ON THE CONTINUITY OF THE GEOMETRIC OPERATIONS IN MINKOWSKI PLANES WITH PARALLEL CLASSES S^2

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Abstract – In a Minkowski plane $\mathcal{M}$ with parallel classes S^2 conditions are investigated that simplify the verification of the continuity of the geometric operations in order to show that $\mathcal{M}$ is a topological Minkowski plane. The point set $S^2 \times S^2$ carries the (euclidean) product topology and circles are closed subsets thereof. It is then shown that continuity of joining suffices with respect to a suitable topology on the set of circles.

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ON THE CONTINUITY OF THE GEOMETRIC OPERATIONS IN MINKOWSKI PLANES WITH PARALLEL CLASSES S^2

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ABSTRACT. In a Minkowski plane $\mathcal{M}$ with parallel classes S^2 conditions are investigated that simplify the verification of the continuity of the geometric operations in order to show that $\mathcal{M}$ is a topological Minkowski plane. The point set $S^2 \times S^2$ carries the (euclidean) product topology and circles are closed subsets thereof. It is then shown that continuity of joining suffices with respect to a suitable topology on the set of circles.

A Minkowski plane $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ consists of a set of points P, a set of at least two circles $\mathcal{K}$ (considered as subsets of P) and two equivalence relations $\|_+$ and $\|_-$ on P (parallelisms) such that three pairwise non-parallel points (that is, neither $(+)$-parallel nor $(-)$-parallel) can be joined uniquely by a circle, such that the circles which touch a fixed circle K at $p \in K$ partition $P \setminus |p|$ (where $|p| = |p|_+ \cup |p|_-$ denotes the union of the two parallel classes of p), such that each parallel class meets each circle in a unique point (parallel projection), such that each $(+)$-parallel class and each $(-)$-parallel class intersect in a unique point, and such that at least one circle contains at least three points (compare [9]). A topological Minkowski plane is a Minkowski plane in which the point set P and the set of circles $\mathcal{K}$ carry topologies such that the geometric operations of joining, touching, the parallel projections, intersecting of parallel classes of different type, and intersecting of circles are continuous operations on their domains of definition (see [9]). A topological Minkowski plane is called (locally) compact, connected, or finite-dimensional if the point space has the respective topological property. For brevity, a topological, locally compact, connected, finite-dimensional Minkowski plane will be called a finite-dimensional Minkowski plane in this paper. According to [6, 2.3] a finite-dimensional Minkowski plane can only be of dimension 2 or 4. In these cases the automorphism group of $\mathcal{M}$ is a Lie group (with respect to the compact-open topology) of dimension at most 7 and 14, respectively, see [10]. The

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classical model of a 2- or 4-dimensional Minkowski plane is obtained as the geometry of non-trivial plane sections of a ruled quadric in the real or complex projective 3-dimensional space respectively; in these cases the topologies on the point set and the set of circles are induced from the surrounding projective 3-space.

Whereas there are many examples of non-classical 2-dimensional Minkowski planes (see [9], [5], and [11]), no non-classical 4-dimensional Minkowski planes are known yet. Furthermore, it was shown in [14] that a 4-dimensional Minkowski plane that admits an 8-dimensional automorphism group must be classical. However, using the results of [15], candidates for 4-dimensional Minkowski planes with a 7-dimensional automorphism groups can be constructed, but it is difficult to verify the axioms of a Minkowski plane. After that one still has to verify the continuity of the geometric operations. For 2-dimensional planes it suffices to know that the point set and lines or circles have the right topological structure to ensure that one has a topological plane (see [8] for projective and affine planes and [9] for Minkowski planes). For stable planes and for certain 4-dimensional Laguerre planes similar results in this direction have been achieved (see [7] and [13, 5.8]). In this note we consider Minkowski planes that have parallel classes S^2, where S^2 denotes the 2-sphere (each 4-dimensional Minkowski plane has parallel classes homeomorphic to S^2 and point set homeomorphic to $S^2 \times S^2$). We then endow $P \simeq S^2 \times S^2$ with the product topology and we assume that circles are compact subsets of P. Furthermore, we endow the set of circles $\mathcal{K}$ with a suitable topology and we finally assume that joining is continuous on its domain of definition. Under these assumptions we prove that the Minkowski plane is indeed a topological Minkowski plane. As the topology of the set of circles in a topological Minkowski plane is uniqely determined by the topology of the point set, the construction is independent of the chosen topology on $\mathcal{K}$.

