Recursion operators
and
nonlocal symmetries

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No. 92 June, 1993

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A new formulation of recursion operators is presented which eliminates difficulties associated with integro-differential operators. This interpretation treats recursion operators and their inverses on an equal footing. Efficient techniques for constructing nonlocal symmetries of differential equations result.

1 Introduction

Recursion operators were first presented in their general form by Olver [6]. They are special linear operators on the algebra of differential functions (functions of the independent variables, dependent variables and derivatives of arbitrary finite order) appropriate to the differential equation under study. Each symmetry of a differential equation can be interpreted as an element of this algebra. Recursion operators map differential functions representing symmetries into differential functions which also represent symmetries. They provide an efficient technique for constructing (usually infinite) families of symmetries for the special differential equations which admit recursion operators.

The symmetries generated by recursion operators are typically "generalized" [7] or "Lie-Bäcklund" [1] symmetries. By allowing differential functions to depend on nonlocal variables, it is possible to use recursion operators to generate nonlocal symmetries [4] [8]. The recursion operators of many differential equations can be inverted and their inverses used to construct more infinite families of nonlocal symmetries, often with complicated algebraic structures [3] [5].

The recursion operators admitted by nonlinear differential equations are typically integro-differential operators. Inverses of differential operators pose problems which are seldom acknowledged. For instance, most authors recognize that an expression such as $D_x^{-1}(Q)$ is only defined when Q is a total x-derivative, say $Q = D_x(P)$. However, there seems to be little appreciation of the difficulties which setting $D_x^{-1}(Q) = P$ can cause. In particular, $D_x^{-1}(Q)$ is not uniquely determined. This lack of precision sometimes leads to bogus symmetries. Given a particular differential equation, there is usually an obvious way to uniquely define $D_x^{-1}(Q)$ such that these "non-symmetries" do not appear. This good fortune is largely a historical accident — as a subsequent example will demonstrate, the so-called "obvious" definitions of $D_x^{-1}(Q)$ do not survive simple coordinate changes.

This paper advocates an alternative definition of recursion operator. In the new approach, each appearance of the inverse of a differential operator is replaced by a system of first order differential equations. The differential function obtained by applying a recursion operator to a symmetry is *guaranteed* to correspond to a symmetry. The difficulties associated with integro-differential operators are eliminated. This formulation has one other important advantage — it unifies the construction of generalized and nonlocal symmetries. In particular, the inverses of recursion operators can be treated on an equal footing with the recursion operators themselves.

Section 2 establishes some notation and terminology and demonstrates the difficulties inherent in the existing interpretation of recursion operators. After introducing the concept of a Wahlquist-Estabrook prolongation [9], Section 3 presents the new recursion operators and proves that they map symmetries into symmetries. Two examples of recursion operators appear in Section 4. The usual recursion operator for the modified Korteweg-de Vries (mKdV) equation is described in the new formulation. Its inverse is also shown to satisfy the definition of the preceding section. Section 5 discusses applications for the new formulation. The ease with which nonlocal symmetries can be constructed is emphasized.

2 Review of symmetries

Consider an n-th order differential equation $\Delta[u] = 0$ with independent variables $x^1, \dots, x^p$ and dependent variables $u^1, \dots, u^q$. Following Olver [7], let $M^{(n)}$ be the Euclidean space with coordinates (x^i, u^α_J), where $i = 1, \dots, p$, $\alpha = 1, \dots, q$ and $J = (j_1, \dots, j_k)$ ranges over all unordered k-tuples of integers $j_l \in \{1, \dots, p\}$ for all $k \in \{0, \dots, n\}$. Δ is represented by the system of equations

$$\Delta^l(x, u^{(n)}) = 0, \quad l = 1, \dots, m,$$

where $(x, u^{(n)})$ denote coordinates for $M^{(n)}$. The algebra of smooth real-valued functions of $(x, u^{(s)})$, for arbitrary finite s, is denoted by $\mathfrak{A}$. $\mathfrak{A}^r$ is the vector space of r-tuples of elements in $\mathfrak{A}$. For each $i = 1, \dots, p$, $D_{x^i} : \mathfrak{A} \to \mathfrak{A}$ is the linear operator

$$P[u] \mapsto \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{\infty} u^\alpha_{iJ} \frac{\partial P}{\partial u^\alpha_J}.$$

For each $P[u] \in \mathfrak{A}^r$, D_P is the $r \times q$ matrix with entries

$$(\mathsf{D}_P)^l_\alpha = \sum_{|J|=0}^{\infty} \frac{\partial P^l}{\partial u^\alpha_J} D_J, \quad \alpha = 1, \dots, q, \quad l = 1, \dots, r,$$

where $D_J = D_{x^{j_1}} \cdots D_{x^{j_k}}$ for $J = (j_1, \dots, j_k)$.

