Determinantal Forms for Symplectic and Orthogonal Schur Functions

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MR Classification Number: Primary 05E05, Secondary 05E10, 20C33.

Determinantal Forms for Symplectic and Orthogonal Schur Functions

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Abstract

Symplectic and orthogonal Schur functions can be defined combinatorially in a manner similar to the classical Schur functions. This paper demonstrates that they can also be expressed as determinants. These determinants are generated using planar decompositions of tableaux into strips and the equivalence of these determinants to symplectic or orthogonal Schur functions is established by Gessel–Viennot lattice path techniques. Results for rational (also called composite) Schur functions are also obtained.

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1 Introduction

Recent work on the symplectic and orthogonal tableaux and the associated symmetric functions has focused on Robinson–Schensted–type algorithms and Cauchy–type identities. See Berele [3], Sundaram [27] [28], Proctor [19] [20] [21] [22], Okada [17] [18], Benkart and Stroomer [2]. Here we develop determinantal expressions for the characters of the symplectic and orthogonal groups $Sp(2n)$ and $SO(2n+1)$ and prove their validity using the techniques of Hamel and Goulden [8]. Some of the determinants generated are symplectic and orthogonal analogues to the Jacobi–Trudi, dual Jacobi–Trudi, and Giambelli determinants defined for the classical Schur functions, and our methods are valid not only for the ordinary symplectic Schur function and *so* Schur function, but for skew versions of these as well (defined below). We follow the notation of Macdonald [16] and Sundaram [27].

Let λ be a partition of k with at most l parts, i.e. $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$ are nonnegative integers and $\lambda_1 + \lambda_2 + \ldots + \lambda_l = k$ (λ_i is the ith part of λ). The empty partition $\emptyset$ of 0 has no parts. A partition can be represented in the plane by an arrangement of boxes called

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a *Ferrers diagram*, or simply a *diagram*. This arrangement is top and left justified with λ_i boxes in the ith row and we say it has standard shape. Given two partitions, λ and μ, we define a Ferrers diagram with *skew shape* λ/μ for $\mu_i \leq \lambda_i$, $i \geq 1$ as an arrangement of boxes where there is a box in row i, column j iff $\mu_i < j \leq \lambda_i$. Geometrically, this is the Ferrers diagram of λ with the Ferrers diagram of μ removed from its upper left hand corner. From this point of view, the standard shape λ is just the skew shape λ/μ with $\mu = \emptyset$. Define the *content* of a box α in a Ferrers diagram as the quantity $j - i$ where α lies in column j and row i of the Ferrers diagram (referred to as box (i,j) where convenient). Associated with each skew shape is a *conjugate* shape. The conjugate of a skew shape λ/μ is defined to be the skew shape λ'/μ' whose Ferrers diagram is the transpose of the Ferrers diagram of λ/μ. More explicitly, the number of boxes in the ith row of λ'/μ' is the number of boxes in the ith column of λ/μ.

Fix a set of elements $1 < \overline{1} < 2 < \overline{2} \ldots < n < \overline{n}$. The following definition is the skew version of one due to King [13].

Definition 1.1 *A symplectic tableau, $SP_{\lambda/\mu}$, of shape λ/μ is a filling of the Ferrers diagram of λ/μ with the integers $1 < \overline{1} < 2 < \overline{2} \ldots < n < \overline{n}$ such that*

1. *the entries weakly increase along the rows and strictly increase down the columns,*
2. *the boxes of content $-i$ contain entries which are greater than or equal to $i+1$.*

We refer to the second condition as the *symplectic condition.* For standard shape tableaux the following condition is usually called the symplectic condition:

$$\text{all entries in row } i \text{ are greater than or equal to } i. \tag{1}$$

The condition (1) and condition 2 of Definition 1.1 are easily seen to be equivalent for standard shape since the first box in row $i+1$ has content $-i$.

The skew symplectic Schur function, $sp_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$sp_{\lambda/\mu}(X) = \sum_{SP_{\lambda/\mu}} \prod_{\alpha \in SP_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\overline{\beta} \in SP_{\lambda/\mu}} x_\beta^{-m(\overline{\beta})},$$

where the sum is over all tableaux $SP_{\lambda/\mu}$ of shape λ/μ, the first product is over all unbarred integers α in $SP_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $SP_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the mulitplicity of α (resp. $\overline{\beta}$) in $SP_{\lambda/\mu}$, i.e. the number of times α (resp. $\overline{\beta}$) appears in a box of the tableau.

There are several equivalent tableau definitions of orthogonal tableaux for $SO(2n+1)$ (see King [13], Proctor [22], and Koike and Terada [14]). The definition we take is a skew version of the one in Sundaram [27], and is very close to the definition of symplectic tableaux.

Definition 1.2 *An so–tableau, $SO_{\lambda/\mu}$, of shape λ/μ is a filling of the Ferrers diagram of λ/μ with the elements $1 < \overline{1} < 2 < \overline{2} \ldots < n < \overline{n} < \infty$ such that*

1. *the entries weakly increase along the rows and, when restricted to $1 < \overline{1} < 2 < \overline{2} \ldots < n < \overline{n}$, strictly increase down the columns,*

2. *the boxes of content $-i$ contain entries which are greater than or equal to $i+1$,*

3. *the entries equal to ∞ form a shape which is such that no two symbols ∞ appear in the same row.*

The skew *so* Schur function, $so_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$so_{\lambda/\mu}(X) = \sum_{SO_{\lambda/\mu}} \prod_{\alpha \in SO_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\overline{\beta} \in SO_{\lambda/\mu}} x_\beta^{-m(\overline{\beta})},$$

where the sum is over all tableaux $SO_{\lambda/\mu}$ of shape λ/μ, the first product is over all unbarred integers α in $SO_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $SO_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the mulitplicity of α (resp. $\overline{\beta}$) in $SO_{\lambda/\mu}$.

Note that the ∞ are in a sense "dummy elements" since they contribute 1 to the weight of the tableau.

Koike and Terada [14] also define skew $SP(2n)$ and $SO(2n+1)$ tableaux; however, their definition differs substantially from ours. They restrict the integers that are allowed to appear so that only those greater than the number of parts of μ are permitted and they use the alternative formulation of the symplectic condition given in (1).

The form of this paper is as follows. Section 2 provides background material from Hamel and Goulden [8], giving the details necessary to define the determinants we generate. Section 3 states and proves two main results, one for symplectic tableaux and one for *so*–tableaux. Section 4 includes some similar results for rational (also called composite) tableaux. As has been pointed out by Stembridge [25], the standard shape symplectic tableaux can be considered to be special cases of standard shape rational tableaux, and hence the results in Section 4 are a generalization of the results in Section 3.

2 Strips and Outside Decompositions

This section gives the tools needed to define classes of determinants equal to the symplectic Schur function and *so* Schur functions. The traditional ways of decomposing a tableau to generate a determinant use decompositions by rows (Jacobi–Trudi), columns (dual Jacobi–Trudi) or hooks (Giambelli). We generalize these notions here to allow decompositions by *strips.* The terminology follows that of Hamel and Goulden [8].

Definition 2.1 *A* strip θ *in a skew shape diagram is a skew diagram with an edgewise connected set of boxes that contains no 2×2 block of boxes.*

Definition 2.2 *The* starting box *of a strip is the box which is bottommost and leftmost in the strip. The* ending box *of a strip is the box which is topmost and rightmost in the strip.*

Figure 1 illustrates these concepts, where the starting box is marked with an x and the ending box is marked with an o. We say a box is approached from the left (resp. from below) if either there is a box immediately to its left or the box is on the left perimeter of the diagram (resp. there is a box immediately below it or the box is on the bottom perimeter of the diagram).

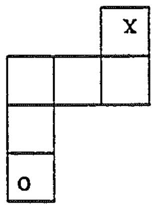

Figure 1: Example of a strip.

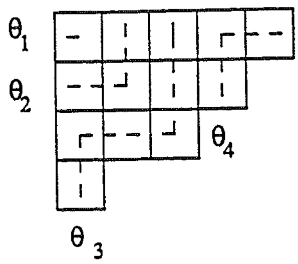

Figure 2: Example of an outside decomposition.

