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Summer Research Project

Lattices in Automorphism Groups of Trees

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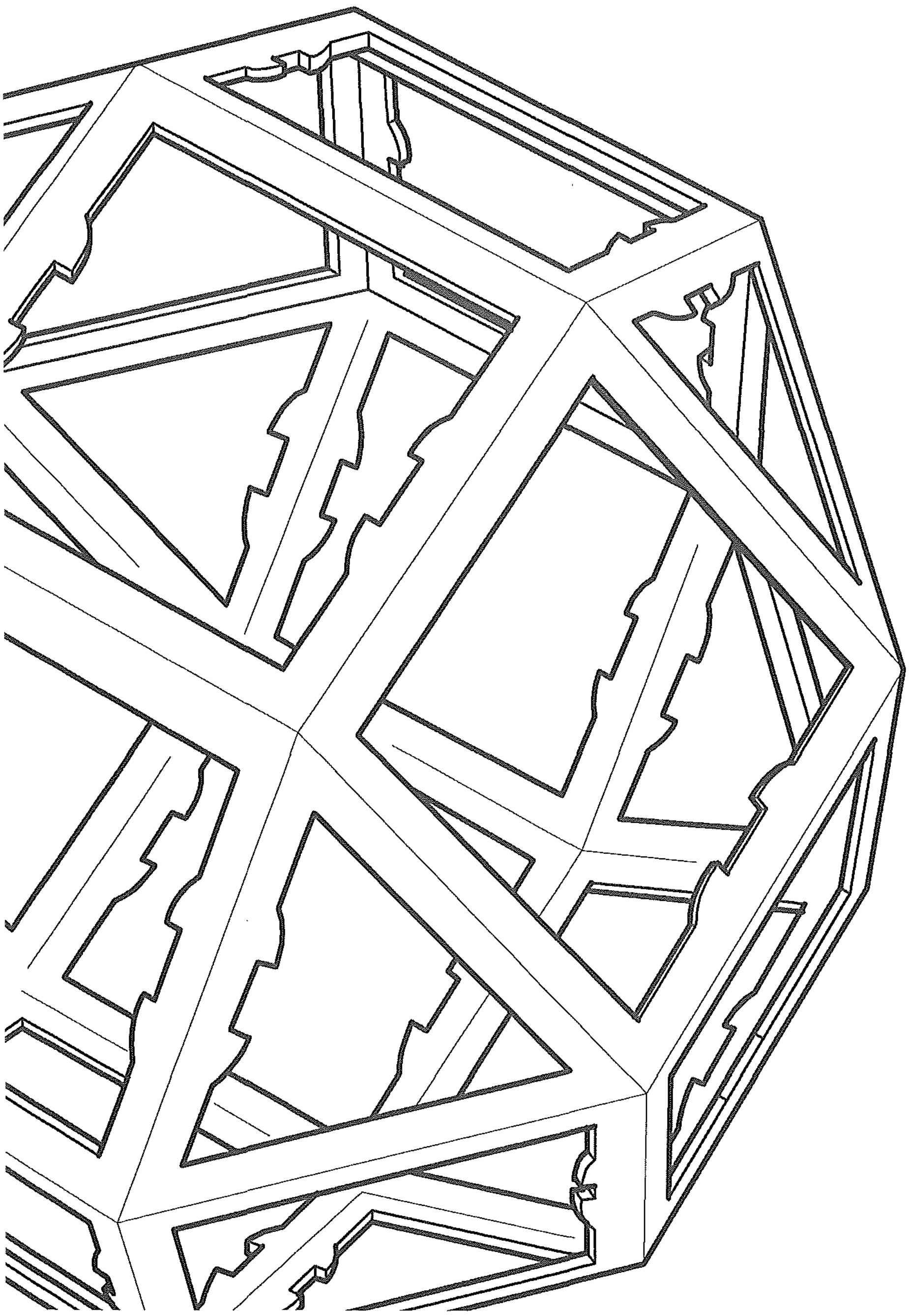

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Lattices in Automorphism Groups of Trees

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1 Introduction

Lattices are intrinsically interesting mathematical objects that arise in diverse and seemingly disparate subfields of pure mathematics. Due to the work of Bass, Serre, and others, it has become clear that one way of investigating these objects is to analyze $\hom(\Lambda, G)$ and $\hom(\Lambda, G)/G$, respectively, the set of all monomorphisms from Λ to G, and the set of orbits under the action of G where Λ is a fundamental group of a graph of groups and $G = \mathrm{GL}_2(\mathbb{C})$.

We shall concern ourselves with the case where Λ is derived from a graph of groups consisting of a single vertex and a single loop. In particular, when considering the quotient space, $\hom(\Lambda, G)/G$, the questions concerning us are: What is the dimension of $\hom(\Lambda, G)/G$? Is $\hom(\Lambda, G)/G$ smooth? These questions will be answered for two particular Λ that arise in the aforementioned way. Along the way we provide the necessary background theory in group presentations, graphs of groups, lattices, dimension, and smoothness.

In what follows we draw on group theory and linear algebra frequently. With respect to the latter, we assume the reader is familiar with the diagonalization process and the closely related topics of eigenvalues and eigenvectors. If not, the relevant material can be found in [5, ch. 4.4]. Acquaintance with group actions, orbits and stabilizers, free groups, and the first isomorphism theorem is assumed regarding group theory. These topics can be found in [2].

