LENZ-BARLOTTI CLASSES OF SEMI-CLASSICAL ORDERED PROJECTIVE PLANES

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No. 156 September, 1997

To appear in Australasian Journal of Combinatorics

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ABSTRACT. This paper concerns a generalization of Moulton planes. We consider these semi-classical projective planes over half-ordered fields and completely determine their Lenz-Barlotti classes in the case of finite planes and of ordered planes. We also obtain a characterization of the Desarguesian planes among the semi-classical planes in terms of linear transitivity. These results are applied to (topological) 2-dimensional semi-classical projective planes.

1. Introduction, notation and examples

In [4] J. Jakóbowski constructed a family of affine planes that generalize Pierce's construction [8] of Moulton planes. The author dualized these planes in [18] and solved the isomorphism and collineation problem posed in [4]. Both types of projective planes are defined over *half-* (or *pseudo-*) *ordered fields*, that is, fields $\mathbb{F}$ with a multiplicative subgroup $\mathbb{P}$ of index two. In particular, $\mathbb{P}$ contains all non-zero squares of $\mathbb{F}$ so that a finite half-ordered field cannot have characteristic two. Elements of $\mathbb{P}$ and of the other coset $\mathbb{N}$ of non-zero elements are called *positive* and *negative* respectively. We use the familiar notation $x > 0$ and $x < 0$ for $x \in \mathbb{P}$ and $x \in \mathbb{N}$ respectively. For finite fields $\mathbb{F} = GF(q)$, the Galois field of order q, $\mathbb{P}$ consists precisely of the non-zero squares of $\mathbb{F}$. A half-ordered field is called an *ordered field*, if $\mathbb{P}$ is closed under addition. In particular, such fields have characteristic zero and -1 is negative.

A mapping f from a half-ordered field $\mathbb{F}$ into itself is called *order-preserving* or *order-reversing* if and only if $(f(x)-f(y))(x-y)^{-1} > 0$ or $(f(x)-f(y))(x-y)^{-1} < 0$, respectively, for all distinct $x, y \in \mathbb{F}$. Note that every order-preserving or order-reversing mapping is injective. Standard examples for order-preserving or order-reversing mappings on $\mathbb{R}$ with the Euclidean ordering are the strictly increasing

1991 *Mathematics Subject Classification.* 51A35, 51G05.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and strictly decreasing functions respectively. For finite fields $\mathbb{F}$ L. Carlitz showed in [2] that the only order-preserving or order-reversing mappings are permutations of the form $x \mapsto a\alpha(x) + b$ for $a, b \in \mathbb{F}$, $a \neq 0$, where α is an automorphism of $\mathbb{F}$. Such a permutation is order-preserving if and only if $a > 0$ (i.e., a is a nonzero square in $\mathbb{F}$) and it is order-reversing if and only if $a < 0$.

Given two permutations h, g of a half-ordered field $\mathbb{F}$ that are either both order-preserving or both order-reversing the incidence structure $\mathcal{P}_{h,g}(\mathbb{F})$ is constructed as follows. The point set is

$$\mathbb{F} \times \mathbb{F} \cup \{(m) \mid m \in \mathbb{F}\} \cup \{(\infty)\}$$

where (m) and (∞) are new symbols not contained in $\mathbb{F}$; lines are the vertical lines

$$L_c = \{(c, y) | \ y \in \mathbb{F}\} \cup \{(\infty)\}$$

for $c \in \mathbb{F}$ and the non-vertical lines

$$\begin{aligned} L_{m,t} = &\{(x, mx + t) | \ x \in \mathbb{F}, x \geq 0\} \\ &\cup \{(x, g^{-1}(h(m)g(x) + g(t))) | \ x \in \mathbb{F}, x < 0\} \cup \{(m)\} \end{aligned}$$

for $m, t \in \mathbb{F}$, and the line at infinity

$$L_\infty = \{(m) \mid m \in \mathbb{F}\} \cup \{(\infty)\}.$$

We denote the collection of all lines through a point p by $\mathcal{L}_p$.

Without loss of generality we may assume that g and h both belong to the collection $\Pi^+_{0,1}(\mathbb{F})$ of all order-preserving permutations of $\mathbb{F}$ which fix 0 and 1; cf. [18, §1]. Then $\mathcal{P}_{h,g}(\mathbb{F})$, $g, h \in \Pi^+_{0,1}(\mathbb{F})$, is a projective plane if and only if

(1) each function $x \mapsto g(ax + b) + ch(-x)$ from $\mathbb{F}$ to itself is surjective for all $a, b, c \in \mathbb{F}$, $c < 0 < a$;

cf. [18, Theorem 1.1] or [4, Theorem 1].

Note that the mapping defined in (1) is a permutation of $\mathbb{F}$. The injectivity follows from the fact that h and g are order-preserving. Furthermore, $\mathcal{P}_{h,g}(\mathbb{F}) = \mathcal{P}_{\alpha h, \alpha g}(\mathbb{F})$ for each order-preserving automorphism α of $\mathbb{F}$. Let $\Pi^{+1}_{0,1}(\mathbb{F})$ be the collection of all pairs (g, h) of order-preserving permutations g, h of $\mathbb{F}$ that fix 0 and 1 and that satisfy (1).

In the usual coordinatization of a projective plane with respect to the frame

$$v = (\infty), \ u = (0), \ o = (0, 0), \text{ and } e = (1, 1)$$

(see [7, 1.5]) the ternary operation is given by

$$\tau(a, x, b) = \begin{cases} ax + b, & \text{if } x \geq 0 \\ g^{-1}(h(a)g(x) + g(b)), & \text{if } x < 0 \end{cases}.$$

Thus non-vertical lines can be described as $\{(x, \tau(a, x, b)) \mid x \in \mathbb{F}\} \cup \{(a)\}$ for $a, b \in \mathbb{F}$.

For $\mathbb{F} = \mathbb{R}$ the planes $\mathcal{P}_{h,g}(\mathbb{R})$ are isomorphic to the planes $\mathcal{P}_{h,g}$ constructed by the author in [17]; cf. [17, 2.3]. Condition (1) from above is satisfied for any two $g, h \in \Pi^+_{0,1}(\mathbb{R})$; see [4, §2 Prop. 1]. We call these planes semi-classical projective planes because the geometries and topologies on $A_+ = \mathbb{R}^+ \times \mathbb{R}$ and $A_- = \mathbb{R}^- \times \mathbb{R}$ are the same as on the corresponding subsets of the (topological) real Desarguesian projective plane.

We also call the planes $\mathcal{P}_{h,g}(\mathbb{F})$ *semi-classical projective planes* since the geometries induced on $A_+ = \mathbb{P} \times \mathbb{F}$ and $A_- = \mathbb{N} \times \mathbb{F}$, where $\mathbb{P}$ and $\mathbb{N}$ denotes the set of positive and negative elements of $\mathbb{F}$ respectively, are the same as on the corresponding subsets of the Desarguesian plane over $\mathbb{F}$. We call A_+ and A_- the *positive* and *negative half-plane* respectively.

Furthermore, if $\mathbb{F}$ is an ordered field the induced order-topologies on A_+ and A_- are the same as on the corresponding subsets of the Desarguesian plane.

The aim of this paper to determine the Lenz-Barlotti classes of the semi-classical projective planes. We say that a projective plane is (p, L)*-transitive*, where p is a point and L is a line, if and only if the group of all central collineations with centre p and axis L is transitive on each central line minus p and the intersection with L. With this notation the Lenz-Barlotti class of a projective plane can be found from the configuration of all point-line pairs (p, L) for which the projective plane admits a linearly transitive group of central collineations with centre p and axis L. A complete list of possible Lenz-Barlotti classes of projective planes can be found in [7, Anhang §6], see also [3], [21] and [22]. The Lenz-Barlotti classes of finite semi-classical projective planes are determined in 4.8 and Theorem 4.9 although this classification may be well-known. Eventually, however, we have to restrict ourselves to semi-classical ordered planes. In particular, Lenz-Barlotti classes of 2-dimensional (compact topological) projective planes are determined. Note that we implicitly have also determined the Lenz-Barlotti classes of the planes constructed in [4] in the respective cases since these planes are duals of our semi-classical projective planes.

