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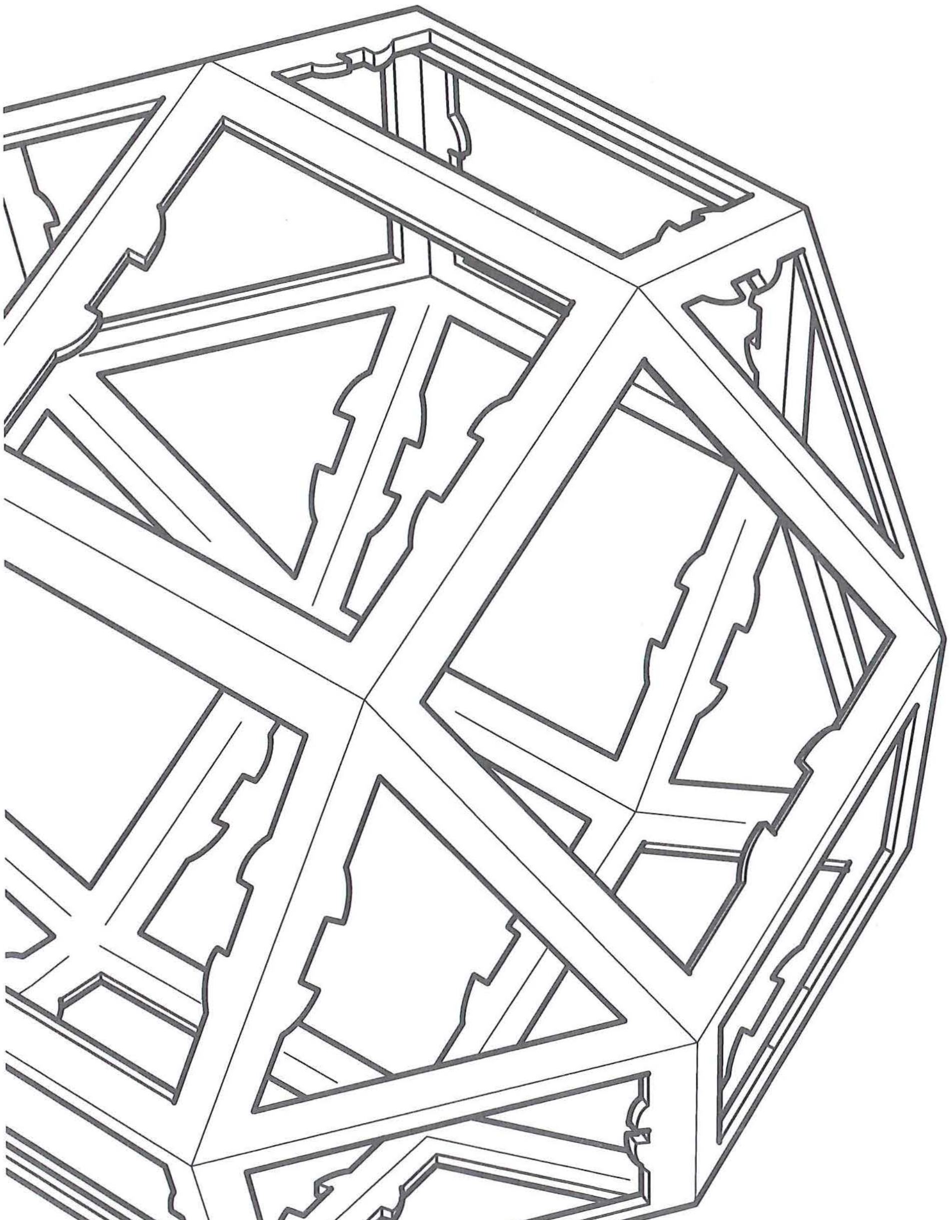
Summer Research Project

Non-Linear Dynamics

by Matthew Botur



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Non-Linear Dynamics

Matthew Botur

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Non-linear dynamics are a vast and complex field, relevant to many scientific disciplines. How systems change within parameter space is fundamental to understanding much real world phenomena, often revealing behaviour more complex than would otherwise be thought. This paper provides an introduction to non-linear dynamics, focusing on fixed points and how systems bifurcate. Three cases studies are provided here, so as to demonstrate these techniques within a framework of particular relevance within New Zealand.

Introduction

We begin our discussion by imagining an empty void, free of all known physical laws and constraints existent in the real world. Since this void is a purely imaginary construct, we may choose any number dimension space, as well as any shape we like, such as a torus (donut shape) or cylinder. For the purpose of this example, a 3 dimensional cube shall satisfy. This empty void is our phase space, to which we entirely define our laws of motion. These laws are continuous everywhere in the phase space, as well as continuously differentiable everywhere in the phase space. Now, imagine a single particle, a phase point, arbitrarily placed within the phase space. How it moves is governed either by a system of differential equations (also known as a flow), which describes motion for continuous time, or by a system of iterated maps, which describes motion for discrete time. Since motion depends entirely on a phase point's position within the phase space, the phase space is completely filled with trajectories, none of which can ever cross. Together, these trajectories are called a phase portrait, the nature of which is the end goal of all further analysis.

From experience within the real world, it is obvious that behaviour is likely to be more complex than simply all trajectories escaping to infinity. Some points in space attract nearby trajectories, others may actively repel. Orbits may exist, as may spirals, as may areas of faster or slower motion. These all depend on how we define motion in our phase space (and, indeed, within which phase space), as a means by which certain behaviour may be modeled and explained. As we develop our understanding, we shall see how as we increase our dimension space we allow for ever more interesting dynamics, culminating in fully chaotic behaviour.

Fixed Points

Let us start with the simplest of phase spaces, that of 1 dimension (i.e. a line), and define motion using a differential equation, which we shall focus on throughout the course of this paper.

$$\dot{X} = \sin(x) \tag{1}$$

Consider a phase point placed anywhere between $-\pi/2$ and $\pi/2$. It has positive velocity, and hence will move forward throughout this interval. At $\pi/2$, the phase point has zero velocity, $\dot{X} = 0$, which we term a fixed point, $x*$. Any phase point at a fixed point will stay at a fixed point for all time, and in this system there are in fact infinitely many fixed points, such as at $-\pi/2$ or $3\pi/2$. However, the fixed points at $\pi/2$ and $-\pi/2$ are of very different nature. If we look at an arbitrarily small perturbation from the fixed point at $\pi/2$, we see that in either direction motion is directed back towards the fixed point. $\pi/2$ is an attracting fixed point, which we shall call stable. At $-\pi/2$, any arbitrarily small perturbation from the point will see the phase point move further away from $-\pi/2$, in the direction of the perturbation, towards either $-3\pi/2$ or $\pi/2$. Phase points are repelled from $-\pi/2$, which we call an unstable fixed point.

Stability depends on the motion of phase points moved an arbitrarily small distance away from a fixed point. If such a disturbance damps out over time, a point is stable, whereas if it grows, a point is unstable. Think of a ball situated at either the top of a hill or at the bottom of a depression, displaced slightly from its resting position. On a hill the ball will roll away from the apex, while within the depression the ball will return to its center. Notice that in either case the displacement from the fixed point must be relatively small; taking the ball from a depression to a flat plain is not a mere disturbance, and would say nothing of the depression's stability. We can define stability more rigorously using a linearization about $x*$.

Let $\nu(t) = x(t) - x*$ represent a small disturbance from a fixed point. To see whether a disturbance grows or decays, we derive a differential equation for $\nu(t)$.

$$\dot{\nu} = d/dt(x - x*) = \dot{x} = f(x) = f(x * + \nu) \tag{2}$$

Using Taylor's expansion

$$f(x * + \nu) = f(x*) + \nu f'(x*) + 0(\nu^2) \tag{3}$$

Since $x*$ is a fixed point, $f(x*) = 0$. Furthermore, if $f'(x*) \neq 0$, quadratic terms are negligible (i.e. insufficient to change the nature of the point), resulting in the linear approximation

$$\dot{\nu} = \nu f'(x*) \tag{4}$$

This shows that if $f'(x*) > 0$ the perturbation ν(t) grows exponentially, whereas if $f'(x*) < 0$ a perturbation decays exponentially. Where $f'(x*) = 0$ a linear approximation cannot be used to tell the stability of the point, as quadratic terms are not negligible. How stable a fixed point is may also now be defined, based on the magnitude of $f'(x*)$. $1/|f'(x*)|$ is the characteristic time scale, which measures the time required for x(t) to vary significantly in the neighbourhood of $x*$. A smaller characteristic time scale means that a system 'bounces back' towards a fixed point more rapidly, and is thus more stable.