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2 Group Presentations

In this section we shall follow closely [2, ch. 26]. Let $S = \{a_1, a_2, ..., a_n\}$ be a set of generators for a group G. Let F_s be the free group on S. Let $R = \{r_1, r_2, ..., r_t\} \subseteq F_s$ and let N be the smallest normal subgroup of F_s containing R. If there is an isomorphism from F_s/N onto G that sends a_iN to a_i, then we say that G has the *presentation* $< a_1, a_2, ..., a_n | r_1 = r_2 = ... = r_t = 1 >$, and write

$$G =< a_1, a_2, ..., a_n | r_1 = r_2 = ... = r_t = 1 > .$$

In other words, G is the largest group that satisfies these conditions.

Remark 2.1 There is no requirement that the set of generators and the set of relations be finite, in fact either or both may be infinite. Following [4, pg. 7], we say a presentation is *finitely generated (finitely related)* if the number of generators (relations) in it is finite. If a presentation is both finitely generated and finitely related (as above), we say the *presentation is finite.* More details on this topic can be found in [4].

Example 2.2 Let $G =< a | a^{12} = 1 >$. Clearly the trivial group and the cyclic groups of orders 2,3,4,6, and 12 satisfy these relations and are given by a single generator, in fact these are the only such groups. The cyclic group of order 12, $\mathbf{C}_{12}$, is the largest, therefore $G = \mathbf{C}_{12}$.

3 Graphs, Graphs of Groups, Tree Lattices

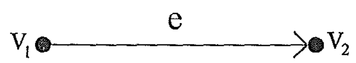
3.1 Graphs, Trees

In this section we shall follow closely [6].

Definition 3.1 A *graph* X consists of two sets $V = vert(X)$, and $E = edge(X)$ along with two functions $E \longrightarrow V \times V$; $e \longmapsto (o(e), t(e))$, and $E \longrightarrow E$; $e \longmapsto \bar{e}$, such that for each $e \in E$, $\bar{\bar{e}} = e$, $e \neq \bar{e}$, and $o(e) = t(\bar{e})$. Any $v \in V$ is a vertex of X, and any $e \in E$ is an oriented edge of X, with $\bar{e}$ being the inverse edge of e, that is e with the reverse orientation. We call $o(e) = t(\bar{e})$ the origin of e, or equivalently the terminus of $\bar{e}$. Furthermore, for $v \in vert(X)$, we define the valence of v, $val(v)$ to be equal to the number of edges whose origin is v, formally $val(v) = Card\{e \in edge(X) | o(e) = v\}$. For more detail see [6, ch. 2].

This is very abstract, and fortunately there is an easy and accurate way of displaying all the relevant information. Using dots for the vertices and arcs for the edges we can depict a graph as the diagram below shows.

Example 3.2



Here $V = \{v_1, v_2\}$, and $E = \{e, \bar{e}\} = \{(v_1, v_2), (v_2, v_1)\}$, as $v_1 = o(e) = t(\bar{e})$ and $v_2 = o(\bar{e}) = t(e)$.

A *path*, $P(v_0, v_n)$ in X joining v_0 to v_n is a sequence of edges $e_1, e_2, ..., e_n$ such that $o(e_1) = v_0$, $o(e_{i+1}) = t(e_i)$ for $1 \leq i \leq n-1$, and $t(e_n) = v_n$. When a path contains a subsequence of the form $e\bar{e}$ *i.e.* an edge followed by its inverse, we shall say the path contains a *reversal.* A graph X is *connected* if there exists a path $P(v', v'')$ for all $v', v'' \in vert(X)$. In other words, there are no isolated vertices. A *tree* is a connected graph, in which any non-empty path beginning and ending at the same vertex contains a reversal. If X is a connected graph, then a *tree* in X is a tree T such that $vert(T) \subseteq vert(X)$ and $edge(T) \subseteq edge(X)$. T is a *maximal tree* in X if T is a tree in X such that T is not contained in any other tree in X. For more details see [6, ch. 2].

3.2 Graphs of Groups, Fundamental Group

Definition 3.3 A graph of groups (G, X), consists of a connected, non-empty graph X, a group G_v for each $v \in vert(X)$, a group G_e for each $e \in edge(X)$ such that $G_e = G_{\bar{e}}$ for each $e \in edge(X)$, along with a monomorphism $G_e \longrightarrow G_{t(e)}$ (denoted by $a \longmapsto a^e$).

Definition 3.4 Here we follow [6, ch. 5].Let T be a maximal tree of X. The *fundamental group* $\pi(G, X, T)$ *of* (G, X) *at* T is the group generated by the groups $G_v(v \in vert(X))$ and the elements $e \in edge(X)$ subject to the relations $\bar{e} = e^{-1}$, $ea^e e^{-1} = a^{\bar{e}}$ when $e \in edge(X)$ and $a \in G_e$, and $e = 1$ if $e \in edge(T)$. The fundamental group is independent of the choice of T.

Remark 3.5 We are interested in the case where (G, X) consists of a single vertex v and a single loop e, in such cases we ignore the condition that $e = 1$ if $e \in edge(T)$, as $edge(T) = \{\ \}$.

3.3 Aut(X)

Definition 3.6 This section follows closely [1, pg. 173] Let X be a graph, furthermore let σ be a permutation of $vert(X)$ and $edge(X)$ such that $\sigma(\bar{e}) = \overline{\sigma(e)}$, $\sigma(o(e)) = o(\sigma(e))$, and $\sigma(e) \neq \bar{e}$ for all $e \in edge(X)$. Such a σ is an *automorphism* of X.

Definition 3.7 Let X, σ be as in definition (3.6). The set of all such σ is called the *automorphism group* of X, denoted $Aut(X)$.

Remark 3.8 It is easily verified (though we will not do so) that $Aut(X)$ is a group under the operation of function composition, and that $Aut(X)$ acts on X.