Following we give simple concrete examples for each of the possible Lenz-Barlotti classes in the case of finite and of ordererd semi-classical projective planes in order to provide the reader with material to work with. However, these examples may be skipped at first. In the general case groups of linearly transitive central collineations can be described very similar to the groups given in the examples.

We begin with finite semi-classical projective planes. These can be of Lenz-Barlotti class VII.2, IV.b.3, IV.b.2 or IV.b.1, see Theorem 4.9. In order to realise these classes let $GF(9)$, $GF(27)$ and $GF(81)$ be the Galois fields of order 9, 27 and 81, respectively, and let $\alpha(x) = x^3$ be the generator of the automorphism group of these fields. Then we obtain the following Lenz-Barlotti classes.

(1) The plane $\mathcal{P}_{id,id}(GF(9))$ is Desarguesian and thus of Lenz-Barlotti class VII.2. It is well known that Desarguesian planes are linearly transitive for every point-line pair.
(2) The plane $\mathcal{P}_{\alpha,id}(GF(27))$ is of Lenz-Barlotti class IV.b.1. This plane is

(∞, L)-transitive for every line L through the point (∞). The collection of all collineations

$$(x,y) \mapsto (x, y+t),\ (m) \mapsto (m)$$

for $t \in GF(27)$ is a linearly transitive group of elations with centre (∞) and axis L_∞. For $c > 0$ the collection of all collineations

$$(x,y) \mapsto \begin{cases} (x, y + a(x-c)), & \text{if } x \geq 0 \\ (x, y + \alpha(a)x - ac), & \text{if } x \leq 0 \end{cases},\ (m) \mapsto (m+a)$$

for $a \in GF(27)$ is a linearly transitive group of elations with centre (∞) and axis L_c and for $c < 0$ the collection of all collineations

$$(x,y) \mapsto \begin{cases} (x, y + ax - c\alpha(a)), & \text{if } x \geq 0 \\ (x, y + \alpha(a)(x-c)), & \text{if } x \leq 0 \end{cases},\ (m) \mapsto (m+a)$$

for $a \in GF(27)$ is a linearly transitive group of elations with centre (∞) and axis L_c. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

(3) The plane $\mathcal{P}_{\alpha^2,id}(GF(81))$ is of Lenz-Barlotti class IV.b.2. This plane is (∞, L)-transitive for every line L through the point (∞) and (p, L_∞)- and (q, L_0)-transitive for all $p \in L_0$, $q \in L_\infty$. Linearly transitive group of elations with centre (∞) and axis L_c, $c \in GF(81) \cup \{(\infty)\}$, are described in exactly the same way as in the preceding example. Furthermore, the collection of all collineations

$$(x,y) \mapsto \begin{cases} (rx, r(y-c)+c), & \text{if } r > 0 \\ (r\alpha^2(x), r\alpha^2(y-c)+c), & \text{if } r < 0 \end{cases},\ (m) \mapsto (m)$$

for $r \in GF(81)$, $r \neq 0$, is a linearly transitive group of homolgies with centre $(0, c)$ and axis L_∞. Likewise the collection of all collineations

$$\begin{aligned} (x,y) &\mapsto \begin{cases} (rx, y + a(r-1)x), & \text{if } x \geq 0 \\ (\alpha^2(r)x, y + \alpha^2(a)(\alpha^2(r)-1)x), & \text{if } x \leq 0 \end{cases} \\ (m) &\mapsto \quad (a + \frac{m-a}{r}) \end{aligned}$$

for $r \in GF(81)$, $r > 0$, and

$$\begin{aligned} (x,y) &\mapsto \begin{cases} (\alpha^2(r)x, y + (\alpha^2(a)\alpha^2(r) - a)x), & \text{if } x \geq 0 \\ (rx, y + (ar - \alpha^2(a))x), & \text{if } x \leq 0 \end{cases}, \\ (m) &\mapsto \quad (a + \frac{\alpha^2(m-a)}{r}) \end{aligned}$$

for $r \in GF(81)$, $r < 0$, is a linearly transitive group of homolgies with centre (a) and axis L_0. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

(4) The plane $\mathcal{P} = \mathcal{P}_{\alpha,id}(GF(9))$ is of Lenz-Barlotti class IV.b.3. (In fact, this is the only projective plane of Lenz-Barlotti class IV.b.3; cf. [3] or [5].) As seen in [9] and [10] the non-Desarguesian semi-classical projective plane of order nine plays a special role. In this plane the distinguished point (∞) is still fixed but the line L_∞ can be moved. One finds that $\mathcal{P}$ is (∞, L)-transitive for every line L through the point (∞) and (p, L_c)-transitive for every point p on the vertical line $L_{\phi(c)}$ where ϕ is the involutory permutation of $GF(9) \cup \{\infty\}$ given by $\phi(\infty) = 0$, $\phi(0) = \infty$, $\phi(m) = -m$ for $m \neq 0, \infty$. There is no other point-line pair (p, L) for which $\mathcal{P}$ is (p, L)-transitive.
Since L_∞ may be moved under these central collineations, it is difficult to write down these collineations explicitly. However, if we dualise this plane we obtain a projective plane $\mathcal{P}^*$, which is coordinatised by a near-field, all of whose collineations fix the infinite line L^*_∞ (= the dual of the point (∞)); for collineations of $\mathcal{P}^*$, see [13, §4.3]. Note that $\mathcal{P}$ is of Lenz-Barlotti class IV.b.3 if and only if $\mathcal{P}^*$ is of Lenz-Barlotti class IV.a.3. Non-vertical lines of $\mathcal{P}^*$ are of the form

$$L^*_{m,t} = \{(x, mx + t) \mid x \in GF(9)\} \cup \{(m)\}$$

for $m, t \in GF(9)$, $m \geq 0$ and of the form

$$L^*_{m,t} = \{(x, mx^3 + t) \mid x \in GF(9)\} \cup \{(m)\}$$

for $m, t \in GF(9)$, $m < 0$. The translations

$$(x, y) \mapsto (x + a, y + b),\ (m) \mapsto (m)$$

for $a, b \in GF(9)$ form a group of elations with axis L_∞ such that the stabilizer of any line in $\mathcal{L}_{(c)}$ for $c \in GF(9) \cup \{\infty\}$ is linearly transitive (with centre (c)). Furthermore the collection of all collineations

$$(x,y) \mapsto \begin{cases} (r(x-c)+c, y), & \text{if } r > 0 \\ (r(x-c)^3 + c, y), & \text{if } r < 0 \end{cases}$$
$$(m) \mapsto \begin{cases} (\frac{m}{r}), & \text{if } rm > 0 \\ (\frac{m}{r^3}), & \text{if } rm < 0 \end{cases}$$

for $r \in GF(9)$, $r \neq 0$ and the collection of all collineations

$$(x, y) \mapsto (x, s(y - d) + d),\ (m) \mapsto (sm)$$

for $s \in GF(9)$, $s > 0$, and

$$(x, y) \mapsto (x, s(y - d)^3 + d),\ (m) \mapsto (sm^3)$$

for $s \in GF(9)$, $s < 0$, are linearly transitive groups of homologies with respective axes L^*_c and $L^*_{0,d}$ and respective centres (0) and (∞). Conjugation by

$$(x, y) \mapsto (x + y, s(x - y))$$
$$(m) \mapsto (-sm) \text{ for } m \in GF(9), m \neq 0, \pm 1$$

for $s > 0$ and

$$(x, y) \mapsto (x + y, s(x - y)^3)$$
$$(m) \mapsto (-sm^3) \text{ for } m \in GF(9), m \neq 0, \pm 1$$

for $s < 0$ where in both cases

$$(0) \mapsto (s), (1) \mapsto (0), (-1) \mapsto (\infty), (\infty) \mapsto (-s)$$

finally yields linearly transitive groups of homologies with centres (s) and axes through $(-s)$ for $s \in GF(9)$, $s \neq 0$.

