UNITALS IN PROJECTIVE PLANES REVISITED

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Abstract

This thesis revisits the topic of unitals in finite projective planes. A unital U in a projective plane of order q^2 is a set of q^3+1 points, such that every line meets U in one or $q+1$ points. Unitals are an important class of point-set in finite projective planes, whose combinatorial and algebraic properties have been the subject of considerable study.

In this work, we summarise, revise, and extend contemporary research on unitals. Chapter 1 covers the necessary prerequisites to study unitals and related objects in finite geometry. In Chapter 2, we focus on Buekenhout-Tits unitals and answer some open problems regarding their equivalence, stabilisers and feet. The results presented in Chapter 2 are also available in a preprint paper [22]. Following this, Chapter 3 summarises recent results on Buekenhout-Metz unitals, and presents a small result on the intersection of ovoidal-Buekenhout-Metz unitals and Buekenhout-Metz unitals. Chapter 4 highlights Kestenband arcs and their relationship to Hermitian unitals, and makes explicit a proof of their equivalence. Finally in Chapter 5, we review our understanding of Figueroa planes. Beyond describing ovals and unitals in Figueroa planes, we also suggest generalisations of their constructions to semi-ovals.

Chapter 1

Background

1.1 Foreword

This thesis contains a number of results, both known and novel, as well as some known results for which we present a novel proof. To distinguish novel theorems, anything presented as a "Result" is known to the literature (see for example Result 1.1). By contrast, lemmas, theorems and corollaries are always novel results (e.g. Theorem 2.3). Where a proof is novel (or significantly clarified), but the result is known, we preface the proof with "Proof*" (see for example Result 4.13).

1.2 Preliminaries

The reader is assumed to be familiar with the basic definitions and results of group theory and field theory (for a reference see [31]). In this section, we highlight the key results we make use of repeatedly.

Given a group G with binary operation $\cdot$ and set X, together with a binary operation $* : G \times X \to X$, we call $*$ a *group action* of G on X if

1. for all $x \in X$ we have $e * x = x$,
2. for all $a, b \in G$ and $x \in X$, the operation $*$ satisfies $a * (b * x) = (a \cdot b) * x$.

We may then define the *orbit* of G on $x \in X$ as $G^x = \mathrm{orb}_G(x) = \{a*x \mid a \in G\}$, and similarly the *stabiliser* of $x \in X$ as $G_x = \mathrm{stab}_G(x) = \{a \in G \mid a*x = x\}$. The Orbit-Stabiliser Theorem relates the orbit of an element to its stabiliser.

Result 1.1 (Orbit-Stabiliser Theorem)**.** *Let G be a finite group acting on a set X. Then for any element $x \in X$*

$$|G| = |\mathrm{orb}_G(x)||\mathrm{stab}_G(x)|. \tag{1.1}$$

Let G be a group acting on a set X. If for all $x, y \in X$, there exists an element $\varphi \in G$ such that $\varphi * x = y$ then we call G a *transitive* group action on x. If the element $\varphi \in G$ mapping to x to y is unique, then we say G is *sharply transitive* on X, or that G acts *regularly* on X. If for two n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ of elements in X, there exists an element $\varphi \in G$ such that $(\varphi * x_1, \varphi * x_2, \dots, \varphi * x_n) = (y_1, y_2, \dots, y_n)$, then we call G n-transitive on X. We may naturally extended the definition of sharply transitive, to sharply n-transitive.

Given two subgroups H, K of a group G, we call H *conjugate* to K if there exists an element x such that $xHx^{-1} = \{xhx^{-1} \,|\, h \in H\} = K$. Conjugacy forms an equivalence relation on subgroups of G. Similarly an element $x \in G$ is conjugate to $y \in G$ if there exists an element a such that $axa^{-1} = y$. Conjugacy of elements forms an equivalence class on G and the equivalence classes are referred to as *conjugacy classes*. Conjugate elements and subgroups are relevant in the study of group actions, as the following result demonstrates.

Result 1.2. *Let G be a group acting on a set X. If $x, y \in X$ and there exists an element $\varphi \in G$ such that $\varphi * x = y$ then* $\operatorname{stab}_G(x) = \varphi \operatorname{stab}_G(y)\varphi^{-1}$.

We can also view conjugacy as a group action on elements (or subgroups) of G, that is the group action $\varphi * x = \varphi x \varphi^{-1}$. In this context the *normaliser* of an element $x \in G$ is the stabiliser of x under the conjugacy group action.

Let G be a group. Then, the *centre* of G, denoted $\mathrm{Z}(G)$ is the set of all elements x such that $xy = yx$ for all $y \in G$; that is the centre of G is the set of elements that commute with all members of G. For a specific element $x \in G$, the *centraliser* $\mathrm{C}(x)$ is the set of elements in G commuting with x. The definition of centraliser naturally extends to the centraliser of a subgroup H of G, we denote the centraliser of H as $C(H)$.

The Sylow theorems concern the existence, conjugacy and size of prime power order subgroups of finite groups.

Result 1.3 (Sylow's First Theorem)**.** *Let G be a finite group of order n and let p be a prime such that $p|n$. If k is the largest integer such that $p^k|n$, then there exists a subgroup H of G having order p^k, called a Sylow p-subgroup.*

Result 1.4 (Sylow's Second Theorem)**.** *Let G be a finite group of order n and p be a prime such that $p|n$. Then the Sylow p-subgroups of G are conjugate.*

Result 1.5 (Sylow's Third Theorem)**.** *Let G be a finite group of order n and p be a prime such that $p|n$ and write $n = p^k m$ where $p \nmid m$. If N is the number of Sylow p-subgroups of G, then:*

1. *the number of Sylow p-subgroups divides m;*

2. *we have $N \equiv 1 \mod p$;*

3. *if $N_G(S)$ is the normaliser of any Sylow p-subgroup of G, then $N = |G : N_G(P)|$.*

Let q be a prime power. Then there exists a unique finite field of order q, which we denote $\mathbb{F}_q$. The following result determines when one finite field may be a subfield of another.

Result 1.6. *Let q be a prime power, then $\mathbb{F}_{q^k} < \mathbb{F}_{q^l}$ for some integers $k, l \geq 1$ if and only if $k|l$.*

Recall that the automorphisms of a finite field $\mathbb{F}_{p^k}$ form a cyclic group generated by the Frobenius automorphism $x \mapsto x^p$. The group of automorphisms of a finite field $\mathbb{F}_{q^k}$ fixing $\mathbb{F}_q$ is denoted $\mathrm{Aut}(\mathbb{F}_{q^k} \,|\, \mathbb{F}_q)$; it is a cyclic group generated by the automorphism $x \to x^q$. Given two finite fields $\mathbb{F}_q < \mathbb{F}_{q^k}$ we define the *trace* function $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q} : \mathbb{F}_{q^k} \to \mathbb{F}_q$ as

$$\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) = \sum_{i=0}^{k-1} x^{q^i}. \tag{1.2}$$

We call $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$ the trace of x over $\mathbb{F}_q$. If p is prime, then $\mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}_p}(x)$ is the *absolute trace* of x and is expressed as $\mathrm{Tr}(x)$. We require a few basic properties of the trace function.

Result 1.7. *The following holds for all prime powers q:*

1. *For all $x \in \mathbb{F}_{q^k}$ we have $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) \in \mathbb{F}_q$.*

2. *If $x, y \in \mathbb{F}_{q^k}$, then $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x+y) = \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) + \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(y)$.*

3. *If $x \in \mathbb{F}_{q^k}$ and $a \in \mathbb{F}_q$, then $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(ax) = a\,\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$.*

4. *If $\mathbb{F}_q < \mathbb{F}_{q^k} < \mathbb{F}_{q^l}$ and $x \in \mathbb{F}_{q^l}$, then $\mathrm{Tr}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(x) = \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(\mathrm{Tr}_{\mathbb{F}_{q^l}/\mathbb{F}_{q^k}}(x))$.*

5. *If $x \in \mathbb{F}_{q^k}$, then $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)^{q^i} = \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$ for all i.*

Similarly the *norm map* $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q} : \mathbb{F}_{q^k} \to \mathbb{F}_q$ is defined as

$$\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) = \prod_{i=0}^{k-1} x^{q^i}. \tag{1.3}$$

We call $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$ the norm of x over $\mathbb{F}_q$ and, when q is prime, the *absolute norm* of x over $\mathbb{F}_q$. The absolute norm of x is expressed as $\mathrm{N}(x)$. We require the following elementary properties of the norm function.

Result 1.8. *The following holds for all prime powers q:*

1. *If $x \in \mathbb{F}_{q^k}$, then $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) \in \mathbb{F}_q$.*

2. *If $x, y \in \mathbb{F}_{q^k}$, then $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(xy) = \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)\,\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(y)$.*

3. *If $x \in \mathbb{F}_{q^k}$ and $a \in \mathbb{F}_q$, then* $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(ax) = a^k \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$.

4. *If $\mathbb{F}_q < \mathbb{F}_{q^k} < \mathbb{F}_{q^l}$ and $x \in \mathbb{F}_{q^l}$, then* $\mathrm{N}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(x) = \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(\mathrm{N}_{\mathbb{F}_{q^l}/\mathbb{F}_{q^k}}(x))$.

5. *If $x \in \mathbb{F}_{q^k}$, then* $\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)^{q^i} = \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)$ *for all i.*

Let σ be a generator of the automorphism group of $\mathrm{Aut}(\mathbb{F}_{q^k} \,|\, \mathbb{F}_q)$. A linearised σ-polynomial (or just σ-polynomial) over $\mathbb{F}_{q^k}$, is a polynomial of the form $f(x) = \sum_{i=0}^{k-1} a_i x^{\sigma^i}$. As σ is an automorphism of $\mathbb{F}_{q^k}$, and σ fixes $\mathbb{F}_q$, we have that $f(x)$ is an $\mathbb{F}_q$-linear function. The roots of f therefore form an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^k}$ which we call the *kernel* of f. Thus, we may define the *nullity* of f to be the rank of its kernel. Likewise, the image of f forms an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^k}$; the *rank* of f may be defined to be the *rank* of its image. We summarise these results in the following theorem.

Result 1.9. *Let $L(x)$ be a linearised σ-polynomial over $\mathbb{F}_{q^k}$. Then $L(x)$ is an $\mathbb{F}_q$-linear function. If $K = \ker(L(x)) = \{x \in \mathbb{F}_{q^k} \mid L(x) = 0\}$ and $I = \mathrm{Im}(L(x))$ then*

1. *the set K is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^k}$, and so $|K| = q^i$ for some $0 \le i \le k$;*

2. *the set S is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^k}$, and so $|S| = q^j$ for some $0 \le j \le k$.*

To determine the rank of a linearised polynomial, we have the following result of Dickson.