$\hookrightarrow$ σ moves away from

Bifurcations

Fixed points form the basis of all further analysis for the simple, yet somewhat surprising, fact that they may be created, destroyed or changed in stability. This is achieved with the introduction of parameters. Let's return to our example of a ball situated on the top of a hill. Physically, that ball rolls down the hill because gravity pulls it downwards. Yet gravity may be thought of as a parameter within this system, which we could increase, set to zero, or reverse entirely. The nature of what happens when the ball is pushed slightly is qualitatively different between positive and negative gravity- zero gravity, then, is a bifurcation point. The term itself is slightly misleading, as a bifurcation point is not a point within a system at all, but the point where changing the parameters of the system changes the system's qualitative dynamics. If by changing a parameter all that is achieved is moving a fixed point to a different place, this is not a bifurcation. The system is still topologically equivalent (qualitatively similar), as it still has a fixed point. If, instead, a fixed point is destroyed as a parameter is varied, then the system has undergone a saddle-node bifurcation (also known as a turning point, or fold). unclear

An important aspect of system analysis is how a bifurcation occurs, before, at, and after a fixed point. Consider the prototypical saddle-node bifurcation example, $\dot{x} = r + x^2$, where r is our parameter. For $r < 0$ the system has two fixed points, $\pm\sqrt{r}$. As r tends to 0, these two fixed points approach each other, until at $r = 0$ they coincide and the fixed point is half-stable (note that at this half-stable fixed point $f'(x*) = 0$; this is a new form of stability, which our earlier linearization could not predict, but not the only possible outcome). $r > 0$ results in mutual annihilation of the two fixed points, following their collision at $r = 0$. In terms of our hills and valleys example, turning off gravity would be a saddle-node bifurcation; as we lessen gravity we retain our fixed points, but once gravity begins to act in reverse, both disappear. At zero gravity, a hill is qualitatively the same as a valley, half-stable to the ball. Once gravity is reversed and no longer pulls objects down, neither has any relevance to a balls motion. Of course from the opposite direction, starting with $r > 0$ and decreasing r past zero, we see the reverse, a half-stable fixed point appearing from nowhere, then splitting into two fixed points, one stable, one unstable. A good way to display this is via a bifurcation diagram, as below, plotting r against x for $\dot{x} = r + x^2$. It should be remembered that this diagram depicts r as an independent variable, towards analysis of the structural changes that occur within the system. I disagree

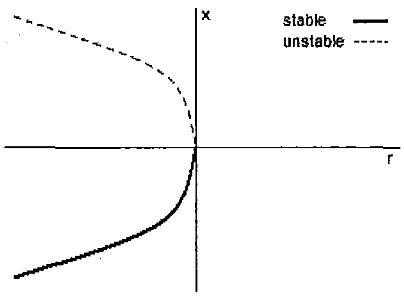


Figure 1: Saddle-Node Bifurcation Diagram

Bifurcations should not be thought limited to linear motion; dimension space is an arbitrary construct, so we may equally describe motion on a circle instead. The non-uniform oscillator $\dot{\theta} = \omega - a\sin(\theta)$ can be used to describe a phase point moving around a circle ($-\pi < \theta < \pi$) at varying speeds. For $a < \omega$, the phase point continually slows down close to $\pi/2$, and accelerates towards $-\pi/2$, producing sinusoidal motion. However, as a $\longrightarrow \omega$, the phase point bottlenecks at $\pi/2$, and takes longer and longer to move past this point. Consider the oscillation period:

$$T = integral(dt) = \int_{0}^{2\pi} ((dt/d\theta)d\theta) \tag{5}$$

$$= \int_{0}^{2\pi} (d\theta/\omega - a(\sin\theta)) \tag{6}$$

$$= 2\pi/\sqrt{\omega^2 - a^2} \tag{7}$$

which for $\omega \approx a$ approaches ∞.

Eventually, when $a = \omega$, the phase point does not move past $\pi/2$ at all, and a half-stable fixed point arises. Where $a > \omega$, this half-stable fixed point splits into a stable fixed point and an unstable fixed point. This, of course, is just a saddle-node bifurcation from the opposite direction.

Case Study: Possums

We are now suitably equipped to develop and analyze our first model, which we shall base around the role of possums in New Zealand. The common Brushtail Possum (*Trichosurus vulpecula*) has become a major problem within New Zealand, currently occupying an estimated 95% of the landmass. Possum damage to native forest varies widely, depending on forest type. Selective browsing may sometimes lead to canopy collapse, other times leading to changes in the overall composition of the forest. They are known to prey on a wide range of indigenous species, such as kokako, kiwi, saddleback, land snails and weta. Current control methods cost between \$45 and \$60 million annually, yet are by no means entirely effective. Possum shyness due to sublethal dosing reduces efficacy, while non-target mortality is always a troublesome issue. Moreover, possum population recovery is typically fast. We therefore seek to understand what effects possum control has on their population.

A good basis for our model is the logistic equation for population growth (also known as the Pearl-Verhulst (1927) equation). While proving less effective for complex, interdependent reproductive cycles, the logistic equation is much better at portraying simple growth, such as that of bacterium. This is because it models a parabolic population growth rate, which tends to zero for some carrying capacity, K, due to high density population effects.

$$\dot{N} = RN(1 - N/K) \tag{8}$$

where r is the intrinsic rate of growth and K is the carrying capacity.

Two steady states exist for the logistic equation, $N* = 0$ and $N* = K$, which we can easily show to be unstable and stable respectively, through linear stability analysis. Yet such a model would be woefully incomplete, requiring the addition of pest control before it becomes of use. Hollings (1965, 1966) postulated 3 types of harvest or predation functions, p(N), which shall be refered to as our pest control term.

Type 1, $p(N) = AN$, is a linear function, where the pest control term is proportional to the total population of possums.

Type 2, $p(N) = AN/(B+N)$, is an upwards tending curve of decreasing slope, where pest control exhibits density dependence, towards an upper maximum.

Type 3, $p(N) = AN^2/(B^2 + N^2)$, is a sigmoid curve, also exhibiting density dependence, except as well as tending towards an upper maximum, it also shows lesser rates of pest control at lesser densities.

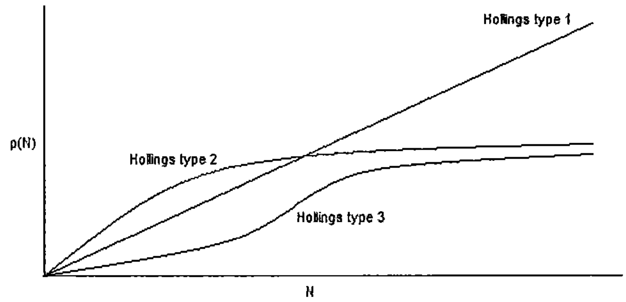


Figure 2: Hollings Functional Response Curves

We shall use a Hollings Type 2 pest control term here, to represent the steep increase in attention paid to the possum problem since they were identified as a pest, approaching an upper maximum due to practical limitations (such as resources for pest control).

$$\dot{N} = RN(1 - N/K) - AN/(B + N) \tag{9}$$

where r is the intrinsic rate of growth, K is the carrying capacity, A is the upper maximum, and B measures how quickly p(N) tends to A.

Analysis in this form is difficult, as we have four parameters, so instead it is helpful to express the equation in dimensionless form. How this is done very much depends on the equation in question, as there are often multiple ways to non-dimensionalize. As B and N both have the same dimension, we take our first dimensionless group as $x = N/B$

$$Bdx/dt = RBx(1 - Bx/K) - Ax/(1 + x) \tag{10}$$

$$(B/A)dx/dt = (RBx/A)(1 - Bx/K) - x/(1 + x) \tag{11}$$

Equation (11) further suggests another 3 dimensionless groups, $\tau = A/Bt$, $r = RB/A$ and $k = K/B$, such that the equation becomes

$$dx/d\tau = rx(1 - x/k) - x/(1 + x) \tag{12}$$

relative to control effort A/B

where r and k are the dimensionless growth rate and carrying capacity respectively.

$x* = 0$ is the most obvious fixed point. Through linear stability analysis, we see that $f'(x*) = r - 2rx/k - 1/(1+x)^2$ gives $x* = 0$ as a stable fixed point for $r < 1$ and unstable for $r > 1$. Simply put, a growth rate above 1 means that any population will grow, while a growth rate less than 1 could potentially lead to extinction. Information as to the existence of other fixed points is necessary here, before the dynamics of a population with a growth rate less than 1 can be correctly determined. These are best found graphically, plotting $r(1 - x/k)$ and $1/(1+x)$ simultaneously and looking for any intercepts.

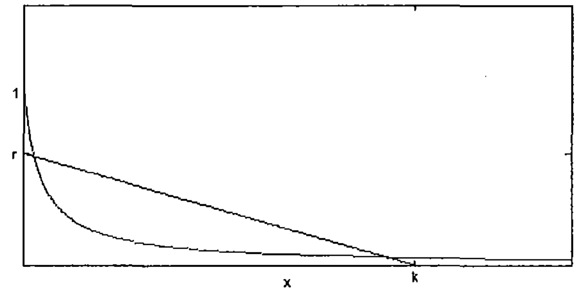


Figure 3: Non-Dimensional Fixed Point Analysis, $r < 1$

Notice that there exist two intercepts, one for low x, one for high x (close to carrying capacity). Logically, fixed points must alternate between stable and unstable (or a phase point would be attracted in two opposite directions, an impossibility). Therefore if $x* = 0$ is a stable fixed point, then the 'low x' fixed point must be unstable, and the 'high x' fixed point must be stable. In this scenario our system has two stable states, separated by an unstable state. Any populations of less than a certain size will always die out, unless they are 'pushed' past this threshold value and naturally grow to a large, stable population. Interestingly, this is similar to our real world experience with possums, where the first brought to New Zealand actually died out. A reduction in breeding capability is an observable phenomenon of species brought to an unfamiliar environment (i.e. a zoo or captive breeding program), at least initially. It took some effort before they became self-sustaining, and subsequently escaped human control. It should also be noted that there does in fact exist a situation where $x* = 0$ is the only fixed point, for very low r. Here any population of possums, even very large ones, will die out over time. This, of course, would be an optimal situation in terms of pest control, but also an extremely difficult one to arrive at, as it would effectively require neutering almost the entire possum population. Now we compare these results with the dynamics of $r > 1$.