As for ordered semi-classical planes it is shown in Theorem 5.3 that only Lenz-Barlotti classes VII.2, IV.b.1, III.2, II.1, I.2 and I.1 can occur. All but class IV.b.1 can be realized over the field of reals with the Euclidean ordering. For this purpose let μ_2 and m_3 be the mappings defined by $\mu_2(x) = x$ for $x \geq 0$ and $\mu_2(x) = 2x$ for $x \leq 0$, cf. Definition 2.1 below, and $m_3(x) = x^3$ for $x \in \mathbb{R}$. These mappings are order-preserving permutations of $\mathbb{R}$.

(5) The plane $\mathcal{P}_{id,id}(\mathbb{R})$ is of Lenz-Barlotti class VII.2. It is linearly transitive for every point-line pair.

(6) The plane $\mathcal{P}_{\mu_2,id}(\mathbb{R})$ is of Lenz-Barlotti class III.2. This plane has many interesting features; here the point (∞) can be moved and it admits the simply connected covering group of the real Lie group $\mathrm{SL}_2(\mathbb{R})$ of all 2×2 matrices of determinant 1 with real entries as a group of collineations (see [14, §34]). Since L_∞ may be moved and given the complicated nature of the covering group of $\mathrm{SL}_2(\mathbb{R})$, it again is difficult to write down these collineations explicitly. We therefore give only the two basic types of linearly transitive groups of central collineations. The collection of all collineations

$$(x, y) \mapsto (rx, y),\ (m) \mapsto (\frac{m}{r})$$

for $r > 0$ and

$$(x, y) \mapsto (r\mu_2(x), y),\ (m) \mapsto (\frac{\mu_2(m)}{2r})$$

for $r < 0$ is a linearly transitive group of homologies with centre (0) and axis L_0. The collection of all collineations

$$(x, y) \mapsto (x, y + t),\ (m) \mapsto (m)$$

for $t \in \mathbb{R}$ is a linearly transitive group of elations with centre (∞) and axis L_∞. Conjugation by suitable collineations shows that $\mathcal{P}_{\mu_2,id}(\mathbb{R})$ is also $((0), L)$-transitive for every line L through (0), cf. [1] and [14, §34]. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

(7) The plane $\mathcal{P}_{m_3,id}(\mathbb{R})$ is of Lenz-Barlotti class II.1. The collection of all collineations

$$(x,y) \mapsto (x, y+t),\ (m) \mapsto (m)$$

for $t \in \mathbb{R}$ is a linearly transitive group of elations with centre (∞) and axis L_∞. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

Note that for each $r > 0$ the collineation

$$(x,y) \mapsto \begin{cases} (rx, y), & \text{if } x \geq 0 \\ (r^3 x, y), & \text{if } x \leq 0 \end{cases},\ (m) \mapsto (\frac{m}{r})$$

is a homology with centre (0) and axis L_0. However, the plane is not $((0), L_0)$-transitive.

(8) The plane $\mathcal{P}_{m_3,m_3}(\mathbb{R})$ is of Lenz-Barlotti class I.2. The collection of all collineations

$$(x,y) \mapsto (x, sy),\ (m) \mapsto (sm)$$

for $s \in \mathbb{R} \setminus \{0\}$ is a linearly transitive group of homologies with centre (∞) and axis $L_{0,0}$. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

(9) The plane $\mathcal{P}_{\mu_2,\mu_2}(\mathbb{R})$ is of Lenz-Barlotti class I.1. In this plane there is no point-line pair (p, L) for which this plane is (p, L)-transitive.

Realising semi-classical ordered projective planes of Lenz-Barlotti class IV.b.1 is more complicated. In order to give an example in this class let $\mathbb{F}$ be the field of all formal power series $\sum_{i=n}^{\infty} a_i X^i$ over the rationals $\mathbb{Q}$ in one indeterminate X where n is an integer and $a_n \neq 0$. $\mathbb{F}$ becomes an ordered field by defining $f(X) = \sum_{i=n}^{\infty} a_i X^i > 0$ if and only if $a_n > 0$, cf. [12, Chapter II, §5]. It readily follows that α defined by $\alpha(f(X)) = f(2X)$ (i.e., substitution of X by $2X$ in every formal power series in X which results in multiplying each coefficient a_i of X^i by 2^i) is an order-preserving automorphism of $\mathbb{F}$. Furthermore, the pair (id, α) satisfies (1) so that $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is a semi-classical projective plane.

(10) The plane $\mathcal{P}_{\alpha,id}(\mathbb{F})$ as defined above is of Lenz-Barlotti class IV.b.1. The collection of all collineations

$$(x,y) \mapsto \begin{cases} (x, y + ax + b), & \text{if } x \geq 0 \\ (x, y + \alpha(a)x + b), & \text{if } x \leq 0 \end{cases},\ (m) \mapsto (m+a)$$

for $a, b \in \mathbb{R}$ is a group of elations with centre (∞). For $a = 0$ the resulting subgroup is a linearly transitive group of elations with axis L_∞. For $b = ca$ with fixed $c < 0$ and $b = c\alpha(a)$ with fixed $c > 0$ the resulting subgroup is a linearly transitive group of elations with axis L_{-c}. There is no other point-line pair (p, L) for which this plane is (p, L)-transitive.

2. Generalized Moulton planes

The generalized Moulton planes constructed by W. A. Pierce [8] can be found among the planes $\mathcal{P}_{h,g}(\mathbb{F})$ with g being the identity, or more generally, g being an order-preserving automorphism of $\mathbb{F}$. Such a plane with $g = id$ is Desarguesian if and only if $h = id$; see [8, Theorem 4]. In order to distinguish between different generalizations of Moulton planes we use the term *Pierce-Moulton plane* for the planes constructed by W.A. Pierce in [8]. What is usually refered to as a Moulton plane and all isomorphic models will be called a *Pickert-Moulton plane* following W.A. Pierce [10]; see Definition 2.3 below. In order to define these planes we need the following

2.1. Definition. *Let $\mathbb{F}$ be a half-ordered field and let $q \in \mathbb{F}$, $q > 0$, such that $(1-x)(q-x) > 0$ for all $x < 0$. Define $\mu_q : \mathbb{F} \to \mathbb{F}$ by*

$$\mu_q(x) = \begin{cases} x, & \text{if } x \geq 0 \\ qx, & \text{if } x < 0 \end{cases}.$$

Then μ_q is an order-preserving permutation of $\mathbb{F}$. Moreover, condition (1) is satisfied for $h = \mu_q$ and $g = id$, that is, we obtain a semi-classical projective plane $\mathcal{P}_{\mu_q,id}(\mathbb{F})$.

It readily follows that μ_q defined as above for arbitrary $q > 0$ is an order-preserving permutation of $\mathbb{F}$ if and only if $(1-x)(q-x) > 0$ for all $x < 0$. In particular, $\mu_1 = id$ is always order-preserving. Furthermore, $\mathbb{F} \neq GF(3)$ is ordered with respect to the given half-ordering if and only if every μ_q, $q > 0$, is order-preserving, see [18, Proposition 1.3].

2.2. Definition. *We call two permutations $f \in \Pi^+_{0,1}(\mathbb{F})$ and $f' \in \Pi^+_{0,1}(\mathbb{E})$ affinely equivalent to each other if and only if there are order-preserving isomorphisms ϕ, ψ from $\mathbb{F}$ onto $\mathbb{E}$ (that is, ϕ and ψ map positive elements of $\mathbb{F}$ to positive elements of $\mathbb{E}$ and negative elements to negative ones) and $a, b, c, d \in \mathbb{F}$, $a, c \neq 0$, such that*

$$f'(\phi(x)) = \psi(cf(ax+b)+d)$$

for all $x \in \mathbb{F}$. This defines an equivalence relation on $\Pi^+_{0,1}(\mathbb{F})$.

Let $\mathcal{A}(\mathbb{F})$ denote the collection of all permutations $f \in \Pi^+_{0,1}(\mathbb{F})$ such that f is affinely equivalent to an order-preserving permutation μ_q (see Definition 2.1) for some $q \in \mathbb{F}$, $q > 0$.

