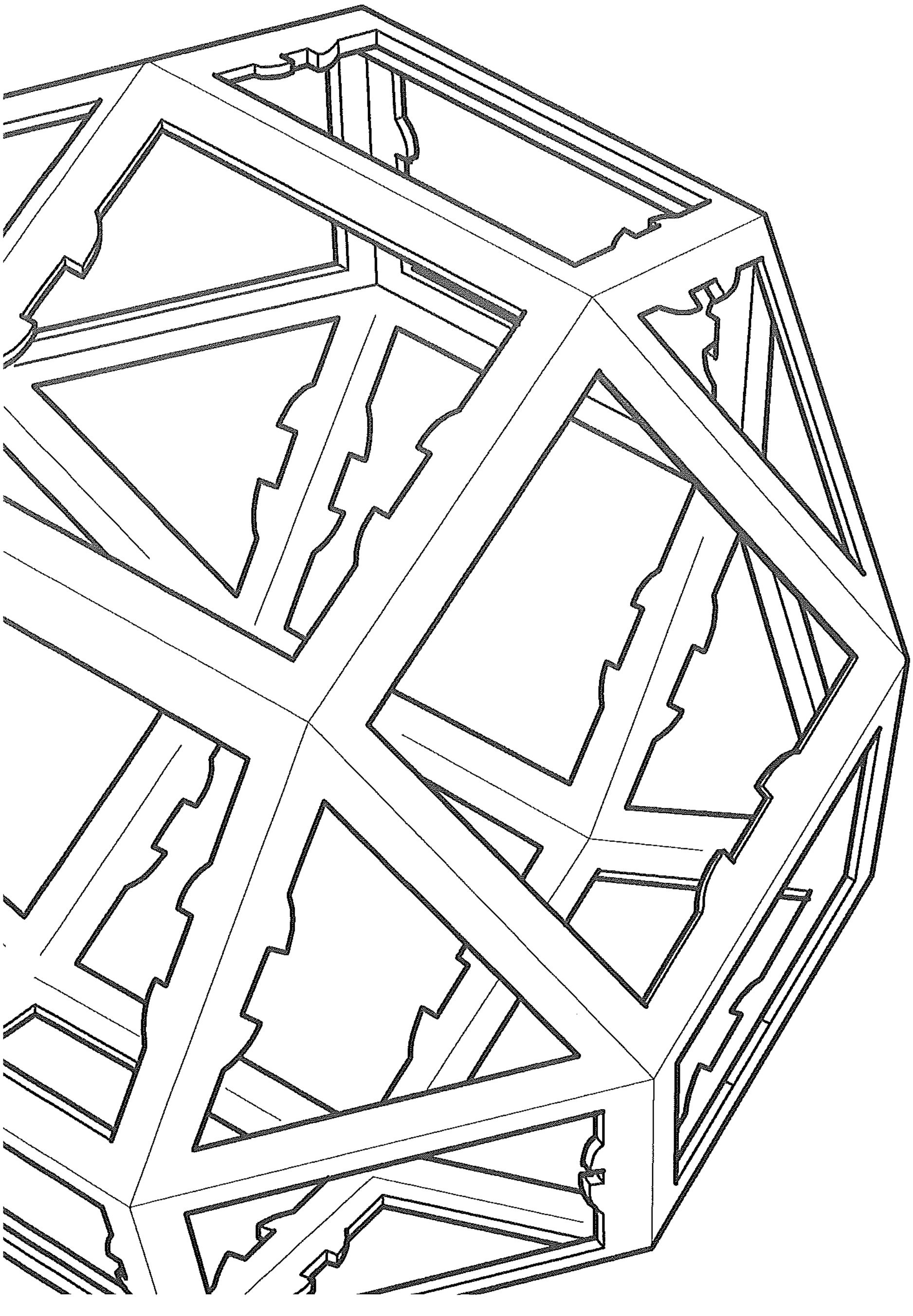
Department of Mathematics and Statistics
College of Engineering