3.4 Quotient Graphs and Tree Lattices

Let X be a tree with infinitely many vertices, furthermore let $val(v) = n$ for some finite, fixed $n \in \mathbb{N}$ for all $v \in vert(X)$. Now consider Γ, where $\Gamma \leq Aut(X)$. Being a subgroup of $Aut(X)$, Γ also acts on X, consequently we may construct the quotient graph, X/Γ in the following way. First let $\Gamma \cdot v$ denote the orbit of the vertex v under the action of Γ, and Γ_v the stabilizer of v under the action of Γ. There is a distinct vertex in X/Γ for every distinct orbit of the vertices of X under the action of Γ, moreover these are the only vertices in X/Γ. If $\Gamma \cdot v \neq \Gamma \cdot u$ correspond to vertices $\dot{v} \neq \dot{u}$ in X/Γ then there is an edge between $\dot{v}$ and $\dot{u}$ if and only if there exist $v' \in \Gamma \cdot v$ and $u' \in \Gamma \cdot u$ such that v' and u' are joined by an edge in X.

Remark 3.9 It is worth noting that if $v', v'' \in \Gamma \cdot v$, then it may be that $\Gamma_{v'} \neq \Gamma_{v''}$, however they will be isomorphic (see theorem 5.8). With this in mind we shall use $\Gamma_{[v]}$ to refer to the abstract group isomorphic to Γ_v for all $v' \in \Gamma \cdot v$. Similarly, $\Gamma_{[e]}$ will refer to the abstract group isomorphic to Γ_e for all $e' \in \Gamma \cdot e$.

If we now assign $\Gamma_{[v]}$ to $\dot{v}$, and do likewise for the other orbits, and also assign the edge stabilizer subgroups $\Gamma_{[e]}$ to the relevant edges in X/Γ then X/Γ is a graph of groups.

Definition 3.10 If X/Γ is finite and Γ_v is finite for all $v \in vert(X)$, then Γ is a *uniform X-lattice.*

Theorem 3.11 Γ *is isomorphic to the fundamental group of* X/Γ. *For the proof see [6, pg. 54,55].*

4 Geometry in $\mathrm{GL}_2(\mathbb{C})$

4.1 Dimension

The notion of dimension, as outlined in this section, is a somewhat vague entity. However mathematically imprecise it may be, it is adequate for our purposes. We shall introduce the concept through some relevant examples.

Example 4.1 Consider the set of scalar matrices in $\mathrm{GL}_2(\mathbb{C})$, *i.e.*

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{C}^*$. This space is parameterizable by one complex variable α, where α can take on any non-zero value in $\mathbb{C}$. It is for this reason that we shall say this space has dimension one, in symbols $\dim(\alpha I) = 1$.

Example 4.2 Now consider the set of diagonal matrices in $\mathrm{GL}_2(\mathbb{C})$, *i.e.*

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

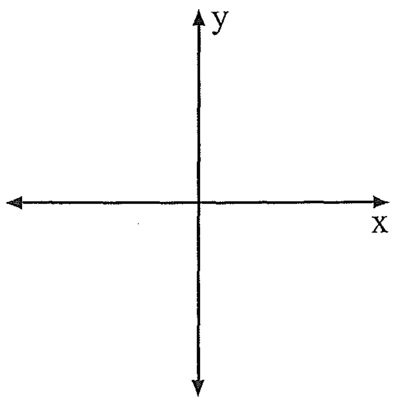
where $\alpha, \beta \in \mathbb{C}^*$. We see that this space is parameterizable by two independent variables α, β, where they can take on any non-zero value in $\mathbb{C}$. Accordingly, we shall say this space has dimension two.

Remark 4.3 We caution the reader that this treatment of dimension is not precise, nor is it the whole story. It is a convenient simplification which omits details that are irrelevant to the calculations carried out later. In short, this treatment will not lead us into trouble here and now. For a more detailed exposition see [3].

4.2 Smoothness

As vaguely as we dealt with the notion of dimension, we shall do so even more with the notion of smoothness. Once again it will be adequate for our purposes, and once again we proceed by examples. The first of which has been chosen for its simplicity, not its relevance.

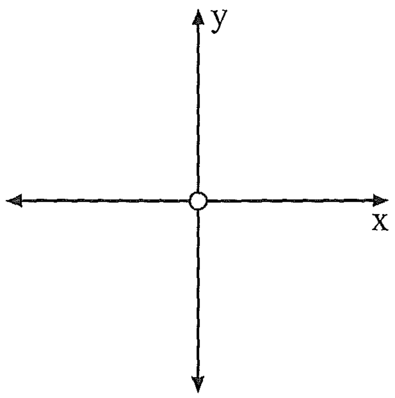
Example 4.4 First let $\mathcal{S}$ be the set of solutions to the equation $xy = 0$, where $x, y \in \mathbb{R}$. When graphed in the usual manner, $\mathcal{S}$ is the set of all points in the x and y axes, as shown below.



At any point $(x_0, y_0) \in \mathcal{S} - \{(0,0)\}$ there exists a non-empty open neighbourhood $\mathcal{X}$, centred at (x_0, y_0), such that the points in $\mathcal{S} \cap \mathcal{X}$ form a line segment. To see this, take any $\delta \neq 0$, then for any point $(0, \delta)$ or $(\delta, 0)$ let $\mathcal{X}$ be the open disk of radius $\frac{|\delta|}{2}$ centred at the point $(0, \delta)$ or $(\delta, 0)$ respectively.

However, any non-empty, open neighbourhood $\mathcal{X}$, centred at the point $(0,0)$, is one in which the points in $\mathcal{S} \cap \mathcal{X}$ form two intersecting line segments, not a single line segment. It is due to this *singularity* at the point $(0,0)$ that we say the space $\mathcal{S}$ is *not smooth.*

If we add the condition that $x \neq y$, then the solution space $\mathcal{S}' = \mathcal{S} - \{(0,0)\}$ looks like this when graphed.