For $r > 1$, there is only ever one other fixed point, a high x close to carrying capacity (k). As $x* = 0$ is unstable, this fixed point must be a stable fixed point, lending the conclusion that a possum population with $r > 1$ will always grow to numbers close to carrying capacity, even despite pest control. As long as possums exist which are able to breed relatively prolifically (i.e. $r > 1$), their numbers will boom again. There is in fact evidence to suggest that those not killed by pest control are those biologically fitter, ensuring a high r value and the species' survival (see www.landcareresearch.co.nz).

From the model we have developed here, we may conclude that the successful control of possums is impossible in the long run without biological control of their reproduction. Consider the characteristic how do you know N is high relative to B?

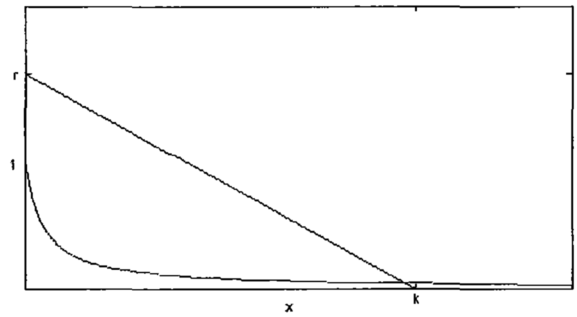


Figure 4: Non-Dimensional Fixed Point Analysis, $r > 1$

time scale for high x, the current situation. If $f'(x*) = r - (2rx/k) - 1/(1+x)^2$, then we approximate $T = 1/|f'(x*)|$ for large $x*$ by T $\approx 1/|r(1 - 2x/k)|$, ignoring small quadratic terms. Since $x* \approx k$, we can again approximate by $T = 1/|r(1-2)|$, or $T = 1/|r|$. Then if r is large, the characteristic time scale must be small, and the possum population will tend back towards $x*$ rapidly. If, however, r is small, then the characteristic time scale must be larger, and it will take longer for possum numbers to rebound. Elimination of this species becomes a much less daunting task when gains made are able to be consolidated, rather than lost to the next generation of possums. Even if possum numbers are still above the unstable fixed point (i.e. will still tend towards a high equilibrium state), lowering their reproductive capabilities will ensure that this occurs much more slowly, aiding control efforts and allowing for more effective forest regeneration.

Real world research is following in just this direction, too. Through a process called immunocontraception, the possum's own biological systems will be used to interfere with the means by which sperm fertilizes the egg. This response will be triggered by proteins that make up part of the coat of either the sperm or egg, ideally towards stopping female possums producing eggs, or interfering with the fertilisation of any eggs produced. Such a biological control agent will ideally be made into baits, and fed to possums from bait stations. For more information, consult www.maf.govt.co.nz and www.landcareresearch.co.nz websites.

Transcritical & Pitchfork Bifurcations

Certain systems may exhibit an interesting phenomenon, the existence of a fixed point which can never be destroyed, regardless of how parameters are varied. We have just encountered this in our above model, for $x* = 0$. What, instead, we saw was that such a fixed point may change its stability as a parameter is varied. The standard mechanism by which this happens is by a transcritical bifurcation, where two fixed points collide and exchange stabilities.

Our prototypical example for this is $\dot{x} = rx - x^2$, where r is our parameter. If $r < 0$, our system is a downwards tending parabola, shifted to the left. Its two fixed points are at $x* = 0$ and $x* = r$. By linear stability analysis, $x* = r$ can be shown to be unstable, $x* = 0$ stable. As r increases to 0, the parabola moves downwards to the right, till at $r = 0$ it touches the x axis once only, at $x = 0$. At this point our system becomes $\dot{x} = -x^2$, such that linear stability analysis cannot predict the nature of the point ($f'(x*) = 0$).

As r increases past zero, the parabola now moves upwards to the right, once again giving two fixed points, $x* = 0$ and $x* = r$. Now, however, these points have exchanged stability, and $x* = 0$ is unstable, whereas $x* = r$ is stable. Note the difference to a saddle-node bifurcation, where the two fixed points collided and mutually annihilated each other.

Our earlier analogy of hills and valleys existing in a situation of adjustable gravity is less useful here, so let us make some alterations. Consider a hot air balloon, struggling to lift off. Its equilibrium state is on the ground, and even though it is possible to raise it slightly, it is too heavy to do anything but return to the ground. We take our system parameter here to be hot air, which once raised high enough in temperature, is eventually able to overcome gravity and lift off. This point, where gravity is finally overcome, is the transcritical bifurcation, where the fixed point of being grounded exchanges stability with that of a small height above the ground. Further increases of the parameter do not alter the instability of the ground as a fixed point, yet increase the height above the ground the balloon will hover at. Below is a bifurcation diagram of a typical transcritical bifurcation.

Good Example

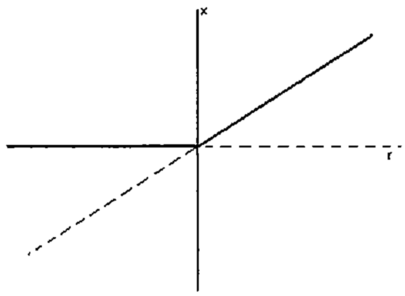


Figure 5: Transcritical Bifurcation Diagram

Structural stability is a method by which the qualitative accuracy of a system may be confirmed or denied. This is done through the introduction of a perturbation term, followed by an examination of whether the system remains topologically equivalent. Our first bifurcation, the saddle-node system $\dot{x} = r + x^2$, is easy to prove structurally stable, when we introduce the perturbation term e.

$$\dot{x} = r + x^2 + e \tag{13}$$

$$\dot{x} = (r + e) + x^2 \tag{14}$$

$$\dot{x} = m + x^2 \tag{15}$$

Equation (15) is of the same form as the prototypical saddle-node system. Introducing a perturbation term has not affected the qualitative structure of the system. Compare this with the typical transcritical bifurcation system.

$$\dot{x} = rx - x^2 + e \tag{16}$$

There is no way to rearrange this new equation into the typical transcritical bifurcation form, and as such we conclude that this type of system is not structurally stable. Any imperfection or error will alter the nature of the system, from a transcritical bifurcation to something else. This would be like trying to lift off in an air balloon on a windy day; the balloon will either take off early, when lift is technically still insufficient (diagram b), or, worse, as the balloon heats up it becomes extremely volatile, randomly taking off and hitting the ground again without finding any steady state at all (diagram a). Only once the balloon heats up well past the bifurcation point would it become stable once again, in this second situation. Our hot air balloon system is clearly only applicable on a windless day, where there is no error to distress our model. As soon as real world imperfections are allowed, the model breaks down, which is hardly ideal.

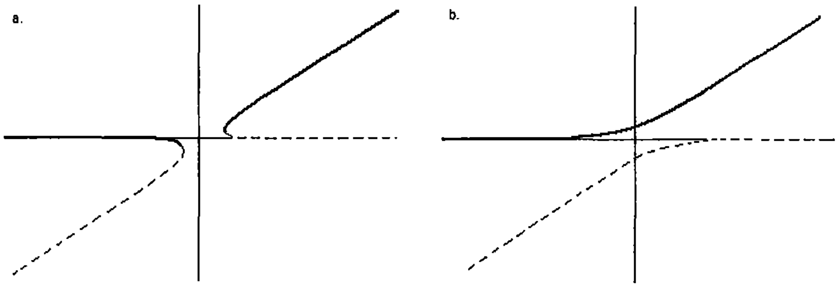


Figure 6: Transcritical Bifurcation Diagrams, With Imperfection Term

Transcritical bifurcations are less useful due to this lack of structural stability, since real world phenomena will not bifurcate smoothly, as originally predicted. Pitchfork bifurcation are similar in this respect, yet remain common for modeling problems exhibiting physical symmetry. For situations where the system is structurally perturbed, the system is said to be symmetry-broken, as an explanation for the structural instability. Let us examine further pitchfork bifurcations so as to understand why this is so, starting with the simpler type, the supercritical pitchfork bifurcation.

Supercritical pitchfork bifurcations take the normal form $\dot{x} = rx - x^3$, meaning that fixed points are found whenever the lines rx and x^3 intersect. For $r < 0$, these two lines only intersect at the origin, for $x* = 0$, which is stable. When $r = 0$ the origin is still the only fixed point, but is now only weakly stable, as $f'(x*) = 0$. In fact, since stability now depends on quadratic terms alone, the characteristic time scale undergoes a process called critical slowing down, where instead of exponential decay perturbations decay algebraically (see appendix A). For $r > 0$ two new fixed points branch off from either side of the origin, both stable, located at $x* = \pm\sqrt{r}$, while the origin becomes unstable. These stable fixed points, symmetric about the origin, are the reason why this type of bifurcation is so useful for problems of physical symmetry. This symmetry is defined mathematically by saying that the equation is invariant (or the vector field is equivariant) under the change of variables $x \rightarrow -x$. good

Metal buckling is probably the most straightforward and intuitively understandable physical analogue. Think of a metal rod standing perfectly upright, with an evenly distributed weight on top. If that weight is sufficiently small, the rod is able to support it without any movement at all. As that weight is increased past a certain point, the rod becomes incredibly unstable. Any slight imperfection or perturbation and the rod buckles to one side, towards a new (bent) equilibrium state. Importantly, in 2 dimensional space, the rod could buckle towards either fixed point, depending on the direction of the perturbation. Now we are able to understand why a structural perturbation is considered symmetry-breaking. If the rod is asymmetrical in one direction, the rod won't buckle equally in either direction, but buckle according to the asymmetry. The rod can be made to buckle in the opposite direction, but must be forced past an unstable fixed point to achieve this.