Result 1.10. *Let σ be an automorphism generating $\mathrm{Aut}(\mathbb{F}_{q^k} / \mathbb{F}_q)$. If $L(x)$ is a σ-polynomial $\sum_{i=0}^{k-1} a_i x^{\sigma^i}$, then the rank of $L(x)$ is equal to the rank of $\mathcal{D}_\sigma(f)$ where*

$$\mathcal{D}_\sigma(f) = \begin{bmatrix} a_0 & a_0 & \dots & a_{k-1} \\ a_{k-1}^\sigma & a_0^\sigma & \dots & a_{k-2}^\sigma \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{\sigma^{k-1}} & a_2^{\sigma^{k-1}} & \dots & a_0^{\sigma^{k-1}} \end{bmatrix}. \tag{1.4}$$

Given a σ-polynomial $L(x)$, the polynomial $L(x)-\alpha$ is called an *affine* polynomial. Given two roots β_1, β_2 of an affine polynomial $L(x) - \alpha$, we have that $\beta_1 - \beta_2$ is a root of $L(x)$. Thus an affine polynomial $A(x) = L(x) - \alpha$ may have at most as many roots as $L(x)$. For more information on linearised polynomials, refer to [31].

1.3 Projective Spaces

A *projective plane* of order n is a set of $n^2 + n + 1$ points and $n^2 + n + 1$ lines together with an incidence relation $\mathcal{I}$ such that

1. Any two points lie on exactly one line.

2. Any two lines meet in an unique point.

3. There exists four points, no three collinear.

The axioms of a projective plane are self-dual; exchanging the roles of points and lines swaps axioms one and two, whilst preserving axiom three.

For some prime power q, the projective plane $\mathrm{PG}(2,q)$ is the plane whose points are the lines of $(\mathbb{F}_q)^3$, and whose lines are the planes of $\mathrm{V}(3,q)$. More generally, the *projective space* $\mathrm{PG}(k,q)$ is the incidence structure whose j-dimensional subspaces are the $(j+1)$-dimensional subspaces of $(\mathbb{F})^{k+1}$, with inherited incidence.

Desargues' Theorem characterises $\mathrm{PG}(2,q)$ as the unique projective plane satisfying the following.

Desargues Axiom: Let ABC and $A'B'C'$ be two distinct triangles in a projective plane Π. Then ABC and $A'B'C'$ are in perspective from a point P if and only if they are in perspective from a line l.

We may thus refer to $\mathrm{PG}(2,q)$ as *Desarguesian*. See Section 1.8 for some examples of *non-Desarguesian* projective planes.

Alongside Desargues' Theorem, Pappus' Theorem is another other fundamental result of classical projective geometry.

Result 1.11 (Pappus' Theorem)**.** *Let $(A,B,C;D,E,F)$ be a tuple of points in $\mathrm{PG}(2,q)$ such that $A,B,C \in l \setminus m$ and $D,E,F \in m \setminus l$ for some distinct lines $l \neq m$. Then, the points $X = AE \cap DB$, $Y = AF \cap DC$ and $Z = BF \cap EC$ are collinear.*

Homogeneous coordinates represent the points of $\mathrm{PG}(k,q)$ as $(k+1)$-tuples, where

$$(x_1, x_2, \ldots, x_{k+1}) \equiv \lambda(x_1, x_2, \ldots, x_{k+1})$$

for any $\lambda \in \mathbb{F}_q^{\times}$ and $x_i \in \mathbb{F}_q$ not all zero. There is a function ρ associated with $\mathrm{PG}(k,q)$ that maps the point $(a_1, a_2, \ldots, a_{k+1})$ to the hyperplane with equation $a_1x_1 + a_2x_2 + \cdots + a_{k+1}x_{k+1} = 0$, and generally maps the subspace spanned by points $p_1, p_2, \ldots, p_j$ to the $k-j$ dimensional subspace $\cap_{i=1}^{j} \rho(p_i)$. The function ρ is a *duality* of $\mathrm{PG}(k,q)$. Thus we use the *dual coordinates* $[a_1, a_2, \ldots, a_{k+1}]$ to represent the hyperplane $a_1x_1 + a_2x_2 + \cdots + a_{k+1}x_{k+1} = 0$.

A collineation ϕ of a projective space S is an incidence-preserving bijection of points. Any non-singular matrix $A \in \mathrm{GL}(k+1,q)$ induces a collineation of $\mathrm{PG}(k,q)$ by mapping the point P with homogeneous coordinates $\mathbf{p}$ to the point with homogeneous coordinates $A\mathbf{p}$ (with $\mathbf{p}$ expressed as a column vector). Such a collineation is called a *homography* or *projectivity* of $\mathrm{PG}(k,q)$, and the group of homographies is $\mathrm{PGL}(k+1,q)$. The collineation induced by A is the same as the collineation induced by λA where $\lambda \in \mathbb{F}_q^\times$. Hence, we find $\mathrm{PGL}(k+1,q) \equiv \mathrm{GL}(k+1,q)/Z$, where $Z = \{\lambda I \mid \lambda \in \mathbb{F}_q^\times\}$.

Any automorphism σ of $\mathbb{F}_q$ induces a collineation $\mathrm{PG}(k,q)$ mapping the point with homogeneous coordinates $\mathbf{p}$ to the point with homogeneous coordinates $\mathbf{p}^\sigma$, where $\mathbf{p}^\sigma$ denotes component-wise exponentiation. We define the group $\mathrm{P\Gamma L}(k+1,q)$ to be semi-direct product of the group of homographies and collineations of $\mathrm{PG}(k,q)$ induced by automorphisms.

The Fundamental Theorem of Projective Geometry establishes that the collineation group of $\mathrm{PG}(k,q)$ is precisely $\mathrm{P\Gamma L}(k+1,q)$.

Result 1.12 (The Fundamental Theorem of Projective Geometry)**.** *Let φ be a collineation of $\mathrm{PG}(k,q)$. Then φ is induced by a semi-linear transformation mapping the point P with homogenous coordinates and σ is an automorphism of $\mathbb{F}_q$.*

A *frame* F of a projective space $\mathrm{PG}(k,q)$ is a set of $k+2$ points, such that no hyperplane contains $k+1$ points of F. A frame of a projective plane is thus four points, no three collinear. From the definition of $\mathrm{PGL}(k+1,q)$ it therefore follows that $\mathrm{PGL}(k+1,q)$ is transitive on frames of $\mathrm{PG}(k,q)$. The *fundamental frame* of $\mathrm{PG}(k,q)$ is the frame $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k, \mathbf{j}\}$ where $(\mathbf{e}_i)_j = \delta_{i,j}$, and $\mathbf{j} = \sum_{i=1}^k \mathbf{e}_i$.

Suppose that φ is a collineation of $\mathrm{PG}(k,q)$ fixing a line ℓ point-wise. Then, considering its action on lines, one finds a that φ must also fix a point P line-wise. Such a collineation is called a *perspectivity* with *centre* P and *axis* ℓ. We further classify perspectivities into *elations* and *homologies* depending on whether $P \in \ell$ or $P \notin \ell$ respectively.

A *polarity* ρ of a projective plane is an involution between the points and lines such that $p \in l$ if and only if $\rho(l) \in \rho(p)$. As with collineations, polarities of $\mathrm{PG}(2,q)$ have a predictable form. We highlight two of these in particular, the *Hermitian polarity* and *orthogonal polarity*. A Hermitian polarity in the plane only exists in $\mathrm{PG}(2,q^2)$, and maps the point P with homogeneous coordinates $\mathbf{p}$ (expressed as a column vector) to the line given dually by $H\mathbf{p}^q$ where $\mathbf{p}^q$, and H is a Hermitian matrix ($H^T = \lambda^{q+1}H^q$ for some $\lambda \in \mathbb{F}_{q^2}$). An orthogonal polarity of $\mathrm{PG}(2,q)$ maps the point P with homogeneous coordinates $\mathbf{p}$ (expressed as a column vector) to the line given dually by $A\mathbf{p}$, where $A^T = A$ (but A is not skew-symmetric if q is even). A point p (line ℓ) is considered *absolute* if $p \in \rho(p)$ ($\rho(\ell) \in \ell$).

1.4 Arcs, Ovals, and Ovoids

A k-arc in a projective space is a set of k points, no three collinear.

Result 1.13 ([34])**.** *Let A be a k-arc in a projective plane of order q. Then, if q is odd, then $k \leq q+1$. If q is odd then $k \leq q+2$.*

Proof. Let A be a k-arc in a projective plane of order q. Consider a point $P \in A$, then each of the $q+1$ lines through P meets A in at most one more point besides P. Hence, there are at most $q+2$ points of A. Now suppose $|A| = q+2$. Because $|A| = q+2$ every line through $P \in A$ must meet A in a further point, so every line meets A in either 0 or 2 points. Let R be a point not on A. As no line may be tangent to A, and because the lines through R partition A, there are $(q+2)/2$ secant lines through R. Hence, q is even. Thus, we have $k \leq q+1$ if q is odd. □

In a projective plane of order q a $(q+1)$-arc is an *oval*. If q is even, a $(q+2)$-arc is a *hyperoval*. There is a fundamental distinction between ovals when q is even versus when q is odd.

Result 1.14 ([34]). *Let $\mathcal{O}$ be an oval in a projective plane π of order q. Then each point $P \in \mathcal{O}$ lies on a unique tangent line. If q is odd, then each point $R \notin \mathcal{O}$ lies on zero or two tangents to $\mathcal{O}$. If q is even, the tangent lines to an oval are concurrent.*

Proof. For each point $P \in \mathcal{O}$ there are q lines through P meeting $\mathcal{O} \setminus \{P\}$ in exactly one point and the remaining line is therefore tangent. Now suppose q is odd and let $R \notin \mathcal{O}$ lie on a tangent t_P meeting $\mathcal{O}$ at P. Then, because $q+1$ is odd, there must exist at least one other tangent through R, and each point of $t_P \setminus \{P\}$. As there are only q tangents to $\mathcal{O}$ besides t_P, every point of t_P must therefore lie on exactly two tangents. Therefore, as t_P is arbitrary, if q is odd every point lies on either zero or two tangents. For q even, every point $R \notin \mathcal{O}$ lies on at least one tangent as $q+1$ is odd, and lines through R partition $\mathcal{O}$. Suppose that N is the point of intersection of two tangents ℓ and m to $\mathcal{O}$. Then, for each line t through N either t is tangent or every point of t lies on a distinct tangent. Because $N = l \cap m \in t$ lies on two tangents, t must therefore be a tangent line. Thus, the tangents to $\mathcal{O}$ are concurrent if q is even. □

The point N at the intersection of all tangents to $\mathcal{O}$ is called the *nucleus* of $\mathcal{O}$. From Result 1.14, it follows that any hyperoval consists of an oval together with its nucleus.

A *complete k-arc* is a k-arc that is not contained in any $(k+1)$-arc. In $\mathrm{PG}(2,q)$ for q odd, an oval is an example of a complete arc. Ovals are not complete arcs when q is even, as they are contained in hyperovals. It is not known what values for what values of k there exist complete k-arcs.

A *conic* in $\mathrm{PG}(2,q)$ is a set of points with homogeneous coordinates (x,y,z) satisfying a homogeneous equation of degree two,

$$ax^2 + by^2 + cz^2 + fyz + gxz + hxy = 0. \tag{1.5}$$

For q odd, we may alternatively describe a conic as the absolute points of an orthogonal polarity, that is points with homogenous coordinates $\mathbf{x} = (x, y, z)$ satisfying

$$\mathbf{x}^T A\mathbf{x} = 0, \tag{1.6}$$

where A is a symmetric matrix, and $\mathbf{x}$ is taken as a column vector. Segre establishes that all ovals in $\mathrm{PG}(2, q)$, with q odd, are conics.