1. Notation and result

Let $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ be a Minkowski plane that has parallel classes S^2. We fix a point $p_\infty \in P$; we then identify both the (+)-parallel class $|p_\infty|_+$ and the (−)-parallel class $|p_\infty|_-$ of p_∞ with the 2-sphere S^2. Two points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ are (+)-parallel or (−)-parallel if and only if $x_1 = x_2$ or $y_1 = y_2$ respectively. With this identification each circle $K \in \mathcal{K}$ is uniquely determined by the map

$$f_K : |p_\infty|_+ \to |p_\infty|_- : z \mapsto |\ |z|_- \cap K\ |_+ \cap |p_\infty|_-.$$

More precisely,

$$K = \{(x, f_K(x)|\ x \in S^2\}$$

is just the graph of f_K. This reflects the well known description of a Minkowski plane by a sharply triply transitive set of permutations on one parallel class.

We say that the Minkowski plane $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ is in *standard representation* if we coordinatize $P \simeq S^2 \times S^2$ and $\mathcal{K}$ as above.

We endow S^2 with the euclidean topology. $P = |p_\infty|_+ \times |p_\infty|_-$ carries the product topology, so that P becomes homeomorphic to $S^2 \times S^2$. Let d_0 bey a metric that induces the topology on S^2. Then

$$d((x_1, y_1), (x_2, y_2)) = d_0(x_1, y_1) + d_0(x_2, y_2)$$

is a metric on $S^2 \times S^2$ that induces the product topology. In particular, P is a 4-dimensional, compact, metric space.

With respect to d we define the parallel metric

$$\begin{aligned} d_+(K, L) &= sup\{d((x, f_K(x)), (x, f_L(x)))|\ x \in S^2\} \\ &= sup\{d_0(f_K(x), f_L(x))|\ x \in S^2\} \end{aligned}$$

and also

$$\begin{aligned} d_-(K, L) &= sup\{d((f_K^{-1}(y), y), (f_L^{-1}(y), y))|\ y \in S^2\} \\ &= sup\{d_0(f_K^{-1}(y), f_L^{-1}(y))|\ y \in S^2\}. \end{aligned}$$

Since circles in topological (4-dimensional) Minkowski planes are homeomorphic to S^2 (see [9, 1.7]), we require that circles are closed and thus compact. Under this assumption each bijection f_K is continuous (see [3, Thm XI.2.7]) and, because S^2 is compact, f_K is even a homeomorphism of S^2. So circles become homeomorphic to S^2.

Let $\mathcal{H}$ denote the set of all homeomorphisms of S^2. We endow $\mathcal{H}$ with the compact-open topology. According to [3, Thm XII.7.2] this topology is induced by the *sup*-metric with respect to d_0. In particular, the topology is independent of the choice of d_0. Since the *sup*-metric with respect to d_0 induces the metric d_+ on $\mathcal{K}$, we can regard $\mathcal{K}$ as a topological subspace of $\mathcal{H}$. Also, as $\mathcal{H}$ is a topological group, see [1, Thm 4], the map $f \to f^{-1}$ is continuous. Therefore the topology on $\mathcal{K}$ does not depend on whether we choose d_+ or d_-.

Theorem. *Let $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ be a Minkowski plane with parallel classes S^2 in standard representation. We endow $P \simeq S^2 \times S^2$ with the product topology τ_P of $S^2 \times S^2$. Assume that circles are closed subsets of P with respect to τ_P and that joining is continuous with respect to the induced topology on the set of all pairwise non-parallel triples of points (a subset of P^3 with the product topology τ_P^3) and the compact-open topology $\tau_{\mathcal{K}}$ on $\mathcal{K} \subseteq \mathcal{H}$, see above. Then $\mathcal{M}$ is a topological (compact 4-dimensional) Minkowski plane.*

2. Proof of the Theorem

The proof is done in a number of steps. The first two steps do not use the continuity of joining. Since all spaces are metric spaces, we can use sequences to prove the continuity of the geometric operations. For that we denote the set of positive integers by $\mathbb{N}$ and a sequence with elements x_n by $(x_n)_{n\in\mathbb{N}}$.

2.1. *The continuity of intersecting two parallel classes.*

Let $\pi_i : S^2 \times S^2 \to S^2 : (x_1, x_2) \mapsto x_i$ for $i = 1, 2$ denote the canonical projection on the first or second factor. Since the 2-sphere is compact, π_i is a continuous open and closed map. For two points $z = (x, y), w = (u, v) \in P$ the unique point of intersection of $|z|_+$ and $|w|_-$ is found as $(x, v) = (\pi_1(z), \pi_2(w))$. This shows that parallel classes of different type intersect continuously.