$Q[u] \in \mathfrak{A}^q$ is a *generalized symmetry* of Δ if $\mathsf{D}_\Delta(Q) = 0$ modulo $\Delta[u] = 0$. A recursion operator of Δ is usually defined to be a linear operator $\mathcal{R} : \mathfrak{A}^q \to \mathfrak{A}^q$ such that whenever $Q[u]$ is a generalized symmetry of Δ, so is $\mathcal{R}(Q)$ (Definition 5.24 of [7]). If there exists a linear differential operator $\mathcal{S} : \mathfrak{A}^m \to \mathfrak{A}^m$ and a linear operator $\mathcal{T} : \mathfrak{A}^q \to \mathfrak{A}^q$ such that

$$\mathsf{D}_\Delta \mathcal{T} = \mathcal{S} \mathsf{D}_\Delta \tag{1}$$

for all solutions to Δ, $\mathcal{T}$ is a recursion operator for Δ (Theorem 5.30 of [7]).

With nonlinear differential equations, recursion operators are frequently integro-differential operators. For example, it is often claimed that

$$\mathcal{R} = D_x^2 - 4v^2 - 4v_x D_x^{-1} \cdot v \tag{2}$$

is a recursion operator for the mKdV equation

$$0 = \Delta[v] = v_t + v_{xxx} - 6v^2 v_x$$

because $\mathsf{D}_\Delta \mathcal{R} = \mathcal{R}\mathsf{D}_\Delta$ on solutions to $\Delta[v] = 0$. In fact, for any $Q \in \mathfrak{A}$, all one can say is that

$$(\mathsf{D}_\Delta \mathcal{R} - \mathcal{R}\mathsf{D}_\Delta)(Q) = v_x h(t)$$

for an arbitrary smooth function h, so that $\mathcal{R}$ does not satisfy the criteria for a recursion operator at all! The reason is that the *differential* operator $\mathcal{S}$ in (1) has been replaced by the *integro-differential* operator $\mathcal{R}$. These difficulties are more apparent when applying $\mathcal{R}$ to the symmetry v_x of the mKdV equation. In order to evaluate $\mathcal{R}(v_x)$, one must first identify $P = D_x^{-1}(vv_x)$. Clearly, $P = \frac{1}{2}v^2 + f(t)$ for an arbitrary smooth function f, so that

$$\mathcal{R}(v_x) = D_x^2(v_x) - 4v^2 v_x - 4v_x D_x^{-1}(vv_x) = v_{xxx} - 6v^2 v_x - 4v_x f(t).$$

On solutions of the mKdV equation, $\mathcal{R}(v_x) = -v_t - 4v_x f(t)$. It is well known that for a function of this form to be a symmetry of the mKdV equation, f must be independent of t. Thus, although v_x is a symmetry of the mKdV equation, $\mathcal{R}(v_x)$ is not, in general, a symmetry. This problem arose because $D_x^{-1}(vv_x) = \frac{1}{2}v^2 + f(t)$. By making the "natural" choice $D_x^{-1}(vv_x) = \frac{1}{2}v^2 + a$, with a constant, $\mathcal{R}(v_x)$ would be a symmetry.

Consider the equation

$$0 = sv_s + s^4 v_{yyy} - 6s^2 v^2 v_y + yv_y \tag{3}$$

and the operator

$$\mathcal{R} = s^2 D_y^2 - 4v^2 - 4v_y D_y^{-1} \cdot v,$$

which would traditionally be regarded as a recursion operator for (3). Applying $\mathcal{R}$ to the symmetry sv_y yields

$$\mathcal{R}(sv_y) = -v_s - s^{-1}yv_y - 4g(s)v_y,$$

with g an arbitrary smooth function introduced via $D_y^{-1}(svv_y) = \frac{1}{2}sv^2 + g(s)$. However, $-v_s - s^{-1}yv_y - 4g(s)v_y$ is a symmetry of (3) if and only if $g(s) = as$, for an arbitrary constant a — not the "natural" choice $g(s) = a$ which one might expect. Note that (3) and its recursion operator are related to the mKdV equation and its recursion operator by the change of coordinates $(x, t, v) \mapsto (y, s, v)$ given by $y = xt$ and $s = t$. Thus, the appropriate definition of $D_{x^i}^{-1}$ depends crucially on the choice of coordinates.

