

**Remark on the characterization of  
weighted  
Besov spaces via temperatures**

by

**Huy-Qui Bui**

*Department of Mathematics and Statistics  
University of Canterbury, Christchurch, New Zealand*

No. 89

May, 1993

# Remark on the characterization of weighted Besov spaces via temperatures

Huy-Qui Bui

5 April 1993

Dedicated to Professor Makoto Ohtsuka on the occasion of his 70th birthday

## 1 Introduction and statements of the results

The aim of this note is to complete the characterization of the weighted homogeneous Besov spaces by means of solutions of the heat equation on  $\mathbf{R}_+^{n+1}$ , which was initiated in [2]. We retain all the notations and terminologies in [1, 2] to which the reader is referred also for background and references to related works in the literature.

In [2, Theorem 1'] we proved that

$$\|f\|_{\dot{B}(s,w,p,q)} \leq C \left( \int_0^\infty \left( t^{k-s/2} \|(\partial/\partial t)^k (W_t * f)\|_{p,w} \right)^q \frac{dt}{t} \right)^{1/q} \quad (1)$$

for any  $f \in S'$ , where  $w \in A_\infty$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbf{R}$ ,  $k$  is a non-negative integer greater than  $s/2$ ,  $W_t(x) = (4t)^{-n/2} e^{-|x|^2/4t}$  is the Gauss-Weierstrass kernel on  $\mathbf{R}_+^{n+1}$ , and as usual, we use  $C, c, \dots$  to denote positive constants whose values might change from one occurrence to the next one; they might depend on the parameters  $s, p, q, k, \dots$ , but not on the distribution  $f$ . If a constant depends also on  $f$ , we use subscript to denote this dependence, e.g.,  $C_f$ .

For the opposite direction to (1), in the same quoted theorem, we had to assume that  $w$  is furthermore in  $\mathcal{M}_d$ . In this note we shall remove this restriction. Namely, we shall prove the following result.

**Theorem** *If  $f \in \dot{B}_{p,q}^{s,w}$  and  $k$  is a non-negative integer greater than  $s/2 + \max(0, 1/p - 1, 1/q - 1)$ , then there is a polynomial  $P$  such that*

$$\left( \int_0^\infty \left( t^{k-s/2} \left\| (\partial/\partial t)^k (W_t * (f - P)) \right\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \|f\|_{\dot{B}(s,w,p,q)}. \quad (2)$$

REMARK. Though we have removed the restriction  $w \in \mathcal{M}_d$ , we have introduced a new restriction on the range of  $k$  in the case either  $p < 1$  or  $q < 1$ . This new restriction is, however, more satisfactory than the other, as it does not depend on the weight function  $w$ , and the result in the Theorem is sufficient for our study of the integrability of the Fourier transform on weighted spaces in [3].

REMARK. In the paper [9], H. Triebel has studied various equivalent quasi-norms on the unweighted Besov and Triebel-Lizorkin spaces. His method was based on a rather sharp multiplier's criterion for spaces of analytic functions of exponential type. It would be an interesting problem to find out how much of his results can be extended to the present weighted case.

## 2 Lemmata

In the proof of the Theorem we shall need a number of Lemmata.

**Lemma 1** *Let  $s < 0$ , and assume that  $u$  is a temperature on  $\mathbf{R}_+^{n+1}$  for which*

$$\|u\|_{s,w,p,q} = \left( \int_0^\infty \left( t^{-s/2} \|u(\cdot, t)\|_{p,w} \right)^q \frac{dt}{t} \right)^{1/q} < \infty.$$

*Then there exist  $\rho > 0$  and  $f \in \mathcal{S}'$  such that  $u = W_t * f$ , and*

$$|u(x, t)| \leq C \|u\|_{s,w,p,q} t^{s/2-n/2\rho} (1+t)^{n/2\rho} (1+|x|)^{n/\rho} \quad (3)$$

*for all  $(x, t) \in \mathbf{R}_+^{n+1}$ . Consequently,  $f$  is in  $\dot{B}_{p,q}^{s,w}$ ,*

$$\|f\|_{\dot{B}(s,w,p,q)} \leq C \|u\|_{s,w,p,q}, \quad (4)$$

*and the semi-group formula holds for  $u$ , i.e.,  $u(\cdot, r+t) = W_t * u(\cdot, r)$  for all  $r, t > 0$ .*

PROOF. The proof of (3) and the existence of  $f$  is similar to that of the sufficient part of [2, Proposition 7(i)] (cf. (29) of [2] for (3)), while (4) follows from (1). The semi-group formula holds for  $u$  because  $u = W_t * f$ .