Result 1.15 ([36])**.** *Let q be an odd prime power. Then, any oval of $\mathrm{PG}(2, q)$ is a conic.*

When $q = 2^e \geq 8$, there exist ovals that are not conics. *Translation ovals* are one class of examples. A translation oval is a set of points projectively equivalent to solutions of the homogenous equation $x^{2^i} - yz = 0$, where $\gcd(i, e) = 1$. A *translation axis* ℓ of a translation oval O is a line tangent to O at P, such that there exists a group of elations with centre $Q \in \ell \setminus \{p\}$ and axis ℓ stabilising O. Conics, being a special case of translation ovals, have an additional identifying property.

Result 1.16. *Let q be an even prime power, and $\mathcal{O}$ an oval in $\mathrm{PG}(2, q)$. Then $\mathcal{O}$ is a conic if and only if $\mathcal{O}$ has two translation axes.*

The remaining translation ovals can be shown to have a unique translation axis.

Result 1.17. *Let $q = 2^h$ with $h \geq 3$, and $\mathcal{O} = \{(t, t^{2^i}, 1) \mid t \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}$ with $\gcd(i, h) = 1$ and $1 < i < h - 1$. Then $\mathcal{O}$ is a translation oval of $\mathrm{PG}(2, q)$, with unique translation axis $z = 0$ and nucleus $(1, 0, 0)$.*

Proof. Let ℓ be a line of $\mathrm{PG}(2, q)$ containing $(0, 1, 0)$. Then, ℓ either has the form $x + az$ or $z = 0$ and these lines are easily seen to meet $\mathcal{O}$ in at most two points. Otherwise, ℓ is given by the equation $ax + y + cz = 0$. A point $(t, t^{2^i}, 1)$ lies on ℓ when

$$at + t^{2^i} + c = 0. \tag{1.7}$$

Because equation (1.7) is an affine polynomial, it has at most as many roots as $at + t^{2^i}$ does. The polynomial $t^{2^i} + at = t(t^{2^i - 1} + a)$ has two roots because $\gcd(2^i - 1, 2^h - 1) = 2^{\gcd(i,h)} - 1 = 1$ so the map $t \to t^{2^i - 1}$ is a bijection on $\mathbb{F}_q$. Hence ℓ contains at most two roots and $\mathcal{O}$ is therefore an oval. The nucleus of this oval is the intersection of the tangent lines $z = 0$ and $y = 0$, that is the point $(1, 0, 0)$.

The elation group stabilising $\mathcal{O}$ with axis $z = 0$ is easily seen to be the collineations induced by the matrices

$$M_a = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a^{2^i} \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.8}$$

for each $a \in \mathbb{F}_q$. This axis is unique as $1 < i < h - 1$, so by Result 1.16, $\mathcal{O}$ is not a conic and hence contains at most one translation axis. $\square$

An *ovoid* of $\mathrm{PG}(3,q)$ is a (q^2+1)-arc. The prototypical example of an ovoid is an *elliptic quadric*. An *elliptic quadric* is a set of points projectively equivalent to the set of points with homogeneous coordinates

$$\mathcal{O} = \{(x_0,x_1,x_2,x_3)\,|\,x_i \in \mathbb{F}_q \text{ and } f(x_0,x_1)+x_2x_3=0\}\,, \tag{1.9}$$

where f is an irreducible binary quadratic form. For q odd, all ovoids are elliptic quadrics [5].

Analogous to ovals, ovoids have the property that at each point $p \in \mathcal{O}$ there exists a unique plane π_p tangent to $\mathcal{O}$ at p. In addition to this, each line of $\mathrm{PG}(3,q)$ tangent to $\mathcal{O}$ lies in π_p. All other planes meet $\mathcal{O}$ in $q+1$ points forming an oval.

If $q = 2^{2e+1} \geq 8$ then a second class of ovoid, the Tits ovoid, arises as the set of points

$$\mathcal{O}_T = \left\{(1,x,y,x^{\sigma+2}+y^{\sigma}+xy)\,\middle|\,x,y \in \mathbb{F}_q\right\} \cup \{(0,0,0,1)\}\,, \tag{1.10}$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ satisfies $\sigma^2 = 2 \mod q-1$. Tits ovoids were first described in [40]. The stabiliser of $\mathcal{O}_T$ is found to be isomorphic to the Suzuki group $\mathrm{Sz}(q)$, which is 2-transitive on its points. The stabiliser is also transitive on secant planes, so that secant sections of the ovoid are projectively equivalent to the translation oval $\left\{(1,t,t^{\sigma})\,|\,t \in \mathbb{F}_q\right\} \cup \{(0,0,1)\}$.

The classification of ovoids in $\mathrm{PG}(3,q)$, q even, is an important open problem. It is believed that all ovoids are either elliptic quadrics or Tits ovoids. Brown provides the strongest characterisation of ovoids with the following theorem.

Result 1.18 ([11])**.** *Let $\mathcal{O}$ be an ovoid in $\mathrm{PG}(3,q)$, q even. If there exists a plane meeting $\mathcal{O}$ in a conic, then $\mathcal{O}$ is an elliptic quadric.*

An *oval cone* in $\mathrm{PG}(3,q)$ is the set of of all points on lines joining the points of an oval O in some plane Π of $\mathrm{PG}(3,q)$ to a point $V \notin \Pi$. Likewise, an *ovoidal cone* $\mathcal{C}$ in $\mathrm{PG}(4,q)$ is the set of all points on lines joining the points of an ovoid $\mathcal{O}$ in some hyperplane H to a point $V \notin H$. The point V is the *vertex* of $\mathcal{C}$, and $\mathcal{O}$ is the *base* of the cone. The lines joining the points of $\mathcal{O}$ to V are called *generator lines*.

The geometry of an ovoidal cone is worth discussing as it is highly relevant to later constructions (see Section 1.9). The following lemma describes the possible intersections of hyperplanes with an ovoidal cone.

Result 1.19. *Let $\mathcal{C}$ be an ovoidal cone in $\mathrm{PG}(4,q)$ with vertex V, and base $\mathcal{O}$ contained in a hyperplane H_0, and generator lines v_i for $1 \leq i \leq q^2+1$. A hyperplane H meets $\mathcal{C}$ in one of four configurations:*

1. *a single generator line;*

2. *an oval cone or;*

3. *an ovoid.*

Moreover for each generator line v_i of $\mathcal{C}$ there exists a unique hyperplane H such that $H \cap \mathcal{C} = v_i$.

Proof. Suppose that $V \in H$. Then because $H \cap H_0$ is a plane contained in H_0, it must meet $\mathcal{O}$ in either 1 or $q+1$ points. In the case where $H \cap H_0$ is tangent to $\mathcal{O}$, then $\mathcal{C} \cap H$ must be a single generator line. In the case where $|(H \cap H_0) \cap \mathcal{O}| = q+1$, we have an oval cone.

Now suppose that $V \notin H$ and let $O = H \cap \mathcal{C}$. By a dimension argument we can see that as $v_i \not\subseteq H$, we must have $|v_i \cap H| = 1$ for all $1 \leq i \leq q^2+1$, so $|O| = q^2+1$. Suppose that there exists three collinear points of O on generator lines v_i, v_j, v_k respectively. Then, $\pi = \langle V, v_i, v_j, v_k \rangle$ forms a plane, so $\pi \cap H_0$ is a line containing three points of $\mathcal{O}$, which is a contradiction as $\mathcal{O}$ is an ovoid. Therefore, no line may contain three points of O, and O is thus an ovoid.

Finally, any hyperplane H meeting $\mathcal{C}$ in precisely the generator line v_i must contain V. Moreover, $H \cap H_0$ is the unique tangent plane to $\mathcal{O}$ at $v_i \cap \mathcal{O}$. As a hyperplane of $\mathrm{PG}(4,q)$ is uniquely determined by a plane and a point not on that plane, H is uniquely determined by v_i. $\square$

The q^2+1 hyperplanes meeting $\mathcal{C}$ in generator lines are called *tangent hyperplanes* of the ovoidal cone $\mathcal{C}$.

We can also describe the intersection of planes of $\mathrm{PG}(4,q)$ with an ovoidal cone $\mathcal{C}$.

Result 1.20. *Let $\mathcal{C}$ be an ovoidal cone in $\mathrm{PG}(4,q)$ with vertex V, base $\mathcal{O}$ contained in a hyperplane H_0, and generator lines v_i for $1 \leq i \leq q^2+1$. A plane π meets $\mathcal{C}$ in one of three configurations:*

1. *a point;*
2. *a single generator line;*
3. *a pair of generator lines or;*
4. *an oval.*

Moreover each plane meeting $\mathcal{C}$ in either a generator line v_i or a single point $p \in v_i$ is contained in the unique tangent hyperplane of $\mathcal{C}$ at v_i.

1.5 Sublines and Subplanes

Let Π be a projective plane of order n with points $\mathcal{P}$ and lines $\mathcal{L}$. A projective plane π with points $\mathcal{P}'$ and lines $\mathcal{L}'$ of order $m \leq n$ is a subplane of Π if $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$. The plane π is a proper subplane if $\mathcal{P}' \neq \mathcal{P}$. The following theorem of Bruck bounds the size of a subplane.

Result 1.21 (Bruck's Theorem, [12]). *Let* Π *be a projective plane of order* n*, and* π *a proper subplane of order* m*. Then either* $m^2 + m \leq n$ *or* $m^2 = n$*.*

Proof. Suppose that $m^2 + m > n$. Let P be a point of Π, there are $n + 1 < m^2 + m + 1$ lines through P and so there exists at least line of Π containing two points of Π. Hence, every point $P \in \Pi$ is contained in at least one line of π. Dually, every line of Π contains at least one point of π. Now assume that $P \in \Pi \setminus \pi$. Then P is incident to at most one, and hence exactly one, line of π. Counting the points of π on each line through P we find $n + m + 1 = m^2 + m + 1$ and so $n = m^2$. □

A subplane $\pi \subset \Pi$ of order m such that $m^2 = n$ is called a *Baer subplane*. From the proof of Bruck's Theorem it follows that Baer subplanes are examples of *blocking sets*, a set of points P in a projective plane Π such that every line contains at least one point of P.

Subplanes of $\mathrm{PG}(2, q^k)$ can be naturally obtained through *subfield embedding*. For any subfield $\mathbb{F}_{q^i} < \mathbb{F}_{q^k}$ we may naturally embed $\mathrm{PG}(2, q^i)$ in $\mathrm{PG}(2, q^k)$ as the set of points with $\mathbb{F}_{q^k}$-homogenous coordinates

$$\{(x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{F}_{q^i}\}. \tag{1.11}$$

It is then clear that this set of points, together with the lines spanning these points form a subplane of $\mathrm{PG}(2, q^k)$ equivalent to $\mathrm{PG}(2, q^i)$. The following result shows that the only subplanes of $\mathrm{PG}(2, q)$ are those obtained from subfield embedding.