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

Summer Research Project

Negation Incompleteness and Paraconsistency

Ty Baen



11/12

Negation Incompleteness and Paraconsistency

Ty Baen

February 7, 2012

1 What's a formal logical system?

A formal axiomatised theory, as set out by Peter Smith[1], has three components: a formalised language, a decidable set of axioms, and a formalised proof-system. The language must consist of a finite set of symbols and have a defined syntax, according to which it is decidable what is a term or sentence of the language. The set of axioms will be a subset of the set of sentences constructable in the language, and it needs to be decidable what's an axiom. We must have a proof system whereby we can derive theorems from the axioms, and it must be decidable whether a given array of sentences does indeed constitute a proof in the theory.

2 What do we do with formal theories?

In the broader context, formal theories enable us to very precisely reason about reason. It is possible to strip away the distracting semantic consequences of our thoughts and arguments, and look to see what all arguments have in common. In a narrower context, formal systems serve as useful models of aspects of our world that we can learn from. It is possible to build on our deductions and to see the sometimes novel and counter-intuitive conclusions our reasoning can lead us to. Alonzo Church, Andrey Markov, and Alan Turing independently created formal (equivalent) models of computation, long before the rise of the personal computer, which eloquently tell us exactly where we don't need to waste the resources of the human race, for example, solving the Halting Problem, or any other problem which is equivalent to it, of which there are many. Formal theories enable us to reason about reason, and to reason about the world.

2.1 What is negation-completeness of a theory?

A theory in which it is decidable about every sentence expressible in the language of the theory whether that sentence is a theorem of the system, i.e. the theory must either prove p or $\neg p$ for every p, would be considered negation-complete.

3 What did Gödel prove?

Gödel proved in his historic 1931 paper that a formal system with these three desirable properties cannot be negation complete:

1. The law of the excluded middle.

2. The Law of Non-Contradiction.

3. A primitive recursive definition of provability.

Gödel developed a method of encoding statements that we call today Gödel Numbering, whereby he was able to effectively translate sentences in the language of the system he was talking about into statements of the the language itself. By a clever trick using the primitive recursive functions, he was then able to translate sentences in the system's metalanguage into sentences of the language, effectively causing the system to talk about itself. Using this method, he constructed what we now call a Gödel Sentence, which essentially says "this statement is unprovable". This showed that theories with the aforementioned properties will always have sentences which, while appearing true and coherent from the outside, cannot be derived either in their assertive or negative form using the proof-system of the theory.

3.1 Why might we want the Law of the Excluded Middle?

The Law of the Excluded Middle, $p \vee \neg p$, simply asserts that every question has an answer, and that such answers must be taken from the realm of values available to a logical theory. Classically, these values are **true** and **false**, although the law could be extended to cover the requirements of a many-valued logic. Since the disjunction is inclusive, this axiom does not classically rule out the possibility of both p and $\neg p$ being given the same value.

3.2 Why might we want the Law of Non-Contradiction?

The Law of Non-Contradiction *prima facie* seems to be a perfectly reasonable assumption about the way things should work. Intuitively, every statement is true or false, and no statement should be both true *and* false. For example, take the statement "I am in the room," it seems clear that it has one answer and one answer only. Formal systems, however, give us the ability to reason *beyond* our intuitions of how things should work, and to explicitly put our intuitions aside and study a system *as it actually is.* Logic is a science.

3.2.1 Why might we want to look at the alternative of excluding the Law of Non-Contradiction from consideration?

The Law of Non-Contradiction, classically, gives rise to the Principle of Explosion, $p \wedge \neg p \rightarrow q$, the consequence of any contradiction being reached. This statement, which is typically understood to mean "if the logic is false, anything is true" seems the prime justification for including the Law of Non-Contradiction in a logical theory. Theories which exclude the Principle of Explosion are called Paraconsistent Logics.