**Lemma 2** *Let  $s < 0$ , and let  $\dot{T}_{p,q}^{s,w}$  denote the space of all temperatures  $u$  on  $\mathbf{R}_+^{n+1}$  for which*

$$\|u\|_{\dot{T}(s,w,p,q)} = \left( \int_0^\infty \left( t^{-s/2} \|u(\cdot, t)\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} < \infty.$$

*Then  $\dot{T}_{p,q}^{s,w}$  is a quasi-Banach space.*

PROOF. First we note that, if  $u \in \dot{T}_{p,q}^{s,w}$ , then, as

$$\|u(\cdot, t)\|_{p,w} \leq \|u(\cdot, t)\|_{H(p,w)}$$

for all  $t > 0$  (by Lebesgue differentiation theorem), we see that

$$\|u\|_{s,w,p,q} \leq \|u\|_{\dot{T}(s,w,p,q)}. \quad (5)$$

Next, observe that an equivalent quasi-norm on  $H_w^p$  is given by

$$\|g\|_{H(p,w)} = \left\| \sup_{\rho > 0} |W_\rho * g| \right\|_{p,w} = \|g^+\|_{p,w}$$

for  $g \in \mathcal{S}'$ . This is the quasi-norm on  $H_w^p$  which we shall use in the rest of this note. By (5) and Lemma 1, the semi-group formula holds for  $u$ , so that

$$\|u(\cdot, t)\|_{H(p,w)} \leq \|u(\cdot, \rho)\|_{H(p,w)}, \quad t \geq \rho > 0.$$

This implies that

$$\|u\|_{\dot{T}(s,w,p,\infty)} = \sup_{t > 0} t^{-s/2} \|u(\cdot, t)\|_{H(p,w)} \leq C \|u\|_{\dot{T}(s,w,p,q)} \quad (6)$$

for  $0 < q \leq \infty$ . (From which it follows that  $\dot{T}_{p,q}^{s,w} \subseteq \dot{T}_{p,r}^{s,w}$ ,  $0 < q \leq r \leq \infty$ , but we do not need this embedding in our note.)

It is obvious that  $\|\cdot\|_{\dot{T}(s,w,p,q)}$  is a quasi-norm. We next prove that  $\dot{T}_{p,q}^{s,w}$  is complete. Let  $(u_j)$  be a Cauchy sequence in  $\dot{T}_{p,q}^{s,w}$ . Then (5) and (3) imply that  $(u_j)$  converges uniformly on each compact subset of  $\mathbf{R}_+^{n+1}$ , so that the

limit function  $u$  is a temperature on  $\mathbf{R}_+^{n+1}$ . Furthermore,  $u_j(\cdot, t)$  converges to  $u(\cdot, t)$  in  $\mathcal{S}'$  for each  $t > 0$ . On the other hand, from (6) we deduce that for each  $t$ ,  $(u_j(\cdot, t))$  is a Cauchy sequence in  $H_w^p$  and thus must converge to an element in  $H_w^p$ . Since  $H_w^p \subset \mathcal{S}'$  (continuous embedding), we conclude that  $u_j(\cdot, t)$  converges to  $u(\cdot, t)$  in  $H_w^p$  for each  $t > 0$ . Fatou's lemma then implies that  $u_j$  converges to  $u$  in  $T_{p,q}^{s,w}$ . The proof of the lemma is thus complete.