Result 1.22. *Let* π *be a subplane of* $\mathrm{PG}(2, p^k)$ *of order* m*, for some prime* p *and* $k \geq 1$*. Then* π *is projectively equivalent to* $\mathrm{PG}(2, p^i)$ *for some* $i|k$*.*

Proof. Let π be a Baer subplane of $\mathrm{PG}(2, p^k)$. Then, there exists a projective frame of $\mathrm{PG}(2, p^k)$ contained in π. Because $\mathrm{PGL}(3, p^k)$ is transitive on frames, there exists a collineation φ such that $\pi' = \varphi(\pi)$ contains the fundamental frame. Then π' must also contain

$$\langle (0, 1, 0), (0, 0, 1) \rangle \cap \langle (1, 0, 0), (1, 1, 1) \rangle = (0, 1, 1).$$

The line $z = 0$ contains some $m + 1$ points of the form $(1, 0, a)$. We will show that the set $F = \{a \in \mathbb{F}_{p^k} \mid (1, 0, a) \in \pi'\}$ forms an additive subgroup of $\mathbb{F}_{p^k}$. Let $a, b \in F$, then

$$\langle (1, 0, b), (0, 1, 0) \rangle \cap \langle (1, 0, 0), (1, 1, 1) \rangle = (1, b, b),$$

and

$$\langle (1, b, b), (0, 1, 0) \rangle \cap \langle (1, 0, 0), (0, 1, 0) \rangle = (1, b, 0)$$

lie in π'. Now the point

$$\langle (1,b,0) \cap (0,0,1) \rangle \cap \langle (0,1,1),(1,0,a) \rangle = (1,b,a+b)$$

is in π' so

$$\langle (0,1,0),(1,b,a+b) \rangle \cap \langle (1,0,0),(0,0,1) \rangle = (1,0,a+b) \in \pi'.$$

Thus F is an additive subgroup of $(\mathbb{F}_{p^k},+)$ and so equal to $(\mathbb{F}_{p^i},+)$ for some $i|k$. In addition, $m = p^i$.

A completely symmetric argument shows that the p^i+1 points of π' the line $z=0$ are $\{(1,a,0) \,|\, a \in \mathbb{F}_{p^i}\}$, and the points on $x=0$ are $\{0,1,a\} \,|\, a \in \mathbb{F}_{p^i}\}$. It now follows that $\langle (1,a,0),(0,0,1) \rangle \cap \langle (1,0,b),(0,1,0) \rangle = (1,a,b) \in \pi'$ for all $a,b \in \mathbb{F}_{p^i}$. Thus $\pi' = \{(1,a,b) \,|\, a,b \in \mathbb{F}_{p^i}\} \cup \{(0,1,a) \,|\, a \in \mathbb{F}_{p^i}\} \cup \{(0,0,1)\}$, which is precisely the subfield embedding of $\mathrm{PG}(2,p^i)$ in $\mathrm{PG}(2,p^k)$. □

The following corollaries are immediate from Result 1.22.

Result 1.23. *All subplanes of* $\mathrm{PG}(2,q)$ *of the same order are uniquely determined by a frame.*

Result 1.24. *If* π *is a Baer subplane of* $\mathrm{PG}(2,q^2)$*, then there exists a unique collineation* $\phi \in \mathrm{P\Gamma L}(3,q^2)$ *of order two whose fixed points are precisely* π*.*

Proof. There is a unique collineation fixing the subplane π_0 obtained by embedding $\mathrm{PG}(2,q)$ in $\mathrm{PG}(2,q^2)$, namely the collineation ϕ mapping the point with homogeonous coordinates $\mathbf{x}$ to the point with homogeneous coordinates $\mathbf{x}^q$. As all Baer subplanes are projectively equivalent to π_0, the result follows. □

The unique collineation fixing the points of a Baer subplane π is called the *Baer involution* of π.

We define a *Baer subline* of $\mathrm{PG}(1,q^2)$ to be a set of points projectively equivalent to $\mathrm{PG}(1,q)$ embedded within $\mathrm{PG}(1,q^2)$. By definition therefore, a Baer subline is uniquely determined by three distinct points of $\mathrm{PG}(1,q^2)$.

1.6 Spreads

A t-spread of $\mathrm{PG}(n,q)$ is a partition of $\mathrm{PG}(n,q)$ into t-dimensional subspaces. Spreads play an important role in the construction of unitals and non-Desarguesian planes (see Sections 1.7 and 1.9).

Result 1.25. *Suppose there exists a* t*-spread of* $\mathrm{PG}(n,q)$*, then* $t+1 \mid n+1$*.*

Proof. Suppose there exists a t-spread of $\mathrm{PG}(n,q)$. Then, as any t-dimensional subspace of $\mathrm{PG}(n,q)$ is equivalent to $\mathrm{PG}(t,q)$ we have,

$$\begin{aligned}
&|\,\mathrm{PG}(t,q)|\mid|\,\mathrm{PG}(n,q)|\\
&\Rightarrow \frac{q^{t+1}-1}{q-1}\mid\frac{q^{n+1}-1}{q-1}\\
&\Rightarrow q^{t+1}-1\mid q^{n+1}-1\\
&\Rightarrow t+1\mid n+1.
\end{aligned}$$

□

The following existence theorem demonstrates *field reduction* as a method to construct spreads.

Result 1.26. *There exists a t-spread of $\mathrm{PG}(n,q)$ if and only if $t+1\mid n+1$.*

Proof. The necessity that $t+1\mid n+1$ is taken care of in Result 1.25.

To construct a t-spread for t such that $t+1\mid n+1$, first let $n=k(t+1)-1$ for some $k\in\mathbb{Z}^+$. Fix some basis $\{\epsilon_0,\epsilon_1,\cdots,\epsilon_t\}$ for $\mathbb{F}_{q^{k(t+1)}}$ over $\mathbb{F}_{q^k}$, so that each $x\in\mathbb{F}_{q^{k(t+1)}}$ may be expressed as $x^{(0)}\epsilon_0+x^{(1)}\epsilon_1+\cdots+x^{(t)}\epsilon_t$ where $x^{(i)}\in\mathbb{F}_{q^k}$. Then consider the following map $\varphi:\mathbb{F}_{q^t}^k\to\mathbb{F}_q^{kt}$

$$\varphi((x_0,x_1,\cdots,x_{k-1}))=(x_0^{(0)},x_0^{(1)},\cdots,x_0^t,\cdots,x_{k-1}^t). \tag{1.12}$$

Clearly, φ is a bijective linear map, and so the image of a subspace of $\mathbb{F}_{q^t}^k$ is a subspace of $\mathbb{F}_q^{kt}$. Now let $\langle x\rangle$ be a point of $\mathrm{PG}(k-1,q^t)$, which is a line of $\mathbb{F}_{q^t}^k$. Then, $\varphi(\langle x\rangle)$ is a subspace of $\mathbb{F}_q^{kt}$ of cardinality q^t and hence dimension t. Thus, $\varphi(\langle x\rangle)$ is a $(t-1)$-dimensional subspace of $\mathrm{PG}(kt-1,q)$. Because φ is a bijection, $S=\{\varphi(\langle x\rangle)\,|\,\langle x\rangle\in\mathrm{PG}(k-1,q)\}$ is a $(t-1)$-spread of $\mathrm{PG}(kt-1,q)=\mathrm{PG}(n,q)$. □

We are most interested in spreads of $\mathrm{PG}(3,q)$, for which the only non-trivial value of t would be $t=1$. A closely related object to a spread in $\mathrm{PG}(3,q)$ is a *regulus.* A *regulus* $\mathcal{R}$ is a set of $q+1$ lines of $\mathrm{PG}(3,q)$, mutually skew, such that any line meeting three lines of $\mathcal{R}$ meets all lines of $\mathcal{R}$. Reguli are uniquely determined by three mutually skew lines (see Theorem 3.5 of [8]).

Let S be the spread of $\mathrm{PG}(3,q)$ obtained by field reduction from $\mathrm{PG}(1,q^2)$. Then S has the additional property that the unique regulus through every three lines of S is contained in S. Such spreads are called *regular.*

Result 1.27. *Any regular spread of $\mathrm{PG}(3,q)$ is projectively equivalent to the regular spread S obtained by field reduction of $\mathrm{PG}(1,q^2)$.*

1.7 The André/Bruck-Bose Construction

Certain projective planes of order q^2 can be represented in $\mathrm{PG}(4,q)$ via the *André/Bruck-Bose construction*. We use the shorthand *ABB construction* for brevity. Fix a hyperplane $H \simeq \mathrm{PG}(3,q)$ within $\Sigma = \mathrm{PG}(4,q)$. Let S be a set of q^2+1 lines of H that are pairwise disjoint (a spread of H). Now define the incidence structure $\mathcal{P}(S)$ as follows.

1. The points of $\mathcal{P}(S)$ are the points of $\mathrm{PG}(4,q) \setminus H$, together with the lines of S.
2. The lines of $\mathcal{P}(S)$ are the planes of $\mathrm{PG}(4,q)$ not contained in H, meeting S in a spread line, together with H.
3. Incidence in $\mathcal{P}(S)$ is inherited from $\mathrm{PG}(4,q)$

Result 1.28. *Let $\mathcal{P}(S)$ be the incidence structure produced by the ABB construction. Then, $\mathcal{P}(S)$ is a projective plane.*

Proof. Firstly, observe that for any two planes Π_1, Π_2 in $\mathrm{PG}(4,q)$, $\dim \langle \Pi_1, \Pi_2 \rangle \leq 4$, so we have $\dim \Pi_1 \cap \Pi_2 \geq 0$; no two planes can be disjoint in $\mathrm{PG}(4,q)$. Let Π_1 and Π_2 be two distinct planes meeting H in spread elements ℓ_1 and ℓ_2. If $\ell_1 = \ell_2$ then $\Pi_1 \cap \Pi_2 = \ell_1$ and so Π_1 meets Π_2 in one point of $\mathcal{P}(S)$. If $\Pi_1 \cap \Pi_2$ is a line m, and $\ell_1 \neq \ell_2$, then m and ℓ_1 are coplanar in Π_1 and so meet, and likewise m and ℓ_2 meet in a point. However, this forces $m \subset H$ and hence $\Pi_1 = \langle \ell, m \rangle \subset H$ which is a contradiction. So $\Pi_1 \cap \Pi_2$ is a point of $\mathcal{P}(S)$. Clearly the line of $\mathcal{P}(S)$ represented by H meets any other line of $\mathcal{P}(s)$ in a point $\mathcal{P}(S)$. Hence, any two distinct lines of $\mathcal{P}(S)$ meet in a unique point.

Let P and Q be any two points of $\mathrm{PG}(4,q) \setminus H$. Then $\ell = \langle P, Q \rangle$ meets H in a point R lying on a unique line $m \in S$. Thus, the plane $\Pi = \langle \ell, m \rangle$ is the unique plane through P and Q. It is clear the plane containing a spread element m and $P \in \mathrm{PG}(4,q) \setminus H$ is unique, and that H is the unique line containing any two elements of S. Hence, any two points of $\mathcal{P}(S)$ are contained in a unique line of $\mathcal{P}(S)$.