Now any point $(x_0, y_0) \in \mathcal{S}'$ admits a non-empty, open neighbourhood $\mathcal{X}$, such that the points in $\mathcal{S}' \cap \mathcal{X}$ form a single line segment, indeed the $\mathcal{X}$ described above will suffice. For this reason, we say the space $\mathcal{S}'$ is *smooth.*

Remark 4.5 It is worth noting that $\mathcal{S}'$ is smooth in virtue of it containing no singularities, there is no requirement that $\mathcal{S}' \cap \mathcal{X}$ form a line segment, that is just a feature of example 4.4.

Example 4.6 With the notion of smoothness in mind, we now return to the set of scalar matrices in $\mathrm{GL}_2(\mathbb{C})$. Recall that this space is parameterizable by one complex variable α, where $\alpha \in \mathbb{C}^*$. It may be helpful to think of this space, call it $\mathbb{C}^*$ as consisting of the complex plane with the origin removed. For any point $\alpha \in \mathbb{C}^*$, let $\mathcal{X}$ be the open disk of radius $\frac{\|\alpha\|}{2}$ centred at α. Such an $\mathcal{X}$ is non-empty, does not contain the origin, and the points in $\mathcal{X}$ form an open disk in $\mathbb{C}^*$. As α was arbitrary, we conclude that $\mathbb{C}^*$ contains no singularities, and is therefore smooth.

Example 4.7 The diagonal matrices in $\mathrm{GL}_2(\mathbb{C})$ are parameterizable by two complex variables $\alpha, \beta \in \mathbb{C}^*$. We shall call this space $\mathbb{C}^* \times \mathbb{C}^*$. No intuitive geometric realization of this space is readily available, but we can still make sense of it. For any $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$, let $\mathcal{X}$ be the open neighbourhood of radius $\frac{\|(\alpha,\beta)\|}{2}$ centred at (α, β). Again, $\mathcal{X}$ is non-empty, does not contain the origin, and the points in $\mathcal{X}$ form some sort of open 4D-ball. As (α, β) was arbitrary, we conclude that $\mathbb{C}^* \times \mathbb{C}^*$ also contains no singularities, and is therefore smooth.

5 Preliminary Results

In the course of analyzing $\hom(\Lambda, G)$ and $\hom(\Lambda, G)/G$ we use several results pertaining to linear algebra and group actions, we now deal with them in that order.

Theorem 5.1 *Every element of* $\mathrm{GL}_2(\mathbb{C})$ *is diagonalizable or triangularizable.*

Proof Assume $M \in \mathrm{GL}_2(\mathbb{C})$ is not diagonalizable, then M has only one eigenvalue $\lambda \in \mathbb{C}^*$. Let $\mathbf{v} \neq \mathbf{0}$ be a corresponding eigenvector, so we have $M\mathbf{v} = \lambda\mathbf{v}$. Let $S = [\mathbf{v} \mid \mathbf{v}']$ where $(M - \lambda I)\mathbf{v}' = \mathbf{v}$, or equivalently $M\mathbf{v}' = \mathbf{v} + \lambda\mathbf{v}'$. Then $MS = M[\mathbf{v} \mid \mathbf{v}'] = [M\mathbf{v} \mid M\mathbf{v}'] = [\lambda\mathbf{v} \mid \mathbf{v} + \lambda\mathbf{v}'] = [\mathbf{v} \mid \mathbf{v}'] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

S is invertible as $\mathbf{v}' \notin \mathrm{null}(M - \lambda I)$, therefore $S^{-1}MS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and M is triangularizable. □

Corollary 5.2 *If* $M \in \mathrm{GL}_2(\mathbb{C})$ *is such that* $M^n = I$ *for some positive integer* n*, then* M *is diagonalizable.*

Proof If not, then $SMS^{-1} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ for some $S \in \mathrm{GL}_2(\mathbb{C})$, and $(SMS^{-1})^n = SM^nS^{-1} = SIS^{-1} = SS^{-1} = I$, whereas

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} = \begin{bmatrix} 1 & n\lambda^{n-1} \\ 0 & 1 \end{bmatrix} \neq I$$

as $n, \lambda \neq 0$, contradiction. □

Remark 5.3 Implicit above are the facts that if λ is an eigenvalue of M, then λ^n is an eigenvalue of M^n for any positive integer n, and that conjugation preserves eigenvalues. See [5, ch. 4.4] for details.

Theorem 5.4 *If* Λ *is a finite cyclic group then* $\hom(\Lambda, \mathrm{GL}_2(\mathbb{C}))/\mathrm{GL}_2(\mathbb{C})$ *is finite.*

Proof Let $\Lambda = < a \mid a^n = 1 >$ *i.e.* the cyclic group of order n for some $n \in \mathbb{N}$. If $\phi \in \hom(\Lambda, \mathrm{GL}_2(\mathbb{C}))$, then $\phi(a) = A$ is such that $A^n = I$. As I has the repeated eigenvalue 1, by remark 5.3 A must have any two of the n distinct roots of unity as its eigenvalues. As $A^n = I$, A is diagonalizable by corollary 5.2. As diagonalization preserves eigenvalues (see remark 5.3), A is in the same conjugacy class as a diagonal matrix D, whose diagonal entries are the eigenvalues of A. We shall see in lemma 5.15 that

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \text{ is conjugate to } \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix},$$

and so we need only count the distinct *unordered* pairs of the n eigenvalues, of which there are precisely $\frac{n(n+1)}{2}$. It follows that A lies in any one of at most $\frac{n(n+1)}{2}$ conjugacy classes. Conversely, any one of the $\frac{n(n+1)}{2}$ diagonal matrices D that represent these classes is such that $D^n = I$, therefore $Card\{\hom(\Lambda, \mathrm{GL}_2(\mathbb{C}))/\mathrm{GL}_2(\mathbb{C})\} = \frac{n(n+1)}{2}$. □