**Lemma 3** *If  $P$  is a non-zero polynomial and  $0 < p < \infty$ , then  $\|P\|_{p,w} = \infty$ .*

PROOF. As  $P$  is non-zero, there exist  $\beta > 0$  and an unbounded cone  $D \subseteq \mathbf{R}^n$  such that  $|P(x)| \geq \beta$  for every  $x$  in  $D$ . In  $D$  we can choose a sequence of closed cubes  $(I_j)$  whose interiors are mutually disjoint such that each  $|I_j| \geq 1$ , and such that there exists  $M > 0$  for which  $\cup_{j=1}^{\infty} I_j^* = \mathbf{R}^n$ , where  $I_j^*$  is the cube  $I_j$  expanded  $M$ -times.

Next, since  $w \in A_\infty$ ,  $w$  satisfies the reverse Hölder's inequality, i.e., there exists  $r$ ,  $1 < r < \infty$ , such that

$$\frac{1}{|I|} \int_I w(x) dx \geq C \left( \frac{1}{|I|} \int_I w(x)^r dx \right)^{1/r}$$

for all cubes  $I$  (cf. [4, Theorem IV]). It follows that

$$\begin{aligned} \|P\|_{p,w}^p &\geq \beta^p \int_D w(x) dx \\ &\geq \beta^p \sum_{j=1}^{\infty} w(I_j) \\ &\geq c \sum_{j=1}^{\infty} |I_j|^{1-1/r} \left( \int_{I_j} w(x)^r dx \right)^{1/r} \\ &\geq c \left( \sum_{j=1}^{\infty} \int_{I_j} w(x)^r dx \right)^{1/r} \\ &\geq cM^{-\alpha} \left( \sum_{j=1}^{\infty} \int_{I_j^*} w(x)^r dx \right)^{1/r} \\ &\geq cM^{-\alpha} \left( \int_{\mathbf{R}^n} w(x)^r dx \right)^{1/r} = \infty, \end{aligned}$$

where we have used the well-known fact that  $w^r \in A_\infty$ , which implies that  $\int_{\mathbf{R}^n} w(x)^r dx = \infty$ , and

$$\int_{I_j^*} w(x)^r dx \leq CM^\alpha \int_{I_j} w(x)^r dx$$

for some  $\alpha > 0$  (cf. [6, Chapter IV] for properties of weight functions). The proof of the lemma is complete.

### 3 Proof of the Theorem

Let  $f \in \dot{B}_{p,q}^{s,w}$ . We retain all the notations in the proof of Theorem 1' in [2, pp. 60–62]. The proof there relied on the representation

$$f - P = \lim_{m \rightarrow \infty} \sum_{j=-m}^{\infty} (\psi_j * f + P_m) \quad \text{in } S',$$

where  $\deg(P_m) \leq N_f$  for all  $m$ , and in that proof we had used the equation

$$(\partial/\partial t)^k (W_t * P_m) = W_t * (-\Delta)^k P_m = 0.$$

In the present case, we do not have control of  $N_f$ , but if we assume that  $k > N_f/2$ , then the above equation holds. A close inspection of the proof in [2] quoted above shows that the inequality

$$\left( \int_0^\infty \left( t^{k-s/2} \left\| (\partial/\partial t)^k (W_t * (f - P)) \right\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} \leq C_f \|f\|_{\dot{B}(s,w,p,q)} \quad (7)$$

holds for all  $k > \max(s/2, N_f/2)$ . The dependence of  $C_f$  on  $f$  comes from the fact that various constants appearing in the proof depend on  $k$  (and hence implicitly on  $f$ ). Fix a non-negative integer  $k$  which satisfies the above condition. We shall prove (7) for  $(k-1)$  (with possibly a different polynomial  $P$ ) if

$$k - 1 > s/2 + \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right)$$

and so on until we reach the desired given  $k$  in the Theorem. (Note that (7) for  $(k+1)$  follows from the result for  $k$  by noticing that

$$\|(\partial/\partial t)(W_t * g)\|_{H(p,w)} \leq Ct^{-1} \|g\|_{H(p,w)}$$

by [1, Lemma 4.6].)