2.2. *The continuity of parallel projection.*

For a point $z = (x, y) \in P$ and a circle $K \in \mathcal{K}$ let z_+ and z_- be the unique points of intersection of K with $|z|_+$ and $|z|_-$ respectively. Obviously, these points are determined by

$$z_+ = (x, f_K(x)) \quad \text{and} \quad z_- = (f_K^{-1}(y), y).$$

That is, $z_+ = (\pi_1(z), \epsilon(\pi_1(z), f_K))$ where ϵ denotes the evaluation map $\epsilon(u, f) = f(u)$. But π_1 and ϵ are continuous (cf. [3, Thm XII.2.4]). This shows that the parallel projection

$$(w, L) \mapsto w_+ = |w|_+ \cap L$$

is continuous.

Analogously, we obtain from $|z|_- = (\epsilon(\pi_2(z), f_K^{-1}), \pi_2(z))$ and by the continuity of $f_K \to f_K^{-1}$ that the other parallel projection

$$(w, L) \mapsto w_- = |w|_- \cap L$$

is continuous.

2.3. *The map α (joining).*

For three pairwise non-parallel points x, y, z we denote the unique circle passing through these points by $\alpha(x, y, z)$. For the moment, we fix three pairwise non-parallel points $p_1, p_2, p_3 \in P$ and consider the restriction $\hat{\alpha}$ of α to the set $A = (|p_1|_+ \times |p_2|_+ \times |p_3|_+) \setminus \Delta_{\|}$, where $\Delta_{\|}$ denotes the set of all triples of points that have at least two parallel members. $\hat{\alpha}$ is a bijection from A onto $\mathcal{K}$ which is continuous by assumption. Also, by 2.2 the map

$$K \mapsto (K \cap |p_i|_+)_{i=1,2,3} \in A$$

is continuous. This is $\hat{\alpha}^{-1}$ and thus $\hat{\alpha}$ is a homeomorphism.

If U is an open subset of $P^3 \setminus \Delta_{\|}$, we choose $p_1, p_2, p_3 \in U$ and open neighbourhoods U_i of p_i in $|p_i|_+$ such that $U_1 \times U_2 \times U_3 \subseteq U$. By the preceding consideration $\alpha(U_1, U_2, U_3)$ is an open neighbourhood of $\alpha(p_1, p_2, p_3)$. This shows that $\{\alpha(x, y, z) | x, y, z \in U \text{ pairwise non-parallel}\}$ is open in $\mathcal{K}$. Hence α is open.

2.4. *The coherence axiom* (K4).

This coherence axiom has the following form, compare [9, 2.1].

(K4) Let $x, y_i, x_n, y_{i,n} \in P$, for $i = 1, 2, 3$ and $n \in \mathbb{N}$, be points such that $y_1 \parallel_+ y_2, y_3 \nparallel_+ y_1, x \nparallel_- y_3$, such that $y_{1,n}, y_{2,n}, y_{3,n}$ are pairwise non-parallel for all $n \in \mathbb{N}$, and such that $x_n \to x$, $y_{i,n} \to y_i$ for $n \to \infty$, $i = 1, 2, 3$. Then $|x_n|_- \cap \alpha(y_{1,n}, y_{2,n}, y_{3,n})$ tends to $|x|_- \cap |y_1|_+$ as n tends to ∞.

A similar axiom holds after interchanginging the role of (+)- and (−)-parallelism.

We essentially follow the proof in [9, 3.8]. Let $x, y_i, x_n, y_{i,n} \in P$ be as in (K4) above. Without loss of generality we further assume that y_2 and y_3 are non-parallel. We set $K_n = \alpha(y_{1,n}, y_{2,n}, y_{3,n})$. These circles cannot accumulate at a circle in $\mathcal{K}$. Otherwise we may assume that $(K_n)_{n\in\mathbb{N}}$ converges to $L \in \mathcal{K}$. As $y_{i,n} = |y_{i,n}|_+ \cap K_n$ converges to $|y_i|_+ \cap L$ for $i = 1, 2$ by 2.2, we find that $y_1 = |y_1|_+ \cap L = |y_2|_+ \cap L = y_2$ contrary to our assumption $y_1 \neq y_2$.

