ALGORITHMS FOR THE COMPUTATION OF APPROXIMATIONS BY ALGEBRAIC FUNCTIONS

by

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1. INTRODUCTION

The application of Padé approximation to problems in mathematical physics was introduced by Baker and Gammel (Baker and Gammel (1961)). Padé approximations, the rational analogue of the Taylor polynomial approximation, could be expected to represent a wider range of behaviour than simple polynomial approximation. Various generalizations of this approach include integral approximants (Hunter and Baker (1979), and Rehr, Joyce and Guttmann (1980)) applied to the theory of critical phenomena, and algebraic approximants (Brak, Guttmann and Enting (1990)) occurring in lattice theory.

In considering approximation from the more general classes of integral or algebraic functions, there are two separate aspects that merit attention. These are the properties of the approximation and the computation of the approximation.

The local approximating properties of the quadratic algebraic function have been studied (Brookes and McInnes (1990)) and some qualitative results have been reported (Brookes (1990)). The results have been extended to general integral functions (McInnes (1989)) and general algebraic functions (McInnes (1991)). The main results established were that a clear formulation of the approximation problem leads to the existence of a unique approximating algebraic form which determines the polynomial coefficients of the algebraic equation for the algebraic function, and the existence of a unique distinguished algebraic function which is a solution of this algebraic equation. In addition the order of approximation is quantified in a variety of circumstances.

However the computation of the general algebraic form which defines the polynomial coefficients of the implicit equation for the algebraic function approximation may involve severe numerical problems, and seems to have received little direct attention. The objective of this paper is to review existing algorithms for this computation and to consider a new approach.

In Section 2 the general formulation of approximation by algebraic functions is summarized. Some existing algorithms for the computation of the algebraic form are reviewed in Section 3 and a new algorithm is described in Section 4.

2. APPROXIMATION BY ALGEBRAIC FUNCTIONS

Consider the problem of approximating a function, $f(x)$, which is locally analytic at the origin. Although this depends on how the approximation is to be determined, we shall consider that the approximations are to be determined by determining a form which has the same structure as a function in the class of approximating functions. Thus if we seek to approximate $f(x)$ by an algebraic function, $Q(x)$, which satisfies the relation

$$P(Q,x) \equiv \sum_{i=0}^{p} a_i(x)Q(x)^i = 0, \tag{1}$$

where the $a_i(x)$ are polynomial coefficients, with $a_i(x) \not\equiv 0$ for at least one value of i for $i = 1(1)p$, then we seek an *algebraic form* for $f(x)$ which has this structure. That is, we seek polynomial coefficients, $a_i(x)$, such that

$$P(f,x) \equiv \sum_{i=0}^{p} a_i(x)f(x)^i \tag{2}$$

is approximately zero in an appropriate sense (McInnes (1991)).

If we seek to approximate $f(x)$ by an integral function, $Q(x)$, which satisfies the relation

$$P(Q,x) \equiv \sum_{i=0}^{p} a_i(x) Q^{(i)}(x) = 0,$$

then we seek an *integral form* for $f(x)$,

$$P(f,x) \equiv \sum_{i=0}^{p} a_i(x) f^{(i)}(x),$$

which is approximately zero in an appropriate sense (McInnes (1989)).

In this paper attention will be restricted to algebraic functions, although the following basic results have been extended to other classes of functions.

Let $\mathbf{n} = (n_0, n_1, \ldots, n_p)$, where, for $i = 0(1)p$, $n_i \geq -1$ are integers such that $\deg\,(a_i(x)) \leq n_i$. Let $N+1 = \sum_{i=0}^{p}(n_i+1)$. If we consider the space $C^N\{0\}$, then the best approximation to zero, of the $\mathbf{n}$ algebraic form of degree p, is given by requiring

$$D^k P(f,x) = 0, \quad k = 0(1)N-1, \tag{3}$$

where the linear functional D^k is defined by $D^k r(x) = r^{(k)}(0)/k!$.

If $p = 1$ and $\deg(a_1(x)) = 0$, then $P(Q,x) = 0$ implies that $Q(x) = -a_0(x)$, and $Q(x)$ is simply the Taylor polynomial approximation. It is trivial that the approximation, $Q(x)$, is locally analytic and that $Q(x) = f(x) + O(x^N)$.