Definition 2.3 *Suppose $\theta_1, \theta_2, \ldots, \theta_m$ are strips in a skew shape diagram of λ/μ and each strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then if the disjoint union of these strips is the skew shape diagram of λ/μ, we say the totally ordered set $(\theta_1, \theta_2, \ldots, \theta_m)$ is a* (planar) outside decomposition *of λ/μ.*

Given a diagram and an outside decomposition of that diagram, then if the diagram is filled with integers to form a symplectic or *so*– tableau, the portion of the tableau that corresponds to a strip in the outside decomposition forms a *symplectic* or *so– tableau of strip shape.* Hence, given an outside decomposition of a shape, a symplectic or *so*– tableau of that shape can be thought of as a union of symplectic or *so*– tableaux of strip shape.

The restrictions of the definition of outside decomposition force the following property:

Property 2.4 *Boxes of the same content are approached from the same direction in their respective strips; that is, they are either all approached from below or all approached from the left.*

Figure 2 gives an example of an outside decomposition into four strips : $\theta_1 = 1, \theta_2 = 22/1, \theta_3 = 3331/22, \theta_4 = 21$. In Figure 2 strips θ_1, θ_2 and θ_3 have boxes of content zero approached from the left, while strips θ_3 and θ_4 have boxes of content two approached from below.

In order to define the determinants in the main results we must define an additional operation on strips. This noncommutative operation was first defined in Hamel and Goulden [8].

Case I: Suppose θ_i and θ_j have some boxes with the same content. Slide θ_i along top-left-to-bottom-right diagonals so that the box of content k in θ_i is superimposed on the box of content k in θ_j for all $k \in Z$. This procedure is well-defined by Proposition 2.4. Define $\theta_i \# \theta_j$ to be the diagram obtained from this superposition by taking all boxes between the ending box of θ_i and the starting box of θ_j inclusive.

Case II: Suppose θ_i and θ_j are two disconnected pieces and thus do not have any boxes of the same content. The starting box of one will be to the right and/or above the ending box of the other. To bridge the gap between θ_i and θ_j, insert boxes from the ending box of one to the starting box of the other so that these inserted boxes follow the approached–from–the–left or approached–from–below arrangement as do other boxes of the same content in the outside decomposition (Property 2.4 ensures the boxes of the same content are arranged in the same way). If there is a content such that there is no box of that content in the diagram (and therefore no determination of the direction from which the box is approached), then arbitrarily choose from which direction boxes of this content should be approached, fix this choice for all boxes of the same content in that particular diagram, and bridge the gap between θ_i and θ_j accordingly. Define $\theta_i\#\theta_j$ as in Case I with the following additional conventions: if the ending box of θ_i is edge connected to the starting box of θ_j, and occurs below or to the left of it, then $\theta_i\#\theta_j = \emptyset$; if the ending box of θ_i is not edge connected but occurs below or to the left of the starting box of θ_j, $\theta_i\#\theta_j$ is undefined.

Note that $\theta_i\#\theta_i = \theta_i$.

As an example consider again the strips in Figure 2. Then

$$\begin{array}{ll}
\theta_1\#\theta_2 = 2 & \theta_2\#\theta_3 = 331/2 \\
\theta_2\#\theta_1 = 11 & \theta_3\#\theta_2 = 222/11 \\
\theta_1\#\theta_3 = 31 & \theta_2\#\theta_4 = \emptyset \\
\theta_3\#\theta_1 = 111 & \theta_4\#\theta_2 = 3222/111 \\
\theta_1\#\theta_4 = \text{undefined} & \theta_3\#\theta_4 = 1 \\
\theta_4\#\theta_1 = 2111 & \theta_4\#\theta_3 = 43331/222
\end{array}$$

In the next section we show how to obtain a determinant from this information.

3 The Main Results

We now state the two main results of this paper:

Theorem 3.1 *Let λ/μ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of λ/μ,*

$$sp_{\lambda/\mu}(X) = \det(sp_{\theta_i\#\theta_j}(X))_{m\times m},$$

where $sp_\emptyset = 1$ and $sp_{undefined} = 0$.

Theorem 3.2 *Let λ/μ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of λ/μ,*

$$so_{\lambda/\mu}(X) = \det(so_{\theta_i\#\theta_j}(X))_{m\times m},$$

where $so_\emptyset = 1$ and $so_{undefined} = 0$.

For the diagram and outside decomposition in Figure 2, Theorem 3.1 gives the following determinant:

$$\det\begin{pmatrix} sp_1 & sp_2 & sp_{31} & 0 \\ sp_{11} & sp_{22/1} & sp_{331/2} & 1 \\ sp_{111} & sp_{222/11} & sp_{3331/22} & sp_{3331} \\ sp_{2111} & sp_1 & sp_{43331/22} & sp_{21} \end{pmatrix}$$

Under the same conditions, Theorem 3.2 gives the following determinant:

$$\det\begin{pmatrix} so_1 & so_2 & so_{31} & 0 \\ so_{11} & so_{22/1} & so_{331/2} & 1 \\ so_{111} & so_{222/11} & so_{3331/22} & so_{3331} \\ so_{2111} & so_1 & so_{43331/22} & so_{21} \end{pmatrix}$$

In Hamel and Goulden [8], the Gessel–Viennot lattice path procedure is used to construct a bijection establishing a family of determinantal results for Schur functions. We now show that this procedure extends easily to a bijection for symplectic Schur functions. We refer the reader to Hamel and Goulden [8] where certain essential details of the proof have been verified. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: Label the y–axis be labeled by $1, \overline{1}, 2, \overline{2}, \ldots$. Before describing the paths we need some guidelines to permissible steps and path restrictions. There are four types of permissible steps: up–vertical steps that *increase* the y–coordinate by 1; down–vertical steps that *decrease* the y–coordinate by 1; right–horizontal (referred to simply as horizontal) steps that increase the x–coordinate by 1; and down–diagonal (referred to simply as diagonal) steps that increase the x–coordinate by 1 *and* decrease the y–coordinate by 1. We specify some additional restrictions: a down–vertical step must not precede an up–vertical step, an up–vertical step must not precede a down–vertical step, a down–vertical step must not precede a horizontal step, and an up–vertical step must not precede a diagonal step. Because of the symplectic condition, we require an additional restriction not present in Hamel and Goulden [8], a *left boundary* in the form of a "backwards lattice path" from $(0,1)$ to $(0,\overline{1})$ to $(0,2)$ to $(-1,2)$ to $(-1,\overline{2})$ to $(-1,3)$ to $(-2,3)$ to $(-2,\overline{3})$, etc. See Figure 3 where this boundary is indicated by a dotted line. A path may touch but not cross the left boundary. This boundary may be interpreted as representing a "phantom" zeroth column in the symplectic tableau, a column containing $1,2,3,4,\ldots$. We also require that all steps between lines $x = c$ and $x = c+1$ for all $c \in Z$ are either all horizontal or all diagonal. The determination of whether these steps are horizontal or diagonal is made by the outside decomposition in the following manner. If boxes of content d are approached from the left, then steps between $x = d$ and $x = d+1$ must be horizontal; if the boxes of content d are approached from below, then steps between $x = d$ and $x = d+1$ must be diagonal. We are now ready to construct paths corresponding to strips.

Consider an outside decomposition $(\theta_1, \ldots, \theta_m)$ of λ/μ. We will construct a nonintersecting m–tuple of lattice paths that corresponds to a symplectic tableau of shape λ/μ with the outside decomposition $(\theta_1, \ldots, \theta_m)$, such that the ith path corresponds to the ith strip and begins at P_i and ends at Q_i, $i = 1, \ldots, m$ as described now. Fix points $P_i = (t-s, -(t-s)+1)$ if strip i has starting box on left perimeter in box (s,t) of the diagram (i.e. P_i is on the left boundary), or $P_i = (t-s, \infty)$ if strip i has starting box on the bottom perimeter in box (s,t) of the diagram ($P_i = (t-s,\infty)$ if both), $i = 1, \ldots, m$. Fix points $Q_i = (v-u+1, 1)$ if strip i has ending box on the top perimeter in box (u,v) of the diagram, or $Q_i = (v-u+1, \infty)$ if strip i has ending box on the right perimeter in box (u,v) of the diagram ($Q_i = (v-u+1,\infty)$ if both), $i = 1, \ldots, m$.