Let $z_n = |x_n|_- \cap K_n$. Since the point set P is compact, the sequence $(z_n)_{n\in\mathbb{N}}$ has an accumulation point $z \in P$ which is $(-)$-parallel to x. Since y_2 and y_3 are non-parallel, z must be parallel to one of them (otherwise $(K_n)_{n\in\mathbb{N}}$ would accumulate at $\alpha(z, y_2, y_3)$). Thus $z \in |y_2| \cup |y_3|$, and because $z \parallel_- x \not\parallel_- y_3$, we even have

$$z \in |y_2|_+ \cup |y_2|_- \cup |y_3|_+.$$

We first consider the generic case $x \not\parallel_- y_1, y_2$. Then also $z \not\parallel_- y_2$. Assume that $z \parallel_+ y_3$. Let $s \in |y_3|_- \setminus \{y_1, y_3\}$ and define $s_n = |s|_+ \cap K_n \in K_n$. Then $(s_n)_{n\in\mathbb{N}}$ has an accumulation point t which is $(+)$-parallel to s. If t were not in $|y_2|_- \cup |y_3|_-$, then $\alpha(t, y_2, y_3)$ would exist and the circles $K_n = \alpha(s_n, y_{2,n}, y_{3,n})$ would accumulate at this circle. Further, if t were $(-)$-parallel to y_2, then $K_n = \alpha(s_n, z_n, y_{1,n})$ would accumulate at $\alpha(t, z, y_1)$; similarly, if t were $(-)$-parallel to y_3, then $K_n = \alpha(s_n, z_n, y_{,n})$ would accumulate at $\alpha(t, z, y_2)$. This proves that $z \not\parallel_+ y_3$ and hence $z \parallel_+ y_2 \parallel_+ y_1$. Thus $z = |y_1|_+ \cap |x|_-$ is uniquely determined and $(z_n)_{n\in\mathbb{N}}$ converges to z.

Assume that $x \in |y_1|_- \cup |y_2|_-$. In this case choose $x' \parallel_+ x$ such that x' is not $(-)$-parallel to neither y_1, y_2, nor y_3. Define $y'_{i,n} = |x'|_- \cap K_n$ and $y'_i = |y_1|_+ \cap |x|_-$ for $i = 1, 2$. Then $(y'_{i,n})_{n\in\mathbb{N}}$ converges to y'_i and $x \not\parallel_- y'_1, y_2$ or $x \not\parallel_- y_1, y'_2$. However, $K_n = \alpha(y'_{i,n}, y_{j,n}, y_{3,n})$ where we choose $\{i, j\} = \{1, 2\}$ accordingly to whether $x \not\parallel_- y'_1, y_2$ or $x \not\parallel_- y_1, y'_2$. By the previous case we then know that $(|x_n|_- \cap K_n)_{n\in\mathbb{N}}$ converges to $|x|_- \cap |y_1|_+$. This proves (K4) in any case.

2.5. *The intersection of two circles.*

For $K, L \in \mathcal{K}$ let $f_{K,L}$ denote the map $f_L^{-1} \circ f_K$. Furthermore, let $Fix(f)$ denote the fixed point set of the map $f : S^2 \to S^2$. With this notation the intersection of K and L becomes

$$K \cap L = \{(x, y) \in P |\ y = f_K(x) = f_L(x)\} = \{(x, f_K(x)) |\ f_{K,L}(x) = x\},$$

that is,

$$\pi_1(K \cap L) = Fix(f_{K,L}).$$

We first show that any two circles have non-empty intersection. Let $K, L \in \mathcal{K}$ and choose $x_1, x_2, x_3 \in S^2$ pairwise distinct. The circles K and L then are uniquely determined by the point triples $(x_i, f_K(x_i))$ and $(x_i, f_L(x_i))$, respectively, for $i = 1, 2, 3$. We choose paths $h_i : [0, 1] \to S^2$ in S^2 from $f_K(x_i)$ to $f_L(x_i)$ such that $h_1(t), h_2(t), h_3(t)$ are pairwise distinct for all $t \in [0, 1]$. By joining corresponding points we obtain a path $h : [0, 1] \to \mathcal{K} : t \mapsto \alpha((x_1, h_1(t)), (x_2, h_2(t)), (x_3, h_3(t)))$ in $\mathcal{K}$ from K to L. Forming the composition of $f_{h(t)}$ with f_L^{-1} this finally yields a homotopy from $f_{K,L} = f_L^{-1} \circ f_{h(0)}$ to $id = f_L^{-1} \circ f_{h(1)}$ in $\mathcal{H}$. Thus the Lefschetz number $\lambda(f_{K,L})$ of $f_{K,L}$ (see [4, 3.10] or [2, I.B and II.a]) can be calculated as $\lambda(f_{K,L}) = \lambda(id) = 2 \neq 0$. According to [4, 3.9], the map $f_{K,L}$ therefore must have a fixed point, that is, K and L intersect in at least one point.