If $p = 1$ and $\deg(a_1(x)) > 0$, then the approximation is a rational function usually referred to as the Padé rational approximation (under the Padé-Frobenius definition but not necessarily under the Baker definition (Baker and Graves-Morris (1981)). It has been shown by Baker (Baker and Graves-Morris (1981)) that $Q(x)$ is locally analytic at the origin and if it is 'normal' ($a_1(0) \neq 0$), then

$$Q(x) = f(x) + O(x^N).$$

However, if $p \geq 2$, it is not obvious either that $Q(x)$ is locally analytic at the origin, or that the derivatives of $Q(x)$ at the origin match those of $f(x)$. It has recently been shown (McInnes (1991)) that the conditions (3) determine a unique algebraic form (2) of maximal order at the origin. In certain circumstances this form may 'over-approximate' and the surplus, S, quantifies this over-approximation. If the algebraic form (2) is normal ($\partial P/\partial f \neq 0$ at the origin), a suitable initial condition identifies a unique, distinguished branch, $Q^*(x)$, satisfying equation (1). However, if the algebraic form is not normal, the order of $\partial P/\partial f$ at the origin defines the deficiency, D, which quantifies the amount by which the approximation to $f(x)$ by the distinguished branch $Q^*(x)$ falls short of the expected order. It is shown that the order of approximation of this unique, distinguished branch is given by

$$Q^*(x) = f(x) + O(x^{N+S-D}).$$

Further details are given in McInnes (1991).

3. COMPUTATION OF THE ALGEBRAIC FORM

The existence of an $\mathbf{n}$ algebraic form of degree p, (2), follows immediately from the conditions (3), since these conditions give a system of N homogeneous linear equations in the $N+1$ coefficients of the polynomials $a_i(x)$, $i = 0(1)p$. The matrix form of this system of linear equations has the coefficient matrix (where I is the identity matrix)

$$F = [F_{n_0} : F_{n_1} : \ldots : F_{n_p}], \tag{4}$$

$$\text{where} \quad F_{n_0} = \begin{bmatrix} I_{n_0+1} \\ 0 \end{bmatrix}, \quad \text{and} \quad F_{n_k} = \begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ \cdot & \cdot & & \\ g_{n_k} & g_{n_k-1} & \cdots & g_0 \\ \cdot & \cdot & & \cdot \\ g_{N-1} & g_{N-2} & \cdots & g_{N-1-n_k} \end{bmatrix},$$

for $1 \leq k \leq p$, where the g_j in this submatrix are defined by $f(x)^k = \sum_{j=0}^{\infty} g_j x^j$.

The matrix F has dimensions $N \times (N+1)$ and hence has a solution space of dimension at least 1. If the solution space dimension is exactly 1 then any constant multiple of the coefficient polynomials $\{a_i(x)\}_{i=0}^{p}$ will also be a solution. A unique representative of this class of essentially unique solutions may be defined by choosing any convenient suitable normalization of the coefficients (see McInnes (1991)).

In this paper, attention will be restricted to the case where the surplus $S = 0$ for all $\mathbf{n}$. (Such a system is called normal by Paszkowski (1987)). In this case the rank of F is N since if $\text{rank}(F) < N$ then it may be shown that $S > 0$ (McInnes (1991)).

The $\mathbf{n}$ algebraic function approximation to $f(x)$ may be represented by considering the N equations arising from conditions (3), together with equation (1), to give a homogeneous system of $N+1$ linear equations in the $N+1$ unknown coefficients. This system has a non-trivial solution which may be represented by the eliminant of these equations, i.e., $\det B = 0$.

Consider the $\mathbf{n} = (n, n, \ldots, n)$ algebraic form of degree p, for the function $f(x) = \sum_{i=0}^{N-1} f_i x^i + O(x^N)$. Since the $(0, 0, \ldots, 0)$ algebraic form of degree p is trivially the form $P(f, x) = (f - f_0)^p = O(x^p) = O(x^N)$, we consider the "diagonal" forms for $n > 0$. The $\mathbf{n}$ algebraic function approximation to $f(x)$ may be represented by considering the N equations arising from conditions (3), together with equation (1), to give a homogeneous system of $N+1$ linear equations in the $N+1$ unknown coefficients. This system has a non-trivial solution which is expressed by the eliminant of these equations, $\det B = 0$, where B is the $N \times (N+1)$ matrix F (equation (4)), augmented by the $(N+1)$st row

$$[1, x, \ldots, x^n, Q(x), xQ(x), \ldots, x^n Q(x), \ldots, Q(x)^p, xQ(x)^p, \ldots, x^n Q(x)^p].$$