To prove the above assertion, let  $v = (\partial/\partial t)^k(W_t * (f - P))$  and  $u = (\partial/\partial t)^{k-1}(W_t * (f - P))$ . As  $v = (\partial/\partial t)u$ , we see that, for  $0 < t_1 < t_2$  and  $x \in \mathbf{R}^n$ ,

$$\begin{aligned} |u(x, t_1) - u(x, t_2)| &\leq \int_{t_1}^{t_2} |v(x, r)| dr \\ &\leq C_x \int_{t_1}^{t_2} r^{-(k-s/2)} dr \rightarrow 0 \end{aligned}$$

as  $t_1, t_2 \rightarrow \infty$ , by (3) and the condition  $k - 1 > s/2$ . It follows that

$$\lim_{t \rightarrow \infty} u(x, t) = \phi(x)$$

exists for every  $x \in \mathbf{R}^n$ . Now since

$$u(x, t) = - \int_0^\infty v(x, r + t) dr + \phi(x), \quad (8)$$

and both  $u$  and the integral on the RHS of (8) are temperatures,  $\phi$  is harmonic in  $\mathbf{R}^n$ . By [5, Theorem 17] and (3) we can find  $M > 0$  for which

$$|\phi(x)| \leq C(1 + |x|)^M$$

for all  $x \in \mathbf{R}^n$ . This implies that  $\phi$  must be a harmonic polynomial. By a result in [7, Proof of Proposition, p.299] we have  $W_t * \phi = \phi$  for all  $t > 0$ , and by [8, Chapter IV, Proof of Theorem 2.1] we can find a polynomial  $Q$  such that  $(-\Delta)^{k-1}Q = \phi$ , so that

$$\phi = (\partial/\partial t)^{k-1}(W_t * Q).$$

If we replace  $P$  by  $P + Q$ , then  $v$  is unchanged ( $(\partial/\partial t)^k(W_t * Q) = W_t * (-\Delta)\phi = 0$ ), and for this new  $P$ , (8) implies the relation

$$u(x, t) = - \int_0^\infty v(x, r + t) dr \quad (9)$$

for every  $x \in \mathbf{R}^n$  and  $t > 0$ . Let  $u_t(x) = u(x, t)$  and  $v_t(x) = v(x, t)$ . Then the semi-group formula and (9) imply that

$$u_t^+(x) \leq tv_t^+(x) + \int_t^\infty v_r^+(x) dr, \quad (10)$$

where  $g^+(x) = \sup_{r>0} |W_r * g(x)|$  for  $g \in \mathcal{S}'$ . We consider two cases.

Case 1:  $1 \leq p < \infty$ .

Integrating (10) and using Minkowski's inequality we obtain

$$\|u_t\|_{H(p,w)} \leq t \|v_t\|_{H(p,w)} + \int_t^\infty \|v_r\|_{H(p,w)} dr. \quad (11)$$

If  $q \geq 1$ , then Hardy's inequality implies that

$$\begin{aligned} & \left( \int_0^\infty \left( t^{k-1-s/2} \|v_t\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} \\ & \leq C \left( \int_0^\infty \left( t^{k-s/2} \|v_t\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} \\ & \leq C_f \|f\|_{\dot{B}(s,w,p,q)} \end{aligned}$$

by (7) for  $k$ . (See [8, Chapter V, Lemma 3.14 for Hardy's inequality].)

Assume next that  $0 < q \leq 1$ . As the estimate for the first term in the RHS of (11) is obvious, we need only estimate the second term. Noting that

$$v_r^+(x) \leq v_t^+(x), \quad (12)$$

for  $r \geq t > 0$  by the semi-group formula, and discretizing the integral in  $r$ , we obtain

$$\begin{aligned} & \int_0^\infty \left( t^{k-1-s/2} \int_t^\infty \|v_r\|_{H(p,w)} dr \right)^q \frac{dt}{t} \\ & \leq \int_0^\infty t^{(k-1-s/2)q} \left( \sum_{j=1}^\infty t^q \|v_{jt}\|_{H(p,w)}^q \right) \frac{dt}{t} \\ & \leq C \int_0^\infty t^{(k-s/2)q} \left( \sum_{j=1}^\infty \int_{jt/2}^{(j+1)t/2} \|v_r\|_{H(p,w)}^q \frac{dr}{t} \right) \frac{dt}{t} \\ & = C \int_0^\infty \left( \int_0^{2r} t^{(k-s/2)q} \frac{dt}{t^2} \right) \|v_r\|_{H(p,w)}^q dr \\ & = C' \int_0^\infty \left( r^{k-s/2} \|v_r\|_{H(p,w)} \right)^q \frac{dr}{r}, \end{aligned}$$

where we have used Fubini's theorem and the condition  $k-1 > s/2 + (1/q - 1)$ . The proof in the case  $1 \leq p < \infty$  is thus complete.