2.6. *The continuity of intersecting two circles.*

Let $K, L \in \mathcal{K}$ be two distinct circles. We distinguish according to whether K and L intersect in one point or two points. We first consider the case that K and L intersect in one point. Here we prove the continuity of intersecting at (K, L) by using sequences. Let $K_n, L_n \in \mathcal{K}$ be circles such that $K_n \to K$ and $L_n \to L$ for $n \to \infty$. Furthermore, let $f_n = f_{K_n,L_n}$. In the topology of $\mathcal{K}$ the convergence of $(K_n)_{n\in N}$ and $(L_n)_{n\in\mathbb{N}}$ to K and L, respectively, means that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f_\infty = f_{K,L}$. Let $A_n = Fix(f_n)$ for $n \in \mathbb{N} \cup \{\infty\}$. According to 2.5 these sets are non-empty, and because $K \neq L$, almost all of these sets contain at most two points.

Let $x_n \in A_n$ for $n \in \mathbb{N}$ and let x be an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. Since the f_n's converge uniformly to f_∞, we find that $x \in A_\infty$. In particular, when A_∞ consists of a single point a_∞, then, as S^2 is compact, each sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in A_n$ for all $n \in \mathbb{N}$ must converge to a_∞. This shows that $(A_n)_{n\in\mathbb{N}}$ converges to A_∞ with respect to the Hausdorff metric on the closed subsets of S^2.

We now consider the case that K and L intersect in two points. Let $K \cap L = \{p, q\}$ and let U be a compact neighbourhood of $p = (p_1, p_2)$ which does not contain q. For a set V we denote the interior of V by V° in the sequel. We further choose compact neighbourhoods U_i of p_i in S^2 such that $U_1 \times U_2 \subseteq U^\circ$ and such that $f_K(U_1), f_L(U_1) \subseteq U_2^\circ$. For later use we also require that U_1 is contractible to p_1 in U_1. Let R_1 denote the boundary of U_1 and let $R = R_1 \times U_2$. Since U_1 is compact and because $p, q \notin R$, the number

$$r = inf\{d_0(f_K(x), f_L(x)) | \; x \in R_1\}$$

is strictly positive. Let $0 < \delta < min(\frac{r}{3}, d_0(f_K(U_1), S^2 \setminus U_2), d_0(f_L(U_1), S^2 \setminus U_2))$. We finally define

$$A_\delta = \{(x, y) \in P \mid d_0(y, f_K(x)) < \delta \text{ and } d_0(y, f_L(x)) < \delta\}.$$

Obviously, we have $A_\delta \cap R = \emptyset$.

We choose points $z_3 \in K$ and $z_4 \in L$ such that $z_1 = p$, $z_2 = q$, z_3, and z_4 are pairwise distinct. Moreover, we choose path connected neighbourhoods V_i of z_i for $i = 1, 2, 3, 4$ such that $W = \alpha(V_1, V_2, V_3) \subseteq B_\delta(K)$ and $W' = \alpha(V_1, V_2, V_4) \subseteq B_\delta(L)$, where $B_\delta(K) = \{K' \in \mathcal{K} \mid d_+(K, K') < \delta\}$ and analogously for $B_\delta(L)$. Since α is open, the sets W and W' are neighbourhoods of K and L respectively.

Let $K' \in W$ and $L' \in W'$ be arbitrary circles. Then there are suitable points $w_1, w_2, w_3 \in K'$ and $w_1', w_2', w_4' \in L'$ such that $w_i, w_i' \in V_i$. Also there are paths $c_i : [0, 1] \to V_i$ in V_i from w_i to z_i and similarly paths $c_i' : [0, 1] \to V_i$ in V_i from w_i' to z_i. We define

$$K_t = \alpha(c_1(t), c_2(t), c_3(t)) \quad \text{and} \quad L_t = \alpha(c_1'(t), c_2'(t), c_3'(t))$$

for $t \in [0,1]$. By construction we have $K_t \in W$ and $L_t \in W'$ for all t. Hence $K_t \cap L_t \subseteq A_\delta$ and $K_t \cap L_t \cap R = \emptyset$. The latter condition shows that $f_t = f_{K_t,L_t}$ has no fixed point in R_1. However, $f_0 = f_{K',L'}$ and $f_1 = f_{K,L}$; so by the homotopy axiom of the fixed point index i as defined in [2] we obtain

$$i(S^2, f_{K',L'}, U_1^\circ) = i(S^2, f_{K,L}, U_1^\circ).$$

If we can show that $i(S^2, f_{K,L}, U_1^\circ) \neq 0$, then $i(S^2, f_{K',L'}, U_1^\circ) \neq 0$ too and according to [2, Cor. IV.A.1] the homeomorphism $f_{K',L'}$ has a fixed point in U_1°, that is, K' and L' intersect in a point of $U_1 \times U_2 \subseteq U^\circ$. Interchanging the roles of p and q we furthermore find that for arbitrary neighbourhoods U_p and U_q of p and q, respectively, any circles K', L' in sufficiently small neighbourhoods of K and L respectively intersect each other at one point of U_p and at one point of U_q. Thus intersecting is continuous at (K, L).