Finally the existence of four points, no three collinear follows from the existence of four points of $\mathrm{PG}(4,q) \setminus H$, no three coplanar. $\square$

We have a standard construction of $\mathrm{PG}(2,q^2)$ from $\mathrm{PG}(4,q)$ using the fixed hyperplane $H_\infty : x_4 = 0$ and the spread S of H_∞ obtained by field reduction of $\mathrm{PG}(1,q^2)$. Associated to this construction is a map ϕ, which we shall call the *ABB map*, that maps points of $\mathrm{PG}(2,q^2)$ to their representation in $\mathrm{PG}(4,q)$. Fix a basis $\{1, \epsilon\}$ of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. For some point $(a,b,c) \in \mathrm{PG}(2,q^2)$, scaled such that $c \in \mathbb{F}_q$, denote $(a,b,c)_{\mathbb{F}_q}$ as the point (a_0, a_1, b_0, b_1, c), where $a = a_0 + a_1\epsilon$ and $b_0 + b_1\epsilon$. We define ϕ as follows:

$$\phi((a,b,1)) = (a,b,1)_{\mathbb{F}_q} \tag{1.13}$$

$$\phi((a,b,0)) = \{(ax,bx,0)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^2}\}. \tag{1.14}$$

Note that $\phi((a,b,0))$ is the line of $\mathrm{PG}(4,q)$ contained in H_∞ obtained from field reduction of (a,b) in $\mathrm{PG}(1,q^2)$. We denote the line of $\mathrm{PG}(2,q^2)$ associated with H_∞ as ℓ_∞.

With a fixed representation of $\mathrm{PG}(2,q^2)$ we can investigate how Baer subplanes and Baer sublines are represented in $\mathrm{PG}(4,q)$.

Result 1.29. *Let Π be a plane of $\mathrm{PG}(4,q)$ meeting H_∞ in a line ℓ not in S. Then the affine points of Π, in addition to the $q+1$ spread elements through the points of ℓ, form a Baer subplane $\mathcal{B}$ of $\mathrm{PG}(2,q^2)$. Conversely, every Baer subplane of $\mathrm{PG}(2,q^2)$ secant to ℓ_∞ has such a representation in $\mathrm{PG}(4,q)$.*

Proof. Clearly, if a plane Π' of $\mathrm{PG}(4,q)$ contains two points of Π it contains a line of $\mathrm{PG}(4,q)$, so each line of $\mathrm{PG}(2,q^2)$ contains 0, 1, or $q+1$ points of $\mathcal{B}$. As no two planes are disjoint in $\mathrm{PG}(4,q)$, no line of $\mathrm{PG}(2,q^2)$ may be disjoint from $\mathcal{B}$. Through two points $P,Q\in\mathcal{B}$ the unique line through P and Q in $\mathrm{PG}(2,q^2)$ meets $\mathcal{B}$ in $q+1$ points. So there are exactly $q+1$ lines through each point of P secant to $\mathcal{B}$, and it follows that $\mathcal{B}$ is a Baer subplane. That every Baer subplane has such a representation follows from a counting argument (see Theorem 3.13 of [8]). $\square$

The following results are then corollaries of Result 1.29.

Result 1.30. *Let ℓ be a line of $\mathrm{PG}(4,q)$ meeting S in a point R on a spread line m. Then ℓ forms a Baer subline of $\mathrm{PG}(2,q^2)$ contained in $\Pi=\langle \ell,m\rangle$. Conversely, every Baer subline tangent to ℓ_∞ has such a representation.*

Result 1.31. *Let R be a regulus contained in S, then R represents a Baer subline of $\mathrm{PG}(2,q^2)$ contained in ℓ_∞.*

1.8 Non-Desarguesian Projective Planes

Whilst we have thus far limited our discussion of projective planes to the Desarguesian plane $\mathrm{PG}(2,q)$, there exist many examples of non-Desarguesian projective planes. The simplest non-Desarguesian planes to describe are the *translation planes*. A translation plane Π has the property that there exists a line ℓ and a group of elations with axis ℓ, acting transitively on $\Pi\setminus\ell$. Translation planes arise from the André/Bruck-Bose construction as the following result demonstrates.

Result 1.32. *Let S be a spread of a hyperplane H contained in $\mathrm{PG}(4,q)$. Then $\Pi=\mathcal{P}(S)$ is a projective plane admitting a group of elations with axis ℓ_∞, that is transitive on the points of $\Pi\setminus\ell_\infty$.*

Proof. Let φ be an elation of $\mathrm{PG}(4,q)$ with axis H and mapping $P \notin H$ to $Q \notin H$. For each point R in Π, let $\tilde{R}$ be the point (or spread line if $R \in \ell_\infty$) representing R in $\mathrm{PG}(4,q)$. Then we may define $\hat{\varphi}$ mapping a point R of Π to the point of Π representing $\varphi(\tilde{R})$ in $\mathrm{PG}(4,q)$. It is clear that $\hat{\varphi}$ must fix ℓ_∞ as ϕ fixes H point-wise. Lines of Π are represented as planes meeting H in spread lines of S, and hence are preserved by $\hat{\varphi}$. Thus, $\hat{\varphi}$ is a collineation of Π. Because φ fixes lines and so planes through $PQ \cap H$, $\hat{\varphi}$ fixes lines through $\tilde{P}\tilde{Q} \cap \ell_\infty$, and is therefore an elation of Π. $\square$

Bruck and Bose [13] show that the converse is also true; every finite translation plane is derived from the ABB construction. Some of the better known translation planes include Nearfield, Semifield and Moufang planes.

The Hughes plane and the Figueroa plane are two of the most well-known examples of non-translation projective planes. The Figueroa plane is discussed in detail Chapter 5, and the Hughes plane is constructed using a nearfield of order p^{2n}. For more information on the Hughes plane see [8].

1.9 Unitals

A *unital* U in a projective plane Π of order q^2 is a set of q^3+1 points such that each line of Π meets U in either 1 or $q+1$ points. The *Hermitian unital*, or *classical unital*, is the set of points that are the absolute points of a non-degenerate Hermitian polarity. That is the set of points in $\mathrm{PG}(2,q^2)$ with homogenous coordinates $\mathbf{x}$ such that

$$\mathbf{x}^T H \mathbf{x}^{(q)} = \mathbf{0}, \tag{1.15}$$

where H is a non-singular Hermitian matrix, $\mathbf{x}$ is taken as a column vector, and $\mathbf{x}^{(q)}$ is component-wise exponentiation.

Result 1.33. *Hermitian unitals are projectively equivalent to the set $\mathcal{H}(2,q^2)$, which is all points with homogenous coordinates (x,y,z) satisfying*

$$x^{q+1} + y^{q+1} + z^{q+1} = 0. \tag{1.16}$$

The group of collineations stabilising $\mathcal{H}(2,q^2)$ is $\mathrm{P\Gamma U}(3,q)$, and the homography subgroup $\mathrm{PGU}(3,q)$ is induced by Hermitian matrices with entries in $\mathbb{F}_q$.

Result 1.34. *The group $\mathrm{PGU}(3,q)$ acts 2-transitively on points of $\mathcal{H}(2,q^2)$.*

It turns out that the secant lines of a Hermitian unital $\mathcal{H}$ meet in $q+1$ points forming Baer sublines.

Result 1.35. *Let $\mathcal{H}$ be a Hermitian unital in* $\mathrm{PG}(2, q^2)$*, and ℓ a line of* $\mathrm{PG}(2, q)$*. Then $\ell \cap \mathcal{H}$ is a Baer subline of ℓ.*

Proof. Using the equivalence of Hermitian curves we may assume that, if q is odd, the unital $\mathcal{H}$ has the equation $\epsilon xy^q - \epsilon x^q y + z^{q+1} = 0$, where $\epsilon = \zeta^{(q+1)/2}$ for some primitive element ζ of $\mathbb{F}_{q^2}$. If q is even, the points of $\mathcal{H}$ may be taken to satisfy $xy^q + x^q y + z^{q+1} = 0$. In either case, we may assume our secant line ℓ is $z = 0$ which meets $\mathcal{H}$ in points satisfying $xy^q - x^q y = 0$. Points on the line ℓ take the form $(1, y, 0)$ or $(0, 1, 0)$, and the points $(1, y, 0)$ on $\mathcal{H}$ have $y^q = y$, so

$$\ell \cap \mathcal{H} = \{(1, y, 0) \mid y \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}. \tag{1.17}$$

This clearly demonstrates $\ell \cap \mathcal{H}$ is a Baer subline of ℓ. □

Given an arbitrary unital U and a point $P \notin U$, we define the *feet* of P to be the set of points on tangent lines through P. We denote the feet of $P \notin U$ as $\tau_P(U)$. The following result shows that $|\tau_P(U)| = q + 1$ for all $P \notin U$.

Result 1.36. *Let U be a unital in a projective plane of order q^2. Then, $|\tau_P(U)| = q + 1$ for all $P \notin U$.*

Proof. Let U be a unital and $P \notin U$. Then, there are $q^2 + 1$ lines through P which partition the points of U into sets of size 1 or $q + 1$. Let s be the number of secants through P to U. We then have

$$s(q + 1) + (q^2 + 1 - s) = q^3 + 1. \tag{1.18}$$

Solving for s yields $s = q^2 - q$. Therefore, there are $q + 1$ tangents of U through P, and so $\tau_P(U) = q + 1$. □

Combinatorial characterisations of unitals are established from the properties of their feet. We note that the Hermitian unital is characterised by the fact that its feet form Baer sublines.

Result 1.37. *Let H be a classical unital in* $\mathrm{PG}(2, q^2)$*. Then the feet of points $P \notin H$ are collinear.*

Proof. Let H be classical unital in $\mathrm{PG}(2, q^2)$. Then there exists a polarity ρ whose absolute points are H. Suppose P is a point not in H. Then, for all $Q \in \tau_P(H)$, we have

$$P \in Q^\rho \tag{1.19}$$

$$\Rightarrow Q \in P^\rho. \tag{1.20}$$

So the feet of P are contained in P^ρ. □

Thas [39] shows the converse is also true; if the feet of all points on two tangent lines to U are collinear, then U is Hermitian.

Result 1.38. *[39] Let U be a unital in* $\mathrm{PG}(2, q^2)$*. If there exists two tangent lines l and m to U, such that $\tau_P(U)$ is collinear for all $P \in (l \cup m) \setminus U$, then U is Hermitian.*

The remaining known unitals in $\mathrm{PG}(2, q^2)$ are *Buekenhout unitals*. Buekenhout unitals are first described in [14]. The *ovoidal-Buekenhout-Metz* unitals arise from ovoidal cones $\mathcal{C}$ in $\mathrm{PG}(4, q)$ via the ABB construction.

Result 1.39. *Let H_∞ be a fixed hyperplane of* $\mathrm{PG}(4, q)$*, fix a regular spread S of H_∞, and let $\mathcal{C}$ be an ovoidal cone such that $H_\infty \cap \mathcal{C}$ is a line p_∞ of S. Then, the point set U obtained via the ABB construction is a unital in* $\mathrm{PG}(2, q^2)$*.*

Proof. Let P_∞ be the point associated with p_∞ in $\mathrm{PG}(2, q^2)$. Each line $\ell \neq \ell_\infty$ of $\mathrm{PG}(2, q^2)$ containing P_∞ therefore corresponds to a plane Π not contained in H_∞ through p_∞ in $\mathrm{PG}(4, q)$. Because Π is not contained in H_∞, it is not a part of the tangent hyperplane of $\mathcal{C}$ at p_∞. Thus, $\Pi \cap \mathcal{C} = p_\infty \cup m$, where $m \neq p_\infty$ is another generator line of $\mathcal{C}$. It therefore follows that $|\ell \cap U| = q + 1$, accounting for P_∞ plus each of the q points associated with $m \setminus H_\infty$.