Case 2:  $0 < p < 1$ .

Again we only estimate the second term in the RHS of (10). Using (12) to discretize the integral in  $r$ , we get

$$I(x) = \int_t^\infty v_r^+(x) dr \leq \sum_{j=1}^\infty t v_{jt}^+(x),$$

so that

$$\|I\|_{p,w} \leq t \left( \sum_{j=1}^\infty \|v_{jt}\|_{H(p,w)}^p \right)^{1/p}.$$

If  $0 < q \leq p$ , then, similar to the proof in the case  $1 \leq p < \infty$  and  $0 < q \leq 1$  given above, we have

$$\begin{aligned} & \left( \int_0^\infty t^{(k-1-s/2)q} \|I\|_{p,w}^q \frac{dt}{t} \right)^{1/q} \\ & \leq C \left( \int_0^\infty \|v_r\|_{H(p,w)}^q \left( \int_0^{2r} t^{(k-s/2)q} \frac{dt}{t^2} \right) dr \right)^{1/q} \\ & = C' \left( \int_0^\infty \left( r^{k-s/2} \|v_r\|_{H(p,w)} \right)^q \frac{dr}{r} \right)^{1/q} \end{aligned}$$

because  $k-1 > s/2 + (1/q - 1)$  by our assumption.

On the other hand, if  $p < q$ , then by using again the discretization of the integral, we obtain

$$\begin{aligned} & \left( \int_0^\infty \left( t^{k-1-s/2} \|I\|_{p,w} \right)^q \frac{dt}{t} \right)^{p/q} \\ & \leq C \left( \int_0^\infty t^{(k-s/2)q} \left( \int_{t/2}^\infty \|v_r\|_{H(p,w)}^p \frac{dr}{t} \right)^{q/p} \frac{dt}{t} \right)^{p/q} \\ & = C' \left( \int_0^\infty t^{(k-s/2)q-q/p} \left( \int_t^\infty \|v_r\|_{H(p,w)}^p dr \right)^{q/p} \frac{dt}{t} \right)^{p/q} \\ & \leq C'' \left( \int_0^\infty \left( r^{k-s/2} \|v_r\|_{H(p,w)} \right)^q \frac{dr}{r} \right)^{p/q} \end{aligned}$$

by Hardy's inequality as  $k-1-s/2 > 1/p - 1$  by our assumption.

Since  $C_f$  in (7) depends on  $f$ , what we have proved is that, for any non-negative integer  $k > s/2 + \max(0, 1/p - 1, 1/q - 1)$ , there is a polynomial  $P$  such that

$$\left( \int_0^\infty \left( t^{k-s/2} \left\| (\partial/\partial t)^k (W_t * (f - P)) \right\|_{H(p,w)} \right)^q \frac{dt}{t} \right)^{1/q} < \infty. \quad (13)$$

To complete the proof of the Theorem, we shall verify the inequality in quasi-norm in (2). Assume first that  $-s/2 > \max(0, 1/p - 1, 1/q - 1)$ , and we shall prove (2) for  $k = 0$ ; the inequality for  $k > 0$  then follows from [1, Lemma 4.6] as at the beginning of Section 3. In this case ( $k = 0$ ) we claim that the polynomial  $P$  for which (13) holds is unique. For if  $Q$  is another such polynomial, then we deduce that

$$\|W_t * (P - Q)\|_{p,w} < \infty$$

for every  $t$ . Fixing a  $t$  and noticing that  $W_t * (P - Q)$  is a polynomial, we derive from Lemma 3 that  $W_t * (P - Q) = 0$ , so that  $P = Q$ . Thus,  $W_t * (f - P)$  in (13) depends only on the equivalent class of  $f$  in  $\dot{B}_{p,q}^{s,w}$ . It follows that

$$E : \dot{B}_{p,q}^{s,w} \longrightarrow \dot{T}_{p,q}^{s,w}$$

given by  $Ef = W_t * (f - P)$  is well-defined and is a linear map.