For strip θ_j construct a path starting at P_j (called the starting point) and ending at Q_j (called the ending point) as follows: if a box containing i (resp. $\overline{i}$) and at coordinates (a,b) in the diagram

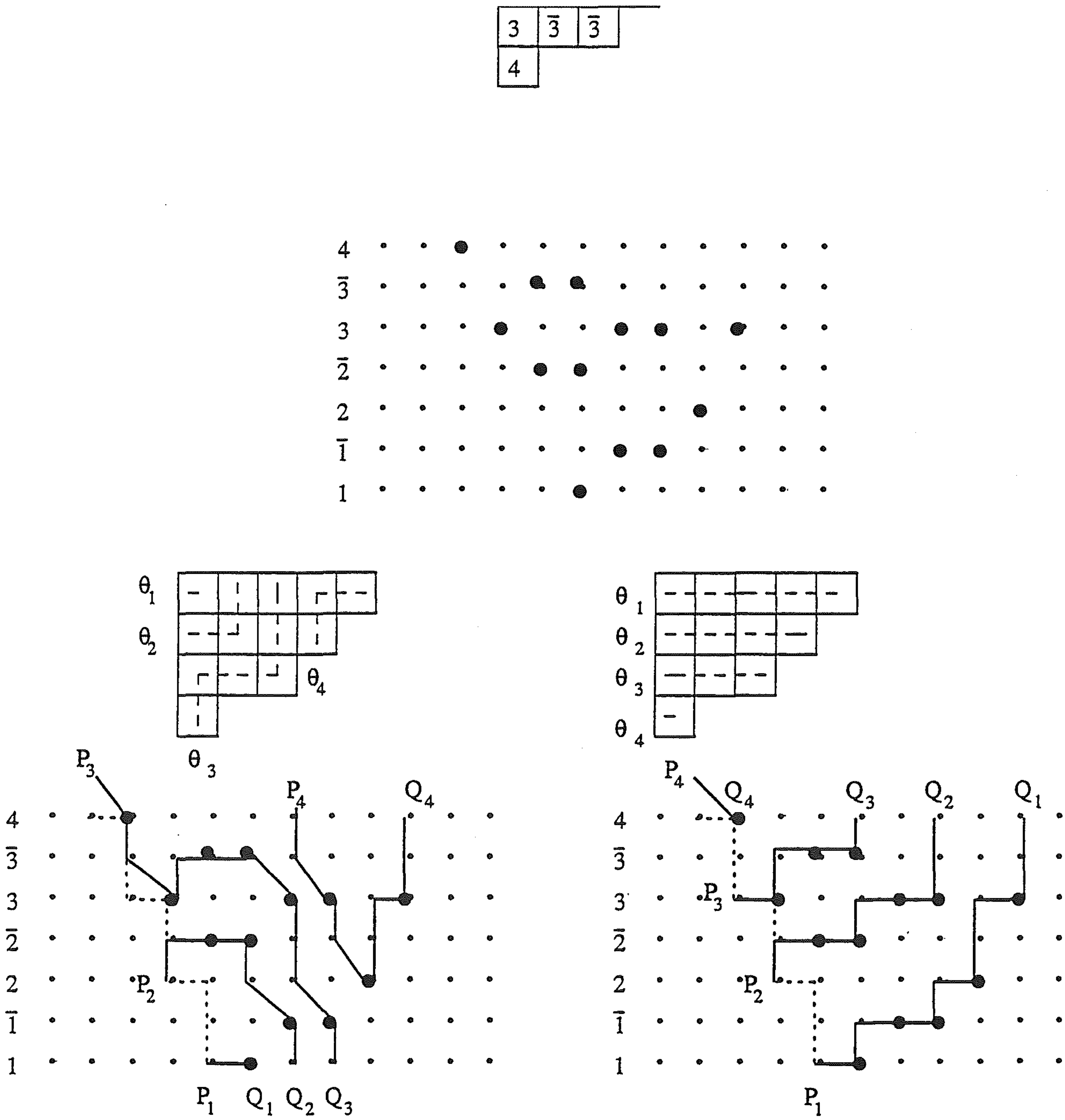

Figure 3: Two outside decompositions and the corresponding lattice paths.

is approached from the left in the strip, put a horizontal step from $(b-a, i)$ to $(b-a+1, i)$ (resp. $(b-a, \bar{i})$ to $(b-a+1, \bar{i})$); if a box containing i (resp. $\bar{i}$) and at coordinates (a, b) in the diagram is approached from below in the strip, put a diagonal step from $(b-a, \bar{i})$ to $(b-a+1, i)$ (resp. $(b-a, i+1)$ to $(b-a+1, \bar{i})$). Notice that the physical locations of the termination points of the steps are independent of the outside decomposition and depend only on the contents of the boxes. See Figure 3 in which first the ending points of steps are shown alone and then complete paths for two different outside decompositions are shown. Note that no two paths can have the same starting and/or ending points, since that would imply two boxes of the same content on the same section of perimeter. Connect these nonvertical steps with vertical steps. It is routine to verify that there is a unique way of doing this.

We must verify that an intersecting m–tuple of lattice paths does not correspond to a symplectic tableau. This follows from the column strictness and row weakness conditions on the symplectic tableau and also from the fact strips are themselves skew diagrams. The argument is a case–by–case analysis which follows exactly as in Hamel and Goulden [8] and we refer the reader to that paper.

The construction described above for producing paths given symplectic tableaux is reversible, and now we verify that a nonintersecting m–tuple of lattice paths obeying the path conditions corresponds to a symplectic tableau and an outside decomposition where each path in the nonintersecting m–tuple gives rise to a symplectic tableau of strip shape. The choice of the starting and ending points and the restrictions on the steps ensure that the m–tuple corresponds to a diagram of the required shape, but we must show that the entries in the tableau obey the column strictness and row weakness rules and also the symplectic condition. We begin by ensuring that a lattice path that starts at P_j and ends at Q_i corresponds to the strips $\theta_i \# \theta_j$. The proof follows exactly as in Hamel and Goulden [8]. Begin with the empty partition. At iteration k, if the kth nonvertical step from the left in the lattice path is horizontal ending at (i, j), then place a box containing j in the symplectic tableau to the right of the previous box; if it is diagonal ending at (i, j), then place a box containing j in the symplectic tableau on top of the previous box. The fact that a down–vertical step does not precede a horizontal step ensures that a horizontal step is at a height higher than or the same as the step just before it. This means the entries in a row of the symplectic tableau are weakly increasing. The fact that an up–vertical step does not precede a diagonal step ensures that a diagonal step ends at a height strictly lower than the step just before it. This means the entries in a column of the symplectic tableau are strictly increasing. Since the symplectic tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from θ_j and θ_i, since boxes of the same content correspond to the same type of step, and since the $\#$ operation is based on boxes of the same content, we know the strip is $\theta_i \# \theta_j$.

Now let $T(l, j)$ denote the entry in box (l, j) of the symplectic tableau. We claim $T(l, j) < T(l+1, j)$ and $T(l, j) \leq T(l, j+1)$. These inequalities are obvious if the boxes in question are in the same strip. Suppose they are not. Then the first claim follows from the fact that the paths are nonintersecting. To see this, suppose that the step starting at line $x = c$ in path i starts at height t. If this step is horizontal, $T(l, j) = t$ (resp. $\bar{t}$), and the step starting at line $x = c-1$ but in path $i+1$ must end at height $\bar{t}$ (resp. $t+1$) or higher to avoid intersection, implying $T(l+1, j) \geq \bar{t}$ (resp. $t+1$). If this step is diagonal, then the box $(l+1, j)$ must be in the same strip as (l, j), and so column strictness is guaranteed by the conditions internal to a path. The second claim follows again by the fact that the paths are nonintersecting. To see this, suppose that the step starting at line $x = c$ in path i starts at height t (resp. $\bar{t}$). If this step is horizontal, $T(l, j) = t$ (resp. $\bar{t}$), and the step starting at line $x = c+1$ but in path $i+1$ must start at height $\bar{t}$ (resp. $t+1$) or higher, implying $T(l, j+1) \geq t$ (resp. $\bar{t}$). If the step is diagonal, $T(l, j) = \overline{t-1}$ (resp. t), and the step starting at line $x = c+1$ in path $i+1$ must start at height t (resp. $\bar{t}$) or higher, implying $T(l, j+1) \geq \overline{t-1}$ (resp. t).