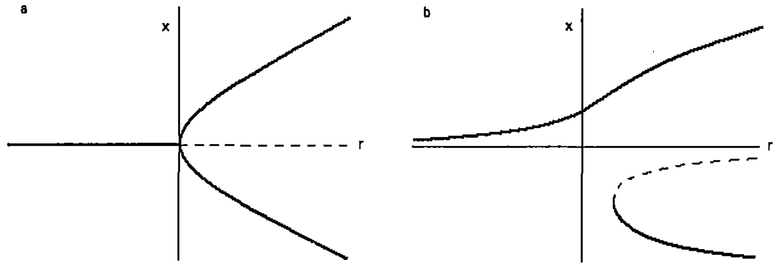


Figure 7: Supercritical Pitchfork Bifurcation Diagrams, Symmetric and Asymmetric

If the asymmetry of a pitchfork bifurcation is taken as a variable parameter, such as the system $\dot{x} = h + rx - x^3$, our system exhibits extremely dangerous dynamics indeed. The above diagram (a) gives the system for $h = 0$, showing a smooth transition from a 1 fixed point system to a 3 fixed point system. (b) shows the system for $h \neq 0$, where there is no longer a smooth transition, two of the branches now discontinuous with the other. Consider a phase point in the system for $h = 0$, for r increasing from $r < 0$ to $r > 0$ (diagram a), further supposing that it happens to travels along the lower stable branch. If h is altered such that $h \neq 0$ (diagram b), the phase point becomes isolated in this lower stable branch. Reversing r from $r > 0$ to $r < 0$ now reaches a point where the phase point suddenly jumps discontinuously from the isolated lower stable branch to the upper stable branch, which we term a catastrophe. Such catastrophic jumps may be used to explain sudden, irreversible changes in equilibrium, applicable to the likes of insect outbreaks or building stability.

Irreversibility in the context of bifurcations is known as hysteresis, a term we describe more fully through looking at subcritical bifurcations. The normal form is $\dot{x} = rx + x^3$, extremely similar to a supercritical pitchfork bifurcation, only the dynamics are entirely reversed. For $r < 0$ there exist 3 fixed points, 2 symmetrical unstable branches around a stable state for the origin. For $r > 0$ there exists only a single fixed point, the origin, which is now unstable. Here the system exhibits blow up, solutions reaching infinity in finite time for any initial condition. Real world examples of subcritical bifurcations usually have this instability opposed by a higher order term, which, if we assume the system still to be symmetric (i.e. under $x \to -x$), must be x^5, giving the system $\dot{x} = h + rx - x^3 - x^5$. Even with these higher order terms, this type of bifurcation is sometimes called hard or dangerous, because of the jump from the stable $x* = 0$ state to a large amplitude stable state when r is increased past zero. Here the hysteresis of the system becomes obvious, since if r is now decreased below zero, the system stays at the large amplitude steady state. Not until r is decreased well past zero does the system jump back to $x* = 0$, which it will then remain at till r is once again increased past zero.

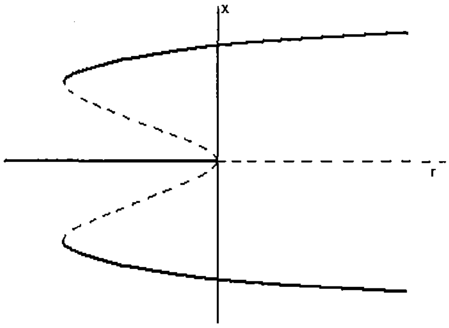


Figure 8: Subcritical Pitchfork Bifurcation Diagram

2D Flows

Up until this point the systems discussed have only been those of one dimension, so now let us move onto those of two. Systems of two or more dimensions provide for entirely new dynamics, capable of much more than remaining constant or monotonic movement. We begin with the general form for linear systems in two dimensions, and develop from there.

$$\dot{x} = ax + by$$

$$\dot{y} = cx + dy \tag{17}$$

where a, b, c, d are parameters.

In a more compact matrix form this system can also be denoted

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{18}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$

Established notions of stability need to be developed upon to meet the added complexity of 2 dimensional space. For example, our systems are now capable of closed orbits and spirals, which are qualitatively different types of motion around a fixed point. Often the entire system may be divided up into basins of attraction, similar to a ridgeline separating two valleys. Or the system might contain a saddle point, exactly alike the real world shape in that it attracts in one direction, and repels in another, exhibiting the dynamics of both a hill and a valley simultaneously. In this context a new form of stability arises, that of a stable or unstable manifold. The stable manifold is defined as the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}*$ as $t \to \infty$, which, somewhat against intuition, is the path leading towards the saddle point. The unstable manifold is the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}*$ as $t \to -\infty$, which is the path leading away from the saddle point.

A fixed point is attracting if $\mathbf{x}(t) \rightarrow \mathbf{x}*$ as $t \rightarrow \infty$; that is, all trajectories that start near $\mathbf{x}*$ approach it over time. For motion in 2-space, however, this does not rule out the possibility of phase points starting very close to $\mathbf{x}*$ taking extremely long paths before they return to $\mathbf{x}*$. Instead, we define a fixed point as being Liapunov stable if all trajectories that start sufficiently close to $\mathbf{x}*$ remain close to it forever. More precisely, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, if $\|x(0)\| < \delta$ then $\|x(t)\| < \epsilon$ for every $t \geq 0$. While these two types of stability often occur together, one does not imply the other. A line of fixed points attracting in one direction are Liapunov stable, but are not stable attractors (called neutrally stable). If both these occur together, we call a fixed point stable, or asymptotically stable.

Most systems can be broken down into a small number of axes of motion, each of which dominates the system to differing degrees. These axes are termed the eigenvectors of the system, their relative importance given by a comparison of their eigenvalues. Furthermore, the sign of the eigenvalue is used to tell the direction of each eigenvector. The previously mentioned saddle point was an example of a mixed situation, where one eigenvalue is positive, the other negative. When both eigenvalues direct towards a fixed point, this point is known as a stable node. Similarly, when both direct away from a fixed point, it is known as an unstable node. In extracting these from the matrix of the system, added terminology is useful. For the matrix A defined in (18), $\tau = trace(A) = a + d$, and $\triangle = det(A) = ad - bc$. In these terms, expanding the characteristic equation, $det(A - \lambda I) = 0$, produces

$$\lambda^2 - \tau\lambda + \triangle = 0 \tag{19}$$

With eigenvalues

$$\lambda_1 = (\tau + \sqrt{\tau^2 - 4\triangle})/2, \lambda_2 = (\tau - \sqrt{\tau^2 - 4\triangle})/2 \tag{20}$$

And general solution

$$X(t) = c_1 \exp^{\lambda_1 \tau} v_1 + c_2 \exp^{\lambda_2 \tau} v_2 \tag{21}$$

It is also possible that the solutions to (20) are complex numbers. Since the complex plane is orthogonal to the real plane, a good interpretation is that these impart a turning force to the system. Consider (21) where $\lambda = (\alpha \pm i\omega)\tau$. By Euler's formula, $\exp^{i\omega\tau} = \cos(\omega\tau) + i\sin(\omega\tau)$. Therefore $\exp^{\lambda\tau} = \exp^{\alpha\tau}(\cos(\omega\tau) + i\sin(\omega\tau))$ (Strogatz (1994)). This represents exponentially decaying oscillations if $\alpha = Re(\lambda) < 0$, and growing oscillations if $\alpha > 0$. In the more intuitive description of the complex plane imparting a turning of the Linear System force, this means a phase point moves inwards while turning if $\alpha < 0$, and moves outwards while turning if $\alpha > 0$, which are stable and unstable spirals respectively. If $\alpha = 0$ (purely imaginary λ), then solutions are periodic. Here the complex plane imparts a turning force without the phase point moving towards or away from the fixed point, which is a center. In this way centers may be though of as a borderline case, between a phase point moving inwards via a stable spiral, and outwards via an unstable spiral.

Four other borderline cases exist. The first lies between a saddle-point and a node, and are systems of non-isolated fixed points, which are neutrally stable. All trajectories approach these fixed points along straight lines. The second and third both lie between nodes and spirals, and are stars and degenerate nodes. Stars exist when both eigenvalues are equal and non-zero, and there are two independent eigenvectors. In linear!

this case every vector is an eigenvector with the same eigenvalue, so that all trajectories are straight lines towards the origin. Degenerate nodes exist when there is only one eigenvector, attracting all trajectories to the single eigendirection. The fourth is the trivial case where A $= 0$, and is an entire plane of fixed points. We are now fully capable of determining the exact nature of a 2D flow based on just the trace and determinant of the matrix A. If $\triangle < 0$, the eigenvalues are real and have opposite signs; the fixed point must be a saddle point. If $\triangle > 0$, the eigenvalues may be real with the same sign (nodes), or complex conjugate (spirals and centers). Nodes satisfy $\tau^2 - 4\triangle > 0$, where spirals < 0. If $\tau^2 - 4\triangle = 0$, then the fixed point is a star or degenerate node. Lastly, if $\triangle = 0$ then there is either a line of fixed points, or an entire plane of fixed points ($A = 0$). The figure below provides a useful visualization as to how these all tie together.