3 A new approach

Introduce $Y = \mathbb{R}^r$ with r arbitrary and $y = (y^1, \ldots, y^r)$ as coordinates. If the smooth functions $\{F_i^a(x, u^{(n-1)}, y) : i = 1, \ldots, p, \quad a = 1, \ldots, r\}$ are such that

$$\left(D_{x^j} F_i^a - D_{x^i} F_j^a - \sum_{b=1}^{r} \left(F_i^b \frac{\partial F_j^a}{\partial y^b} - F_j^b \frac{\partial F_i^a}{\partial y^b} \right) \right) (x, u^{(n)}, y) = 0,$$

for all $i, j = 1, \ldots, p$ and $a = 1, \ldots, r$, whenever $\Delta(x, u^{(n)}) = 0$, then the equations

$$0 = \Xi_i^a = \frac{\partial y^a}{\partial x^i} - F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r, \tag{4}$$

are said to define a *Wahlquist-Estabrook prolongation* (Δ, Ξ) *of* Δ. The new dependent variables $y^1, \ldots, y^r$ are called *pseudopotentials* and (4) are termed *prolongation equations* [9]. Let $\tilde{\mathfrak{A}}$ denote the algebra of functions of $(x, u^{(s)}, y)$, for arbitrary finite s. $\tilde{\mathfrak{A}}$ is, in some sense, the prolongation of $\mathfrak{A}$. The operators $D_{x^i} : \mathfrak{A} \to \mathfrak{A}$ can be prolonged to $\tilde{D}_{x^i} : \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ by

$$\tilde{D}_{x^i} = D_{x^i} + \sum_{a=1}^{r} F_i^a \partial_{y^a}, \quad i = 1, \ldots, p,$$

allowing one to prolong any differential operator $\mathcal{A} : \mathfrak{A}^q \to \mathfrak{A}^q$ to $\tilde{\mathcal{A}} : \tilde{\mathfrak{A}}^q \to \tilde{\mathfrak{A}}^q$ by replacing every appearance of D_{x^i} by $\tilde{D}_{x^i}$.

The introduction of a Wahlquist-Estabrook prolongation allows the concept of symmetry to be extended. $Q[u, y] \in \tilde{\mathfrak{A}}^q$ is a *generalized symmetry* of Δ if $\tilde{\mathsf{D}}_\Delta(Q) = 0$ modulo $\Delta[u] = 0$ and $\Xi[u, y] = 0$. When Q depends nontrivially on a pseudopotential it is called a *nonlocal symmetry.* (This definition generalizes the "partial" symmetries of [2].)

Definition 1 Let Δ denote a system of differential equations, involving p independent and q dependent variables, with Wahlquist-Estabrook prolongation (Δ, Ξ). A *recursion operator* for Δ comprises the matrices of elements of $\tilde{\mathfrak{A}}$ (with dimensions in brackets) $\{\mathcal{A}_i(s \times s) : i = 1, \ldots, p\}$ together with the matrices of differential operators on $\tilde{\mathfrak{A}}$ $\{\mathcal{B}_i(s \times q), \mathcal{C}(q \times s), \mathcal{D}(q \times q) : i =$

$1,\ldots,p\}$, where s is some nonnegative integer, provided that they satisfy

$$\begin{aligned} \tilde{D}_{x^i}(\mathcal{A}_j) - \tilde{D}_{x^j}(\mathcal{A}_i) + \mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j &= 0, \\ \tilde{D}_{x^i}\mathcal{B}_j - \tilde{D}_{x^j}\mathcal{B}_i - \mathcal{A}_i\mathcal{B}_j + \mathcal{A}_j\mathcal{B}_i &= \sum_{|J|=0}^{\infty} f_{ij}^J \tilde{D}_J \cdot \tilde{\mathsf{D}}_\Delta, \end{aligned} \tag{5}$$

modulo $\Delta[u]=0$, $\Xi[u,y]=0$, for suitable $f_{ij}^J \in \tilde{\mathfrak{A}}$, and that

$$\tilde{\mathsf{D}}_\Delta(\mathcal{C}(P)+\mathcal{D}(Q)) = 0 \tag{6}$$

for all $P \in \tilde{\mathfrak{A}}^s$ and $Q \in \tilde{\mathfrak{A}}^q$, modulo the equations

$$\tilde{D}_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1,\ldots,p,$$

$\Delta[u]=0$, $\Xi[u,y]=0$ and $\tilde{\mathsf{D}}_\Delta(Q)=0$. □

Recursion operators map generalized symmetries into generalized symmetries.