To show that $i(S^2, f_{K,L}, U_1^\circ) = 1$ we choose a point $z = (x, y) \in P$ non-parallel to neither p nor q. Furthermore, we choose disjoint simple paths c_K, c_L in S^2 from $f_K(x)$ to p_2 and from $f_L(x)$ to $\pi_2(q)$. We define

$$M_t = \alpha(p, q, (x, c_K(t))) \quad \text{and} \quad N_t = \alpha(p, q, (x, c_L(t)))$$

for $0 \leq t < 1$. As $M_t \cap N_t = \{p, q\}$, no homeomorphism $h_t = f_{M_t,N_t}$ has a fixed point in R_1. By the homotopy axiom of the fixed point index i we therefore infer

$$i(S^2, f_{K,L}, U_1^\circ)) = i(S^2, h_t, U_1^\circ) \tag{1}$$

for all $0 \leq t < 1$ — note that $M_0 = K$ and $N_0 = L$ and thus $h_0 = f_{K,L}$. Let $Z_i \subseteq U_i^\circ$ be compact neighbourhoods of p_i for $i = 1, 2$. We claim that there is a $t' < 1$ such that $f_{M_{t'}}(U_1) \subseteq Z_2$ and such that also $f_{N_{t'}}^{-1}(U_2) \subseteq Z_1$. To match the notation in the coherence axiom (K4) in 2.4 we pass over to sequences. Let $t_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$, so that $(t_n)_{n\in\mathbb{N}}$ converges to 1 as n tends to infinity. For brevity we write $M_n = M_{t_n}$ and $N_n = N_{t_n}$. According to (K4), when n tends to infinity, $M_n \setminus \{p\}$ and $N_n \setminus \{q\}$ tend pointwise to $|p|_+ \setminus \{|p|_+ \cap |q|_-)\}$ and $|q|_+ \setminus \{|p|_- \cap |q|_+\}$ respectively. Hence for each $u \in U_1$ there is a neighbourhood W_u of u and a positive integer n_u such that $|w|_- \cap M_n \in U_1 \times Z_2$ for all $w \in W_u$ and all $n \geq n_u$. Since U_1 is compact, it can be covered by finitely many of the neighbourhoods W_u. Thus there is a positive integer n_1 such that $|u|_- \cap M_n \in U_1 \times Z_2$ for all $u \in U_1$ and all $n \geq n_1$, i.e. $f_{M_n}(U_1) \subseteq Z_2$ for all $n \geq n_1$. We similarly find a positive integer n_2 such that $f_{N_n}^{-1}(U_2) \subseteq Z_1$ for all $n \geq n_2$. So, if we let $n = max(n_1, n_2)$, we can take $t' = t_n$. For this t' we have $h_{t'}(U_1) \subseteq f_{N_{t'}}^{-1}(Z_2) \subseteq f_{N_{t'}}^{-1}(U_2) \subseteq Z_1 \subseteq U_1^\circ$.

Let $k : U_1 \times [0,1] \to U_1$ be a contraction from U_1 to p_1 within U_1. Since S^2 has the homotopy extension property, we can extend $h_{t'} \circ k$ to a homotopy $H : S^2 \times [0,1] \to S^2$. The restriction $H_s = H|_{S^2 \times \{s\}}$ has no fixed point in the boundary R_1 of U_1 because for $r \in R_1$ we have $H_s(r) = H(r,s) = h_{t'} \circ k(r,s) \in h_{t'}(U_1) \subseteq Z_1 \subseteq U_1^\circ$ by constuction of H. So

$$i(S^2, h_{t'}, U_1^\circ) = i(S^2, H_1, U_1^\circ) \tag{2}$$

as $H_0|_{U_1} = h_{t'}$. But $H_1|_{U_1}$ is the constant map $c : z \mapsto p_1$. By the localisation axiom of the fixed point index we therefore infer

$$(3) \qquad i(S^2, H_1, U_1^\circ) = i(S^2, c, U_1^\circ) = 1$$

From (1)–(3) it finally follows that $i(S^2, f_{K,L}, U_1^\circ) = 1$.