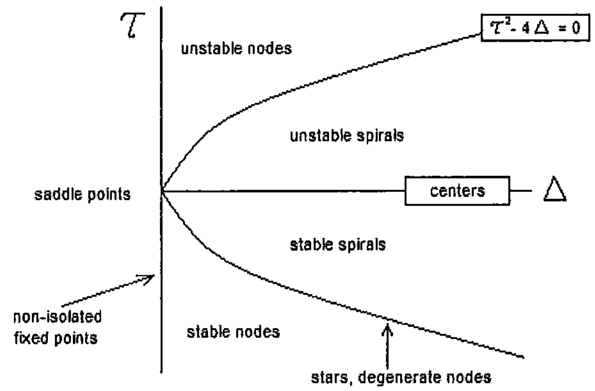


Figure 9: Classification Diagram for Fixed Points of 2D Flows

how?

Non-linear 2D flows are a step further in complexity, yet we are able to adapt certain previously developed analytical techniques. Determining the nullclines (sometimes called isoclines) is always the simplest first step, which are the curves where $\dot{x}$ or $\dot{y} = 0$, just like the stable and unstable manifolds earlier described. From finding where these intersect, the fixed points of the system are easily extractable. The nature of these fixed points may be found by extending the linearization technique we used earlier. For the system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y) \tag{22}$$

with the fixed point $(x*, y*)$, let $\mu = x - x*$, and $\nu = y - y*$ represent a small disturbance from the fixed point. To see whether a disturbance grows or decays, we derive a differential equation for μ and ν, starting with μ.

$$\dot{\mu} = \dot{x} = f(x * + \mu, y * + \nu) \tag{23}$$

Using Taylor's expansion

$$\dot{\mu} = f(x*, y*) + \mu df/dx_{x*,y*} + \nu df/dy_{x*,y*} + 0(\mu^2, \nu^2, \mu\nu) \tag{24}$$

Since $x*$ is a fixed point, $f(x*, y*) = 0$, and since μ and ν are both small, quadratic terms are negligible. When the same is done for $\dot{\nu}$, we get the following

$$\begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} df/dx & df/dy \\ dg/dx & dg/dy \end{pmatrix}_{x*,y*} \begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix}$$

The matrix is called the Jacobian matrix, and is the multivariate equivalent of the derivative. Matrix analysis techniques developed for linear 2D flows may now be applied to determine the nature of the system at each fixed point. There do, however, arise situations where the quadratic terms are not negligible, where the linearization predicts one of the borderline cases. These are similar to where $f'(x*) = 0$ for the 1 dimensional linearization. We define a fixed point as being hyperbolic if $Re(\lambda) \neq 0$ for each eigenvalue, which is a point whose stability is unaffected by small, non-linear terms. More specifically, the Hartman-Grobman theorem (1966) states that the local phase portrait near a hyperbolic fixed point is topologically equivalent to the phase portrait of the linearization (here topologically equivalent means that there is a homomorphism (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time is preserved).

good

Case Study: Wasps and Bellbirds

Introduced plants and animals have proven a major problem for many New Zealand species, even where a predator-prey relationship does not exist. Food and territorial competition has been devastating to many species unused to aggressive invaders, such that extinction has become a very real threat. The list of vanished species since human settlement is very sobering reading indeed, with many more extremely close to the edge. With our new arsenal of 2 dimensional analytical tools we turn to the Lotka-Volterra model of competition (Murray (2001)), as a means to understanding the dynamics of such a relationship.

Korimako, the Bellbird (*Anthornis melanura*), are endemic to New Zealand. They were originally found throughout New Zealand, but since the 1860s have been virtually extinct in the Auckland and Northland regions. Podocarps, which produce large amounts of fruit and seed, are a main source of food for the Bellbird and other frugivorous birds. However it is honeydew, the sugary extract produced by native scale insects (*Ultracoelostoma*) infesting beech trees, that is the most abundant nectar-like food resource. Two alien species of wasp have become established in New Zealand, the German wasp (*Vespula germanica*) and the common wasp (*Vespula vulgaris*), each of which are especially prevalent in regions where there is honeydew. It is estimated that about 70% of annual production is consumed by wasps, though during their peak abundance during summer and autumn this jumps to 90-99% (Pearson, 2006). Bellbirds remain in the forest during the high wasp season, but either reduce their foraging on honeydew or increase their foraging time by sacrificing time spent on other activities, such as singing, social interactions and grooming. Changes in the foraging behaviour of these birds could affect their breeding success and survival.

If honeydew is assumed to be the main source of food for wasps and bellbirds, then a competitive interaction between the two species lowers the growth rate of each, but more severely for the bellbird. Such interactions are thus proportional to the size of each species, but weighted more heavily against the bellbird. Each species is given an intrinsic reproductive rate, which even after compensating for size differences is taken as higher for the wasp, which is capable of 12000 - 15000 wasps (queens and workers) per nest per season. From these assumptions we create our model, coefficients taken as indicative but arbitrary.

$$\dot{x} = x(1 - x - 5y)$$

$$\dot{y} = y(2 - 3x - y) \tag{25}$$

where x are bellbirds and y are wasps.

Fixed points are found for where $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four are found: (0, 0), (0, 2), (1, 0) and (9/14, 1/14). For their classification, we turn to the Jacobian for each point in turn.

$$\mathrm{J} = \begin{pmatrix} 1-2x-5y & -5x \\ -3y & 2-3x-2y \end{pmatrix}$$

$$(0, 0)\text{: J} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

By inspection, it is obvious that both eigenvalues are positive real numbers, 1 and 2, so (0, 0) must be an unstable node. Alternatively, we might compute the determinant, trace and $\tau^2 - 4\triangle$ values. The determinant is positive, so this point cannot be a saddle point. The trace is also positive, so the point is unstable. Lastly, its $\tau^2 - 4\triangle$ value is positive as well, giving the point as an unstable node. To derive the trajectories leaving the node, we note that there is a fast/dominant eigenvalue and slow/lesser eigenvalue for this point. In forward time, trajectories always tend towards the direction of the fast eigenvalue. However, in reverse time trajectories are parallel to the slow eigenvalue at a node, meaning that it is the eigenvector associated with $\lambda = 1$ that gives the direction trajectories leave the unstable node by. This we can derive to be $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, using $A\mathbf{v} = \lambda\mathbf{v}$.

What does this mean biologically?

$$(0, 2)\text{: J} = \begin{pmatrix} -9 & 0 \\ -6 & -2 \end{pmatrix}$$

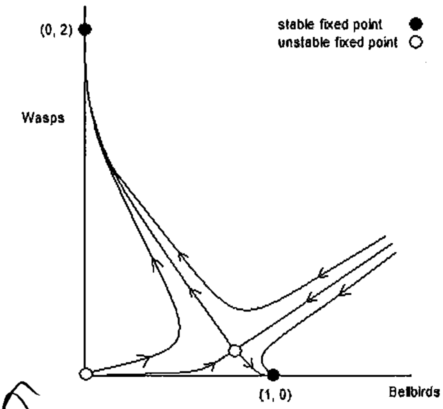
Once more we may proceed by inspection or by matrix analysis. Inspection is much easier here, so we take that route. By this method, there are two real negative eigenvalues, so we deduce that the fixed point (0, 2) must be a stable node. The slow eigenvalue is -2, with an associated eigenvector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$(1, 0)\text{: J} = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}$$

Proceeding by matrix analysis gives a positive determinant and negative trace, indicating that the fixed point is not a saddle and is stable. But this is all the linearization can conclusively tell us, as we find that $\tau^2 - 4\triangle = 0$ at this point, such that small quadratic terms are no longer negligible. We are faced with one of the borderline cases, where the linearization suggests that the point is either a star or degenerate node, but could possibly be either a stable node or stable spiral instead. Inspecting the Jacobian directly proves to be more informative in this instance. As matrix analysis has already told us, the single, repeated eigenvalue suggests either a star or a degenerate node. Already we eliminate the possibility of the fixed point being a spiral or a node, as the eigenvalue is not complex, and there exists only one eigenvalue. This fixed point truly is a borderline case, as the linearization suggested. Next note that the Jacobian is in upper triangular form, which tells us that there can only be a single eigenvector; the fixed point must therefore be a degenerate node, with associated eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 9/14 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -0.6429 & -3.2143 \\ -0.2143 & -0.0714 \end{pmatrix}$$

Inspection proves less transparent in this case, so we return to matrix analysis. By calculating the determinant to be negative, we see that the fixed point is a saddle point. By combining the local portraits of each of these fixed points, the phase portrait for the system may now be drawn, as below.



Of x^*

Of the saddle

Figure 10: Lotka-Volterra Competition Model

Our model provides a good example of a system with two basins of attraction, the stable manifold acting as a basin boundary. We define a basin of attraction as the set of initial conditions such that $x(t) \to x*$ as $t \to \infty$. Depending on which basin the system begins at, either bellbirds or wasps will eventually tend to zero. In biological terms, this is known as the principle of competitive exclusion, stating that two species competing for the same limited resource typically cannot coexist. At first such a system seems to defy our real world scenario, as wasps are an invasive species, moving into an environment where bellbirds are much more numerous. It would seem that this model predicts that in such a situation wasps should never be capable of establishing themselves, at least without their introduction on a suitably large scale. But this model is necessarily limited, primarily in that it assumes honeydew is the sole food source for each species. Wasps feed on a wide range of invertebrates, including spiders, caterpillars, ants, bees, and flies, only collecting honeydew where available. Bellbirds, as earlier mentioned, are frugivorous, feeding on the fruit of native podocarps. We must therefore interpret this model solely as indicative of whether two species can coexist in harvesting a single resource; our model predicts that either wasps or bellbirds will eventually exclude the other collecting honeydew, and says nothing of potential extinction. It would simply exceed the reach of such a model to predict the dynamics of the entire wasp or bellbird population. For further discussion, see Pianka (1981) or Edelstein-Keshet (1988).