Note that the definition of $\mathcal{A}(\mathbb{F})$ uses only those permutations μ_q that are order-preserving.

The equivalence class of the identity is the set $Aut^+(\mathbb{F})$ of all order-preserving automorphisms of $\mathbb{F}$. Furthermore, it readily follows that if a permutation $f' \in \Pi^+_{0,1}(\mathbb{E})$ is affinely equivalent to an order-preserving automorphism of $\mathbb{F}$ then f' is an order-preserving automorphism of $\mathbb{E}$.

Since $\mu_1 = id$ is the only additive permutation among the mappings μ_q, $q > 0$, the additive permutations in $\mathcal{A}(\mathbb{F})$ are precisely the order-preserving automorphisms of $\mathbb{F}$.

After these preliminaries we can define Pickert-Moulton and Pierce-Moulton planes.

2.3. Definition. *We call a semi-classical plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ a Pickert-Moulton plane if and only if $g \in Aut^{+}(\mathbb{F})$, $h \in \mathcal{A}(\mathbb{F})$ or $h \in Aut^{+}(\mathbb{F})$, $g \in \mathcal{A}(\mathbb{F})$.*

We call a semi-classical plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ a Pierce-Moulton plane if and only if $g \in Aut^{+}(\mathbb{F})$ or $h \in Aut^{+}(\mathbb{F})$.

Pickert-Moulton planes are direct generalizations of Moulton's original plane over $\mathbb{R}$ (the plane $\mathcal{P}_{\mu_2,id}(\mathbb{R})$), see [6]. Each such plane is isomorphic to a plane $\mathcal{P}_{\mu_q,id}(\mathbb{F})$ by means of isomorphisms of the form 3.1, 3.2, 3.3, 3.4; see section 3. The planes described in [10, §4, Theorem 2] are the Pickert-Moulton planes with $g = id$ over an ordered field.

3. Some basic facts about semi-classical projective planes

There are four fundamental types of isomorphisms between semi-classical projective planes all of which map the point (∞) in one plane to the corresponding point (∞') in the other plane; cf. [18, §2].

3.1. Isomorphisms induced by linear maps:

$$(x,y) \mapsto \begin{cases} (a_1x, a_2y + a_3x + a_4), & x \geq 0 \\ \left((g')^{-1}\left(\frac{h(\frac{a_1-a_3}{a_2})-h(\frac{-a_3}{a_2})}{g(\frac{1-a_4}{a_2})-g(\frac{-a_4}{a_2})}g(x)\right), (g')^{-1}\left(\frac{g(y)-h(\frac{-a_3}{a_2})g(x)-g(\frac{-a_4}{a_2})}{g(\frac{1-a_4}{a_2})-g(\frac{-a_4}{a_2})}\right)\right), & x < 0 \end{cases}$$

$$(m) \mapsto \left(\frac{a_2m + a_3}{a_1}\right)$$

$$(\infty) \mapsto (\infty)$$

where $a_i \in \mathbb{F}$, $a_1 > 0$, $a_2 \neq 0$, and

$$h'(x) = \frac{h(\frac{a_1x-a_3}{a_2}) - h(\frac{-a_3}{a_2})}{h(\frac{a_1-a_3}{a_2}) - h(\frac{-a_3}{a_2})},$$

$$g'(x) = \frac{g(\frac{x-a_4}{a_2}) - g(\frac{-a_4}{a_2})}{g(\frac{1-a_4}{a_2}) - g(\frac{-a_4}{a_2})}.$$

This mapping yields an isomorphism from $\mathcal{P}_{h,g}(\mathbb{F})$ to $\mathcal{P}_{h',g'}(\mathbb{F})$.

3.2. Isomorphisms induced by isomorphisms from a half-ordered field $\mathbb{F}$ to a half-ordered field $\mathbb{E}$:

$$(x,y) \mapsto (\alpha(x), \alpha(y))$$
$$(m) \mapsto (\alpha(m))$$
$$(\infty) \mapsto (\infty)$$

where α is an order-preserving isomorphism from $\mathbb{F}$ to $\mathbb{E}$. This mapping yields an isomorphism from $\mathcal{P}_{h,g}(\mathbb{F})$ to $\mathcal{P}_{\alpha h \alpha^{-1}, \alpha g \alpha^{-1}}(\mathbb{E})$.

3.3. Isomorphisms that interchange the roles of the two half-planes: Let $n \in \mathbb{F}$, $n < 0$. Then

$$(x,y) \mapsto \begin{cases} (g(h^{-1}(n)x), g(y)), & \text{if } x \geq 0 \\ (ng(x), g(y)), & \text{if } x < 0 \end{cases}$$
$$(m) \mapsto (\frac{h(m)}{n})$$
$$(\infty) \mapsto (\infty)$$

is an isomorphism from $\mathcal{P}_{h,g}(\mathbb{F})$ to $\mathcal{P}_{\tilde{h},g^{-1}}(\mathbb{F})$ where the permutation $\tilde{h}$ is defined by $\tilde{h}(x) = \frac{1}{h^{-1}(n)} h^{-1}(nx)$.

3.4. Isomorphisms that interchange the roles of the two lines L_0 and L_∞:

$$(x,y) \mapsto \begin{cases} (\frac{1}{x}, \frac{y}{x}), & \text{if } x > 0 \\ (h^{-1}(\frac{1}{g(x)}), h^{-1}(\frac{g(y)}{g(x)})), & \text{if } x < 0 \\ (y), & \text{if } x = 0\,\cdot \end{cases}$$
$$(m) \mapsto (0, m)$$
$$(\infty) \mapsto (\infty)$$

This mapping yields an isomorphism from $\mathcal{P}_{h,g}(\mathbb{F})$ to $\mathcal{P}_{g,h}(\mathbb{F})$.

Note that all four types of isomorphisms yield planes whose describing permutations again are in $\Pi_{0,1}^{+1}(\mathbb{F})$ or $\Pi_{0,1}^{+1}(\mathbb{E})$. Furthermore, corresponding describing permutations under isomorphisms of type 3.1 and 3.2 are affinely equivalent to each other.

We summarize some of the results obtained in [18] as far as they are needed for the determination of Lenz-Barlotti classes. We begin with a characterization of Desarguesian planes.

3.5 Theorem ([18, Theorem 3.9]). *The projective plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ is Desarguesian if and only if $h = g$ is an order-preserving automorphism of $\mathbb{F}$.*

The following proposition characterizes those planes which possibly admit collineations that do not map half-planes to half-planes but fix the distinguished point (∞) and the line L_∞.

3.6 Proposition ([18, Proposition 4.1]). *If an isomorphism γ from $\mathcal{P}_{h,g}(\mathbb{F})$ to $\mathcal{P}_{h',g'}(\mathbb{E})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ and $(g',h') \in \Pi_{0,1}^{+1}(\mathbb{E})$ maps the point (∞) onto (∞') and maps L_∞ onto L'_∞ but fails to map L_0 onto L'_0 then g, h, g' and h' must be additive.*

Recall that a projective plane is (p,L)-transitive, where p is a point and L is a line, if and only if the group of all central collineations with centre p and axis L is transitive on each central line minus p and the intersection with L. With this notation we have the following.

3.7 Proposition. *The projective plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ is*

(1) $((\infty), L_\infty)$*-transitive if and only if g is additive; cf.* [18, Corollary 3.2].

(2) $((0,0), L_\infty)$*-transitive if and only if*
- $h \in Aut^+(\mathbb{F})$,
- $(g^{-1}h)^2 = \mu_q$ *for some* $q > 0$ *where* μ_q *is as in Definition 2.1, and*
- $g(xy) = \begin{cases} g(x)g(y), & \text{if } x \geq 0 \text{ or } y \geq 0 \\ g(q)g(x)g(y), & \text{if } x, y < 0 \end{cases}$;

cf. [18, Corollary 3.5]. *In this case, $\mathcal{P}_{h,g}(\mathbb{F})$ is a Pierce-Moulton plane.*

Finally, for semi-classical ordered projective planes and their collineations we found in [18] Theorem 5.3 and Corollary 5.6 the following.