Although the mathematical solution of the problem of determining the $\mathbf{n}$ algebraic form is easily expressed by the equation $\det B = 0$, the computational solution of this problem may prove more difficult. If the coefficient polynomials, $a_i(x)$, are expressed in terms of the usual basis functions, x^k, numerical problems are not surprising in view of the near dependence of the basis functions. The straightforward approach is simply to carry out a Gaussian elimination procedure to reduce the matrix F. However, a recent investigation (Balakrishnan and McInnes (1991)) has shown that the condition number (using the 2-norm) of the matrix F may be quite large, even for moderate values of $\mathbf{n}$. For example, if $\mathbf{n} = (6,6,6)$ then the condition number of F for $f(x) = \exp(x)$ is 6.1882×10^{19}, for $f(x) = \log(1+x)$ is 3.2681×10^{15}, and for $f(x) = (1+x+x^2)^{(1/3)}$ is 1.5616×10^9. In view of these values, it is not surprising that the normalized solutions show a range of from only 2 correct significant figures to 10 correct significant figures, out of 15 significant figures computed.

In the rational case ($p = 1$), various identities (the Frobenius identities) may be established for the recursive calculation of the polynomial coefficients (Baker and Graves-Morris (1981)). These were extended to general Padé-Hermite approximations by Paszkowski (1987). A similar, but simplified, recurrence algorithm was developed for the diagonal types of the quadratic algebraic forms, but this algorithm was shown to be numerically unstable (Brookes and McInnes (1989)).

In order to compare the computational performance of these various methods, Paszkowski's two algorithms have been implemented as algorithms to compute a diagonal sequence (n,n,n) of quadratic algebraic forms. The numerical results vary for different functions, $f(x)$, but the primary conclusion is that these recursive methods offer no advantage in numerical stability over the direct solution of the matrix system. Further details may be found in Balakrishnan and McInnes (1991).

It is apparent that the primary problem is the form in which the algebraic form is expressed. For example, the $(7,7,7)$ quadratic form for $f(x) = \exp(x)$ is given by

$$\begin{aligned}
(\ & 130945815 + 53531415x + 9882810x^2 + 1070685x^3 \\
&\qquad + 73710x^4 + 3234x^5 + 84x^6 + x^7) \\
&+(154828800x + 7741440x^3 + 86016x^5 + 256x^7)f(x) \\
&+(-130945815 + 53531415x - 9882810x^2 + 1070685x^3 \\
&\qquad - 73710x^4 + 3234x^5 - 84x^6 + x^7)f(x)^2 = O(x^{23}).
\end{aligned}$$

With the coefficients having such a large variation in this form, it is difficult to see how a numerically stable computation might be achieved. In the following section an alternative set of basis functions for the algebraic form is derived.

4. FORMULATION OF AN ALTERNATIVE BASIS

A Newton formulation for the problem of finding the collocating quadratic algebraic form was given in McInnes (1990). While the spirit of this formulation is similar, the details are significantly different.

If column operations are applied to the matrix, B, to reduce this matrix to lower triangular form, a vector of new basis functions $[\varphi_0(x), \varphi_1(x), \ldots, \varphi_N(x)]$ is obtained in the final row. If the diagonal elements are scaled to 1 then the eliminant $\det B = 0$ becomes $\varphi_N(x) = 0$, which is the expression for the algebraic function analogous to (1).

The details of this elimination are tedious but routine algebra. However, the pattern that emerges produces the following recursion for the new basis elements.

$$\begin{aligned}
\varphi_0(x) &= 1, \\
\varphi_{k+1}(x) &= x\varphi_k(x), && \text{for } k = 0(1)n-1, \\
\varphi_{k+n+1}(x) &= Q(x)\varphi_k(x) - \textstyle\sum_{j=0}^{n} R_{kj}\varphi_{k+j}(x), && \text{for } k = 0(1)N-1-n. \quad (5)
\end{aligned}$$

The remaining question is the determination of the constants R_{kj}. Since $R_{k0} = f_0$ for all k, the remaining values may be determined by setting up the following matrix and applying the coefficient algorithm so that the required values may be read off the resulting matrix. Let

$$F = \begin{bmatrix} f_1 & & & & \\ f_1 & f_2 & & & \\ f_1 & f_2 & f_3 & & \\ \cdot & \cdot & \cdot & & \\ f_1 & f_2 & f_3 & \dots & f_{N-1} \end{bmatrix}.$$

Coefficient Algorithm.