Orbits & Hopf Bifurcations

Conservative systems are an important form of second order system, especially where Newton's laws are concerned. These systems are generally phrased in terms of conserved energy, but this need not be so. Any system for which a conserved quantity exists is called a conservative system. Specifically, given a system $\dot{x} = f(x)$, a conserved quantity is a real-valued continuous function $E(x)$ that is constant on trajectories (i.e. $dE/dt = 0$) (Strogatz (1994)). Furthermore, $E(x)$ is non-constant on every open set, to avoid trivial examples like $E(x) = 0$. An important result of this definition is that a conservative system cannot have any attracting fixed points. This is because for a fixed point to be attracting in a conservative system, all points in its basin of attraction would have to be at the same energy level, $E(x)$, which contradicts the requirement that $E(x)$ be non-constant on every open set. Without attracting fixed points, conservative systems are instead comprised mainly of orbits, of which the two most important types are homoclinic and heteroclinic.

and closed orbits

A homoclinic orbit is one where trajectories begin and end at the same fixed point, looping around to where they began. Heteroclinic orbits instead loop towards another fixed point, sometimes called saddle connections for this reason. Closely linked with homoclinic orbits is the idea of reversibility, where the dynamics of a system are the same whether time runs forwards or backwards. We define the system $\dot{x} = f(x)$ as reversible if it is invariant under the change of variables $t \rightarrow -t$, $x \rightarrow R(x)$, where $R^2(x) = x$.

Non-conservative systems, on the other hand, have limit cycles as the most important type of orbit. Orbits now display dynamics similar to systems previously dealt with, in that they may be stable (attracting), unstable (repelling) or half-stable (attracting in one direction, repelling in another). Stable limit cycles are of particular importance, as they reflect self-sustained oscillations. The beating of a heart is an excellent example of such a system, which even once stopped may be excited back to a self sustaining cycle through electrical stimulation. Limit cycles are inherently non-linear phenomena; we have already encountered centers, but the difference is that centers are not isolated. They are a fixed point surrounded by a family of orbits. Trajectories close to a limit cycle will always either spiral towards or away from the limit cycle.

To establish that a closed orbit exists in a particular system, we turn to the Poincare-Bendixson theorem. Suppose that: 1) R is a closed, bounded subset of the plane, 2) $\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R, 3) R does not contain any fixed points, and 4) there exists a trajectory, C, that is confined in R (starts within R and does not exit for all time). Then either C is a closed orbit, or it spirals towards a closed orbit. Here we shall take this the theorem as self-evident, as actual proof goes beyond the scope of this paper; see Perko (1991) or Cesari (1963) for further details. In applying this theorem, conditions 1 - 3 are generally easily satisfied. Only ensuring a confined trajectory presents any real difficulty, which is tackled via the construction of a trapping region. A trapping region may be of any shape, as long as it is closed and that the vector field points into the trapping region everywhere along the boundary. This ensures that any trajectory within the region does not leave. The trapping region may even have 'holes' within it, so long as the vector field at these holes points back into the trapping region. A useful application of this is that if an unstable fixed point is within the trapping region, the region may be considered 'punctured' at the fixed point, such that the theorem still applies.

good

Closed orbits are capable of taking any number of shapes, but it is often extremely difficult to quantitatively tell what they are. It may instead be possible to derive an approximation, where some parameter is especially large or small. Consider the van der Pol equation, where $\mu >> 1$

$$\ddot{x} + \mu\dot{x}(x^2 - 1) + x = 0 \tag{26}$$

If we let $F(x) = 1/3x^3 - x$ and $w = \dot{x} + \mu F(x)$ (Strogatz(1994)), then the van der Pol equation may be rewritten as the system

$$\dot{x} = w - \mu F(x)$$

$$\dot{w} = -x \tag{27}$$

Or, by the substitution $y = w/\mu$,

$$\dot{x} = \mu(y - F(x))$$

$$\dot{y} = -x/(\mu) \tag{28}$$

Since μ is large, $\dot{y}$ is generally very small, so that $\dot{x}$ dominates any trajectory, and a phase point arbitrarily placed moves across the phase plane rapidly. Once the phase point is close enough to $F(x) = 1/3x^3 - x$, $\dot{x}$ becomes extremely small, so that $\dot{y}$ now dominates the trajectory. The phase point will slowly follow the outer edge of $F(x)$ closely, as whenever it begins to stray then $\dot{x}$ starts to becomes larger, and the phase point is pushed back towards $F(x)$. When the phase point reaches the cubic maxima or minima, $\dot{x}$ stops forcing the phase point back towards $F(x)$, instead heading rapidly across till it hits $F(x)$ again, and the motion is repeated. Oscillations of this type, of slow buildup followed by sudden discharge, are known as relaxation oscillations. Note that such a limit cycle effectively has two time scales, where most of the motion is spent on the slow part of the cycle.

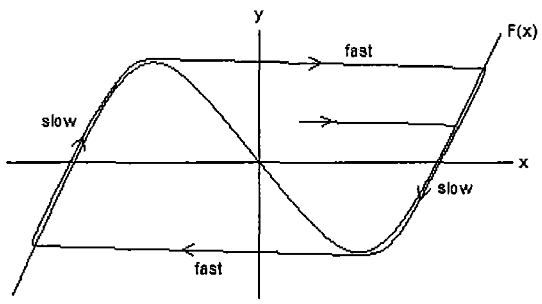


Figure 11: Relaxation Oscillation

Up till this point we have described 3 general types of bifurcation; saddle-node, transcritical and pitchfork bifurcations. All of these involve a fixed point losing stability, as some parameter r varies. To understand how this occurs, consider the eigenvalues of the Jacobian. If a fixed point is stable, then both λ are real and < 0. Bifurcations occur as one or both of the eigenvalues cross the $\lambda = 0$ threshold, becoming positive. For this reason they are known as zero-eigenvalue bifurcations. Closed orbits, as earlier described, have complex conjugate eigenvalues. This tells us that we should expect a new form of bifurcation where the eigenvalues pass through $Re(\lambda) = 0$, the Hopf bifurcation (while this crossing must be smooth, it does not have to be linear). Let us look at the two types of Hopf bifurcation further, as well as the third degenerate type.

The supercritical Hopf bifurcation is the first we shall deal with. In this case, a stable spiral changes into an unstable spiral, surrounded by a small, nearly elliptical limit cycle. Imagine a ball on the end of a piece of string held by a person turning in a circle. As the person spins at low speeds, the ball stays on the ground as there is insufficient force to raise it up. Even if physically raised, it will simply spiral back down to the ground. As the person spins faster, a threshold will be reached where the ball now gains some height above the ground, following around in a small orbit. In fact, the ball will always tend to that particular orbit, when spun at that particular speed. Raising the ball further will only see the ball spiral back downwards to its stable orbit. And as the speed of spin parameter is raised further, the ball will take a higher, wider orbit still. A crucial feature of supercritical Hopf bifurcations is that the size of the limit cycle grows continuously from zero, and may be reversed similarly. I didn't get this example

not necessarily

Subcritical Hopf bifurcations are potentially much more dangerous, as systems of this type always exhibit hysteresis. They occur when an unstable limit cycle, surrounded by a stable limit cycle, shrinks to engulf a stable fixed point. This renders the fixed point unstable, making the stable limit cycle suddenly the only attractor. A phase point previously at the fixed point will then jump to the stable limit cycle, and will not return to the fixed point if we reverse the bifurcation. In terms of a ball on the end of a string, a subcritical Hopf bifurcation would be the difference between the ball languishing on the ground and suddenly orbiting at arm's reach, a drastic change indeed. Super and subcritical bifurcations are differentiable analytically, though not always easily. Any system at a Hopf bifurcation can be put into the following form:

$$\dot{x} = -\omega y + f(x,y)$$

$$\dot{y} = \omega x + g(x,y) \tag{29}$$

Guckenheimer and Holmes (1983) then showed that in this form one can determine the nature of the bifurcation by finding the sign of the first Lyapunov coefficient, given here by the quantity:

$$16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + (1/\omega)(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \tag{30}$$

where the subscripts denote partial derivatives evaluated at (0,0). If $a < 0$, the bifurcation is supercritical, if $a > 0$ the bifurcation is subcritical.

There also exists the degenerate case. On either side of the bifurcation point, it resembles a typical Hopf bifurcation, with a fixed point changing from a stable to unstable spiral. At the bifurcation point, however, there is a continuous band of closed orbits surrounding the origin, instead of an isolated limit cycle. Generally this is found where a non-conservative system suddenly becomes conservative, the fixed point becoming a non-linear center.

Case Study: Zooplankton

To a country as geographically isolated as New Zealand, our oceans are vastly important to us. The role it has played in our history has defined us as a people in many ways, from the food we gather to the cities we build to the international ties we create. Increasingly, we are also realizing the relationship it has with the health of our land. Seabirds, for example, bring nutrients far inland, greatly increasing soil productivity. In terms of our biodiversity, many species endemic to New Zealand have only been saved from extinction by use of offshore islands, acting as small biological havens. It is with this growing concern for the ocean that deep sea fisheries have proven contentious in recent years, worries growing as to their sustainability. When selling goods for a hungry overseas market, companies seek to take as much fish as possible, potentially without allowing numbers to rebuild.