3.8. Theorem. *If γ is an order-preserving isomorphism between semi-classical ordered projective planes that are not Pickert-Moulton planes, then γ maps the point (∞) onto (∞') and $\{L_0, L_\infty\}$ onto $\{L'_0, L'_\infty\}$. Furthermore, such an isomorphism is a composition of isomorphisms of types 3.1 to 3.4.*

A semi-classical ordered projective plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ that is not a Pickert-Moulton plane admits a non-trivial collineation if and only if there are order-preserving automorphisms ϕ, ψ of $\mathbb{F}$ and $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{F}$, $a, c, \tilde{a}, \tilde{c} \neq 0$, such that one of the following holds:

(i) $h(\phi(x)) = \psi(ch(ax+b)+d)$ *and* $g(\phi(x)) = \psi(\tilde{c}g(\tilde{a}x+\tilde{b})+\tilde{d})$ *for all* $x \in \mathbb{F}$ *with* $a\tilde{a} > 0$;

(ii) $h(\phi(x)) = \psi(cg(ax+b)+d)$ *and* $g(\phi(x)) = \psi(\tilde{c}h(\tilde{a}x+\tilde{b})+\tilde{d})$ *for all* $x \in \mathbb{F}$ *with* $a\tilde{a} > 0$;

(iii) $h(\phi(x)) = \psi(ch^{-1}(ax+b)+d)$ *and* $g(\phi(x)) = \psi(\tilde{c}g^{-1}(\tilde{a}x+\tilde{b})+\tilde{d})$ *for all* $x \in \mathbb{F}$ *with* $a\tilde{a} < 0$;

(iv) $h(\phi(x)) = \psi(cg^{-1}(ax+b)+d)$ *and* $g(\phi(x)) = \psi(\tilde{c}h^{-1}(\tilde{a}x+\tilde{b})+\tilde{d})$ *for all* $x \in \mathbb{F}$ *with* $a\tilde{a} < 0$.

4. Some linearly transitive groups of central collineations in semi-classical projective planes

In this section we investigate some transitivity properties of semi-classical planes over half-ordered fields. This leads to the characterizaton of (p, L)-transitivity of semi-classical planes in terms of the describing permutations g and h for all centres p on $L_\infty \cup L_0$ and all axes L such that the distinguished point (∞) is fixed. Under these restrictions $p = (\infty)$ or, up to isomorphism (use isomorphisms of type 3.1 and 3.4), $p = (0)$. In the latter case L must pass through (∞) in order that (∞) remains fixed under all respective central collineations.

4.1. Proposition. *$\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((\infty), L_1)$-transitive if and only if g and h are both additive. In this case $\mathcal{P}_{h,g}(\mathbb{F})$ is a dual translation plane with translation centre (∞).*

Proof. Suppose that $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_1)$-transitive. An $((\infty), L_1)$-elation which maps (0) to (c) must have the form

$$(x,y) \mapsto \begin{cases} (x, y+cx-c), & \text{if } x \geq 0 \\ (x, g^{-1}(h(c)g(x)+g(y-c))), & \text{if } x < 0\,. \end{cases}$$
$$(m) \mapsto (m+c)$$

Since the line $L_{m,t}$ is mapped to $L_{m+c,t-c}$, one obtains

$$h(c)g(x) + g(g^{-1}(h(m)g(x)+g(t)) - c) = h(m+c)g(x) + g(t-c)$$

or

$$(2) \qquad (h(m+c) - h(c))g(x) = g(g^{-1}(h(m)g(x)+g(t)) - c) - g(t-c)$$

for all $c, m, t, x \in \mathbb{F}$, $x < 0$. Let

$$f_c(z) = g(g^{-1}(z+g(c)) - c)$$

for $z \in \mathbb{F}$. Since the left-hand side in (2) is independent of t, we obtain for $t = c$ that

$$f_c(h(m)g(x)) = (h(m+c) - h(c))g(x).$$

Substituting $m = 1$ yields

$$f_c(z) = \phi(c)z$$

for all $z, c \in \mathbb{F}$ where

$$\phi(c) = h(1+c) - h(c).$$

Therefore

$$h(m+c) = h(c) + \phi(c)h(m).$$

Here the left-hand side is symmetrical in m and c so that

$$h(c) + \phi(c)h(m) = h(m) + \phi(m)h(c).$$

Substituting $m = 1$ yields $\phi(c) = (\phi(1) - 1)h(c) + 1$. Thus

$$(3) \qquad h(m+c) = h(m) + h(c) + (\phi(1) - 1)h(c)h(m).$$

Suppose that $\phi(1) \neq 1$. Let $c = h^{-1}(\frac{1}{1-\phi(1)})$; then $h(m+c) = h(c)$ for all $m \in \mathbb{F}$ – a contradiction to the injectivity of h. Thus $\phi(1) = 1$. Now (3) shows that h is additive and that $f_c(z) = z$ for all $c, z \in \mathbb{F}$. Then

$$g(g^{-1}(z+g(c)) - c) = z$$

which implies that g^{-1} and thus g is additive too.

Conversely, assume that g and h are both additive. By Proposition 3.7 the plane $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_\infty)$-transitive. Using an isomorphism of the form 3.4 we see that $\mathcal{P}_{h,g}(\mathbb{F})$ is also $((\infty), L_0)$-transitive; cf. [18, Corollary 3.3]. Hence, the plane is $((\infty), (\infty))$-transitive and thus a dual translation plane. In particular, $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_1)$-transitive. □

4.2. Proposition. *$\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((\infty), L_{0,0})$-transitive if and only if $g = h$ is multiplicative.*

Proof. Suppose that $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_{0,0})$-transitive and let γ be a $((\infty), L_{0,0})$-homology which maps the infinite point (1) to (c), $c \neq 0$. γ must have the form

$$(x, y) \mapsto (x, \alpha(y))$$

on the affine part of $\mathcal{P}_{h,g}(\mathbb{F})$ for some permutation α. Thus $\gamma((1, m)) = (1, \alpha(m))$ and therefore $\gamma(L_{m,t}) = L_{\alpha(m),\alpha(t)}$ for all $m, t \in \mathbb{F}$. This gives us

$$\alpha(mx + t) = \alpha(m)x + \alpha(t)$$

for $x \geq 0$. One readily infers that $\alpha(y) = cy$ for all $y \in \mathbb{F}$. For $x < 0$ and $m = 1$ we now have

$$cg^{-1}(g(x) + g(t)) = g^{-1}(h(c)g(x) + g(ct)). \tag{4}$$

Substituting $x = g^{-1}(-g(t))$ for $t \geq 0$ gives us $g(ct) = h(c)g(t)$. Thus $g = h$ and $g(ct) = g(c)g(t)$ for all $t \geq 0$. Substituting $t = 0$ in (4) yields $g(cx) = g(c)g(x)$ for all $x < 0$. This proves that $g = h$ is multiplicative.

Conversely, if $g = h$ is multiplicative, it readily follows that each mapping $(x, y) \mapsto (x, cy)$ for $c \neq 0$ extends to a central collineation of $\mathcal{P}_{h,g}(\mathbb{F})$ with centre (∞) and axus $L_{0,0}$. Hence $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_{0,0})$-transitive. □

4.3. Lemma. *Let $\mathbb{F} \neq GF(3)$ be a half-ordered field.*

(1) *For each $p \in \mathbb{F}$, $p > 0$ there exists an $x < 0$ such that $p + x < 0$.*
(2) *For each $x \in \mathbb{F}$, $x < 0$ there exists a $p > 0$ such that $p + x < 0$.*

Proof. Given $p > 0$ there always exists an $x < 0$, $x \neq -p$. Then $\frac{p}{x}, \frac{p^2}{x} < 0$ and $(p + x)(p + \frac{p^2}{x}) = \frac{p}{x}(p + x)^2 < 0$. Therefore, $p + x < 0$ or $p + \frac{p^2}{x} < 0$.