Step 1: Initialize by dividing each row after the nth by its diagonal element.

For $i = n+1(1)N-1$
  $K \leftarrow F(i,i)$;
  For $j = 1(1)i$
    $F(i,j) \leftarrow F(i,j)/K$;
  end
end

Step 2: Main recursion.

For $j = n+2(1)N-1$
  For $k = N-1(-1)j$
    $K \leftarrow F(k, k-j+n+1) - F(j-1, n)$;
    For $i = 2(1)n$
      $F(k, k-j+n+1-(i-1)) \leftarrow (F(k, k-j+n+1-(i-1)) - F(j-1, n-i+1))/K$;
    end
    $s \leftarrow 0$;
    For $i = 1(1)k-j+1$
      $s \leftarrow s + f_i * (F(k,1)/F(k-i,1))$;
    end
    $F(k, k-j+1) \leftarrow s/K$;
    If $k > j$ then $F(k,1) \leftarrow f_1 * (F(k,1)/F(j-1,1))/K$; end
  end
end

Step 3: Form a new $N \times (n+1)$ matrix H by taking the first column of all f_0, and the remaining n columns to be the first n columns of F.

$$H = \begin{bmatrix} f_0 & & & & \\ f_0 & f_1 & & & \\ f_0 & f_1 & f_2 & & \\ \cdot & \cdot & \cdot & & \\ f_0 & f_1 & f_2 & \dots & f_n \\ f_0 & F_{n+1,1} & F_{n+1,2} & \dots & F_{n+1,n} \\ f_0 & F_{n+2,1} & F_{n+2,2} & \dots & F_{n+2,n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ f_0 & F_{N-1,1} & F_{N-1,2} & \dots & F_{N-1,n} \end{bmatrix}$$

The constants R_{kj} are the elements along the kth diagonal of this matrix H. If the successive values of $\varphi_k(x)$ are substituted into the equation $\varphi_N(x) = 0$, and the result rearranged into the form (1), the leading coefficient of $a_p(x)$ will be normalized to 1.

The set of basis function obtained from this algorithm in fact form a sequence of algebraic forms in the following order:

For $k = 0(1)n$, $\varphi_{n+1+k}(x) = 0$ is the (n, k) algebraic form.
For $k = 0(1)n$, $\varphi_{2(n+1)+k}(x) = 0$ is the (n, n, k) algebraic form.
For $k = 0(1)n$, $\varphi_{3(n+1)+k}(x) = 0$ is the (n, n, n, k) algebraic form.

That is, the sequence of basis functions $\{\varphi_k(x)\}_{k=0}^{N}$ is made up of p blocks of $n+1$ functions, with the blocks being algebraic forms of increasing degree. After the first block of the powers of x, the next block consists of rational (Padé) forms, followed by a block of quadratic algebraic forms, followed by a block of cubic algebraic forms and similarly to the block of algebraic forms of degree p. In view of this relationship, the coefficients of this sequence of algebraic forms in the usual basis may be recovered in the columns of a matrix C by the following algorithm:

$C \leftarrow I_{n+1}$;
For $i = 1(1)p$
  $D \leftarrow \begin{bmatrix} C \\ O_{n+1} \end{bmatrix}$;
  For $k = 1(1)n+1$
    $l \leftarrow (i-1) * (n+1) + k$;
    $s \leftarrow 0$;
    For $j = 1(1)n+1$
      $m \leftarrow (k+j-2) \mod (n+1) + 1$;
      $s \leftarrow s - H(l+j-1, j) * D(:, m)$;
    end
    $D(:, k) \leftarrow s + \begin{bmatrix} 0 \\ C(:, k) \end{bmatrix}$;
  end
  $C \leftarrow D$
end

In his thesis on the properties of the rational function approximation, Padé placed special emphasis on the exponential function, so this is an appropriate function with which to illustrate this process by a specific example.

Example 1: Consider the (2,2,2,2) cubic algebraic form for $f(x) = e^x$.