Our model here does not concern fish directly, but the creatures they feed on, zooplankton. Zooplankton are an excellent indicator of a fisheries health, as virtually every other creature that lives in the sea in some way derives their sustenance from them. Our other population we model are phytoplankton, the creatures zooplankton feed upon. Phytoplankton are even more basic yet, deriving their energy from the sun, through photosynthesis (phytoplankton are autotrophs). For this reason we define their growth as logistic, and use a Hollings type-3 function to model their predation by zooplankton (Brindley (1994)). Analysis follows similarly to that by Kot (2001), where a Hollings type-2 function was used.

$$\dot{P} = rP(1 - P/K) - cP^2Z/(a^2 + P^2), \dot{Z} = bP^2Z/(a^2 + P^2) - mZ, \tag{31}$$

Where P and Z stand for Plankton and Zooplankton respectively, and r is the phytoplankton growth rate, K is their carrying capacity, a measures the density at which zooplankton predation increases rapidly, c gives the maximum zooplankton predation, b measures zooplankton growth (in relation to phytoplankton eaten), and m gives zooplankton mortality.

Let us non-dimensionalize this equation, using $P = ax, Z = (ra/c)y, t = (1/r)\tau$.

$$dx/d\tau = x(1 - ax/K) - x^2y/(1 + x^2), dy/d\tau = bx^2y/r(1 + x^2) - (m/r)y \tag{32}$$

The final three dimensionless groups are $\alpha = m/b$, $\beta = b/r$, and $\gamma = K/a$, giving

$$dx/d\tau = x(1 - x/\gamma) - x^2y/(1 + x^2), dy/d\tau = \beta((x^2/(1 + x^2) - \alpha)y \tag{33}$$

To find the fixed points of the system, we set $dx/d\tau = 0$, and $dy/d\tau = 0$, which gives us zero-growth isoclines. Where these meet are the fixed points, which turn out to be $(x*, y*) = (0,0)$, $(\gamma, 0)$, and $(x*, g(x*))$, if $x* = \sqrt{\alpha/(1-\alpha)}$ and $g(x) = ((1 + x^2)/x)(1 - x/\gamma)$. Furthermore, we can simplify things if we define $f(x) = x/(1 + x)$, which lets us rewrite our non-dimensionalized equations as

$$dx/d\tau = f(x)[g(x) - y], dy/d\tau = \beta[f(x) - \alpha]y \tag{34}$$

We can then write the Jacobian as

$$\mathrm{J} = \begin{pmatrix} f(x)g'(x)+f'(x)g(x)-yf'(x) & -f(x) \\ \beta f'(x)y & \beta[f(x)-\alpha] \end{pmatrix}$$

For (0,0), the Jacobian is $\begin{pmatrix} 0 & 0 \\ 0 & -\alpha\beta \end{pmatrix}$, which has a single eigenvalue, $\lambda = -\alpha\beta$. This point is therefore a non-isolated fixed point.

Why?

For $(\gamma, 0)$, $g(\gamma) = 0$, so the Jacobian is $\left(\begin{smallmatrix} -1 & -\gamma^2/(1+\gamma^2) \\ 0 & \beta(\gamma^2/(1+\gamma^2)-\alpha) \end{smallmatrix}\right)$, which has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \beta[\gamma^2/(1+\gamma^2) - \alpha]$. It is therefore a stable node if $\gamma^2/(1+\gamma^2) < \alpha$ (or $x* > \gamma$).

For $(x*, g(x*))$, $f(x*) = \alpha$ (as $dy/d\tau(x*) = 0$), which lets us write the Jacobian as $\left(\begin{smallmatrix} \alpha g'(x*) & -\alpha \\ \beta f'(x*)g(x*) & 0 \end{smallmatrix}\right)$. λ here is difficult to compute directly, so instead we determine the determinant and trace. The determinant $= \alpha\beta f'(x*)g(x*)$, which, as α, β and $f'(x*)$ are all strictly positive, requires only that $g(x*)$ be positive for the point to not be a saddle. This is satisfied for $-1 < x < \gamma$. Stability, then, is determined by the sign of $g'(x*)$, as the trace $= \alpha g'(x*)$. The point is stable if $g'(x*) < 0$, unstable if $g'(x*) > 0$. At $g'(x*) = 0$ the point is a center, where the eigenvalues are purely imaginary, which means our system undergoes a hopf bifurcation as $g'(x*)$ changes sign.

$g'(x*)$ is the slope of the phytoplankton zero-growth isocline at its intersection with the zooplankton zero-growth isocline. A negative value gives a stable equilibrium, while a positive value means that the point is unstable, trajectories attracted to a stable limit cycle. Here a stable limit cycle means that relative populations of phyto and zooplankton oscillate over time. However, such a limit cycle means that populations may periodically come dangerously close to extinction, at risk of some external force suddenly depleting numbers entirely. It is actually by increasing the phytoplankton carrying capacity K that drives the system to such a bifurcation, known as the 'paradox of enrichment' (Rosenzweig, 1971). With more human waste emptying into the sea than ever, we run the real risk of just this situation. An enriched ocean could see fisheries, based around their plankton food source, move to a cyclical population structure. The worry here is that when numbers appear to boom, fishing would increase in tandem, creating huge strain once they again fall. As it is extremely difficult to know just how many fish there are, their accidental extinction would become highly likely.

Chaos

The previously discussed Poincare-Bendixson theorem is one of the most important results of non-linear dynamics, as it says that the possibilities within a 2 dimensional phase plane are extremely limited. If a trajectory is bounded within a region free of fixed points, then it must eventually approach a closed orbit. This is essentially due to the 2 dimensionality of the plane; there isn't space for anything more complicated. If, instead, we consider a 3 dimensional system, then the Poincare-Bendixson theorem no longer applies. Trajectories are now capable of wandering the phase space endlessly, without ever settling down to a fixed point or closed orbit. Though virtually any 3 dimensional system is capable of chaotic dynamics, illustrated here are but two: the Lorenz equations (1963), and the Logistic map (May (1976). These shall serve as an introduction to this highly complex topic. We define the Lorenz equations as follows:

not quite!

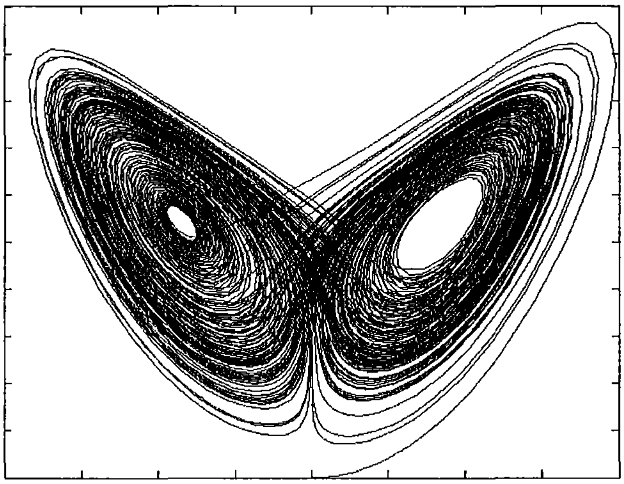
$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz \tag{35}$$

where $\sigma, r, b > 0$ are parameters.

For $r < 1$, the system has only one fixed point, $(x*, y*, z*) = (0, 0, 0)$. At $r = 1$ a supercritical pitchfork bifurcation occurs, the origin losing stability as two new fixed points branch off. These are $(x*, y*, z*) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$, generally called C^+ and C^-. C^+ and C^- are however only stable until $r = r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, for $\sigma > b + 1$ (see appendix B). At this point a subcritical Hopf bifurcation occurs, and trajectories are repelled from one unstable object after another, while confined to a bounded set of zero volume (without intersecting). This limiting set is called a strange attractor, and is pictured below.



How did you calculate this?

Figure 12: Strange Attractor for Lorenz Equations

We can define an attractor as a closed set A, where: A is an invariant set, A attracts an open set of initial conditions (basin of attraction), and A is minimal (no smaller subset). While this appears to be a pair of merging surfaces, it is really something quite different. Lorenz (1963) gives an eloquent summary: "It would seem, then, that the two surfaces merely appear to merge, and remain distinct surfaces. Following these surfaces along a path parallel to a trajectory, and circling C^+ and C^-, we see that each surface is really a pair of surfaces, so that, where they appear to merge, there are really four surfaces. Continuing this process for another circuit, we see that there are really eight surfaces, etc., and we finally conclude that there is an infinite complex of surfaces, each extremely close to one or the other of two merging surfaces". Such an 'infinite complex of surfaces' is now known as a fractal.

Chaotic motion is extremely difficult to define exactly, as it is extremely difficult to rule out the possibility that motion will not eventually repeat itself. Despite this, there are three main features that typify chaos: sensitive dependence on initial conditions, aperiodic long term behaviour, and a deterministic system. It is generally obvious when these occur, but how is much more subtle. Beginning with a 'blob' of initial conditions, Henon (1976) showed each circuit of the attractor involves repeated stretching and folding, such that two points initially close together diverge rapidly. This is why such systems are dissipative (volumes in phase space contract under flow): all initial volumes are continually squashed and isolated by the massive number of unstable objects, towards an infinity of zero volumes. To see how this might be so, we finally consider the logistic map, a discrete-time analogue of the logistic equation.

$$x_{n+1} = rx_n(1 - x_n) \tag{36}$$

where $0 \leq r \leq 4$ and $0 \leq x \leq 1$, so that (36) maps the interval $0 \leq x \leq 1$ into itself.