Theorem 2 *Suppose that $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}, \mathcal{D} : i = 1,\ldots,p\}$ form a recursion operator for Δ involving the Wahlquist-Estabrook prolongation (Δ,Ξ). Let Q be a generalized symmetry of Δ. Then the system of equations*

$$\tilde{D}_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1,\ldots,p, \tag{7}$$

is integrable, in the sense that

$$\tilde{D}_{x^i}\tilde{D}_{x^j}(P) = \tilde{D}_{x^j}\tilde{D}_{x^i}(P), \quad i,j = 1,\ldots,p,$$

and for every solution $P \in \tilde{\mathfrak{A}}^s$ of this system, $Q' = \mathcal{C}(P) + \mathcal{D}(Q)$ is a generalized symmetry of Δ.

PROOF: Integrability of (7) is guaranteed by (5) and the fact that Q' is a generalized symmetry follows from (6) implying that $\tilde{\mathsf{D}}_\Delta(Q') = 0$ on solutions of (Δ,Ξ). □

4 Two examples

All integro-differential recursion operators known to the author can be expressed in terms of Definition 1. In converting an integro-differential operator into this form, one essentially replaces each appearance of $D_{x^i}^{-1}$ by a system of equations taking the form of (7). Thus s equals the number of $D_{x^i}^{-1}$ terms appearing in a factorization of the integro-differential operator.

For instance, $Q \in \mathfrak{A}$ is a generalized symmetry of the mKdV equation if and only if

$$0 = \mathsf{D}_\Delta(Q) = (D_t + D_x^3 - 6v^2 D_x - 12vv_x)(Q),$$

implying that the system of equations

$$D_x(P) = vQ, \quad D_t(P) = (-vD_x^2 + v_x D_x + (-v_{xx} + 6v^3))(Q), \tag{8}$$

is integrable. The solution P to (8), if it exists, is determined up to an additive constant. Using (2), it is anticipated that

$$Q' = \mathcal{R}(Q) = \mathcal{R} \cdot v^{-1} D_x(P) = D_x \cdot (D_x \cdot v^{-1} D_x - 4v)(P)$$

will be a generalized symmetry of the mKdV equation. When P satisfies (8), it is easy to show that $\mathsf{D}_\Delta(Q') = 0$. As expected, Q' is a generalized symmetry of the mKdV equation. In terms of Definition 1, the differential operators

$$\begin{aligned} \mathcal{A}_x &= 0, \\ \mathcal{A}_t &= 0, \\ \mathcal{B}_x &= v, \\ \mathcal{B}_t &= -vD_x^2 + v_x D_x + (-v_{xx} + 6v^3), \\ \mathcal{C} &= D_x \cdot (D_x \cdot v^{-1} D_x - 4v), \\ \mathcal{D} &= 0, \end{aligned} \tag{9}$$

comprise a recursion operator for the mKdV equation involving the null Wahlquist-Estabrook prolongation.

Definition 1 is sufficiently general to include new recursion operators. For the remainder of this section, let $\mathcal{R}$ denote the recursion operator of the mKdV equation described by (9). If $\mathcal{R} : Q \mapsto Q'$ then $Q' = D_x \mathcal{E}(P)$, where P

satisfies (8) and $\mathcal{E} = D_x \cdot v^{-1} D_x - 4v$. Traditionally, one writes $P = D_x^{-1}(vQ)$, so that $Q' = D_x \mathcal{E} D_x^{-1}(vQ)$ and

$$Q = v^{-1} D_x \mathcal{E}^{-1} D_x^{-1}(Q').$$

The mapping $Q' \mapsto Q$ motivates the new recursion operator. Following the construction outlined at the beginning of this section, $D_x^{-1}(Q')$ is replaced by the system of equations