A similar argument proves the second statement. □

Since every Desarguesian projective plane is (p, L)-transitive for every point-line pair (p, L), one direction in the following propositions is trivially true. Furthermore, a projective plane of order 3 is Desarguesian.

4.4. Proposition. *$\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((0), L_\infty)$-transitive if and only if $g = h \in Aut^+(\mathbb{F})$. In this case $\mathcal{P}_{h,g}(\mathbb{F})$ is Desarguesian.*

Proof. Suppose that $\mathcal{P}_{h,g}(\mathbb{F})$, $\mathbb{F} \neq GF(3)$, is $((0), L_\infty)$-transitive. Then there is an $((0), L_\infty)$-elation that fixes (∞) and L_∞ but moves L_0. Hence g and h must be additive by Proposition 3.6. An $((0), L_\infty)$-elation which maps $(0, 0)$ to $(p, 0)$, $p > 0$, must have the form

$$(x, y) \mapsto (x + p, y)$$

on the affine part of $\mathcal{P}_{h,g}(\mathbb{F})$ because $L_{1,0}$ is mapped to $L_{1,-p}$. Moreover, a line $L_{m,0}$ is mapped to $L_{m,-mp}$. For $x < 0$ and $x + p < 0$ one therefore obtains

$$h(m)g(x) = h(m)g(x + p) + g(-mp);$$

thus

$$g(mp) = h(m)g(p) \tag{5}$$

for all $m \in \mathbb{F}$ and all $p > 0$ such that there is an $x < 0$ with $x + p < 0$. However, such an x always exists by Lemma 4.3. For $p = 1$ in (5) one obtains $g = h$. But then

$$g(mp) = g(m)g(p)$$

for all $m, p \in \mathbb{F}$, $p \geq 0$. It follows from [18, Lemma 2.5] that $g = h \in Aut^+(\mathbb{F})$. Hence $\mathcal{P}_{h,g}(\mathbb{F}) \cong \mathcal{P}_{id,id}(\mathbb{F})$ is Desarguesian. $\square$

4.5. Proposition. *$\mathcal{P}_{h,g}(\mathbb{F})$, $(g, h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((0), L_1)$-transitive if and only if $g = h \in Aut^+(\mathbb{F})$. In this case $\mathcal{P}_{h,g}(\mathbb{F})$ is Desarguesian.*

Proof. Suppose that $\mathcal{P}_{h,g}(\mathbb{F})$, $\mathbb{F} \neq GF(3)$, is $((0), L_1)$-transitive. Then there is an $((0), L_1)$-homology that fixes (∞) and L_∞ but moves L_0. Hence g and h must be additive by Proposition 3.6. An $((0), L_1)$-homology which maps $(0, 0)$ to $(p, 0)$, $1 \neq p > 0$, takes $L_{1,0}$ to $L_{\frac{1}{1-p}, -\frac{p}{1-p}}$. Hence, on the affine part, one finds

$$(x, y) \mapsto \begin{cases} ((1-p)x + p, y), & \text{if } (1-p)x + p \geq 0 \\ (g^{-1}(\frac{g(x+\frac{p}{1-p})}{h(\frac{1}{1-p})}), y), & \text{if } (1-p)x + p < 0 \end{cases}.$$

A line $L_{m,0}$ is mapped to $L_{\frac{m}{1-p}, -\frac{mp}{1-p}}$. For $x < 0$ and $(1-p)x + p \geq 0$ one therefore obtains $g^{-1}(h(m)g(x)) = \frac{m}{1-p}((1-p)x + p) - \frac{mp}{1-p} = mx$. Thus

$$g(mx) = h(m)g(x) \tag{6}$$

for all $m \in \mathbb{F}$ and all $x < 0$ such that there is a $p > 0$ with $(1-p)x + p \geq 0$.

We now show that such a $p > 0$ exists for each $x < 0$. We note that $(1-p)x+p \geq 0$ if and only if $1 + \frac{1-x}{x}p \leq 0$. If $\frac{1-x}{x} < 0$ then, by Lemma 4.3, there exists a $p > 0$ such that $1 + \frac{1-x}{x}p \leq 0$. On the other hand, if $\frac{1-x}{x} > 0$ and we assume that $1 + \frac{1-x}{x}p > 0$ for all $p > 0$, then $\mathbb{F}$ is ordered and $\frac{1-x}{x} < 0$ - a contradiction. Therefore (6) is valid for all $m \in \mathbb{F}$ and all $x < 0$.

Let $q > 0$. Substituting mq for m in (6) yields

$$h(mq)g(x) = g(mqx) = h(m)g(qx) = h(m)h(q)g(x).$$

Therefore

$$h(mq) = h(m)h(q) \quad \text{for all } m, q \in \mathbb{F}, q > 0. \tag{7}$$

From (6) we find $1 = g(\frac{1}{x}x) = h(\frac{1}{x})g(x)$. Thus $h(\frac{1}{x}) = \frac{1}{g(x)}$ for all $x < 0$. For $p > 0 > x$ we now obtain $g(p) = g(\frac{p}{x}x) = h(\frac{p}{x})g(x) = h(\frac{1}{x})h(p)g(x) = h(p)$. This shows that $g(p) = h(p)$ for all $p \geq 0$. It follows from (7) that $h \in Aut^+(\mathbb{F})$ by [18, Lemma 2.5]. Hence $g = h \in Aut^+(\mathbb{F})$ and $\mathcal{P}_{h,g}(\mathbb{F})$ is Desarguesian. $\square$

We now have to make the first additional assumption and eventually have to consider semi-classical ordered projective planes.

4.6. Proposition. *Suppose that -1 is negative. $\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((0),L_0)$-transitive if and only if g is an order-preserving automorphism of $\mathbb{F}$ and $g^{-1}h = \mu_r$, that is, $\mathcal{P}_{h,g}(\mathbb{F})$ is a Pickert-Moulton plane $\mathcal{P}_{\mu_r,id}(\mathbb{F})$.*

Note that $\mathcal{P}_{h,g}(\mathbb{F})$ is still a Pierce-Moulton plane if $\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is $((0),L_0)$-transitive even if -1 is positive.

Proof. Suppose that $\mathcal{P}_{h,g}(\mathbb{F})$ is $((0),L_0)$-transitive. Using an isomorphism of type 3.4, it follows from Proposition 3.7.(2) that g is an automorphism of $\mathbb{F}$, that $(h^{-1}g)^2 = \mu_q$ for some $q > 0$, and that $h(xy) = \begin{cases} h(x)h(y) & \text{if } x \geq 0 \text{ or } y \geq 0 \\ h(q)h(x)h(y) & \text{if } x,y < 0 \end{cases}$. Since g is an automorphism, we have $\mathcal{P}_{h,g}(\mathbb{F}) = \mathcal{P}_{g^{-1}h,id}(\mathbb{F})$. In particular, we obtain a Pierce-Moulton plane.

If $-1 < 0$, then [18, Remark 3.6] shows that $g^{-1}h = \mu_r$ where $r = -g^{-1}h(-1) > 0$. Hence we have a Pickert-Moulton plane $\mathcal{P}_{\mu_r,id}(\mathbb{F})$. □

Propositions 4.4, 4.5 and 4.6 can be summarized in the following.

4.7. Theorem. *The Pierce-Moulton planes are the only semi-classical projective planes that are (p,L)-transitive for some $p \in (L_\infty \cup L_0) \setminus \{(\infty)\}$ and $(\infty) \in L$.*

The Desarguesian planes are the only semi-classical projective planes that are (p,L)-transitive for some $p \in (L_\infty \cup L_0) \setminus \{(\infty)\}$ and $L \in \mathcal{L}_{(\infty)}$ such that $\{L, L_c\} \neq \{L_0, L_\infty\}$ where L_c, $c \in \mathbb{F} \cup \{\infty\}$, is the vertical line containing p.