Lemma 5.5 *If $M \in \mathrm{GL}_2(\mathbb{C})$ is diagonalizable and has only one eigenvalue, then M is scalar.*

Proof Let M have the repeated eigenvalue λ. If M is diagonalizable, then by remark 5.3 M is in the same conjugacy class as λI. For any $G \in \mathrm{GL}_2(\mathbb{C})$ we have $G\lambda IG^{-1} = \lambda GIG^{-1} = \lambda GG^{-1} = \lambda I$, which shows that the conjugacy class in question contains only λI. We conclude that $M = \lambda I$. □

Theorem 5.6 *If $A, B \in \mathrm{GL}_2(\mathbb{C})$ are diagonalizable, then $AB = BA$ if and only if A and B are simultaneously diagonalizable.*

Proof $\Longleftarrow$: Let $G \in \mathrm{GL}_2(\mathbb{C})$ be such that $GAG^{-1} = D$ and $GBG^{-1} = \overline{D}$, where D and $\overline{D}$ are diagonal. Being diagonal, D and $\overline{D}$ commute, so we have $GABG^{-1} = GAG^{-1}GBG^{-1} = D\overline{D} = \overline{D}D = GBG^{-1}GAG^{-1} = GBAG^{-1}$. Left and right cancelation by G^{-1} and G respectively gives $AB = BA$.

$\Longrightarrow$: If either A or B is scalar then the result is trivial, consequently we shall assume that A and B each have two distinct eigenvalues. For any $G \in \mathrm{GL}_2(\mathbb{C})$ we have $GAG^{-1}GBG^{-1} = GABG^{-1} = GBAG^{-1} = GBG^{-1}GAG^{-1}$, so GAG^{-1} and GBG^{-1} commute for all $G \in \mathrm{GL}_2(\mathbb{C})$. Choose a G such that $GAG^{-1} = D$, where D is diagonal, then $D(GBG^{-1}) = (GBG^{-1})D$. We now show that GBG^{-1} is also diagonal.

$$D(GBG^{-1}) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda_1 a & \lambda_1 b \\ \lambda_2 c & \lambda_2 d \end{bmatrix} = \begin{bmatrix} a\lambda_1 & b\lambda_2 \\ c\lambda_1 & c\lambda_2 \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = (GBG^{-1})D$$

implies that $b\lambda_1 = b\lambda_2$ and $c\lambda_1 = c\lambda_2$. By hypothesis $\lambda_1 \neq \lambda_2$, and $\lambda_1, \lambda_2 \neq 0$ as D is invertible. It follows that $b = c = 0$, hence GBG^{-1} is diagonal, with G simultaneously diagonalizing A and B. □

The next result is a classification of matrices in $\mathrm{GL}_2(\mathbb{C})$ that square to some scalar matrix λI.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $M^2 = \lambda I$ for some $\lambda \in \mathbb{C}^*$. Then

$$M^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

gives the following system of equations:

$$a^2 + bc = \lambda \tag{1}$$

$$d^2 + bc = \lambda \tag{2}$$

$$b(a+d) = 0 \tag{3}$$

$$c(a+d) = 0 \tag{4}$$

Consider first the case where $a + d \neq 0$. By equations (3) and (4), $b = c = 0$, which when substituted into equations (1) and (2) gives $a^2 = d^2 = \lambda$. Clearly $a = \pm\sqrt{\lambda}$ and $d = \pm\sqrt{\lambda}$ are the only candidate solutions, and $a \neq -d$ by hypothesis so $a = d = \pm\sqrt{\lambda}$ and

$$M = \begin{bmatrix} \pm\sqrt{\lambda} & 0 \\ 0 & \pm\sqrt{\lambda} \end{bmatrix}.$$

Next we consider the case where $a+d = 0$. First set $a = k$ and $d = -k$ with $k \neq \pm\sqrt{\lambda}$. By (1) and (2), $bc = \lambda - k^2 \neq 0$, which implies that $b, c \neq 0$, so $c = \frac{\lambda - k^2}{b}$ and

$$M = \begin{bmatrix} k & b \\ \frac{\lambda-k^2}{b} & -k \end{bmatrix},$$

with $b \neq 0, k \neq \pm\sqrt{\lambda}$. Finally let $k = \pm\sqrt{\lambda}$. By (1) and (2), $bc = \lambda - k^2 = 0$, which implies that $b = 0$ or $c = 0$ (or both). Hence M is in the form

$$\begin{bmatrix} \pm\sqrt{\lambda} & b \\ 0 & \mp\sqrt{\lambda} \end{bmatrix} \text{ or } \begin{bmatrix} \pm\sqrt{\lambda} & 0 \\ c & \mp\sqrt{\lambda} \end{bmatrix} \text{ where } b, c \in \mathbb{C}.$$

This exhausts the possibilities and we conclude that M must be in one of the forms above. Conversely, it is easily verified by direct calculation that all these forms do indeed square to give λI. While less elegant than classifying these matrices into their three conjugacy classes, this result is useful when simultaneous diagonalization is not possible, as we shall see in the second set of calculations.

We now look at some results related to the situation where a group G acts on a set Λ.

Theorem 5.7 *The function* $f : G \times \hom(\Lambda, G) \longrightarrow \hom(\Lambda, G)$ *given by* $f(g, \rho) := g \cdot \rho$, *where* $(g \cdot \rho) = g\rho(\gamma)g^{-1}$, *is an action of* G *on* $\hom(\Lambda, G)$.