We next show that  $E$  has closed graph. Let  $f, f_j, \dots$ , be in  $\dot{B}_{p,q}^{s,w}$  such that  $f_j \rightarrow f$  in  $\dot{B}_{p,q}^{s,w}$  and  $u_j = Ef_j = W_t * (f_j - P_j) \rightarrow u$  in  $\dot{T}_{p,q}^{s,w}$ . Lemma 1 and (5) imply that  $u = W_t * g$  for some  $g \in \dot{B}_{p,q}^{s,w}$ . It then follows from (1) that

$$\begin{aligned} \|f_j - g\|_{\dot{B}(s,w,p,q)} &= \|f_j - P_j - g\|_{\dot{B}(s,w,p,q)} \\ &\leq C \|u_j - u\|_{\dot{T}(s,w,p,q)}, \end{aligned}$$

so that  $f = g$  in  $\dot{B}_{p,q}^{s,w}$ , i.e., there exists a polynomial  $P$  such that  $g = f - P$ , or  $u = W_t * (f - P) = Ef$ . Thus the graph of  $E$  is closed. It follows from the closed graph theorem that  $E$  is continuous, and hence there is  $C > 0$  for which

$$\|Ef\|_{\dot{T}(s,w,p,q)} \leq C \|f\|_{\dot{B}(s,w,p,q)}$$

for all  $f \in \dot{B}_{p,q}^{s,w}$ , which is (2) in this case.

In the general case, let  $f \in \dot{B}_{p,q}^{s,w}$  and  $k > s/2 + \max(0, 1/p - 1, 1/q - 1)$ . Then, as  $(-\Delta)^k f$  is in  $\dot{B}_{p,q}^{s-2k,w}$  and

$$-(s - 2k)/2 > \max(0, 1/p - 1, 1/q - 1),$$

the result in the previous case yields a polynomial  $Q$  for which

$$\begin{aligned} \|W_t * ((-\Delta)^k f - Q)\|_{\dot{T}(s-2k,w,p,q)} &\leq C \|(-\Delta)^k f\|_{\dot{B}(s-2k,w,p,q)} \\ &\leq C \|f\|_{\dot{B}(s,w,p,q)}. \end{aligned}$$

By [8, Chapter IV, Proof of Theorem 2.1], we can find a polynomial  $P$  such that  $(-\Delta)^k P = Q$ . This fact and the above inequality imply the desired result (2). The proof of the Theorem is thus complete.

The results in this paper were obtained during my stay at University of New South Wales and Flinders University from mid-January to mid-March 1993. The writing was done when I was at Washington University. I wish to thank my colleagues at these institutions, in particular, Tony Dooley, Garth Gaudry and Mitch Taibleson, for their hospitality and support.

## References

- [1] H.-Q. Bui, Weighted Besov and Triebel spaces: Interpolation by the real method, *Hiroshima Math. J.* 12(1982), 581–605.
- [2] H.-Q. Bui, Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures, *J. Funct. Anal.* 55(1984), 39–62.
- [3] H.-Q. Bui, Bernstein's theorem on weighted Besov spaces, in preparation.
- [4] R.R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* 51(1974), 241–250.
- [5] T.M. Flett, Temperatures, Bessel potentials and Lipschitz spaces, *Proc. London Math. Soc.* 22(1971), 385–451.
- [6] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North Holland, Amsterdam, New York, 1985.

- [7] R. Johnson, Temperatures, Riesz potentials, and the Lipschitz spaces of Herz, Proc. London Math. Soc. 27(1973), 290–316.
- [8] E.M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, New Jersey, 1971.
- [9] H. Triebel, Characterizations of Besov-Hardy-Sobolev spaces: A unified approach, J. Approx. Theory 52(1988), 162–203.