Proof. Suppose that $\mathcal{P} = \mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, is (p,L)-transitive where $p \in L_\infty \setminus \{(\infty)\}$ and $(\infty) \in L$. Using an isomorphism of type 3.4, if necessary, we can assume that $p \in L_\infty$. Then, using an isomorphism of type 3.1, if necessary, we can further assume that $p = (0)$. If $L = L_\infty$, then $\mathcal{P}$ is Desarguesian by Proposition 4.4. If $L = L_0$, then $\mathcal{P}$ is a Pierce-Moulton plane by Proposition 4.6 after using an isomorphism of type 3.4. If $L = L_c$, $c \neq \infty, 0$, we can use an isomorphism of type 3.1 and 3.3 to obtain $c = 1$. But then $\mathcal{P}$ is Desarguesian by Proposition 4.5.

In particular, one obtains Desarguesian planes if the line through p and (∞) is not the one distinguished line L_∞ or L_0 that does not contain p. □

4.8 Finite semi-classical planes.

The results obtained so far suffice to determine the Lenz-Barlotti classes of finite semi-classical projective planes. As mentioned in the introduction in a finite half-ordered field $\mathbb{F}$ the set of positive elements consists precisely of the non-zero squares of $\mathbb{F}$. Furthermore, $\mathbb{F}$ has odd characteristic. By [2] the only order-preserving permutations that fix 0 and 1 are automorphisms of $\mathbb{F}$. If $\mathbb{F}$ has characteristic p and $q = p^n$, then every automorphism of $\mathbb{F} = GF(q)$ is of the form $x \mapsto x^{p^m}$ where $0 \leq m < n$.

Every finite semi-classical projective plane is isomorphic to a plane $\mathcal{P}_{\alpha,id}(\mathbb{F})$ for some automorphism α of $\mathbb{F}$. Moreover, in this case, each isomorphism of type 3.1 actually is a collineation of the plane. In particular, the collineation group of $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is transitive on the set of lines L_c with $c > 0$ and on the set of lines L_c with $c < 0$. Furthermore, any such plane is a dual translation plane and thus of Lenz-Barlotti class at least IV.b.1. According to [7, Figure 63], $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is of class IV.b.1, IV.b.2, IV.b.3, V, VII.1 or VII.2. Class VII.2 consists precisely of the Desarguesian planes; thus $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is of class VII.2 if and only if $\alpha = id$. No finite plane is of class VII.1, see [3]. Class IV.b.3 consists precisely of the non-Desarguesian dual translation plane of order 9, i.e., the plane $\mathcal{P}_{\alpha,id}(GF(9))$ where α is the Frobenius automorphism $\alpha(x) = x^3$, see Example (4) in section 1.

In planes of Lenz-Barlotti class IV.b.2 there is a point p and two lines L_1 and L_2 through p such that the plane is (p, L)-transitive for each line L through p and such that it is (q, L_i)-transitive for each point q on L_{3-i}, $i = 1, 2$. Given this configuration it follows in our situation that $p = (\infty)$. If $\{L_1, L_2\} \neq \{L_0, L_\infty\}$, then $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is (q, L_c)-transitive for some point $q \notin L_c$ and some $c \neq 0, \infty$. Using the collineation group of $\mathcal{P}_{\alpha,id}(\mathbb{F})$ we then obtain that the plane is also $(q(r), L_{rc})$-transitive for each $r > 0$ and some point $q(r)$. Since this cannot occur in class IV.b.2, we must have that without loss of generality $L_1 = L_0$ and $L_2 = L_\infty$. In particular, $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is $((0,0), L_\infty)$-transitive and thus $\alpha^2 = id$ by Proposition 3.7.2. Conversely, one obtains Lenz-Barlotti class IV.b.2 unless $\alpha = id$ or $\mathbb{F}$ has order nine.

In planes of Lenz-Barlotti class V there is a flag (p, L) such that the plane is (q, L)- and (p, K)-transitive for each point q on L and each line K through p, i.e., we have a translation plane (with translation axis L) which also is a dual translation plane (with translation centre p). These planes are coordinatised by semifields. For our semi-classical planes we must have $p = (\infty)$. By Proposition 4.4 the line L cannot be equal to L_∞ unless the plane is Desarguesian and thus of class VII.2. The dualization of Proposition 4.4 similarly excludes that $L = L_0$. Hence $L = L_c$ for some $c \neq 0, \infty$. As before this implies that $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is $(\infty), L_{rc})$-transitive for each $r > 0$. Hence, in this case, $\mathcal{P}_{\alpha,id}(\mathbb{F})$ must again be of class VII.2.

In summary we obtain the following classification of Lenz-Barlotti classes of semi-classical finite planes.

4.9. Theorem. *A semi-classical plane $\mathcal{P}_{\alpha,id}(\mathbb{F})$ with $\alpha \in Aut(\mathbb{F})$ over a finite half-ordered field $\mathbb{F} = GF(p^n)$, p an odd prime, is of Lenz-Barlotti class*

- *VII.2 if and only if $\alpha = id$; $\mathcal{P}_{\alpha,id}(\mathbb{F})$ is the Desarguesian plane over $\mathbb{F}$.*
- *IV.b.3 if and only if $\mathbb{F} = GF(9)$ is the field with nine elements and α is the unique automorphism of $GF(9)$ of order 2.*
- *IV.b.2 if and only if α has order 2 and $\mathbb{F}$ has order greater than 9. This case only occurs if n is even.*
- *IV.b.1 if and only if α has order greater than 2.*

Note that all the classes in the above theorem actually occur as Lenz-Barlotti classes for some finite semi-classical planes by choosing suitable finite fields $\mathbb{F}$ and automorphisms α of $\mathbb{F}$, see examples (1) to (4).

5. Lenz-Barlotti classes of semi-classical ordered planes

Although many transitivity properties of semi-classical planes can be characterized in terms of the describing functions g and h, as we have seen in the preceding section, it is difficult to do so in general, since the distinguished point (∞) may be moved under a collineation. Only under the stronger assumption of an ordered plane comprehensive results were obtained in [18], see in particular §4.

In order to obtain a semi-classical ordered projective plane let $\mathbb{F}$ be an ordered field. Then the corresponding projective plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ is an ordered plane in the sense of [12]; see also [7, §9], [16] or [19]. The relation of separation between pairs of points on a projective line is invariant under projectivities. It is naturally inherited from the ordering of the coordinatizing field with which affine lines can be identified. A collineation γ of $\mathcal{P}_{h,g}(\mathbb{F})$ is order-preserving if it preserves the relation of separation between pairs of points on a line. This extends to isomorphisms between semi-classical ordered projective planes. In particular, the isomorphisms of types 3.1 to 3.4 are order-preserving.

We now restrict our attention to semi-classical ordered planes and apply the results of sections 3 and 4 to such planes. Using isomorphisms of types 3.1 and 3.4 we obtain an immediate corollary of Proposition 4.6.

5.1. Proposition. *A semi-classical plane $\mathcal{P}_{h,g}(\mathbb{F})$, $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$, over a half-ordered field $\mathbb{F}$ with $-1 < 0$ is $((m), L_0)$- or $((0,t), L_\infty)$-transitive for some $m, t \in \mathbb{F}$ if and only if $\mathcal{P}_{h,g}(\mathbb{F})$ is a Pickert-Moulton plane.*

We now assume that $\mathbb{F}$ is an ordered field and apply the results of the previous sections in order to determine the possible Lenz-Barlotti classes for those semi-classical planes; see [7, Anhang §6] for the Lenz-Barlotti classification.

Collineations of ordered Pickert-Moulton planes were determined in [8], [11], see [20] for their Lenz-Barlotti classes.

5.2. Proposition. *A Pickert-Moulton plane $\mathcal{P}_{\mu_q,id}(\mathbb{F})$ over an ordered field $\mathbb{F}$ is Desarguesian if and only if $q = 1$. The Desarguesian plane is of Lenz-Barlotti class VII.2. The proper Pickert-Moulton planes, that is, $q \neq 1$, are of Lenz-Barlotti class III.2.*

Let $\mathcal{P} = \mathcal{P}_{h,g}(\mathbb{F})$ be a semi-classical plane over an ordered field $\mathbb{F}$ with $g, h \in \Pi_{0,1}^{+1}(\mathbb{F})$. We first note that each central collineation of $\mathcal{P}$ is order-preserving; cf. [12, V, Satz 10].