Proof Although we omit the details, f is well-defined. Let 1 denote the identity in G, $\rho \in \hom(\Lambda, G)$, and $\gamma \in \Lambda$. Then we have $f(1, \rho)(\gamma) = 1\rho(\gamma)1^{-1} = \rho(\gamma)$. As ρ, γ were arbitrary we have that $f(1, \rho) = \rho$ for all $\rho \in \hom(\Lambda, G)$. For any $g_1, g_2 \in G$, and ρ, γ as before, we have $f(g_1g_2, \rho)(\gamma) = (g_1g_2)\rho(\gamma)(g_1g_2)^{-1} = (g_1g_2)\rho(\gamma)(g_2^{-1}g_1^{-1}) = g_1(g_2\rho(\gamma)g_2^{-1})g_1^{-1} = g_1(f(g_2, \rho)(\gamma))g_1^{-1} = f(g_1, f(g_2, \rho))(\gamma)$. Again these were arbitrary, so we have $f(g_1g_2, \rho) = f(g_1, f(g_2, \rho))$ for all $g \in G$ and $\rho \in \hom(\Lambda, G)$. $\square$

The above result ensures that when we have a homomorphic image of Λ in G, simultaneous conjugation by any $g \in G$ takes us to another (not necessarily distinct) homomorphic image. That simultaneous conjugation is an action of G on $\hom(\Lambda, G)$ makes relevant the next results concerning orbits and stabilizers.

Theorem 5.8 *If a group G acts on a set X, and $g \cdot x = y$ for some $g \in G$, $x, y \in X$ then the stabilizers of x and y, denoted G_x and G_y are isomorphic.*

Proof First we define $\phi : G_x \longrightarrow G_y$ by $\phi(h) = ghg^{-1}$. Although we do not provide the details, ϕ is well-defined. If $ghg^{-1} = g\hat{h}g^{-1}$, then $h = \hat{h}$ after left and right cancelation, so ϕ is injective. If $a \in G_y$, then $(g^{-1}ag)\cdot x = (g^{-1}a)\cdot(g\cdot x) = (g^{-1}a)\cdot y = g^{-1}\cdot(a\cdot y) = g^{-1}\cdot y = x$, so $g^{-1}ag = h$ for some $h \in G_x$. It follows that $a = ghg^{-1}$ for some $h \in G_x$, so ϕ is surjective. Finally, $\phi(a,b) = gabg^{-1} = gag^{-1}gbg^{-1} = \phi(a)\phi(b)$, so ϕ is operation-preserving, therefore $G_x \approx G_y$. $\square$

Theorem 5.9 *If a group G acts on a set X of n-tuples, in a setting where dimension is suitably defined, then the dimension of G equals the dimension of the orbit of x, $G \cdot x$ plus the dimension of the stabilizer of x, G_x for all $x \in X$. See [3] for the proof.*

Corollary 5.10 *If $g \cdot x = y$ for some $g \in G$, $x, y \in X$ then* $\dim(G_x) = \dim(G_y)$.

Proof As $g \cdot x = y$, we have $G \cdot x = G \cdot y$, and the result follows from theorem (5.9).

Finally we present a few useful results concerning conjugation and stabilizers in $\mathrm{GL}_2(\mathbb{C})$.

Lemma 5.11 *If $M \in \mathrm{GL}_2(\mathbb{C})$ is of the form*

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$$

with $\alpha \neq \delta$, then the stabilizer of M under the conjugation action of $\mathrm{GL}_2(\mathbb{C})$, denoted G_M, is the diagonal matrices.

Proof In any group, including $\mathrm{GL}_2(\mathbb{C})$, $MX = XM \iff M = XMX^{-1}$ holds. So

$$MX = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \delta c & \delta d \end{bmatrix},$$

whereas

$$XM = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} a\alpha & b\delta \\ c\alpha & d\delta \end{bmatrix}.$$

Setting $MX = XM$ gives $b\alpha = b\delta$ and $c\alpha = c\delta$. As $\alpha \neq \delta$ and $\alpha, \delta \neq 0$ we have

$$b = c = 0 \text{ and } X = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

with $a, d \neq 0$ so as to be invertible. It is easily verified that any X of this form does in fact stabilize M. $\square$

Remark 5.12 This space is two dimensional and smooth, as discussed in examples (4.2) and (4.7).

Lemma 5.13 *If $M, N \in \mathrm{GL}_2(\mathbb{C})$ with M as in lemma 5.11 and N not diagonal, then the stabilizer of the ordered pair (M, N) under simultaneous conjugation by $\mathrm{GL}_2(\mathbb{C})$, denoted $G_{(M,N)}$, is the scalar matrices.*

Proof It is easily seen that $G_{(M,N)} = G_M \cap G_N$, from which it follows that $G_{(M,N)}$ is all matrices of the form

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \text{with } \alpha, \delta \in \mathbb{C}^*,$$

such that

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{bmatrix} = \begin{bmatrix} a & \alpha b \delta^{-1} \\ \delta c \alpha^{-1} & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Which gives $\alpha b \delta^{-1} = b$ and $\delta c \alpha^{-1} = c$. As N is not diagonal, at least one of b or c is non-zero, which implies that $\alpha = \delta$ and $G_{(M,N)} \subseteq \mu I$ with $\mu \in \mathbb{C}^*$. As $\mu I = \mathrm{Z}(G)$ (the centre of G), we have $G_{(M,N)} = \mu I$. $\square$

Remark 5.14 This space is one dimensional and smooth, as discussed in examples (4.1) and (4.6).