2.7. *A homeomorphism.*

Let $K \in \mathcal{K}$ and let $x, y, z \in P$ such that $x \in K$, $y \notin K$, and such that z is neither parallel to x nor to y. Furthermore, let L be the circle through y which touches K at x and let $y_+ = |y|_+ \cap K$ and $y_- = |y|_- \cap K$. We consider the map $\phi : K \to |z|_+$ defined by

$$u \mapsto \begin{cases} |z|_+ \cap |x|_-, & \text{for } u = y_+ \\ |z|_+ \cap |y|_-, & \text{for } u = y_- \\ |z|_+ \cap \alpha(u,x,y), & \text{for } u \in K \setminus \{x, y_+, y_-\} \\ |z|_+ \cap L, & \text{for } u = x. \end{cases}$$

Obviously, ϕ is a bijection. Since α is continuous and because of the coherence axiom (K4), the restriction ϕ' of ϕ to $K \setminus \{x\}$ is continuous. Hence ϕ' is a homeomorphism from $K \setminus \{x\} \cong \mathbb{R}^2$ onto $|z|_+ \setminus \{|z|_+ \cap L\} \cong \mathbb{R}^2$ by the invariance of domain (cf. [4, 18.9]). But K and $|z|_+$ are both homeomorphic to S^2, that is, they are the one-point-compactifications of $K \setminus \{x\}$ and $|z|_+ \setminus \{|z|_+ \cap L\}$ respectively. Being a homeomorphism ϕ' is a proper (or perfect) map. Therefore ϕ' has a continuous extension onto the respective one-point-compactifications (cf. [3, Problem XI.8.12]). However, this extension is ϕ and so ϕ is a homeomorphism from K onto $|z|_+$.

In particular, given a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in K \setminus \{x\}$ and which converges to x, then the circles $\alpha(x_n, x, y)$ converge to L for $n \to \infty$. This is a special case of the coherence axiom (K1) below.

2.8. *The coherence axiom* (K1).

This coherence axiom has the following form, compare [9, 2.1].

(K1) Let $x, y, x_n, y_n, z_n \in P$, for $n \in \mathbb{N}$, be points and let $K, K_n \in \mathcal{K}$ be circles such that $x \not\parallel y$, x_n, y_n, z_n are pairwise non-parallel for all $n \in \mathbb{N}$, $x_n, z_n \in K_n$ and $x_n \to x, y_n \to y, z_n \to x$, $K_n \to K$ for $n \to \infty$. Then the circles $\alpha(x_n, y_n, z_n)$ through x_n, y_n, z_n converge to the circle through y which touches K at x for $n \to \infty$.

We essentially follow the proof in [9, 3.16]. Let $x, y \in P$, let $K \in \mathcal{K}$ such that $x \in K$ and $y \notin K$, and let L be the circle through y which touches K at x. To prove (K1) we show that given a neighbourhood V_L of L we can find neighbourhoods U_x, U_y, and V_K of x, y, and K, respectively, such that $\alpha(u, v, w) \in V$ for all $u \in U_y$ and $v, v' \in U_x$ such that $v, v' \in K'$ for some circle $K' \in V_K$. To this end let

$$\mu(U_x, V_K) = \{(z, w) \in U_x \times U_x |\ z, w \in K' \text{ for some } K' \in V_K\} \quad \text{and}$$
$$\nu(U_y, U_x, V_K) = \{\alpha(u, v, w) |\ u \in U_y, (v, w) \in \mu(U_x, V_K)\}$$

for neighbourhoods U_y, U_x, and V_K of y, x, and K, respectively.

Let V_L be a neighbourhood of L and let $p = (p_1, p_2) \in L \setminus \{x, y\}$. Since α is open and because the restiction $\hat{\alpha}$ of α (see 2.3) is a homeomorphism, we can find parallel-disjoint neighbourhoods U'_x and U'_y of x and y respectively and a neighbourhood $U \subseteq |p|_+$ of p_1 such that $\alpha(U'_y, U'_x, U) \subseteq V_L$ is a neighbourhood of L. Let R denote the boundary of U in $|p|_+$. Because R is compact and does not contain the point p, we can find neighbourhoods $U''_x \subseteq U'_x$, $U''_y \subseteq U'_y$, and V'_K of x, y, and K, respectively, such that no circle in $\nu(U''_y, U''_x, V'_K)$ intersects R. (Otherwise one can construct a sequence of points and circles as in (K1) such that $(\alpha(x_n, y_n, z_n))_{n\in\mathbb{N}}$ converges to a circle $L' \neq L$ through x, y, and a point of R contrary to the continuity of intersection.)