$$D_x(F) = Q', \quad D_t(F) = (-D_x^2 + 6v^2)(Q'), \tag{10}$$

which is integrable whenever $\mathsf{D}_\Delta(Q') = 0$. It follows that $Q = v^{-1} D_x \mathcal{E}^{-1}(F)$, so the next step involves constructing a function E such that

$$(D_x \cdot v^{-1} D_x - 4v)(E) = \mathcal{E}(E) = F. \tag{11}$$

Equation (11) simplifies to

$$((v^{-1} D_x) \cdot (v^{-1} D_x) - 4)(E) = v^{-1} F,$$

so that $\mathcal{E}(e^{2w}) = \mathcal{E}(e^{-2w}) = 0$, where w is the pseudopotential of the mKdV equation described by

$$w_x = v, \quad w_t = -v_{xx} + 2v^3.$$

An elementary calculation verifies that $E = -\frac{1}{4} e^{-2w} G + \frac{1}{4} e^{2w} H$, where G and H must satisfy

$$\begin{aligned} \tilde{D}_x(G) &= e^{2w} F, \quad \tilde{D}_t(G) = e^{2w}(-Q'_x + 2vQ' - 2(v_x - v^2)F), \\ \tilde{D}_x(H) &= e^{-2w} F, \quad \tilde{D}_t(H) = e^{-2w}(-Q'_x - 2vQ' + 2(v_x + v^2)F). \end{aligned} \tag{12}$$

Cross-differentiation confirms that (12) is integrable. Putting

$$Q = v^{-1} D_x(E) = \tfrac{1}{2} e^{-2w} G + \tfrac{1}{2} e^{2w} H, \tag{13}$$

one finds that $\tilde{\mathsf{D}}_\Delta(Q) = 0$ whenever $\tilde{\mathsf{D}}_\Delta(Q') = 0$ and all of equations (10), (12) and (13) are satisfied. In terms of Definition 1, the differential operators

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 \\ e^{2w} & 0 & 0 \\ e^{-2w} & 0 & 0 \end{pmatrix},$$

$$
\begin{aligned}
\mathcal{A}_t &= \begin{pmatrix} 0 & 0 & 0 \\ -2e^{2w}(v_x - v^2) & 0 & 0 \\ 2e^{-2w}(v_x + v^2) & 0 & 0 \end{pmatrix}, \\
\mathcal{B}_x &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
\mathcal{B}_t &= \begin{pmatrix} -\tilde{D}_x^2 + 6v^2 \\ -e^{2w}(\tilde{D}_x - 2v) \\ -e^{-2w}(\tilde{D}_x + 2v) \end{pmatrix}, \\
\mathcal{C} &= \begin{pmatrix} 0 & \frac{1}{2}e^{-2w} & \frac{1}{2}e^{2w} \end{pmatrix}, \\
\mathcal{D} &= 0,
\end{aligned}
\tag{14}
$$

form a recursion operator for any prolongation of the mKdV equation involving the pseudopotential w. Application of the recursion operator described by (9) to Q returns Q', modulo a term $-4av_x$ which arises from the arbitrary constant introduced when solving (8). For this reason, the mapping $Q' \mapsto Q$ introduced here will be referred to as an application of the inverse of the usual recursion operator for the mKdV equation.

5 Applications to nonlocal symmetries

In addition to removing the ambiguity associated with integro-differential recursion operators, the formulation developed above assists in the construction of nonlocal symmetries.

The generalized symmetry Q' of Theorem 2 occurs only when a solution to (7) can be found. Suppose that $Q \in \tilde{\mathfrak{A}}^q$ is a generalized symmetry of Δ, but that equations (7) cannot be solved for any $P \in \tilde{\mathfrak{A}}^s$. One can augment the Wahlquist-Estabrook prolongation (Δ, Ξ) of Δ to a prolongation (Δ, Ξ, Υ) by introducing new pseudopotentials $\{z^1, \ldots, z^s\}$ defined by

$$
\frac{\partial z^a}{\partial x^i} = G_i^a = \sum_{b=1}^{s} (\mathcal{A}_i)^a_b \, z^b + \sum_{\alpha=1}^{q} (\mathcal{B}_i)^a_\alpha \, (Q^\alpha), \quad i = 1, \ldots, p, \quad a = 1, \ldots, s.
$$