Lemma 5.15 *If $M \in \mathrm{GL}_2(\mathbb{C})$ is of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$ with $\alpha \neq \delta$, then the set of all $X \in \mathrm{GL}_2(\mathbb{C})$ such that $XMX^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \alpha \end{bmatrix}$ is $\left\{ \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \middle| \beta, \gamma \in \mathbb{C}^* \right\}$.*

Proof Setting $X \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} X^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \alpha \end{bmatrix}$, we have

$$\frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad\alpha - bc\delta & ab(\delta - \alpha) \\ cd(\alpha - \delta) & ad\delta - bc\alpha \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \alpha \end{bmatrix},$$

which gives the following equations:

$$ad\alpha = ad\delta \tag{5}$$

$$ab(\delta - \alpha) = 0 \tag{6}$$

$$cd(\alpha - \delta) = 0 \tag{7}$$

It follows from (5) that $a = 0$ or $d = 0$, as $\alpha \neq \delta$ and $\alpha, \delta \neq 0$. We also have that $a = 0$ or $b = 0$ by (6), and $c = 0$ or $d = 0$ by (7). If $b = 0$ then $a, d \neq 0$ as X is invertible. As (5) implies that $a = 0$ or $d = 0$ we conclude that $b \neq 0$. Then by (6) we have $a = 0$. If $c = 0$ then the same argument applies, and we conclude that $c \neq 0$. Then by (7) we have $d = 0$, and X is of the form $\begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$ with $\beta, \gamma \in \mathbb{C}^*$. Again, direct calculations show that any matrix of this form has this effect on M.

Remark 5.16 Lemma (5.15) allows us to count only the unordered pairs of distinct eigenvalues in theorem (5.4).

6 Calculations

We now proceed to describe $\hom(\Lambda, G)$ and $\hom(\Lambda, G)/G$ for two such Λ that are derived from a graph of groups composed of a single vertex and a single loop. In both cases $G = \mathrm{GL}_2(\mathbb{C})$ and we only consider injective homomorphisms.

Example 1: The vertex group is $\mathrm{C}_2 \times \mathrm{C}_2 =< a, b \mid a^2 = b^2 = 1, ab = ba >$. The edge group is $\mathrm{C}_2 =< c \mid c^2 = 1 >$. We have the monomorphisms $\phi, \psi : \mathrm{C}_2 \longrightarrow \mathrm{C}_2 \times \mathrm{C}_2$ given by $\phi(c) = a, \psi(c) = b$. It follows from definition (3.4) and remark (3.5) that the fundamental group of this graph of groups has the presentation

$$\Lambda =< a, b, y \mid a^2 = b^2 = 1,\ ab = ba,\ yay^{-1} = b >$$

We require all possible ordered triples (A, B, Y) in $(\mathrm{GL}_2(\mathbb{C}))^3$ such that $A^2 = B^2 = I$, $AB = BA$, and $YAY^{-1} = B$

As $A^2 = B^2 = I$, by theorem (5.2) A and B are diagonalisable, as $AB = BA$, by theorem (5.6) they are simultaneously so. After diagonalising A and B, by theorem (5.4) they must be in one of the following forms:

$$\begin{aligned}(A,B) = (i) &\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right), (ii) \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right), \\ (iii) &\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right), (iv) \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right), \\ (v)(&\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right).\end{aligned}$$

Being interested in monomorphisms only, we may ignore (i) and (iii). This is also why $A, B \neq I$. The condition that $YAY^{-1} = B$ entails that A and B have the same eigenvalues (see remark (5.3)), consequently we may also discard (iv) and (v).

This leaves (A, B) conjugate to $\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)$, as our only possible candidate. We now require Y such that $Y \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Y^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

As shown in lemma (5.15), the set of all such Y is $\left\{ \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \mid \beta, \gamma \in \mathbb{C}^* \right\}$

So we have established that every element in $hom(\Lambda, G)$ is conjugate to

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}\right) \text{ for some } \beta, \gamma \in \mathbb{C}^*.$$

Conversely, direct calculations show that any triple of this form satisfies the required conditions. By lemma (5.11) the stabilizer of any of these triples is precisely the scalar matrices, which was shown in examples (4.1) and (4.6) to be a smooth one-dimensional space. Given that $GL_2(\mathbb{C})$ is a four dimensional space, it follows from theorem (5.9) and corollary (5.10) that any given orbit has dimension three.

We now require a set of representatives that meets every orbit exactly once. For any given $\beta, \gamma \in \mathbb{C}^*$ we can simultaneously conjugate

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}\right) \text{ by } \begin{bmatrix} \alpha\beta^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \text{ for any } \alpha \in \mathbb{C}^*.$$

This lies in the stabilizer of A and B, while its effect on Y is seen below:

$$\begin{bmatrix} \alpha\beta^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \alpha^{-1}\beta & 0 \\ 0 & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha\gamma & 0 \end{bmatrix} \begin{bmatrix} \alpha^{-1}\beta & 0 \\ 0 & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix}$$

So every orbit is met by an element of the form

$$(A, B, Y) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix}\right); \gamma \in \mathbb{C}^*.$$

Moreover, every orbit is met exactly once as

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \delta\gamma & 0 \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \alpha\delta^{-1} \\ \delta\gamma\alpha^{-1} & \end{bmatrix},$$

$$\text{and } \alpha\delta^{-1} = 1 \Longrightarrow \alpha = \delta \Longrightarrow \alpha^{-1}\delta\gamma = \gamma \Longrightarrow \begin{bmatrix} 0 & \alpha\delta^{-1} \\ \delta\gamma\alpha^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix}.$$

Therefore, for each choice of $\gamma \in \mathbb{C}^*$,

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix}\right)$$

is in a distinct orbit, and we have a set of representatives that meets each orbit exactly once. The quotient space has the same dimension as this set, namely one, due to the single complex variable γ. The space is also smooth, as γ can take on any non-zero value. In other words, this space is in some sense the "same" as the space $\mathbb{C}^*$, which we discussed in example (4.6).