We must verify that both the individual strips and the entire tableau are symplectic. In both cases this follows from the left boundary and from the content–based nature both of the symplectic condition and the lattice path environment. The left boundary effectively implies that a step between lines $x = -c$ and $x = -c+1$ must occur at a height of $c+1$ or higher, i.e. the corresponding box of content $-c$ must contain an integer greater than or equal to $c+1$—the symplectic condition. This makes the situation clear for entire tableaux and also for individual strips, if we make this additional proviso: strips θ_i and θ_j for all i and j are to be considered as retaining their original contents (and passing them on to $\theta_i\#\theta_j$) and are not to be reinitialized with content 0 for the upper left hand corner, so that

$$\begin{array}{|c|}\hline 1 \\ \hline \bar{1} \\ \hline \end{array}$$

can still satisfy the symplectic condition if the contents of the boxes are, say, 2 and 1 respectively and not 0 and -1. The same content–intact provision has also been used previously in the case of factorial Schur functions, symmetric functions whose variables are modified by content. See Hamel [9].

For each horizontal or diagonal step that ends at (i,j), we choose a weight of x_j. For each horizontal or diagonal step that ends at $(i,\bar{j})$, we choose a weight of x_j^{-1}. For each up–vertical or down–vertical step, regardless of position, we choose a weight of one. Since there is a one–to–one correspondence between lattice paths and symplectic skew tableaux whose shape is a strip, the generating function for these lattice paths is the symplectic Schur function for the shape of a strip.

The proof now follows by the well–known Gessel–Viennot lattice path procedure. To obtain the full generality we require, we invoke the broader result of Stembridge [26, Theorem 1.2]. To do this we must insure that the only m–tuples of nonintersecting paths from starting points $P_1,\ldots,P_m$ to ending points $Q_1,\ldots,Q_m$ must connect P_i to Q_i for $i = 1,\ldots,m$; however, this is routine. Note that the introduction of a left boundary does not interfere with the intersecting/nonintersecting properties of the lattice paths. As has been demonstrated in Stembridge [26], the underlying structure does not have to be a lattice at all, but may be as general a structure as an acyclic digraph. Note additionally that although Stembridge does not impose conditions on which steps may follow each other (as we do in this proof), his theorem is still applicable since it is stated in terms of generating functions and is without reference to specific types of steps allowed. $\diamond$

We now present two corollaries to Theorem 3.1. One is an identity involving a determinantal form which has appeared previously in the literature; the other is a version of Theorem 3.1 for odd symplectic groups.

The literature contains some determinantal forms for $Sp(2n)$, although the subject does not appear to be as well–developed as for the classical Schur functions. There are bideterminantal forms dating back to Weyl [29] and Littlewood [15] and also more recent results due to Proctor [22]. Determinantal results in which each matrix element is expressed as a difference of symmetric functions can also be found in King [12], El Samra and King [5], Koike and Terada [14], Sagan [23], Stembridge [26] and Proctor [21] [22]. In addition El Samra and King [5] give a determinant which is a special case of Theorem 3.1 above for an outside decomposition into hooks (a Giambelli–type result, see Macdonald [16, p. 30]).

Corollary 3.3 (El Samra and King [5]) *Let λ be a partition. Then*

$$sp_\lambda = \det(sp_{\lambda_i-i+1,1^{\lambda'_j-j}}(X)).$$

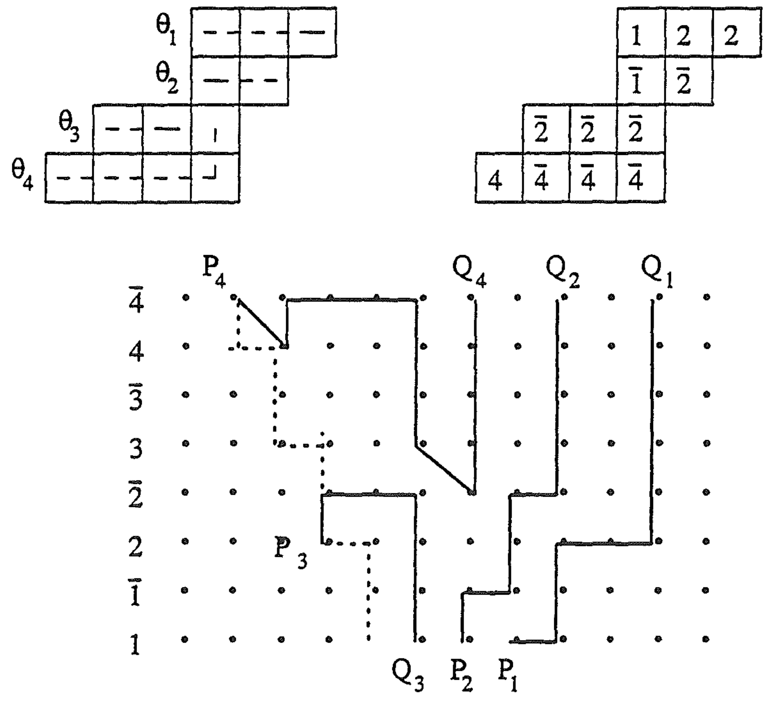

Figure 4: An outside decomposition of a skew symplectic tableau and the associated 4–tuple of lattice paths.

Proof: Theorem 3.1 with outside decomposition $\theta_1 = \lambda_1, \theta_2 = 1^{\lambda'_1-1}, \theta_3 = \lambda_2 - 1, \theta_4 = 1^{\lambda'_2-2}, \ldots,$ $\theta_{2r-1} = \lambda_r - r + 1, \theta_{2r} = 1^{\lambda'_r-r}$ where there are r boxes on the main diagonal of λ (i.e. r boxes of content 0). ◇

A second corollary to Theorem 3.1 concerns the odd symplectic groups as defined by Proctor [19]. The odd symplectic tableaux are an easy generalization of the symplectic tableaux we defined in Section 1. We generalize Proctor's original definition to skew shape.

Definition 3.4 *An odd symplectic tableau, $SPO_{\lambda/\mu}$, of shape λ/μ is a filling of the Ferrers diagram of λ/μ with the integers $1 < \overline{1} < 2 < \overline{2} \ldots < n < \overline{n} < n+1$ such that*

1. *the entries are weakly increasing along the rows and strictly increasing down the columns,*
2. *the boxes of content $-i$ contain entries which are greater than or equal to $i+1$.*

We can define a Schur-type function for these tableaux. The odd skew symplectic Schur function, $spo_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}, x_{n+1}$, is given by

$$spo_{\lambda/\mu}(X) = \sum_{SPO_{\lambda/\mu}} \prod_{\alpha \in SPO_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\overline{\beta} \in SPO_{\lambda/\mu}} x_\beta^{-m(\overline{\beta})},$$

where the sum is over all odd symplectic tableaux $SPO_{\lambda/\mu}$ of shape λ/μ, the first product is over all unbarred integers α in $SPO_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $SPO_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the mulitplicity of α (resp. $\overline{\beta}$) in $SPO_{\lambda/\mu}$.

Corollary 3.5 *Let λ/μ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of λ/μ,*

$$spo_{\lambda/\mu}(X) = \det(spo_{\theta_i \# \theta_j}(X))_{m \times m},$$

where $spo_\emptyset = 1$ and $spo_{undefined} = 0$.

Proof: Use the same lattice path set–up as for Theorem 3.1 with the y–axis labeled $1, \overline{1}, 2, \overline{2}, \ldots, n,$ $\overline{n}, n+1$. Then Stembridge's generalization [26] of the Gessel–Viennot lattice path argument provides the proof, as in Theorem 3.1. ◇

The proof of Theorem 3.2 is quite similar to that of Theorem 3.1. Since an *so*–tableau consists of a symplectic tableau adjoined to a (possible discontinuous) strip filled with ∞'s, the only difference between the proofs of Theorem 3.2 and Theorem 3.1 will be accounting for the presence of ∞. The ∞ has special characteristics which distinguish it from the integers filling the *so*–tableau. It is permitted to appear twice in the same column but not more than once in each row. This can be translated as "∞ is inserted row strictly and column weakly." Hence an *so*–tableau has two types of entries, those inserted row weakly and column strictly, and those inserted row strictly and column weakly. But this is precisely the arrangement for *supersymmetric tableaux*, tableaux which contain $1, 2, \ldots$ forming a row weak, column strict "inside shape," and $1', 2', \ldots$ forming a row strict, column weak "outside shape." These tableaux can be weighted by x_i for each entry i and y_i for each entry i', and supersymmetric Schur functions can be defined using this weighting. Results similar to Theorems 3.1 and 3.2 but for supersymmetric Schur functions can also be obtained in two different ways, either indirectly by replacing $\{x_1, x_2, \ldots\}$ by $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ in the main result of Hamel and Goulden [8] and applying ω_y where ω_y is the operator $\omega_y s_\lambda(Y) = s_{\lambda'}(Y)$, or directly using lattice paths as has been done in Hamel [10].