Many people have given good accounts of the motivations for studying a paraconsistent logic. Priest and Routley[5] say that since there are interesting, non-trivial, paraconsistent theories, the study of a paraconsistent reasoning is a necessity to understand them. An inconsistent database is often cited as an example of such a system, or the wave-particle duality of quantum mechanics.

The move to construct Logics of Relevance in which the statement "the moon is made of green cheese, therefore it is either raining or not in Ecuador"[9] is not a valid inference because the consequent is simply *not relevant* to the antecedent, necessarily excludes the Principle of Explosion. I find the argument from relevance to be the more moving of the two, since it could conceivably be a matter of some debate as to the actual existence of interesting inconsistent physical or logical systems.

3.3 Why might we want a recursive definition of proof?

In considering the provability of statements to be a purely syntactic endeavour, accomplished only through the manipulation of symbols according to the rules of the theory, we give ourselves an explicit definition of proof. In being so explicit, we free ourselves from the liability of needing to consider the semantic consequences of the system in order to be able to tell what the system says. If a sentence is derivable through a purely mechanical application of system's rules of inference to the axioms and theorems of the system, then there is no room for uncertainty, and no need for recourse to the potentially troubling notion of truth.

4 Primitive recursive functions

More central to the notion of incompleteness than specific axioms seems to be the definition of proof as a primitive recursive function. A primitive recursive function is a function that is defined either by recursion or by the composition of other p.r. functions. For example, the addition function is recursively defined in terms itself, and is composed with the successor function S(x) and the identity function(=):

$x + 0 = x$

$x + Sy = S(x + y)$

Classically, it is provable that in any (classical) theory in which the successor, addition, and multiplication functions can be defined, barring some axiom which forbids it, all other primitive recursive functions can be defined. If all other p.r. functions can be constructed in a theory, then in particular the characteristic function can be defined. The characteristic function is simply a function which partitions a domain into two sets: the objects with a particular property, and the objects without the property. It is from this function that p.r. properties and relations can be defined, and it is in this way Gödel constructed the provability relation which is the basis for the Gödel sentence "this statement is not provable".

5 Paraconsistent Reasoning

A classical line of reasoning, starts from some sentences (premises) and proceeds by transforming sentences by applying the rules of inference. If a contradiction is reached and all the transformations occurred via valid applications of the rules, then the premises are deemed to be inconsistent and the line of reasoning halts.

A paraconsistent line of reasoning, however, would utilise the fact that there's still an unbroken line of truth, from the premises to the contradiction, and be able to continue through the contradiction and see what is capable of being drawn out.

Meyer claims[8] that since all Peano Arithmetic theorems are theorems of the theory R#, and the entire set of primitive recursive functions are capturable in PA, R# contains all the primitive recursive functions, and is therefore subject to the construction of a Gödel sentence, and hence negation-incomplete. From this conclusion it seems that even paraconsistent theories which hope to perform arithmetic are subject to Gödel's Incompleteness Theorem.

As I have outlined above, not even the ability to introduce a contradiction into the definitional chain of p.r. functions to a paraconsistent theory through the addition of some limiting axiom would be able to prevent the construction of the full chain of p.r. functions, as while there would both be and not be some functions, there would still *be* those functions. At the final interesting level, we'll either be able to construct a Gödel sentence, or we will be able to construct both the sentence and its negation, although how we are to interpret that result is unclear.

The last real hope against incompleteness is that there may be some problem in defining the primitive recursive functions themselves, due to a fundamental inability to express them with the paraconsistent connectives. At this time Zach Webber of Otago University is working on that project; preliminary results suggest the p.r. functions are indeed constructable.

6 Further Inquiry

6.1 The Law of the Excluded Middle in multi-valued logic

A theory is said to be sound if its logic is truth-preserving. This means that any transformation of a true statement using the rules of inference along a line of reasoning must result in another true statement. Soundness is generally a property of two-valued logics, although the concept does generalise to multi-valued logics. Ignoring the notion of *truth*, what is really being propagated is the *value* assigned to a given statement. In Mortensen's RM3[4] those values would be **true**, **false**, or **both** (RM3 is a paraconsistent theory). In a four-valued logic, it's natural to denote the fourth value as **neither**.