Example 2: The vertex group is $S_3 =< a, b \mid a^2 = b^2 = (ab)^3 = 1 >$. The edge group is $C_2 =< c \mid c^2 = 1 >$. We have the monomorphisms $\phi, \psi : C_2 \longrightarrow S_3$ given by $\phi(c) = a, \psi(c) = b$. It follows from definition (3.4) and remark (3.5) that the fundamental group of this graph of groups has the presentation

$$\Lambda =< a, b, y \mid a^2 = b^2 = (ab)^3 = 1,\ yay^{-1} = b > .$$

We require all possible ordered triples $(A, B, Y) \in (\mathrm{GL}_2(\mathbb{C}))^3$ such that $A^2 = B^2 = (AB)^3 = I$, and $YAY^{-1} = B$.

As $A^2 = B^2 = I$, by corollary (5.2) A and B are diagonalisable. By theorem (5.4) and lemma (5.15) it follows that either $A = -I$, or A is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, likewise for B. Being interested in monomorphisms only, we may assume that $A, B \neq I$.

Assume first that $A = -I$. Any such monomorphic image is such that $B \neq -I$, consequently $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, or B is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In either case, A and B are simultaneously diagonalizable as A is scalar and B is diagonalizable. However,

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow (AB)^3 = AB \neq I,$$

and we cannot satisfy all the conditions. Therefore $A \neq -I$, and by reversing the roles of A and B we have that $B \neq -I$.

We now have that A and B are both conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. A monomorphic image is not one where this happens simultaneously, so having diagonalised A, using our classification result in section 5, B must be in one of the following forms:

$$(i) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, (ii) \begin{bmatrix} \pm 1 & b \\ 0 & \mp 1 \end{bmatrix}, (iii) \begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}, (iv) \begin{bmatrix} k & b \\ \frac{1-k^2}{b} & -k \end{bmatrix};\ b, c \neq 0, k \neq \pm 1.$$

In cases (i),(ii), and (iii) the product $(AB)^3 \neq I$, which leaves only case (iv):

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k & b \\ \frac{1-k^2}{b} & -k \end{bmatrix} = \begin{bmatrix} k & b \\ \frac{k^2-1}{b} & k \end{bmatrix} \Longrightarrow (AB)^3 = \begin{bmatrix} 4k^3 - 3k & 4k^2b - b \\ \frac{4k^4-5k^2+1}{b} & 4k^3 - 3k \end{bmatrix},$$

which equals I if and only if $k = \frac{-1}{2}$. So having diagonalized A to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and having substituted $\frac{-1}{2}$ for k, B is in the form $\begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{4b} & \frac{1}{2} \end{bmatrix}$ for any $b \in \mathbb{C}^*$.

Given any $b \in \mathbb{C}^*$ we may conjugate $\begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{4b} & \frac{1}{2} \end{bmatrix}$ by $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & b \end{bmatrix}$ to give $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ while stabilizing A. A word of explanation as to where $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & b \end{bmatrix}$ came from. Taking some $\begin{bmatrix} \alpha & 0 \\ 0 & b \end{bmatrix} \in G_A$, i.e. $\alpha \neq 0$, and conjugating B gives
$\begin{bmatrix} \alpha & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{4b} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{2} & \alpha b \\ \frac{3}{4} & \frac{b}{2} \end{bmatrix}\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \alpha \\ \alpha^{-1}\frac{3}{4} & \frac{1}{2} \end{bmatrix}$.
While setting $\alpha = \alpha^{-1}\frac{3}{4}$ gives $\alpha^2 = \frac{3}{4} \Longrightarrow \alpha = \pm\frac{\sqrt{3}}{2}$. So every orbit contains at least one element whose first and second components are $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ respectively. All that remains is to find the set of all Y such that $YAY^{-1} = B$, or equivalently $Y^{-1}BY = A$. Diagonalizing B shows that this set is

$$\left\{ \begin{bmatrix} \alpha & -\beta\sqrt{3} \\ \alpha\sqrt{3} & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{C}^* \right\}.$$

So we have established that every element of $\hom(\Lambda, G)$ is conjugate to

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \alpha & -\beta\sqrt{3} \\ \alpha\sqrt{3} & \beta \end{bmatrix} \right) \text{ for some } \alpha, \beta \in \mathbb{C}^*.$$

Conversely, direct calculations show that any triple of this form satisfies the required conditions. By lemma (5.11) the stabilizer of any of these triples is precisely the scalar matrices, which was shown in examples (4.1) and (4.6) to be a smooth one-dimensional space. Given that $\mathrm{GL}_2(\mathbb{C})$ is a four dimensional space, it follows from theorem (5.9) and corollary (5.10) that any given orbit has dimension three.

We now require a set of representatives that meets every orbit exactly once. We have just shown that every orbit is met at least once by an element of the form

$$(A, B, Y) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \alpha & -\beta\sqrt{3} \\ \alpha\sqrt{3} & \beta \end{bmatrix} \right) \text{ for some } \alpha, \beta \in \mathbb{C}^*.$$

By lemma (5.13), $G_{(A,B)}$ is the scalar matrices, which also stabilize Y. In other words, any change in either variable, α, β will lie in a different orbit. As a result every orbit is met exactly once by an element of this form. The quotient space has the same dimension as this set, namely two, due to the two independent complex variables α, β. The space is also smooth, as α and β can take on any non-zero value. In other words, this space is in some sense the "same" as the space $\mathbb{C}^* \times \mathbb{C}^*$, which we discussed in example (4.7).

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