Proof of Theorem 3.2: Label the y–axis with $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}, \infty$. We call heights corresponding to any one of $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}$ *integer levels.* Fix a left boundary as in Theorem 3.1. Define lattice paths with five types of permissible steps—the four as in Theorem 3.1, and up–diagonal steps from height $\overline{n}$ to height ∞ that increase the x–coordinate by 1 and increase the y–coordinate by 1. We distinguish between horizontal steps at integer levels and horizontal steps at ∞. The steps are subject to same restrictions as in Theorem 3.1 plus the following additional restrictions: an up–vertical step must not precede a horizontal step at a ∞ level, and a down–vertical step must not precede an up–diagonal step. We also require that all steps between lines $x = c$ and $x = c+1$ for all c are either 1) horizontal at ∞ or down–diagonal, or 2) horizontal at integer levels or up–diagonal. The determination of whether the steps are of type 1) or 2) is made by the outside decomposition: if boxes of content d are approached from the left, then steps between $x = d$ and $x = d+1$ must be of type 2); if the boxes of content d are approached from below, then steps between $x = d$ and $x = d+1$ must be of type 1). Fix starting points and ending points as in Theorem 3.1 with the adjustment that the y–coordinate of the highest points is $\infty + 1$ instead of ∞ (this is so there is no conflict with the ∞ used here as a symbol). Given an *so*–tableau of shape λ/μ with an ouside decomposition, we can construct an m–tuple of nonintersecting lattice paths. For each strip construct a path as follows: if a box contains i or $\overline{i}$, place a step as in the proof of Theorem 3.1. If a box contains ∞, is at coordinates (a, b) in the diagram, and is approached from the left in the strip, put an up–diagonal step from $(a-b, \overline{n})$ to $(a-b+1, \infty)$; if it is approached from below, put a horizontal step from $(a-b, \infty)$ to $(a-b+1, \infty)$. Connect these steps with vertical steps. It is routine to verify that there is a unique way of doing this.

We verify that an intersecting m–tuple of lattice paths does not correspond to an *so*–tableau. This can be verified by a case–by–case analysis as in Theorem 3.1. Precise details on cases for lattice paths with the five distinct types of steps used here can be found in Hamel [10] where decomposition results for the Schur Q–functions and supersymmetric functions are proved.

The construction described above for generating paths given *so*–tableaux is reversible, and now we verify that a nonintersecting m–tuple of lattice paths obeying these conditions corresponds to an so–tableau with the given outside decomposition. We begin by ensuring that a lattice path that starts at P_j and ends at Q_i corresponds to the strip $\theta_i \# \theta_j$. The proof is as follows. Begin with the empty partition. At iteration k, if the kth nonvertical step from the left is horizontal or down–diagonal, proceed as in Theorem 3.1. If it is horizontal ending at (i, ∞), then place a box containing ∞ on top of the previous box. If it is up–diagonal ending at (i, ∞), then place a box containing ∞ in the *so*–tableau beside the previous box. As in the proof of Theorem 3.1, the path restrictions ensure the entries in a row of the *so*–tableau are weakly increasing, and integer entries in a column of the *so*–tableau are strictly increasing. Since the *so*–tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from θ_j and θ_i, since boxes of the same content correspond to the same type of step, and since the $\#$ operation is based on boxes of the content, we know the strip is $\theta_i \# \theta_j$.

Let $T(l, j)$ denote the entry in box (l, j) of the *so*–tableau. The inequalities $T(l, j) \leq T(l, j+1)$ and $T(l, j) < T(l+1, j)$ for $T(l, j)$ integer (row weakness and column strictness) follows from the arguments in the proof of Theorem 3.1 and from the fact ∞ is greater than $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}$. Now consider the case where $T(l, j)$ is ∞. We must show $T(l, j+1)$ does not exist and $T(l, j) = T(l+1, j)$. These assertions are obvious if the boxes in question are in the same strip, so suppose they are not. Consider $T(l, j+1)$. Suppose the step in path i starting at line $x = c$ and representing $T(l, j)$ ends at height ∞. Then the step starting at line $x = c+1$ but in path $i+1$ must start at a height higher than ∞ to avoid intersection. This is impossible and hence $T(l, j+1)$ does not exist. Consider now $T(l+1, j)$. Suppose again the step in path i starting at line $x = c$ and representing $T(l, j)$ ends at height ∞. If this step is horizontal, the step starting at line $x = c-1$ but in path $i+1$ must end at a height higher than ∞ to avoid intersection, implying $T(l+1,j)$ does not exist. If this step is up–diagonal, the step starting at the line $x = c-1$ but in path $i+1$ must end at height ∞ and $T(l+1,j) = \infty$.

The verification that the symplectic condition is satisfied for these *so*–tableaux follows from the same argument as in Theorem 3.1.

For each horizontal or diagonal step that ends at (i,j), we choose a weight of x_j. For each horizontal or diagonal step that ends at $(i,\bar{j})$, we choose a weight of x_j^{-1}. For each horizontal or diagonal step that ends at (i,∞), choose a weight of one. For each up–vertical or down–vertical step, regardless of position, we choose a weight of one. Since there is a one–to–one correspondence between lattice paths and skew *so*–tableaux whose shape is a strip, the generating function for these lattice paths is the *so*–Schur function for the shape of a strip.

The proof now follows as in Theorem 3.1 by Stembridge's generalization of Gessel–Viennot [26, Theorem 1.2]. ◇

4 Rational Tableaux

This final section gives determinantal results for rational Schur functions (also called composite Schur functions). The tableaux underlying these functions are *rational tableaux* defined originally for standard shape by King [11]. We take a modified version due to Stembridge [25]. First, however, we define a new type of shape. A Ferrers diagram of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$ is defined as follows. Take the Ferrers diagram of ν/ρ and reflect it first about a vertical axis along its left perimeter and then about a horizontal axis along its top perimeter. Place it to the left of the Ferrers diagram of λ/μ such that the content zero boxes form a continuous diagonal. See the diagrams on the left side of Figure 6.

Definition 4.1 *A rational tableau, $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$, of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$, where we let $T_{\overline{\nu}/\overline{\rho}}$ denote the $\overline{\nu}/\overline{\rho}$ portion and $T_{\lambda/\mu}$ denote the λ/μ portion and where we let $T_{\overline{\nu}/\overline{\rho}}(i,j)$ (resp. $T_{\lambda/\mu}(i,j)$) denote the entry in box (i,j) of $T_{\overline{\nu}/\overline{\rho}}$ (resp. $T_{\lambda/\mu}$), is a filling of the Ferrers diagram of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$ such that*

1. *$T_{\overline{\nu}/\overline{\rho}}$ is filled with integers from $\overline{1} < \overline{2} < \ldots < \overline{n}$.*
2. *$T_{\lambda/\mu}$ is filled with integers from $1 < 2 < \ldots < n$.*
3. *The entries in $T_{\overline{\nu}/\overline{\rho}}$ strictly decrease in the columns and weakly decrease in the rows.*
4. *The entries in $T_{\lambda/\mu}$ strictly increase in the columns and weakly increase in the rows.*
5. *$|\{\overline{j} : T_{\overline{\nu}/\overline{\rho}}(j,1) \leq \overline{i}| + |\{j : T_{\lambda/\mu}(j,1) \leq i\}| \leq i$ for $1 \leq i \leq n$.*

The skew rational Schur function, $s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X) = \sum_{T_{\overline{\nu}/\overline{\rho};\lambda/\mu}} \prod_{\alpha \in T_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\overline{\beta} \in T_{\overline{\nu}/\overline{\rho}}} x_\beta^{-m(\overline{\beta})},$$