Already we have seen that populations under this type of growth tend to a maximum value, where $r > 1$. If r is increased just past $r = 3$, a period-2 cycle is born about the former steady state. The system contains a single unstable object, which pushes trajectories away from the steady state. Increasing r past 3.449 gives birth to a period-4 cycle; the system now has 3 unstable objects, such that the previous steady states are each split in two. Increasing r further sees each former steady state split more rapidly each time (though, interestingly, tending towards a constant ratio- see Feigenbaum (1978, 1979)), giving birth to 2^n cycles each time, until at r = 3.569946 there exists an infinity of unstable objects, and motion becomes chaotic, as below.

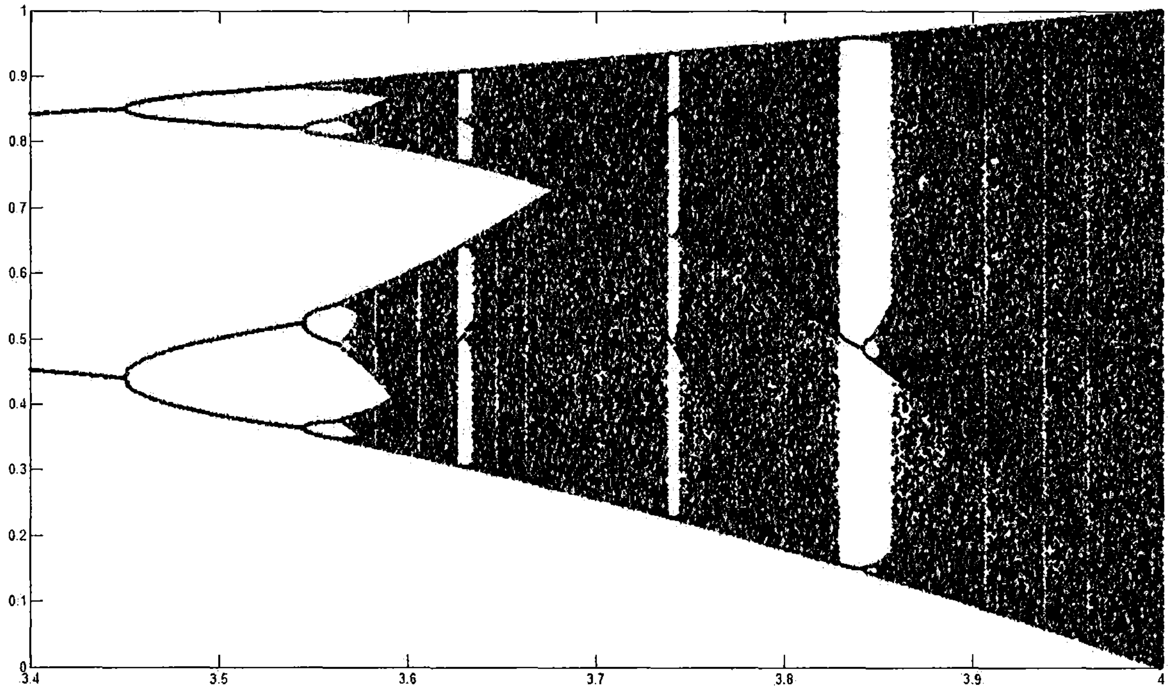


Figure 13: Logistic Map

Conclusion

Non-linear dynamics is a far broader topic than may be fully described here. Spatial models allow for an exploration of many new systems, such as disease outbreaks, animal territorial interactions, and reaction diffusion systems. In comparison with the ODE systems developed here, these require PDE analysis, and are as such of a higher order of complexity. Many systems, however, can be adequately described with some detail using ODE maps and flows, given certain assumptions. We have seen here applications relating to pest control, resource competition and oscillating populations, each of which revealed insight into real world dynamics, despite seeming simplicity. It is for this reason that non-linear dynamics prove so fascinating, in their ability to model real world phenomena with a high degree of qualitative accuracy. And it is for this reason that an understanding of this subject is truly an interdisciplinary study, from biologists to economists to chemists and beyond.

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Appendix A - Critical Slowing Down

Consider the system $\dot{x} = -x^3$, for an arbitrary initial condition. Normally our linearization gives $\dot{\nu} = \nu f'(x*)$, but for this system the only $x*$ exists at the origin; pertubations are an exact function of x. Instead of

$$\nu(t) = x(t) - x*$$

we have

$$\nu(t) = x(t)$$

This lets us rewrite the decay of any pertubation as

$$\dot{\nu} = \dot{x}(t)$$

Pertubations now decay algebraically, instead of exponentially. For $\dot{x} = -x^3$, this means that $\dot{\nu} = -3x^2$.

Compare this to the linearization for $\dot{x} = rx - x^3$, where $r < 0$. Here there exists one real $x*$, the origin, as well as two imaginary, for $x* = \pm i\sqrt{r}$. Pertubations decay at a rate of $\dot{\nu} = -\nu(r + 3x^2)$, an exponential decay rate, which is much faster than than the algebraic rate of $\dot{\nu} = -3x^2$.

Appendix B - C^+ and C^- Stability

The Lorenz equations are:

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

where $\sigma, r, b > 0$ are parameters.

The fixed points, as already shown, are $(x*, y*, z*) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$, which we call C^+ and C^-.

Matrix analysis has to this point been for 2 dimensional systems, but it proceeds exactly the same for 3 dimensional systems. From the Lorenz equations, we find the Jacobian as

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

We wish to know the stability of either C^+ or C^-. These are symmetrical, so their stability will be identical, letting us arbitrarily choose to analyse C^+. The Jacobian at this point then is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

For determining the stability of this point, we need to know the eigenvalues. Thus we put the Jacobian into the characteristic equation, and set the determinant equal to zero (i.e. $det(A - \lambda I) = 0$).

$$0 = det \begin{pmatrix} -(\sigma + \lambda) & \sigma & 0 \\ r - z & -(1 + \lambda) & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -(b + \lambda) \end{pmatrix}$$

Then, expanding the determinant...

$$0 = (\sigma + \lambda)((1 + \lambda)(b + \lambda) + b(r - 1)) + \sigma(b(r - 1) - (b + \lambda)) + 0$$

$$0 = (\sigma + \lambda)(\lambda^2 + \sigma(b + 1) + br) + \sigma(-\lambda + br - 2b)$$

$$0 = \lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(r + \sigma) + 2\sigma b(r - 1)$$

Stability, then, depends on the solutions of the above equation, and is lost when there are a pair of purely imaginary eigenvalues. As such, we seek solutions of the form $\lambda = i\sigma$

$$0 = (i\sigma)^3 + (i\sigma)^2(\sigma + b + 1) + (i\sigma)b(r + \sigma) + 2\sigma b(r - 1)$$

$$0 = -i\sigma^3 - \sigma^2(\sigma + b + 1) + i\sigma b(r + \sigma) + 2\sigma b(r - 1)$$

Which can be solved first for i terms

$$i\sigma^3 = i\sigma b(r + \sigma)$$

$$\sigma^2 = b(r + \sigma)$$

Then substituted back to solve for non-i terms

$$\sigma^2(\sigma + b + 1) = 2\sigma b(r - 1)$$

$$b(r + \sigma)(\sigma + b + 1) = 2\sigma b(r - 1)$$

$$r(\sigma + b + 1) - 2\sigma r = -2\sigma - \sigma(\sigma + b + 1)$$

$$r = \sigma(\sigma + b + 3)/(\sigma - b - 1)$$

At this point a Hopf bifurcation occurs, as the pair of eigenvalues become purely imaginary. This defines C^+ and C^- as stable for $1 < r < \sigma(\sigma + b + 3)/(\sigma - b - 1)$, where $\sigma > b + 1$.

Appendix C - Selected Matlab Files

```
%Plot For Possum Model Stability
%Matt Botur
%Thurs, Jan 31th

clear;

r = 1.6;            %parameters
k = 25;
h = 0.001;
lim = 35;
count = 0;
x = zeros(1,(lim/h)+1);        %predefine vectors
y1 = zeros(1,(lim/h)+1);
y2 = zeros(1,(lim/h)+1);

for I = 0:h:lim
    count = count+1;
    x(count) = I;
    y1(count) = r*(1-(x(count)/k));     %Logistic growth term
    y2(count) = 1/(1+x(count));         %Pest control term
end

plot(x,y1,x,y2)             %plot results
    xlabel('x')
    title('Non-Dimensional Fixed Point Analysis, r>1')
    ylim([0 r+.5])
    set(gca,'XTick',0:k:lim)
    set(gca,'XTickLabel',{'0','k'})
    set(gca,'YTick',0:r:2)
    set(gca,'YTickLabel',{'0','r'})



%Plotting Logistic Map
%Matt Botur
%Tues, Jan 15th

clear;

t = 500;
n = 1;
X = zeros(t,t);         %predefine vectors
R = zeros(1,t);
x = 0.05;               %initial value
h = 0;

for r = 3.4:(0.6/t):4       %increasing r values
    h = h+1;
    R(h) = r;
    for I = 1:300        %initial transients (disregard)
        x = r*x*(1-x);      %logistic map
    end
    X(1,h) = x;
    for I = 2:t      %good number of data points (keep)
    X(I,h) = r*X(I-1,h)*(1-X(I-1,h));       %logistic map
    end
end

plot(R,X,'.')       %plot results as discrete data points
xlim([3.4 4])
ylim([0 1])


%Test Attempt At Plotting Lorenz Eq
%Matt Botur
```

```
%Tues, Jan 15th

clear;

t0 = 0;
t1 = 200;                                   %number of data points
y0 = [0 1 0];                               %arbitrary initial values

[t,Y] = ode23('lorenz',[t0,t1],y0);         %Matlab ODE solver (23) used

plot(Y(:,1),Y(:,3))                         %plot results (2D only)
```