Now suppose that $\ell' \neq \ell_\infty$ is a line of $\mathrm{PG}(2, q^2)$ that does not contain P_∞. Then the line ℓ' is represented by a plane Π' of $\mathrm{PG}(4, q)$ through a spread line of $S \setminus \{p_\infty\}$. Because ℓ' does not contain P_∞, the plane Π' is disjoint from p_∞. Therefore Π' does not contain a generator line of $\mathcal{C}$, as generator lines are concurrent at a point of p_∞. Hence, Π' meets $\mathcal{C}$ in either an affine point or an oval consisting of affine points, and $|\ell' \cap U| = 1$ or $q + 1$.

Lastly it is clear that ℓ_∞, represented by H_∞ in $\mathrm{PG}(4, q)$, meets $\mathcal{C}$ at precisely p_∞ and is thus tangent to U at P_∞. □

We classify ovoidal-Buekenhout-Metz unitals by the base of the ovoidal cone.

1. If the base of $\mathcal{C}$ is an elliptic quadric, the resulting unital is a *Buekenhout-Metz* unital.

2. If $q = 2^{2e+1}$, and the base of $\mathcal{C}$ is a Tits ovoid, then the resulting unital is a *Buekenhout-Tits* unital.

An additional class of Buekenhout unital arises via the ABB construction applied to an elliptic quadric meeting the spread at infinity in a regulus.

Result 1.40 ([14])**.** *Let H_∞ be a fixed hyperplane of* $\mathrm{PG}(4, q)$*, fix a regular spread S of H_∞, and let $\mathcal{E}$ be an elliptic quadric such that $H_\infty \cap \mathcal{E}$ is a regulus. Then $\mathcal{E}$ represents a classical unital U secant to ℓ_∞ in* $\mathrm{PG}(2, q^2)$*.*

We finish our overview of unitals by describing the feet of ovoidal-Buekenhout-Metz unitals for points $P \in \ell_\infty$.

Result 1.41. *Let U be a ovoidal-Buekenhout-Metz unital constructed as per Result 1.39. The feet of all points $P \in \ell_\infty \setminus U$ are Baer sublines.*

Proof. This immediately follows from Result 1.30. □

1.10 Codes of Projective Planes

Let F be a finite set of q symbols. A q-ary *code* C is a set of sequences of elements in F, called *codewords*. If all sequences in C have the same length, then C is a q-ary *block code*. We are interested in q-ary block codes $\mathcal{C}$ where q is some prime power, and C is a subspace of $(\mathbb{F}_q)^n$. Such a code is called a *linear code*.

Given a q-ary linear code C, and a codeword $v \in C$ we may define the *weight* of v to be the number of non-zero entries in v. If codewords in C have length n, the subspace $C \subseteq (\mathbb{F}_q)^n$ has dimension k, and the minimum weight codeword in C has weight d, then C is an $[n, k, d]$-linear q-ary code (or just an $[n, k, d]$-code if q is clear from context).

Let Π be a projective plane with order $q = p^k$ for some prime p, and integer $k \geq 1$. Fix an ordering of the points of Π as $P_1, P_2, \cdots, P_{q^2+q+1}$. We define the *characteristic vector* of a set S of points in Π to be the vector $\mathbf{v}^S \in \mathbb{F}_q$ where $(\mathbf{v}^S)_i = 1$ if $P_i \in S$ and $(\mathbf{v}^S)_i = 0$ otherwise. The p-ary linear code $C_p(\Pi)$ is the subspace of $\mathbb{F}_p^{q^2+q+1}$ spanned by the characteristic vectors of all lines in Π.

We now present a few theorems regarding $C_p(\mathrm{PG}(2, p^k))$. The results discussed here are explored in detail in [2].

Result 1.42. *Let Π be a projective plane of order $q = p^k$ for some prime p and integer $k \geq 1$. Then $C_p(\Pi)$ is an $[q^2 + q + 1, k, q + 1]$-code. Moreover, the minimum weight vectors of $C_p(\Pi)$ are the scalar multiples of the characteristic vectors of lines of Π.*

Result 1.43. *The dimension of $C_p(\mathrm{PG}(2, p^k))$ is $\binom{p+1}{2}^k + 1$.*

The determination of codewords in $C_p(\mathrm{PG}(2, q))$ is an area of ongoing research. The following result characterises the Hermitian as the only unital that is a codeword of $C_p(\mathrm{PG}(2, q))$.

Result 1.44 ([9])**.** *Let $q = p^k$ for some prime p and integer $k \geq 1$. Then a unital U of $\mathrm{PG}(2, q^2)$ is a codeword if and only if U is classical.*

We do not have an explicit expression for $\mathbf{v}^{\mathcal{H}(2,q^2)}$ as a sum of characteristic vectors of lines in $\mathrm{PG}(2, q^2)$ for arbitrary q; an expression for q even has since been discovered (see Section 4.4).

Chapter 2

Buekenhout-Tits Unitals

The aim of this chapter will be to summarise what is known about Buekenhout-Tits unitals, and to present the following results:

1. A proof that all Buekenhout-Tits unitals are projectively equivalent. This addresses an open problem in [8].

2. A description of the stabiliser group of a Buekenhout-Tits unital in $\mathrm{P\Gamma L}(3, q^2)$. Ebert [20] only provides a description of stabiliser of the Buekenhout-Tits unital in $\mathrm{PGL}(3, q^2)$. The stabiliser of the classical unital is $\mathrm{P\Gamma U}(3, q)$, and the stabiliser of the Buekenhout-Metz unital in $\mathrm{P\Gamma L}(3, q^2)$ is described in [21] for q even and [4] for q odd.

3. If U is a Buekenhout-Tits unital, then a line ℓ meets the feet of a point $P \notin \ell_\infty \cup U$ in at most 4 points. Moreover, there exists a point P and line ℓ such that the feet of P meet ℓ in exactly three points. This highlights a difference between Buekenhout-Metz unitals and Buekenhout-Tits unitals.

2.1 Prior Results

It was Ebert [20] who first gave coordinates for the Buekenhout-Tits unital in $\mathrm{PG}(2, q^2)$, here we reproduce his work. Let $q = 2^{2e+1}$, for some $e \geq 1$. Recall from equation (1.10) that the points

$$\mathcal{O} = \{(0,0,0,1)\} \cup \left\{(1, s, t, s^{\sigma+2} + t^\sigma + st) \,\middle|\, s, t \in \mathbb{F}_q\right\}, \tag{2.1}$$

form a Tits ovoid in $\mathrm{PG}(3, q)$. The following is then an ovoidal cone with vertex $Q = (0,0,0,1,0)$ with base a Tits ovoid, tangent to $H_\infty : x_0 = 0$:

$$\begin{aligned} \mathcal{C} = &\{(0,0,0,0,1)\} \cup \{(0,0,0,1,\lambda) \mid \lambda \in \mathbb{F}_q\} \\ &\cup \left\{(1, s, t, r, s^{\sigma+2} + t^\sigma + st) \,\middle|\, r, s, t \in \mathbb{F}_q\right\}. \end{aligned} \tag{2.2}$$

Now fix the hyperplane at infinity $H_\infty : x_0 = 0$. Via the ABB construction described in Section 1.7, the cone $\mathcal{C}$ cosrresponds to the following Buekenhout-Tits unital in $\mathrm{PG}(2, q^2)$,

$$\mathcal{U}_{\mathrm{BT}} = \{(0,0,1)\} \cup \left\{(1, s + t\epsilon, r + (s^{\sigma+2} + t^\sigma + st)\epsilon) \,\middle|\, r, s, t \in \mathbb{F}_q\right\}. \tag{2.3}$$

Where $\{1, \epsilon\}$ forms a basis for $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Without loss of generality we may assume that ϵ is a root of $x^q + x + 1$, and that $1 \neq \delta = \epsilon^2 + \epsilon$ is an element of $\mathbb{F}_{q^2}$ with absolute trace one. This assumption can be made because $x^q + x + 1$ factors over $\mathbb{F}_q$ as the product of $x^2 + x + \alpha$ for each α in $\mathbb{F}_q$ with absolute trace one (see [31]).

Using the coordinates of the Buekenhout-Tits unital in Equation (2.3), Ebert describes the homography stabiliser of U. We provide here an expanded proof of his result,

Result 2.1 (Ebert [20]). *Let G denote the group of projectivities stabilising U. Then G is an abelian group of order q^2, consisting of projectivities induced by the matrices*

$$M_{u,v} = \left\{ \begin{bmatrix} 1 & u\epsilon & v + u^\sigma\epsilon \\ 0 & 1 & u + u\epsilon \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, u, v \in \mathbb{F}_q \right\}. \tag{2.4}$$

Proof. By direct calculation we see that

$$\begin{aligned}&(1, s + t\epsilon, r + (s^{\sigma+2} + t^\sigma + st)\epsilon)M_{u,v} = \\ &\qquad (1, s + (t+u)\epsilon, r + v + su + tu\delta + (s^{\sigma+2} + (t+u)^\sigma + s(t+u))\epsilon),\end{aligned} \tag{2.5}$$

and thus the collineation induced by $M_{u,v}$ is in G.

Now suppose a matrix M induces a collineation $\varphi \in G$. As φ stabilises U, we know that φ must fix $P_\infty = \{(0,0,1)\}$ and the line at infinity $\ell_\infty : x = 0$. Therefore, M is of the following form,

$$M = \begin{bmatrix} 1 & a & b \\ 0 & e & c \\ 0 & 0 & f \end{bmatrix}. \tag{2.6}$$

where $a, b, e, c, f \in \mathbb{F}_{q^2}$. Let $a = a_1 + a_2\epsilon$, $b = b_1 + b_2\epsilon$ and so on for e, c and f. The points $(1,0,1)$ and $(1,0,0)$ lie on U, so $(1,0,1)M = (1, a, b + f)$ and $(0,0,1)M = (1, a, b)$ lie on U. Hence $f \in \mathbb{F}_q$ and $f_2 = 0$.