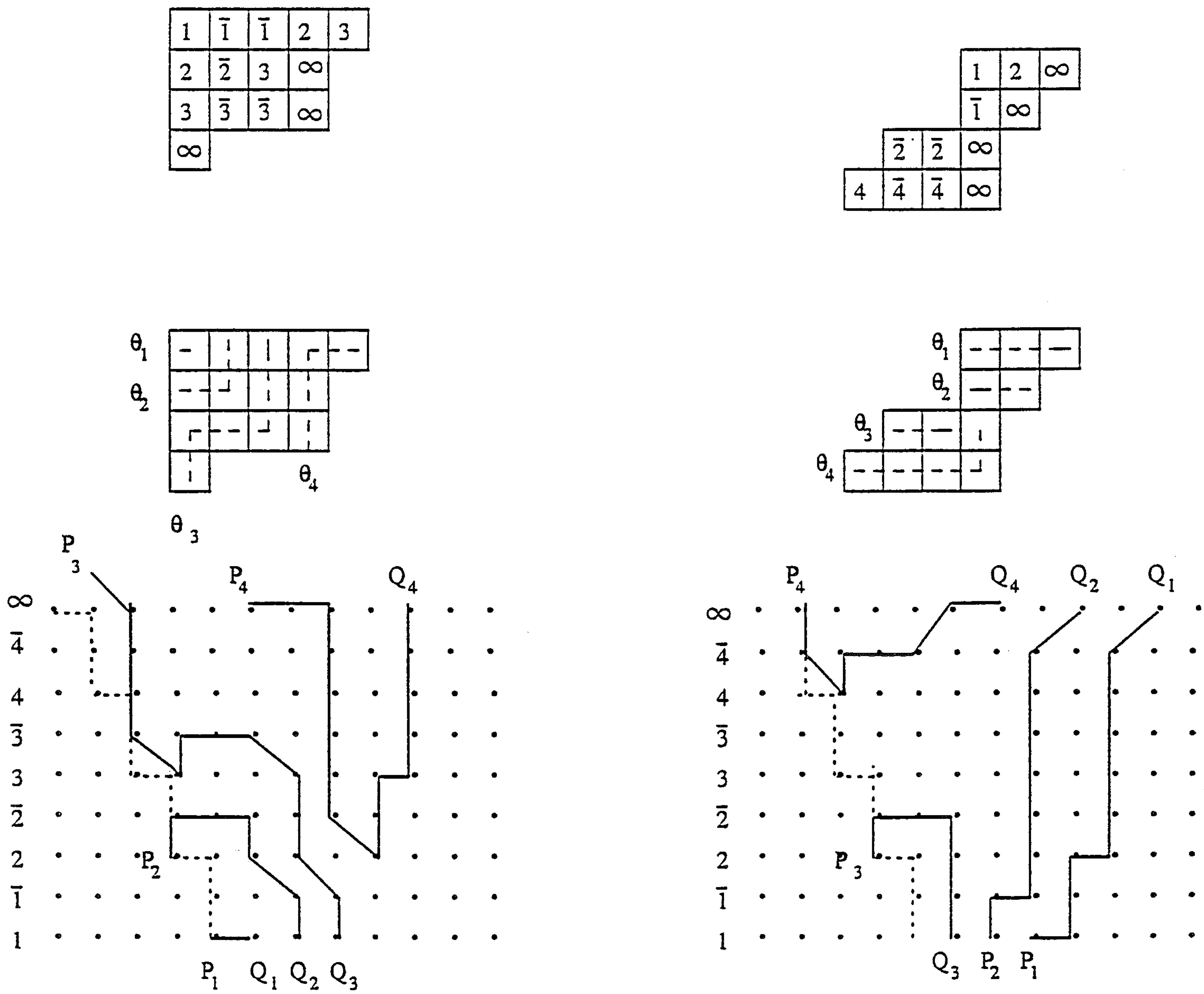

Figure 5: Outside decompositions of two *so*–tableaux and the corresponding paths.

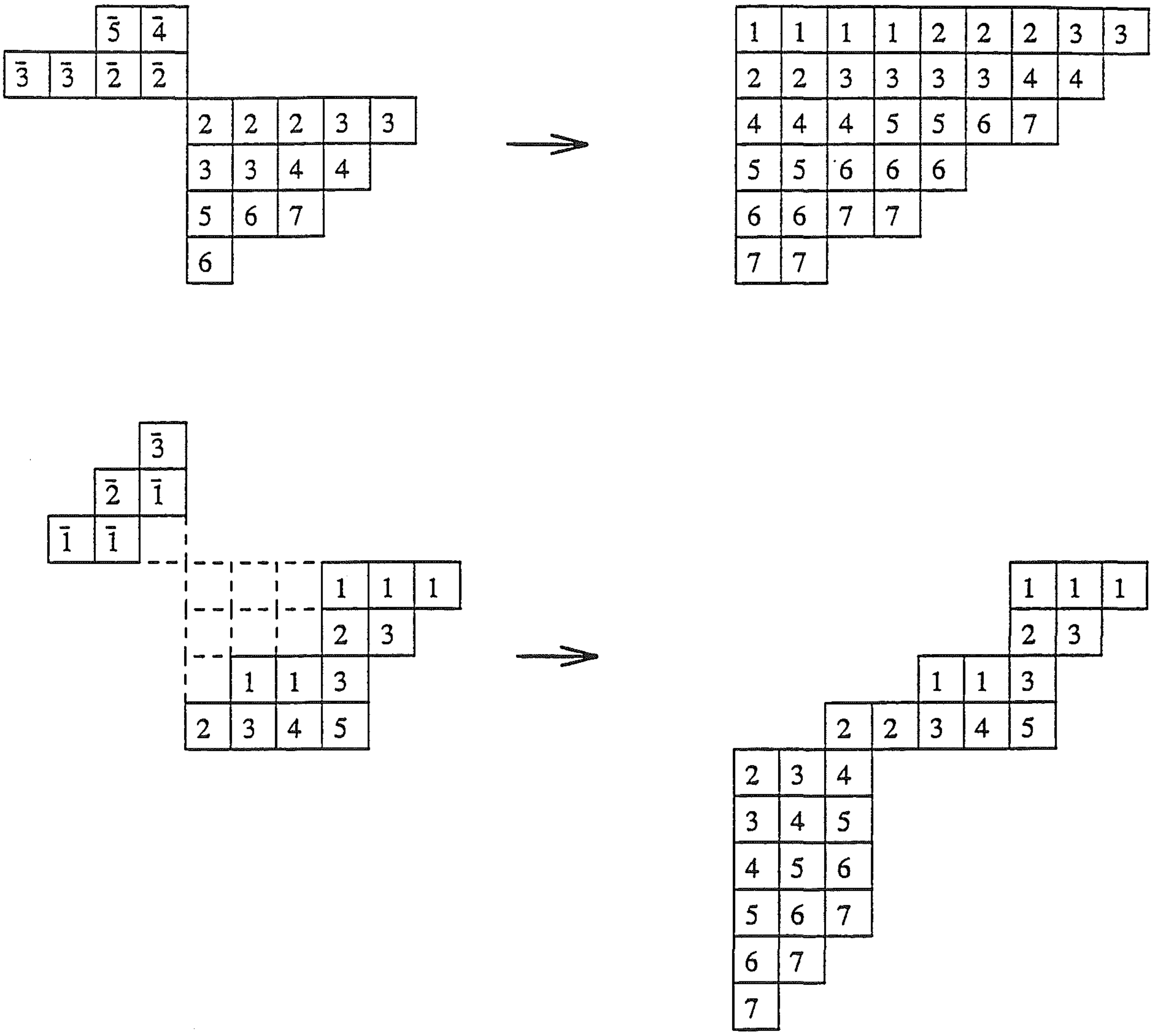

Figure 6: Example of rational tableaux and their complements.

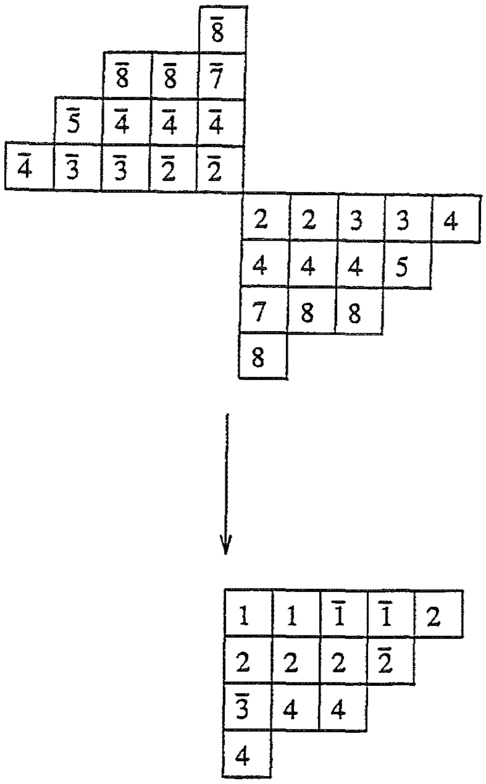

Figure 7: Transformation from rational to symplectic tableau.

where the sum is over all tableaux $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$ of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$, the first product is over all unbarred integers α in $T_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $T_{\overline{\nu}/\overline{\rho}}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the mulitplicity of α (resp. $\overline{\beta}$) in $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$.