In this case $n = 2$; $p = 3$ and hence $N = 11$. As a result of the coefficient algorithm we obtain

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \frac{1}{2} \\ 1 & 6 & 3 \\ 1 & 4 & 6 \\ 1 & \frac{5}{2} & \frac{7}{2} \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & 7 & 3 \\ 1 & 4 & \frac{19}{2} \\ 1 & \frac{2}{3} & 1 \\ 1 & 10 & 2 \end{bmatrix}$$

Substituting in the recurrence relations (5) for the basis functions gives, by reading off the coefficients from the diagonals of the matrix H,

$$\begin{aligned} \varphi_3(x) &= Q(x) - 1 - x - \tfrac{1}{2}x^2, \\ \varphi_4(x) &= xQ(x) - x - x^2 - 3\varphi_3(x), \\ \varphi_5(x) &= x^2Q(x) - x^2 - 6\varphi_3(x) - 6\varphi_4(x), \end{aligned}$$

and $\varphi_5(x) = 0$ is the (2,2) rational (Padé) approximation. Continuing in this way

$$\varphi_{11}(x) = \varphi_8(x)Q(x) - \varphi_8(x) - \tfrac{2}{3}\varphi_9(x) - 2\varphi_{10}(x),$$

which is the (2,2,2,2) cubic algebraic form for $f(x) = e^x$.

Applying the algorithm to obtain the coefficients in the usual basis gives the following three forms of the matrix C after the three loops

$$C = \begin{bmatrix} -1 & 3 & -12 \\ -1 & 2 & -6 \\ -\frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -3 & 12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 31 & -\frac{39}{2} & 24 \\ 14 & -8 & 9 \\ 2 & -1 & 1 \\ -32 & 24 & -48 \\ 16 & -8 & -0 \\ -4 & 4 & -8 \\ 1 & -\frac{9}{2} & 24 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -\frac{245}{2} & 46 & -\frac{103}{3} \\ -\frac{87}{2} & \frac{31}{2} & -11 \\ -\frac{9}{2} & \frac{3}{2} & -1 \\ 351 & -\frac{405}{2} & 243 \\ 54 & -54 & 81 \\ 54 & -27 & 27 \\ -\frac{459}{2} & 162 & -243 \\ \frac{189}{2} & -\frac{135}{2} & 81 \\ -\frac{27}{2} & \frac{27}{2} & -27 \\ 1 & -\frac{11}{2} & \frac{103}{3} \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix}.$$

Reading off the coefficients from the columns of the first matrix C, the (2,0), (2,1), (2,2) Padé approximations satisfy, respectively,

$$\begin{aligned} -1 - x - \tfrac{1}{2}x^2 + Q(x) &= 0, \\ 3 + 2x + \tfrac{1}{2}x^2 - 3Q(x) + xQ(x) &= 0, \\ -12 - 6x - x^2 + 12Q(x) - 6xQ(x) + x^2Q(x) &= 0. \end{aligned}$$

Similarly from the final column of the third matrix C, the (2,2,2,2) cubic algebraic approximation satisfies

$$\begin{aligned} &-(\tfrac{103}{3} + 11x + x^2) + (243 + 81x + 27x^2)Q(x) \\ &\qquad + (-243 + 81x - 27x^2)Q(x)^2 + (\tfrac{103}{3} - 11x + x^2)Q(x)^3 = 0. \end{aligned}$$

Note that this formulation of the algorithm implies an implicit normalization so that $a_p(x)$ is a monic polynomial at each stage of the algorithm, and this is possible only if all members of the computed sequence of algebraic forms have non-zero surplus. This is also illustrated in a further example of a more complex function.

Example 2: Consider the (2,2,2) quadratic algebraic form for $f(x) = \exp(\tan(x))$. In this case $n = 2$; $p = 2$ and hence $N = 8$. For this function

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \frac{37}{120}x^5 + \frac{59}{240}x^6 + \frac{137}{720}x^7 + O(x^8),$$

and as a result of the coefficient algorithm we obtain

$$H = \begin{bmatrix} 1.0000 & 0 & 0 \\ 1.0000 & 1.0000 & 0 \\ 1.0000 & 1.0000 & 0.5000 \\ 1.0000 & 2.0000 & 1.0000 \\ 1.0000 & 4.0000 & -2.0000 \\ 1.0000 & 7.5000 & 21.5000 \\ 1.0000 & 0.0526 & 0.2105 \\ 1.0000 & 0.0303 & -0.4286 \end{bmatrix}.$$