Next, let $P_{r,s,t} = (1, s + t\epsilon, r + (s^{\sigma+2} + t^\sigma + st)\,\epsilon)$ denote any point of $U \setminus \{P_\infty\}$, where $r, s, t \in \mathbb{F}_q$ are arbitrary. Then $P_{r,s,t}M \in U$ implies that

$$\left(1, a + e(s + t\epsilon), b + c\,(s + t\epsilon) + f\left(r + s^{\sigma+2} + t^\sigma + st\right)\epsilon\right) \in U. \tag{2.7}$$

We now have,

$$a + e\,(s + t\epsilon) = (a_1 + e_1 s + e_2 t\delta) + (a_2 + (e_1 + e_2)t)\,\epsilon. \tag{2.8}$$

Let $s' = a_1 + e_1 s + e_2 t\delta$ and $t' = a_2 + e_2 s + (e_1 + e_2)\, t$. Then if $P_{r,s,t}M \in U$ we have

$$b_2 + (c_1 + c_2)\, t + c_2 s + f_1 \left(s^{\sigma+2} + t^{\sigma} + st\right) = s'^{\sigma+2} + t'^{\sigma} + s't',$$

This must hold for all values of s and t. Letting $s = t = 0$, so that $s' = a_1$ and $t' = a_2$ we find

$$a_1 a_2 + a_1^{\sigma+2} + a_2^{\sigma} = b_2.$$

Now letting $t = 0$ and leaving s arbitrary so that $s' = a_1 + e_1 s$ and $t' = a_2 + e_2 s$ we find

$$\left(e_1^{\sigma+2} + f_1\right) s^{\sigma+2} + \left(e_2^{\sigma} + a_1^{\sigma} e_1^{\sigma}\right) s^{\sigma} + \left(e_1 e_2 + a_{\sigma} e_1^2\right) s^2 + \left(a_2 e_1 + a_1 e_2 + c_2\right) s = 0 \qquad (2.9)$$

which must hold for all $s \in \mathbb{F}_q$. The last line of the reduction is a polynomial in s, with degree at most $2^{(e+1)/2} + 2 < q$ as $e \geq 3$. To have q roots the polynomial must be identically zero. This forces

$$\begin{aligned} f_1 &= e_1^{\sigma+2} \\ e_2^{\sigma} &= a_1^2 e_1^{\sigma} \\ e_1 e_2 &= a_1^{\sigma} e_1^2 \\ a_2 e_1 + a_1 e_2 &= c_2. \end{aligned}$$

Similarly, letting $s = 0$ and t be arbitrary we find the following polynomial in t,

$$\begin{aligned} e_1^{\sigma+2}\delta^{\sigma+2} t^{\sigma+2} &+ \left(f_1 + (e_1 + e_2)^{\sigma} + e_2^{\sigma} a_1^2 \delta^{\sigma}\right) t^{\sigma} \\ &+ \left(e_2 (e_1 + e_2)\, \delta + a_1^{\sigma} e_2^2 \delta^2\right) t^2 + \left(c_1 + c_2 + (e_1 + e_2)\, a_1 + e_2 a_2 \delta\right) t = 0 \qquad (2.10) \end{aligned}$$

Thus, we get a second set of constraints,

$$\begin{aligned} e_2^{\sigma+2}\delta^{\sigma+2} &= 0 \\ f_1 &= e_1^{\sigma} + e_2^{\sigma} + a_1^2 e_2^{\sigma} \delta^{\sigma} \\ a_1^{\sigma} e_2^2 \delta^2 &= (e_1 + e_2)\, e_2 \delta \\ c_1 + c_2 &= a_2 e_2 \delta + a_1 (e_1 + e_2)\,. \end{aligned}$$

From both sets of constraints, and because $\delta \neq 0$ we have $e_2^{\sigma+2} = 0$, and so $e_1 \neq 0$ because $e \neq 0$ (this is required so M is invertible). As $e_2^{\sigma} = a_1^2 e_1^{\sigma}$ and $e_2 = 0$ we have $a_1 = 0$ because $e_1 \neq 0$. Then we find $c_1 + c_2 = 0$ so $c_1 = c_2$. On the one hand we have $f_1 = e_1^{\sigma+2}$ and on the other we have $f_1 = e_1^{\sigma} + e_2^{\sigma} + a_1^2 e_2^{\sigma} \delta^{\sigma} = e_1^{\sigma}$, so $e_1^{\sigma+2} = e_1^{\sigma}$. Simplifying this implies $e_1^2 = e_1$, and as $e_1 \neq 0$ this implies $f_1 = e_1 = 1$. Because $a_2 e_1 + a_1 e_2 = c_2$ we have $c_1 = c_2 = a_2 e_1 = a_2$. Finally from $a_1 a_2 + a_1^{\sigma+2} + a_2^{\sigma} = b_2$ we have $b_2 = a_2^{\sigma}$. Letting $u = c_1 = c_2 = a_2$, $b_2 = u^{\sigma}$, and $b_1 = v$ for some arbitrary $\mathbb{F}_q$ we find

$$M = \begin{bmatrix} 1 & u\epsilon & v + u^{\sigma}\epsilon \\ 0 & 1 & u + u^{\sigma}\epsilon \\ 0 & 0 & 1 \end{bmatrix}.$$

This completes the proof. □

We now obtain the following corollary.

Corollary 2.1. *Let G be the homography stabiliser group of U as described in Result 2.1. Then, G fixes P_∞, has q orbits of size q^2 on $U \setminus \{P_\infty\}$, has q orbits of size q on $\ell_\infty \setminus \{P_\infty\}$, and has $q^2 - q$ orbits of size q^2 on $\mathrm{PG}(2, q^2) \setminus (U \cup \ell_\infty)$.*

Proof. From Result 2.1 we know G fixes P_∞. If $(0, 1, z) \in \ell_\infty \setminus \{P_\infty\}$, then the G-stabiliser of this point is all homographies of the form

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.11}$$

where $b \in \mathbb{F}_q$. Hence we have q orbits of size q on $\ell_\infty \setminus P_\infty$. Otherwise, the point $(1, a, b)$ is mapped to $(1, a+u\epsilon, (a+1)u+(v^\sigma+au)\epsilon+b)$, so points not on ℓ_∞ have a trivial G-stabiliser. Hence the q^4 points of $\mathrm{PG}(2, q^2) \setminus \ell_\infty$ are partitioned into q^2 orbits of order q^2, q of these orbits being contained in $U \setminus P_\infty$. □

2.2 On the Projective Equivalence of Buekenhout-Tits Unitals

In this section, we show that all Buekenhout-Tits unitals are projectively equivalent to the unital $\mathcal{U}_{\mathrm{BT}}$ given in equation (2.3).

Remark 2.1. *The authors of [23] give this result without proof and state it can be derived using the same techniques employed by Ebert in [20]. Ebert however, lists the equivalence of Buekenhout-Tits unital as an open problem in [8] which appeared about ten years after his original paper [20].*

It is easy to see that the Buekenhout-Tits unital $\mathcal{U}_{\mathrm{BT}}$ is tangent to the line $\ell_\infty : x = 0$ in the point $P_\infty = (0, 0, 1)$. From the ABB construction it follows that P_∞ has the following property with respect to $\mathcal{U}_{\mathrm{BT}}$.

Property 2.1. *Given any unital U, a point $P \in U$ has Property 2.1 if all secant lines through P meet U in Baer sublines.*

It is shown in [7] that if two different points of U have Property 2.1, then U is classical. Hence, the point P_∞ is the unique point of $\mathcal{U}_{\mathrm{BT}}$ admitting this property. We will count all Buekenhout-Tits unitals tangent to ℓ_∞ at a point P_∞ having Property 2.1.

Lemma 2.1. *There are $q^4(q^2-1)^2$ unitals projectively equivalent to $\mathcal{U}_{\mathrm{BT}}$ in $\mathrm{PG}(2,q^2)$ tangent to $\ell_\infty : x = 0$, and containing the point $P_\infty = (0,0,1)$ with Property 2.1.*

Proof. First note that any projectivity mapping $\mathcal{U}_{\mathrm{BT}}$ to a unital tangent to ℓ_∞ in P_∞ necessarily is contained in the group H of projectivities fixing ℓ_∞ line-wise and P_∞ point-wise. The elements of H are induced by all matrices of the following form,

$$\begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix},$$

where $x_{22}x_{33} \neq 0$ and matrices act on homogeneous coordinates by multiplication on the right. It follows that $|H| = (q^2-1)^2 q^6$. Furthermore, from the description of $G = \mathrm{PGL}(3,q^2)_{\mathcal{U}_{\mathrm{BT}}}$ in Result 2.1, we know that $H_{\mathcal{U}_{\mathrm{BT}}} = G$, and hence, $H_{\mathcal{U}_{\mathrm{BT}}}$ has order q^2. By the orbit-stabiliser theorem, we find that there are $(q^2-1)^2 q^4$ unitals in the orbit of $\mathcal{U}_{\mathrm{BT}}$ under H. □

Consider $\mathrm{PG}(2,q^2)$ modelled using the ABB construction with fixed hyperplane H_∞. Let p_∞ be the spread line corresponding to P_∞. Then any Buekenhout-Tits unital U tangent to ℓ_∞ at P_∞ with Property 2.1 corresponds uniquely to an ovoidal cone $\mathcal{C}$ meeting H_∞ at p_∞.

Lemma 2.2. *There are $q^4(q^2-1)^2$ ovoidal cones $\mathcal{C}$ in $\mathrm{PG}(4,q)$ with base a Tits ovoid, such that $\mathcal{C}$ meets H_∞ in the spread element p_∞.*

Proof. Let V be a point on the line p_∞, and $H \neq H_\infty$ a hyperplane not containing V. Then, H meets H_∞ in a plane containing a point $R \in p_\infty \setminus \{V\}$. Any ovoidal cone $\mathcal{C}$ with vertex V and base a Tits ovoid, such that $\mathcal{C}$ meets H_∞ precisely in p_∞, meets H in a Tits ovoid tangent to $H \cap H_\infty$ at the point R. We will count all cones of this form, for all $V \in p_\infty$.

Consider the pairs of planes Π and Tits ovoids $\mathcal{O}$, $(\Pi,\mathcal{O})$, where $\Pi, \mathcal{O} \subset H$ and Π is tangent to $\mathcal{O}$. On the one hand, there are $|\mathrm{PGL}(4,q)|/|\mathcal{O}_{\mathrm{PGL}(4,q)}| = (q+1)^2 q^4 (q-1)^2 (q^2+q+1)$ Tits ovoids in $\mathrm{PG}(3,q)$, and each has q^2+1 tangent planes. On the other hand, $\mathrm{PGL}(4,q)$ is transitive on hyperplanes of $\mathrm{PG}(3,q)$, so each plane is tangent to the same number of Tits ovoids. It thus follows, that there are

$$\frac{(q+1)^2 q^4 (q-1)^2 (q^2+q+1)(q^2+1)}{q^4+q^3+q^2+q+1} = (q-1)^2 q^4 (q+1)(q^2+q+1)$$

Tits ovoids tangent to $H \cap H_\infty$ conitained in H.

Furthermore, since $\mathrm{PGL}(4,q)_{H\cap H_\infty}$ is transitive on points of $H \cap H_\infty$, each point of $H \cap H_\infty$ is contained in the same number of Tits ovoids $\mathcal{O}$, so it follows that the number of Tits ovoids tangent to $H \cap H_\infty$ at $R = p_\infty \cap H$ is $(q-1)^2 q^4 (q+1)$. Hence, there is an equal number of ovoidal cones with base a Tits ovoid, vertex V, and meeting H_∞ at p_∞. As the choice of V was arbitrary, and there are $q+1$ points on p_∞, there are $(q^2-1)^2 q^4$ ovoidal cones with base a Tits ovoid, and meeting H_∞ at p_∞. □

Theorem 2.1. *All Buekenhout-Tits unitals in* $\mathrm{PG}(2, q^2)$ *are* PGL-*equivalent.*

Proof. From Lemmas 2.1 and 2.2, we see that the number of ovoidal cones with vertex a Tits ovoid, tangent to H_∞ at p_∞ is equal to the number of Buekenhout-Tits unitals that are PGL equivalent to $\mathcal{U}_{\mathrm{BT}}$ and tangent to l_∞ at P_∞ with Property 2.1. The result follows. □

Corollary 2.2. *Let U be a Buekenhout-Tits unital, then the projectivity group stabilising U is isomorphic to the group G in Theorem 2.1.*

In showing that all Buekenhout-Tits unitals are projectively equivalent, we may use $\mathcal{U}_{\mathrm{BT}}$ to verify statements about general Buekenhout-Tits unitals.