As mentioned in Section 1, the standard shape symplectic tableaux are actually a special case of the standard shape rational tableaux. This correspondence has been outlined by Stembridge [25] and proceeds as follows. Let $T_{\overline{\lambda};\lambda}$ be a standard shape rational tableau such that, if we ignore the bars on elements in $T_{\overline{\lambda}}$ and rotate $\overline{\lambda}$ to the same orientation as λ, then $T_{\overline{\lambda}} = T_{\lambda}$, and such that the largest entry is less than or equal to $2n+1$. Let T'_{λ} be the tableau obtained from T_{λ} by replacing $2 < 3 < \ldots < 2n < 2n+1$ by $1 < \overline{1} < \ldots < n < \overline{n}$ (note T_{λ} will not contain 1 because of restriction 5 in Definition 4.1). Then T'_{λ} satisfies the symplectic condition and is a symplectic tableau. See Figure 7.

Rational tableaux are also related to ordinary tableaux, and a rational tableau gives rise to an ordinary tableau in the following manner. Take the complement in $1, 2, \ldots, n$ of the entries in each column of $T_{\overline{\nu}/\overline{\rho}}$ and place these complements to the left of the columns in $T_{\lambda/\mu}$ such that the resulting diagram has shape γ/α, where it is most natural to define γ/α in terms of columns:

$$\gamma'_i = \begin{cases} n - \nu'_{\nu_1-i+1} & 1 \leq i \leq \nu_1 \\ \lambda'_{i-\nu_1} & \nu_1 i + 1 \leq i \leq \nu_1 + \lambda_1 \end{cases}$$

$$\alpha'_i = \begin{cases} \mu'_1 + \rho'_1 - \rho'_{\nu_1-i+1} & 1 \leq i \leq \nu_1 \\ \mu'_{i-\nu_1} & \nu_1 + 1 \leq i \leq \nu_1 + \lambda_1. \end{cases}$$

Call the tableau of shape γ/α the *complement* of the tableau of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$. See Figure 6.

It is then obvious that, for $n \geq \lambda'_1 + \nu'_1$,

$$(x_1 x_2 \ldots x_n)^{\nu_1} s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X) = s_{\gamma/\alpha}(X), \tag{2}$$

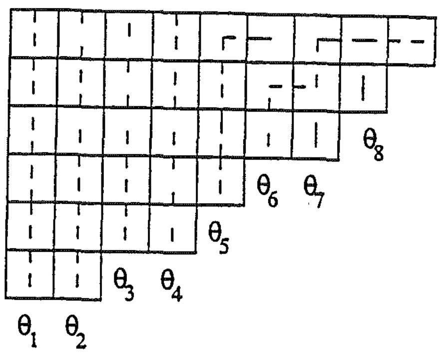

Figure 8: Example of a columns–first outside decomposition.

where $s_{\gamma/\alpha}(X)$ is the ordinary skew Schur function of shape γ/α as defined in Macdonald [16] or Sagan [24].

A special case of (2) is

$$(x_1 \dots x_n) s_{\overline{1^k}} = s_{1^{n-k}}(X). \tag{3}$$

We can use (2) and (3) and the main result of Hamel and Goulden [8] to prove a determinantal result, Theorem 4.3, for rational Schur functions. However, this result does not apply to all outside decompositions, but only to those having the form described in the next definition (see Figure 8). The restriction to such outside decompositions is a restriction necessitated by a transformation performed in the proof, and it is likely that a more general form of Theorem 4.3—one as general as Theorems 3.1 and 3.2—can be proved.

Definition 4.2 *A columns–first outside decomposition $(\theta_1, \dots, \theta_m)$ of shape $\overline{\nu}/\overline{\rho}; \lambda/\mu$ is an outside decomposition such that $\theta_1 = \nu'_{\nu_1} - \rho'_{\nu_1}, \theta_2 = \nu'_{\nu_1-1} - \rho'_{\nu_1-1}, \dots, \theta_{\nu_1} = \nu'_1 - \rho'_1$ and such that $(\theta_{\nu_1+1}, \dots, \theta_m)$ is an outside decomposition of λ/μ where the only strip allowed to start in the first column of λ/μ is the strip starting in the $(\lambda'_1, 1)$ box.*

Given any columns–first outside decomposition of $\overline{\nu}/\overline{\rho}; \lambda/\mu$ there is a related outside decomposition, θ^c, of the complement, γ/α, where strips $\theta^c_i = 1^n/\theta_i$ for $1 \le i \le \nu_1$ and $\theta^c_i = \theta_i$ for $\nu_1 + 1 \le i \le m$.

Theorem 4.3 *Let $\overline{\nu}/\overline{\rho}; \lambda/\mu$ be a shape. Then for any columns–first outside decomposition $(\theta_1, \dots, \theta_m)$ of $\overline{\nu}/\overline{\rho}; \lambda/\mu$ and corresponding outside decomposition $(\theta^c_1, \dots, \theta^c_m)$ of γ/α,*

$$s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X) = \det \begin{pmatrix} (s_{\overline{1^n/(\theta^c_i \# \theta^c_j)}}(X))_{1 \le i \le \nu_1; 1 \le j \le \nu_1} & \vdots & (s_{\theta^c_i \# \theta^c_j}(X))_{1 \le i \le \nu_1, \nu_1+1 \le j \le m} \\ \dots & \vdots & \dots \\ (s_{\overline{1^n/(\theta^c_i \# \theta^c_j)'_1}; (\theta^c_i \# \theta^c_j)/(\theta^c_i \# \theta^c_j)'_1}(X))_{\nu_1+1 \le i \le m, 1 \le j \le \nu_1} & \vdots & (s_{\theta_i \# \theta_j}(X))_{\nu_1+1 \le i \le m, \nu_1+1 \le j} \end{pmatrix}$$

where $(\theta^c_i \# \theta^c_j)'_1$ is the first column of $\theta^c_i \# \theta^c_j$.

Proof: From Theorem 3.1 of Hamel and Goulden [8], the following identity holds:

$$s_{\gamma/\alpha}(X) = \det\left(s_{\theta_i\#\theta_j}(X)\right). \tag{4}$$

Apply (2) to the left hand side of (4); apply (3) to each of the first ν_1 columns on the right hand side of (4). The result follows. $\diamond$.

An example can be found in Figure 9. The determinants corresponding to the outside decomposition in Figure 9 are as follows (note $s_{1^k} = 0$ for $k > n$).

$$s_{987542} = \det\begin{pmatrix} s_{111111} & s_{11111} & s_{111} & s_{11} & 1 & 0 & 0 & 0 \\ s_{1111111} & s_{111111} & s_{1111} & s_{111} & s_1 & 0 & 0 & 0 \\ 0 & s_{1111111} & s_{11111} & s_{1111} & s_{11} & 1 & 0 & 0 \\ 0 & 0 & s_{111111} & s_{11111} & s_{111} & s_1 & 1 & 0 \\ 0 & 0 & s_{2111111} & s_{211111} & s_{2111} & s_{21} & s_2 & 0 \\ 0 & 0 & s_{42111111/1} & s_{4211111/1} & s_{42111/1} & s_{421/1} & s_{42/1} & s_3 \\ 0 & 0 & s_{1111111} & s_{111111} & s_{1111} & s_{11} & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_{22111/1} & s_{221/1} & s_{22/1} & s_1 \end{pmatrix}$$

$$s_{\overline{42};5431} = \det\begin{pmatrix} s_{\overline{1}} & s_{\overline{11}} & s_{\overline{1111}} & s_{\overline{11111}} & 1 & 0 & 0 & 0 \\ 1 & s_{\overline{1}} & s_{\overline{111}} & s_{\overline{1111}} & s_1 & 0 & 0 & 0 \\ 0 & 1 & s_{\overline{11}} & s_{\overline{111}} & s_{11} & 1 & 0 & 0 \\ 0 & 0 & s_{\overline{1}} & s_{\overline{11}} & s_{111} & s_1 & 1 & 0 \\ 0 & 0 & s_{\emptyset;1} & s_{\overline{1};1} & s_{2111} & s_{21} & s_2 & 0 \\ 0 & 0 & s_{\emptyset;31} & s_{\overline{1};31} & s_{42111/1} & s_{421/1} & s_{42/1} & s_3 \\ 0 & 0 & 1 & s_{\overline{1}} & s_{1111} & s_{11} & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_{22111/1} & s_{221/1} & s_{22/1} & s_1 \end{pmatrix}$$

The following corollary due to Balankentin and Bars [1] for standard shape has been proved by Cummins and King [4] using the same complementing transformation technique as in the proof of Theorem 4.3.