With respect to multi-valued logics, it seems a more complete account of incompleteness should be the inability for a logical system to give a value to some expressible statement. A four valued LEM may well be better expressed as **provable, the negation is provable, both the negation and the assertion are provable**, and **neither**. In such an inconsistent system it would be derivable that neither a statement nor its negation were able to be given a value by the system. It may of course be that the **both** and **neither** values are more properly *meta-values*, but that consideration is certainly beyond the scope of this paper.

6.2 Wff is not a real property

Another related line of thought concerns the Provability relation and its constituent relation, Wff (for well-formed formula). It seems to me that Wff is a nonsensical predicate, for two reasons:

1. Every sentence in the language is Wff. This is true by definition, since a string of symbols is not a sentence in the language unless it conforms to the syntactical rules of the language, in other words, it's well-formed.
2. The sentence $\neg Wff(\varphi)$ is classically unconstructable. It is simply not possible to substitute φ with anything that would yield a true sentence, since anything substituted for φ is by definition well-formed.

The Provability predicate may still be constructed by ignoring the Wff component, however, since it always returns "yes". It may be possible to construct a theory in which the language was the closure of the alphabet, and had axioms concerning the structure of sentences the theory was interested in examining (wffs). Then the Wff predicate (of a sort) would certainly apply (if constructable), and since the theory would know the rules for construction of sentences, it would know the rules of the characteristic function (hidden deep inside) which would enable the construction of a provability predicate which wouldn't need to rely upon information being fed into the system from outside as I will describe below.

6.3 The applicability of the Characteristic Function in terms of primitive recursion

It seems to me that there is a fundamental difference between a p.r. function like *addition* and the *characteristic* function. I understand the characteristic function is decidable, or rather, any property we care to characterise is theoretically decidable. It's decidable what's a proof because it's decidable what's a wff. How is it decidable what's a wff, though? *We* decide, and tell the system though the characteristic function. The theory PA can quite easily decide whether 2+2=4 or not. A theory like PA is incapable of even representing a non-wff, so it cannot make decisions about what is or is not a wff. It seems to me no surprise that having handed the system information it would not otherwise be in possession of, we encounter an area further down the logical stream where the system again cannot make a decision.

If the characteristic function is only provably p.r. from the bottom up, by proving it's decidable and computable with a bounded search (and that all such functions are p.r.), then perhaps there may be two classes of p.r. functions. If it is indeed provable from the top down, i.e. that the characteristic function itself is p. r., via the construction of its definitional chain showing how it is defined from the initial functions through primitive recursion and composition, then perhaps as I have suggested above all that is needed is to give the deciding ability to the theory, by including in its axioms the syntactic rules of the sentences you are interested in examining.

References

[1] Smith, Peter. "Gödel Without Tears". University of Cambridge. 2011

[2] Priest, Graham. "An Introduction to Non-Classical Logic". University of Cambridge. 2001

[3] Rescher, Nicholas and Robert Brandom. "The Logic of Inconsistency". 1979

[4] Mortensen, Chris. "Inconsistent Mathematics". Kluwer Academic Publishers. 1995

[5] Priest, Graham, Richard Routley, and Jean Norman, eds. "Paraconsistent Logic: Essays on the Inconsistent". Philosophia Verlag. 1989

[6] McKubre-Jordens, Maarten and Zach Webber. "Real Analysis in Paraconsistent Logic". Springer. 2011

[7] McKubre-Jordens, Maarten. "This is Not a Carrot: Paraconsistent Mathematics". http://plus.maths.org/content/not-carrot

[8] Friedman, Harvey, and Robert K. Meyer. "Whither Relevant Arithmetic?" The Journal of Symbolic Logic. Volume 57, Issue 3, pp. 824 - 831

[9] Mares, Edwin. "Relevance Logic". The Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/archives/win2011/entries/logic-relevance/