If $\mathcal{P}$ is not a Pickert-Moulton plane, then each order-preserving collineation of $\mathcal{P}$ must fix (∞) and $\{L_0, L_\infty\}$ according to Theorem 3.8. Consequently, if $\mathcal{P}$ is (p,L)-transitive, then L_0 and L_∞ must be fixed under every central collineation with centre p and axis L. This can only occur when both lines L_0 and L_∞ are central lines or one of them is a central line and the other is the axis. Hence $p = (\infty)$ or $p \in L_\infty \setminus \{(\infty)\}$, $L = L_0$ or $p \in L_0 \setminus \{(\infty)\}$, $L = L_\infty$. The last two cases lead to Pickert-Moulton planes by Proposition 5.1.

This leaves the classes I.1, I.2, II.1, and IV.b.1 as possible Lenz-Barlotti classes of semi-classical ordered planes that are not Pickert-Moulton planes. Furthermore $p = (\infty)$ except in class I.1. Class IV.b.1 comprises precisely the dual translation planes; thus $\mathcal{P}$ is of Lenz-Barlotti class IV.b.1 if and only if g and h are both additive by [18, Corollary 3.3]. Planes of Lenz-Barlotti class II.1 have precisely one flag (p,L), $p \in L$, such that the projective plane is (p,L)-transitive. By Proposition

4.1 the axis L must be one of the lines L_0 or L_∞ in such planes, otherwise we would have Lenz-Barlotti class IV.b.1.

Planes of Lenz-Barlotti class I.2 have precisely one anti-flag (p, L), $p \notin L$, such that the projective plane is (p, L)-transitive. This leads to $p = (\infty)$ and $L = L_{0,0}$ which has been dealt with in Proposition 4.2. This situation can only occur if g and h are affinely equivalent to the same multiplicative permutation of $\mathbb{F}$.

In summary we obtain the following classification of Lenz-Barlotti classes of semi-classical ordered planes.

5.3. Theorem. *A semi-classical plane $\mathcal{P}_{h,g}(\mathbb{F})$ with $(g,h) \in \Pi_{0,1}^{+1}(\mathbb{F})$ over an ordered field $\mathbb{F}$ is of Lenz-Barlotti class*

- *VII.2 if and only if $g = h$ is an order-preserving automorphism of $\mathbb{F}$; $\mathcal{P}_{h,g}(\mathbb{F})$ is the Desarguesian plane over $\mathbb{F}$.*
- *IV.b.1 if and only if g and h are both additive but not as in class VII.2; $\mathcal{P}_{h,g}(\mathbb{F})$ is a non-Desarguesian dual translation plane with translation centre (∞).*
- *III.2 if and only if $g \in Aut^+(\mathbb{F})$, $h \in \mathcal{A}(\mathbb{F}) \setminus Aut^+(\mathbb{F})$ or $h \in Aut^+(\mathbb{F})$, $g \in \mathcal{A}(\mathbb{F}) \setminus Aut^+(\mathbb{F})$; $\mathcal{P}_{h,g}(\mathbb{F})$ is a proper Pickert-Moulton plane.*
- *II.1 if and only if g or h is additive but not both and g, h are not as in class III.2; in this case $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_0)$- or $((\infty), L_\infty)$-transitive according to h or g being additive.*
- *I.2 if and only if (h, g) is affinely equivalent to (ν, ν) where $\nu \notin Aut^+(\mathbb{F})$ is a multiplicative permutation of $\mathbb{F}$; in this case the plane $\mathcal{P}_{h,g}(\mathbb{F})$ is $((\infty), L_{m,t})$-transitive for some line $L_{m,t}$.*

In all other cases $\mathcal{P}_{h,g}(\mathbb{F})$ is of Lenz-Barlotti class I.1.

All the classes in the above theorem actually occur as Lenz-Barlotti classes for some semi-classical prdered projective planes, see examples (5) to (10). However, note that for a given ordered field $\mathbb{F}$ the possible Lenz-Barlotti classes IV.b.1 and I.2 may be empty. For example, for $\mathbb{F} = \mathbb{R}$ with the Euclidean ordering class IV.b.1 is empty, see 5.4 and 5.5 below, and for $\mathbb{F} = \mathbb{Q}$ with the Euclidean ordering both these Lenz-Barlotti classes are empty. Since there always are the Desarguesian plane and proper Picker-Moulton planes, classes VII.2 and III.2 are never empty. Furthermore, classes II.1 and I.1 are not empty either. For $q > 0$, the map μ_q' defined by $\mu_q'(x) = \mu_{\frac{1}{q}}(\mu_q(x) + 1) - 1$ is in $\Pi_{0,1}^{+}(\mathbb{F})$ and $(id, \mu_q') \in \Pi_{0,1}^{+1}(\mathbb{F})$ so that $\mathcal{P}_{\mu_{q,r}, id}(\mathbb{F})$ is a semi-classical projective plane of class II.1 if $q \neq 1$. Finally, the plane $\mathcal{P}_{\mu_q,\mu_q}(\mathbb{F})$, $q > 0$, $q \neq 1$, is a semi-classical projective plane of class Lenz-Barlotti I.1.

5.4. Semiclassical 2-dimensional planes.

We apply the results obtained for semi-classical ordered planes to topological 2-dimensional planes. A topological projective plane is a projective plane in which the point set and the set of lines carry Hausdorff topologies such that the geometric operations of joining two distinct points by a line and intersecting two distinct lines in a point are continuous; cf. [14] or [15, §3]. In topological compact 2-dimensional planes both the point space and the line space are homeomorphic to the point space of the topological real Desarguesian projective plane. (This is a 2-dimensional manifold.) When the point set is constructed from the Euclidean 2-sphere by identifying antipodal points the topology is just the quotient topology under this identification map. Compact 2-dimensional semi-classical topological projective planes have been classified in [17]. The planes $\mathcal{P}_{h,g}$ constructed there are isomorphic to the planes considered here over $\mathbb{R}$, i.e. $\mathcal{P}_{h,g} \simeq \mathcal{P}_{h,g}(\mathbb{R})$.

Since each order-preserving additive mapping of $\mathbb{R}$ that fixes 1 is the identity – in particular, the identity is the only automorphism of $\mathbb{R}$ – one readily obtains from Theorem 5.3 the following.

5.5. Theorem. *A semi-classical 2-dimensional plane $\mathcal{P}_{h,g}(\mathbb{R})$ with $g, h \in \Pi^+_{0,1}(\mathbb{R})$ is of Lenz-Barlotti class*

- *VII.2 if and only if $g = h = id$; $\mathcal{P}_{h,g}(\mathbb{R})$ is the Desarguesian plane.*
- *III.2 if and only if $g = id$, $h \in \mathcal{A}(\mathbb{R}) \setminus \{id\}$ or $h = id$, $g \in \mathcal{A}(\mathbb{R}) \setminus \{id\}$; $\mathcal{P}_{h,g}(\mathbb{R})$ is a proper Pickert-Moulton plane.*
- *II.1 if and only if $g = id$, $h \in \Pi^+_{0,1}(\mathbb{R}) \setminus \mathcal{A}(\mathbb{R})$ or $h = id$, $g \in \Pi^+_{0,1}(\mathbb{R}) \setminus \mathcal{A}(\mathbb{R})$; in this case $\mathcal{P}_{h,g}(\mathbb{R})$ is $((\infty), L_0)$- or $((\infty), L_\infty)$-transitive according to h or g being the identity.*
- *I.2 if and only if g and h are both affinely equivalent to the same multiplicative permutation $\neq id$ of $\mathbb{R}$; in this case $\mathcal{P}_{h,g}(\mathbb{R})$ is $((\infty), L_{m,t})$-transitive for some line $L_{m,t}$.*

In all other cases $\mathcal{P}_{h,g}(\mathbb{R})$ is of Lenz-Barlotti class I.1.

Note that all the classes in the above theorem actually occur as Lenz-Barlotti classes for some semi-classical 2-dimensional planes by choosing suitable homeomorphisms g and h of $\mathbb{R}$, see examples (5) to (9).

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