oplus gR$ and $bfR = r(a) \cap bR$. Now

$$fR \cong r(a) \cap bR \tag{2}$$

$$\text{and} \qquad r(ab) = r(b) \oplus fR. \tag{3}$$

By (1), $R = r(a) \oplus cR \oplus eR$ whence $aR = acR \oplus aeR = abR \oplus aeR$. Hence

$$\begin{aligned}
R/abR &\cong (R/aR) \oplus aeR \\
&\cong r(a) \oplus eR && \text{since } a \text{ is unit-regular} \\
&= (r(a) \cap bR) \oplus dR \oplus eR \\
&\cong fR \oplus dR \oplus eR && \text{by (2)} \\
&\cong fR \oplus (R/bR) && \text{by (1)} \\
&\cong fR \oplus r(b) && \text{since } b \text{ is unit-regular} \\
&= r(ab) && \text{by (3),}
\end{aligned}$$

which shows ab is unit-regular.

Remark. The Lemma fails for a regular semigroup S (so there is no purely multiplicative proof of the Lemma). For example, take S to be any regular semigroup with 1, generated by its idempotents but containing non-idempotents, and with only the trivial unit 1 (such as the semigroup of all singular $n \times n$ matrices ($n > 1$) over a field, together with the identity matrix). The unit-regular elements are then just the idempotents. (This example was suggested to us by Peter Jones and Karl Byleen). The Lemma also fails for general rings.

Proposition 3.3 *Let R be a regular ring.*
(1) If $a \in R$ is a product of idempotents, then a is unit-regular.
(2) If $(*)$ characterizes products of idempotents in R, then R satisfies the following "weak unit-regularity" property: for all idempotents $e, f \in R$,

$$eR \cong fR \quad \text{and} \quad R(1-e)R = R(1-f)R = R \Rightarrow (1-e)R \cong (1-f)R.$$

Proof. (1) is immediate from Lemma 3.2.

(2) Assume $(*)$ characterizes products. Let $e, f \in R$ be idempotents such that $eR \cong fR$ and $R(1-e)R = R(1-f)R = R$. Then there exists $a \in R$ with $aR = eR$ and $Ra = Rf$, and so

$$Rr(a) = \ell(a)R = R = R(1-a)R.$$

Hence a satisfies $(*)$, whence a is a product of idempotents and, consequently, unit-regular by (1). Now $(1-e)R \cong R/aR \cong r(a) = (1-f)R$, as required.

Thus from Proposition 3.3 and Corollary 1.5 we see that a directly finite, simple regular ring is unit-regular if and only if each non-unit is a product of idempotents. This provides another perspective to [G, Problem 3, p344].

A question of interest to semigroup theorists is: what is the minimum number of idempotents needed to express a general element of an idempotent-generated semigroup S as a product of idempotents (the so-called "depth" of S)? In a forthcoming paper "Depth of idempotent-generated subsemigroups of a regular ring", we address this question in the case where S is the semigroup generated by the idempotents of a regular ring R which is directly finite or right self-injective. For directly finite regular rings, for instance, it turns out that the depth of S equals the index of nilpotence of R.

References

[B] C.S. Ballantine, Products of idempotent matrices, *Linear Algebra Appl.* 19(1978)81-86.

[D] R.J.H. Dawlings, The idempotent generated subsemigroup of the semigroup of continuous endomorphisms of a separable Hilbert space, *Proc. Roy. Soc. Edinburgh*, Section A, 94 (1983) 351-360.

[E] J.A. Erdos, On products of idempotent matrices, *Glasgow Math. J.* 8(1967)118-122.

[G] K.R. Goodearl, *Von Neumann Regular Rings*, (Pitman, 1979).

[H] J.M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* 41(1966)707-716.

[OM] K.C. O'Meara, Products of idempotents in regular rings, *Glasgow Math. J.* 28(1986)143-152.

[RS] M.A. Reynolds and R.P. Sullivan, Products of idempotent linear transformations, *Proc. Roy. Soc. Edinburgh*, Section A, 100(1985)123-138.

[ZN] A.E. Zalesskii and O.M. Neroslavskii, There exists a simple noetherian ring with divisors of zero, but without idempotents, *Commun. in Algebra* 5(1977)231-244. (For an English version of this see K.R. Goodearl's paper "Simple Noetherian rings, the Zalesskii-Neroslavskii examples" in *Ring Theory, Waterloo 1978*, Lecture Notes in Mathematics, No. 734 (Springer-Verlag, 1979)).