Corollary 4.4 *Let $\overline{\nu};\lambda$ be a shape. Then*

$$s_{\overline{\nu};\lambda}(X) = \det\left(\left(e_{\nu'_{\nu_1-j+1}-j+i}(X)\right)_{1\le i\le\nu_1+\lambda_1,1\le j\le\nu_1} \vdots \left(e_{\lambda'_{j-\nu_1}-j+v_1+i}(X)\right)_{1\le i\le\nu_1+\lambda,\nu_1+1\le j\le\nu_1+\lambda_1}\right).$$

Proof: Theorem 4.3 with outside decomposition $\theta_i = \nu'_i$ for $1 \le i \le \nu_1$ and $\theta_i = \lambda'_{\nu_1-i}$ for $\nu_1 + 1 \le i \le \lambda_1 + \nu_1$. $\diamond$

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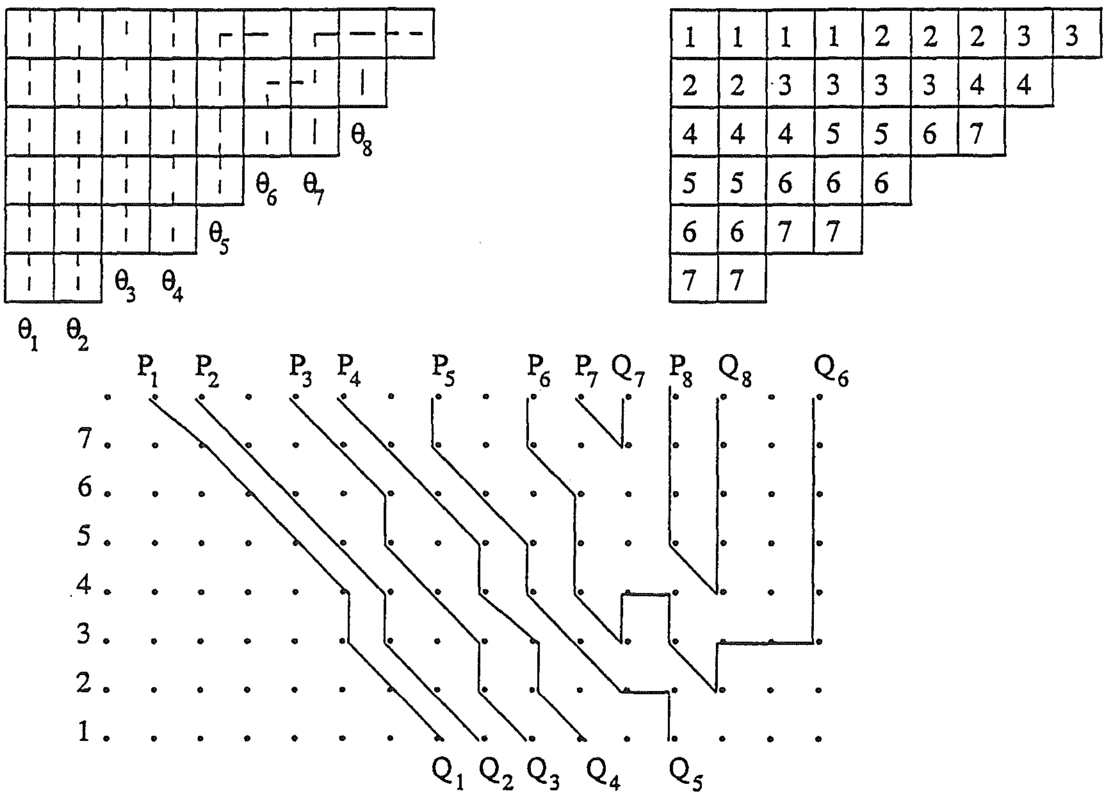

Figure 9: An outside decomposition of the complement of a rational tableau and a corresponding 8–tuple of lattice paths.

References

[1] A.B. Balantekin and I. Bars, *Representations of supergroups*, J. Math. Phys. 22 (1981), 1810–1818.

[2] G. Benkart and J. Stroomer, *Tableaux and insertions schemes for spinor representations of the orthogonal Lie algebra* $so(2r+1, C)$, J. Combin. Theory A 57 (1991), 211–237.

[3] A. Berele, *A Schensted-type correspondence for the symplectic group*, J. Combin. Theory A 43 (1984), 320–328.

[4] C.J. Cummins and R.C. King, *Composite Young diagrams, supercharacters of* $U(M/N)$ *and modification rules*, J. Phys. A 20 (1987), 3121–3133.

[5] N. El Samra and R.C. King, *Reduced determinantal forms for characters of the classical Lie groups*, J. Phys. A. 12 (1979), 2305–2315.

[6] I. Gessel and G.X. Viennot, *Determinants, paths, and plane partitions*, preprint.

[7] I.P. Goulden and D.M. Jackson, *Combinatorial Enumeration*, Wiley, New York, 1983.

[8] A.M. Hamel and I.P. Goulden, *Planar Decompositions of Tableaux and Schur Function Determinants*, University of Waterloo Research Report CORR 94-09.

[9] A.M. Hamel, *Algebraic and Combinatorial Methods for Symmetric and Supersymmetric Functions*, Ph.D. Dissertation, University of Waterloo, Waterloo, Ont. Canada. April 1994.

[10] A.M. Hamel, *Pfaffians and determinants for Schur Q-functions*, University of Canterbury Research Report No. 111 (1994).

[11] R.C. King, *Generalized Young tableaux and the general linear group*, J. Math. Phys. 11 (1970), 280–293.

[12] R.C. King, *Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups*, J. Math. Phys. 12 (1971), 1588–1598.

[13] R.C. King, *Weight multiplicities for the classical groups*, in Lecture Notes in Physics, Vol. 50, pp. 490–499, Springer–Verlag, New York, 1975.

[14] K. Koike and I. Terada, *Young-diagramatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank*, Adv. in Math 79 (1990), 104–135.

[15] D.E. Littlewood, *The Theory of Group Characters*, 2nd ed., Oxford Univ. Press, Oxford, 1950.

[16] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, Oxford. 1979.

[17] S. Okada, *A Robinson-Schensted-type algorithm for* $SO(2n, C)$, J. Algebra 143 (1991), 334–372.

[18] S. Okada, *Robinson–Schensted-type algorithms for the spinor representations of the orthogonal Lie algebra* $so(2n, C)$, J. Algebra 158 (1993), 155–200.

[19] R. Proctor, *Odd symplectic groups*, Invent. Math. 92 (1988), 307–332.

[20] R. Proctor, *A Schensted algorithm which models tensor representations of the orthogonal group*, Canad. J. Math. 42 (1990), 28–49.

[21] R. Proctor, *A generalized Berele-Schensted algorithm and conjectured Young tableaux for intermediate symplectic groups*, Trans. Amer. Math. Soc. 324 (1991), 655–692.

[22] R. Proctor, *Young tableaux, Gelfand patterns, and branching rules for classical groups*, J. Algebra 164 (1994), 299–360.

[23] B.E. Sagan, *The ubiquitous Young tableau*, in Invariant Theory and Tableau, IMA Volumes in Mathematics and its Applications, No. 19, Dennis Stanton ed., Springer–Verlag, New York, 1990, 262–298.

[24] B.E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Wadsworth & Brooks/Cole, Pacific Grove, Calif. 1991.

[25] J.R. Stembridge, *Rational tableaux and the tensor algebra of gl_n*, J. Combin. Theory A 46 (1987), 79–120.

[26] J.R. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. in Math. 83 (1990), 96–131.

[27] S. Sundaram, *The Cauchy identity for $Sp(2n)$*, J. Combin. Theory A 53 (1990), 209–238.

[28] S. Sundaram, *Orthogonal tableaux and an insertion algorithm for $SO(2n+1)$*, J. Combin. Theory A 53 (1990), 239–285.

[29] H. Weyl, *The Classical Groups*, 2nd ed., Princeton Univ. Press, Princeton, 1946.