Reading off the values of successive diagonals, and substituting in the recurrence relations (5) for the basis functions gives,

$$\begin{aligned}\varphi_3(x) &= Q(x) - 1 - x - \tfrac{1}{2}x^2,\\ \varphi_4(x) &= xQ(x) - x - x^2 - \varphi_3(x),\\ \varphi_5(x) &= x^2Q(x) - x^2 - 2\varphi_3(x) + 2\varphi_4(x).\end{aligned}$$

Hence the (2,2) rational (Padé) approximation is $\varphi_5(x) = 0$. Continuing in this way, the final diagonal implies that the (2,2,2) quadratic form for $f(x)$ is

$$\varphi_8(x) = \varphi_5(x)Q(x) - \varphi_5(x) - 0.0526\varphi_6(x) + 0.4286\varphi_7(x).$$

Applying the algorithm to obtain the coefficients in the usual basis gives the following two forms of the matrix C after the two loops

$$C = \begin{bmatrix} -1.0000 & 1.0000 & 4.0000 \\ -1.0000 & 0.0000 & 2.0000 \\ -0.5000 & -0.5000 & -1.0000 \\ 1.0000 & -1.0000 & -4.0000 \\ 0 & 1.0000 & 2.0000 \\ 0 & 0 & 1.0000 \end{bmatrix},$$

$$C = \begin{bmatrix} -89.0000 & -12.2632 & -4.5714 \\ -42.0000 & -6.1579 & -2.4286 \\ 24.0000 & 2.9474 & 1.0000 \\ 88.0000 & 13.4737 & 9.1429 \\ -48.0000 & -5.8947 & -0.0000 \\ -22.0000 & -3.3684 & -2.2857 \\ 1.0000 & -1.2105 & -4.5714 \\ 0 & 1.0000 & 2.4286 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

Reading off the coefficients from the columns of the first matrix C, the (2,0), (2,1), (2,2) Padé approximations satisfy, respectively,

$$\begin{aligned} -1 - x - \tfrac{1}{2}x^2 + Q(x) &= 0,\\ 1 - \tfrac{1}{2}x^2 - Q(x) + xQ(x) &= 0,\\ 4 + 2x - x^2 - 4Q(x) + 2xQ(x) + x^2Q(x) &= 0.\end{aligned}$$

Similarly from the final column of the second matrix C, the (2,2,2) quadratic algebraic approximation satisfies

$$-4.5714 - 2.4286x + x^2 + (9.1429 - 2.2857x^2)Q(x) + (-4.5714 + 2.4286x + x^2)Q(x)^2 = 0,$$

which, scaled by 7, gives the more usual integer coefficient form

$$-32 - 17x + 7x^2 + (64 - 16x^2)Q(x) + (-32 + 17x + 7x^2)Q(x)^2 = 0.$$

Since the new basis functions, $\varphi_k(x)$, satisfy the relationship $\varphi_k(x) = O(x^k)$ (because the matrix B was transformed into lower triangular form), it follows that $D^j\varphi_k(x) = 0$ for $0 \le j < k$, and that the set $\{\varphi_k(x)\}_{k=0}^{N}$ is linearly independent.

A dual basis may be constructed for the basis functions $\{\varphi_k(x)\}_{k=0}^{N-1}$ by adding two statements to the coefficient algorithm. The set of linear functionals $\{L_k\}_{k=0}^{N-1}$ is a dual (biorthogonal) basis if $L_k\varphi_j(x) = \delta_{kj}$. The initial set of $n+1$ functionals is clearly $L_k = D^k$ for $k = 0(1)n$. In step 1, after the assignment of K, insert the statement: $L_k \leftarrow D^k/K$;, and in step 2, after the assignment of K, insert the statement: $L_k \leftarrow (L_k - L_{j-1})/K$;. The resulting set of linear functionals, L_k, will be a dual basis for the basis functions $\varphi_k(x)$.

The interesting feature about these results is that a set of basis functions have been obtained, which, in a loose sense, are an extension of the powers of x for polynomial functions. Further, they are easily defined by a simple recursion.

However, some features require further investigation. An obvious problem arises if any of the coefficients f_1 or $\{f_k\}_{k=n+1}^{N-1}$ are zero. It is conjectured that this problem is related to the problem of non-zero surplus, which is itself related to the structure of the table of algebraic forms. It has been shown (McInnes (1991)) that a non-zero surplus generates blocks of identical algebraic forms in the table. The relationship between these problems is the subject of further investigation.

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