Theorem 2 confirms that these equations define a Wahlquist-Estabrook prolongation, as equations (5) imply that the functions

$$D_{x^j}(G_i^a)-D_{x^i}(G_j^a)+\left(\sum_{b=1}^r F_j^b\frac{\partial G_i^a}{\partial y^b}+\sum_{b=1}^s G_j^b\frac{\partial G_i^a}{\partial z^b}\right)-\left(\sum_{b=1}^r F_i^b\frac{\partial G_j^a}{\partial y^b}+\sum_{b=1}^s G_i^b\frac{\partial G_j^a}{\partial z^b}\right)$$

vanish on solutions to Δ. Furthermore, $\{\mathcal{A}_i,\mathcal{B}_i,\mathcal{C},\mathcal{D} : i=1,\ldots,p\}$ prolong to a recursion operator of (Δ,Ξ,Υ) by replacing the old $\tilde{D}_{x^i}$ operators with

$$\tilde{D}_{x^i}=D_{x^i}+\sum_{a=1}^r F_i^a\frac{\partial}{\partial y^a}+\sum_{b=1}^s G_i^b\frac{\partial}{\partial z^b},\quad i=1,\ldots,p.$$

Equations (7) now have solution $P=(z^1,\ldots,z^s)^T$ and a new generalized symmetry of Δ is $Q'=\mathcal{C}(P)+\mathcal{D}(Q)$. Thus, not only do the recursion operators of Definition 1 generate new symmetries from existing ones, they also naturally yield Wahlquist-Estabrook prolongations of Δ which lead to nonlocal symmetries. A demonstration of this process is included below.

In order to use recursion operators one must have a seed symmetry to which these operators can be applied. One possibility often overlooked, but which is especially effective when combined with Definition 1, is to begin with the trivial symmetry. The first step in applying the recursion operator given by (9) to $Q=0$ is to solve $D_x(P)=D_t(P)=0$, which clearly has general solution $P=a$ for an arbitrary constant a. It follows that

$$Q'=D_x\cdot(D_x\cdot v^{-1}D_x-4v)(a)=-4av_x.$$

Continuing to apply the recursion operator in this way yields the well known family of generalized symmetries of the mKdV equation [6]. Consequently, the action of this recursion operator on zero yields an infinite-dimensional Abelian symmetry algebra (Theorem 5.20 of [7] confirms the Abelian nature of the Lie algebra). The recursion operator described by (14) acts on zero to generate another infinite-dimensional Lie algebra — this one is non-Abelian and is a subalgebra of $\mathfrak{sl}(2,\mathbb{R})\otimes\mathbb{R}[\lambda,\lambda^{-1}]$. The complete set of symmetries and the resulting algebraic structure can be found in [3]. The first few symmetries in this algebra are constructed below.

In applying the recursion operator of (14) to the symmetry $Q=0$, one must first solve (10). That is, $\tilde{D}_x(F)=\tilde{D}_t(F)=0$, so take $F=c_1$ for an arbitrary constant c_1. G and H are determined by (12), in this case

$$\begin{aligned}\tilde{D}_x(G)&=c_1e^{2w},\quad \tilde{D}_t(G)=-2c_1(v_x-v^2)e^{2w},\\ \tilde{D}_x(H)&=c_1e^{-2w},\quad \tilde{D}_t(H)=2c_1(v_x+v^2)e^{-2w}.\end{aligned}\tag{15}$$

There are no smooth functions $G(x,t,v,w)$, $H(x,t,v,w)$ satisfying these equations. Following the discussion which began this section, it is possible to introduce new pseudopotentials of the mKdV equation which will yield a complete solution to the above system. In this case, the pseudopotentials y and z are defined by

$$y_x = e^{2w}, \quad y_t = 2e^{2w}(-v_x + v^2), \quad z_x = e^{-2w}, \quad z_t = 2e^{-2w}(v_x + v^2).$$

$G = c_1 y + c_2$ and $H = c_1 z + c_3$ give the general solution to (15) and the resulting symmetry of the mKdV equation is

$$Q' = \tfrac{1}{2}c_1(ye^{-2w} + ze^{2w}) + \tfrac{1}{2}c_2 e^{-2w} + \tfrac{1}{2}c_3 e^{2w}.$$

It corresponds to the vector field $\frac{1}{4}c_1\mathbf{U}_{-1} - \frac{1}{2}c_2\mathbf{V}_{-1} + \frac{1}{2}c_3\mathbf{W}_0$ of [3].

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