Let $x = (x_1, x_2)$ and let U_i be a neighbourhood of x_i in S^2 such that $U_1 \times U_2 \subseteq U''_x$. Since the evaluation map $\epsilon : (t, C) \to f_C(t)$ is continuous, there is a neighbourhood $U'_1 \subseteq U_1$ of x_1 and a neighbourhood V''_K of K such that U'_1 is homeomorphic to the open unit disc D and such that $\epsilon(U'_1, V''_K) \subseteq U_2$. We set $U_x = U'_1 \times U_2$ and we choose a connected neighbourhood $U_y \subseteq U''_y$ of y. Finally, we choose three pairwise non-parallel points $t_i \in U_x$, $i = 1, 2, 3$, and open connected neighbourhoods $W_i \subseteq |t_i|_+ \cap U_x$ of t_i such that $V_K = \{\alpha(w_1, w_2, w_3) |\ w_i \in W_i\}$ is contained in V''_K. We claim that $\nu(U_y, U_x, V_K) \subseteq V_L$ for U_x, U_y, and V_K as above.

We first show that $\mu(U_x, V_K)$ is connected. Fix a circle $K' \in V_K$. Then $U_x \cap K' \setminus \{z\}) \cong U'_1 \setminus \{\pi(z)\} \cong D \setminus \{0\}$ is connected for every $z \in U_x \cap K'$ and

$$\begin{aligned} M(U_x, K') &= \{(z, w) \in U_x \times U_x |\ z, w \in K'\} \\ &= \bigcup_{z\in U_x} \{z\} \times (U_x \cap K' \setminus \{z\}) \\ &= \bigcup_{w\in U_x} (U_x \cap K' \setminus \{w\}) \times \{w\} \end{aligned}$$

is the union of connected sets. Furthermore, since $\{z\} \times (U_x \cap K' \setminus \{z\})$ and $(U_x \cap K' \setminus \{w\}) \times \{w\}$ intersect at (z, w) if $z \neq w$, it follows that $M(U_x, K')$ is connected. By construction of V_K we have that the connected set $W_1 \times W_2$ intersects each $M(U_x, K')$. Thus $\mu(U_x, V_K) = \bigcup_{K'\in V_K} M(U_x, K')$ is connected. But then $\nu(U_y, U_x, V_K)$ is also connected by the continuity of α.

Since $\nu(U_y, U_x, V_K) \subseteq \nu(U''_y, U''_x, V'_K)$, no circle in $\nu(U_y, U_x, V_K)$ intersects R. According to 2.7, the set $\nu(U_y, U_x, V_K)$ contains circles that intersect $|p|_+$ in U and by connectedness we finally obtain $\nu(U_y, U_x, V_K) \subseteq \alpha(U_y, U_x, U) \subseteq \alpha(U'_y, U'_x, U) \subseteq V_L$.

2.9. *The continuity of touching.*

Let $x, x_n, y, y_n \in P$ be points and let $K, K_n \in \mathcal{K}$ be circles such that $x_n \in K_n$ for all n, such that $x \in K$, and such that $x_n \to x$, $y_n \to y$, and $K_n \to K$ for $n \to \infty$. Let L_n and L be the circle through y_n which touches K_n in x_n and let L be the circle through y which touches K in x. We have to prove that L_n tends to L for $n \to \infty$.

By the coherence axiom (K1), see 2.8, we can choose a point $z_n \neq x_n, y_n$ on K_n sufficiently close to x_n such that $d_+(\alpha(z_n, x_n, y_n), L_n) < \frac{1}{n}$. Then $(z_n)_{n\in\mathbb{N}}$

converges to x. Applying (K1) a second time we find that $\alpha(x_n, y_n, z_n)$ converges to L as n tends to ∞. Hence, given $\delta > 0$, there is an integer $n_0 > \frac{2}{\delta}$ such that $d_+(\alpha(z_n, x_n, y_n), L) < \frac{\delta}{2}$ for all $n \geq n_0$. The triangle inequality finally yields

$$d_+(L_n, L) \leq d_+(L_n, \alpha(x_n, y_n, z_n)) + d_+(\alpha(x_n, y_n, z_n), L) < \delta.$$

Thus $(L_n)_{n\in\mathbb{N}}$ converges to L. This shows that touching is continuous.

2.10. *Conclusion.*

The continuity of the geometric operations is proved in 2.1, 2.2, 2.6, and 2.9 asuming the continuity of joining. Thus $\mathcal{M}$ becomes a topological Minkowski plane, which indeed is compact and 4-dimensional.

As already mentioned in the introduction there are candidates for 4-dimensional Minkowski planes. Those candidates are in standard representation and it is easy to see that circles are indeed compact. So, with the present theorem, it suffices to verify the geometric axioms of a Minkowski plane for those candidates and the continuity of joining.

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