2.3 On the Stabiliser of the Buekenhout-Tits Unital

We now describe the stabiliser of the Buekenhout-Tits unital $\mathcal{U}_{\mathrm{BT}}$ in $\mathrm{P\Gamma L}(3, q^2)$.

Lemma 2.3. *Let $M_{u,v}, M_{s,t}$ be matrices inducing collineations of G as defined in Result 2.1, then $M_{u,v}M_{s,t} = M_{u+s,t+v+su\delta}$.*

Proof. Using equation (2.4), we find

$$M_{u,v}M_{s,t} = \begin{bmatrix} 1 & (s+u)\epsilon & (t+v+su\delta)+(s+u)^\sigma \\ 0 & 1 & (u+s)+(u+s)\epsilon \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

Thus, we have $M_{u,v}M_{s,t} = M_{u+s,t+v+su\delta}$. □

Corollary 2.3. *The order of any collineation of G induced by a matrix $M_{u,v}$ as defined in Result 2.1 is four if and only if $u \neq 0$, and two if and only if $u = 0$ and $v \neq 0$.*

Proof. Firstly note that $M_{0,0} = I$. Direct calculation shows that $M^2_{u,v} = M_{0,u^2\delta}$, $M^3_{u,v} = M_{u,v+u^2\delta}$ and $M^4_{u,v} = M_{0,0}$. □

Corollary 2.4. *The stabiliser group G as defined in Result 2.1 is isomorphic to $(C_4)^{2e+1}$.*

Proof. Recall from Result 2.1 that $|G| = q^2 = 2^{4e+2}$. From Corollary 2.3, we have that $G \equiv (C_4)^k(C_2)^l$ for some integers k, l such that $2^{2k+l} = |G| = 2^{4e+2}$, and hence,

$$l = 2(e+1-k). \tag{2.13}$$

Furthermore, we see that the number of elements of order four in G is q^2-q as they correspond to all matrices $M_{u,v}$ with $u, v \in \mathbb{F}_q$ and $u \neq 0$. The number of elements of order four in a group isomorphic to $(C_4)^k(C_2)^l$ is $(4^k-2^k)2^l$. Thus,

$$(4^k-2^k)2^l = 4^{2e+1} - 2^{2e+1}. \tag{2.14}$$

Using (2.13), we find that $k = 2e+1$, and therefore $G \equiv (C_4)^{2e+1}$. □

Theorem 2.2. *Let $q = 2^{2e+1}$. Then the stabiliser group of $\mathcal{U}_{\mathrm{BT}}$ in $\mathrm{P\Gamma L}(3, q^2)$ is the order $q^2(4e+2)$ group GK, where K is a cyclic subgroup of order $16e+8$ generated by*

$$\psi : \mathbf{x} \mapsto \mathbf{x}^2 \begin{bmatrix} 1 & 1 & \epsilon \\ 0 & \delta^{\sigma/2}(1+\epsilon) & \delta^{\sigma/2}(1+\epsilon) \\ 0 & 0 & \delta^{\sigma+1} \end{bmatrix}. \tag{2.15}$$

(Here, $\mathbf{x}$ denotes the row vector containing the three homogeneous coordinates of a point, and $\mathbf{x}^2$ denotes its elementwise power.)

Proof. From Lemma 2.2, we have that the number of Buekenhout-Tits unitals is $q^4(q^2-1)^2$. Since all of those unitals are PGL-equivalent by Theorem 2.1, and $\mathrm{PGL}(3,q^2) \triangleleft \mathrm{P\Gamma L}(3,q^2)$, we have that

$$q^4(q^2-1)^2 = \frac{|\,\mathrm{PGL}(3,q^2)|}{|\,\mathrm{PGL}\,(3,q^2)_U|} = \frac{|\,\mathrm{P\Gamma L}(3,q^2)|}{|\,\mathrm{P\Gamma L}\,(3,q^2)_U|}. \tag{2.16}$$

So $\mathrm{P\Gamma L}(3,q^2)_U$ must have order $q^2(4e+2)$. Direct calculation shows that ψ stabilises $\mathcal{U}_{\mathrm{BT}}$. We have $\psi^{4e+2} \in G$ as $\mathbf{x}^{2^{4e+2}} = \mathbf{x}^{q^2} = \mathbf{x}$. Hence, $|\psi| = (4e+2)|\psi^{4e+2}|$. From Corollary 2.3, it follows that $|\psi^{4e+2}| \in \{1,2,4\}$, with $|\psi^{4e+2}| = 4$ if and only if ψ^{4e+2} is induced by $M_{u,v}$ for some $u \neq 0$. Hence, $|\psi^{4e+2}| = 4$ if and only if $\psi^{4e+2}(0,1,0) \neq (0,1,0)$ as $(0,1,0)M_{u,v} = (0,1,u+u\epsilon)$. Consider the point $(0,1,z)$ for some arbitrary $z \in \mathbb{F}_q$. Direct calculation shows that $\psi(0,1,z) = (0,1,1+\mu z^2)$, where $\mu = \frac{\delta^{\sigma+1}}{\delta^{\sigma/2}(1+\epsilon)} = \delta^{\sigma/2}\epsilon$. Thus,

$$\psi^k(0,1,z) = (0,1,\sum_{i=0}^{k} \mu^{2^i-1} + zg(z)) \tag{2.17}$$

for some polynomial $g(z)$ depending on k. If $z=0$ and $k=4e+2$ we thus find:

$$\psi^{4e+2}(0,1,0) = (0,1,\sum_{i=0}^{4e+2} \mu^{2^i-1}) \tag{2.18}$$

$$= (0,1,\frac{\mathrm{Tr}(\mu)}{\mu}). \tag{2.19}$$

Recall that $\epsilon^q = \epsilon+1$, so $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\epsilon) = 1$. We have that, $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_2}(\delta^{\sigma/2}\epsilon) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\delta^{\sigma/2}\epsilon)) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta^{\sigma/2}\,\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\epsilon)) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta^{\sigma/2}) = 1$. Hence, $\psi((0,1,0)) \neq (0,1,0)$, so $|\psi^{4e+2}| = 4$ and $|\psi| = 16e+8$. Let $K = \langle\psi\rangle$, because $|K \cap G| = 4$, it follows that $|GK| = q^2(4e+2)$ and thus $GK = \mathrm{P\Gamma L}\,(3,q^2)_U$. □

2.4 On the Feet of the Buekenhout-Tits Unital

The feet of the Buekenhout-Tits unital $\mathcal{U}_{\mathrm{BT}}$ are first described by Ebert in [20]. He shows that the feet of a point $P = (1, y_1 + y_2\epsilon, z_1 + z_2\epsilon)$ is the following set of points:

$$\tau_P(\mathcal{U}_{\mathrm{BT}}) = \{(1, s + t\epsilon, s^2 + t^2\delta + st + y_1 s + y_1 t + y_2\delta t + z_1 + (s^{\sigma+2} + t^{\sigma} + st)\epsilon) \\ \mid s, t \in \mathbb{F}_q,\ s^{\sigma+2} + t^{\sigma} + st = y_2 s + y_1 t + z_2\}. \quad (2.20)$$

If the line ℓ has equation $\alpha x + y = 0$, where $\alpha \in \mathbb{F}_{q^2}$, Ebert shows that $|\ell \cap \tau_P(\mathcal{U}_{\mathrm{BT}})| \leq 1$. Otherwise, ℓ has equation $(a_1 + a_2\epsilon)x + (b_1 + b_2\epsilon)y + z = 0$, with $a_1, a_2, b_1, b_2 \in \mathbb{F}_q$, and Ebert shows that ℓ meets $\tau_P(\mathcal{U}_{\mathrm{BT}})$ in the points $P_{r,s,t} \in \mathcal{U}_{\mathrm{BT}}$, where $r, s, t \in \mathbb{F}_q$ satisfy

$$s^2 + \delta t^2 + st + (y_1 + b_1)s + (y_1 + y_2\delta + b_2\delta)t + z_1 + a_1 = 0, \quad (2.21)$$

$$s^{\sigma+2} + t^{\sigma} + st = b_2 s + (b_1 + b_2)t + a_2, \quad (2.22)$$

$$y_2 s + y_1 t + z_2 = b_2 s + (b_1 + b_2)t + a_2. \quad (2.23)$$

We will show that for all choices of points $P \notin \ell_\infty$ and lines ℓ, $|\tau_P(\mathcal{U}_{\mathrm{BT}}) \cap \ell| \leq 4$.

Lemma 2.4. *Let G be the group of projectivities stabilising the Buekenhout-Tits unital as described in Result 2.1. Then, the set of $q^2 - q$ points $\{P_{a,b} = (1, a, b\epsilon) \mid a, b \in \mathbb{F}_q,\ b \neq a^{\sigma+2}\}$ are points from $q^2 - q$ distinct point orbits of order q^2 under G.*

Proof. Suppose there exists a collineation of G induced by a matrix $M_{u,v}$ such that $P_{a,b}M_{u,v} = P_{c,d}$. Then,

$$(1, a, b\epsilon) \begin{bmatrix} 1 & u\epsilon & v + u^{\sigma}\epsilon \\ 0 & 1 & u + u\epsilon \\ 0 & 0 & 1 \end{bmatrix} = (1, c, d\epsilon).$$

However, it is clear that $P_{a,b}M_{u,v} = (1, a + u\epsilon, v + u^{\sigma}\epsilon + a(u + u\epsilon) + b\epsilon)$, so $a + u\epsilon = c$. Therefore, $a = c$ and $u = 0$. If $u = 0$, then $v + b\epsilon = d\epsilon$, and we have $b = d$. Hence, $P_{a,b} = P_{c,d}$ and the lemma follows. □

There are $q^4 - q^3 = q^2(q^2 - q)$ points of $\mathrm{PG}(2, q)$ not on ℓ_∞ or $\mathcal{U}_{\mathrm{BT}}$. By Lemma 2.4, each of these points lies in the orbit of a point of the form $(1, a, b\epsilon)$. Therefore, in order to study the feet of a point P, we may assume that the point $P = (1, y_1 + y_2\epsilon, z_1 + z_2\epsilon)$ has $y_2 = z_1 = 0$.

The following lemma shows that the feet of a point $P = (1, y_1, z_2\epsilon)$ meets almost all lines in at most 2 points.

Lemma 2.5. *Let $\ell : \alpha x + \beta y + z$ be a line in $\mathrm{PG}(2, q^2)$, where $\alpha = a_1 + a_2\epsilon$, $\beta = b_1 + b_2\epsilon$ and $a_1, a_2, b_1, b_2 \in \mathbb{F}_q$. Let $P = (1, y_1, z_2\epsilon)$, with $y_1, z_2 \in \mathbb{F}_q$ such that $z_2 \neq y_1$. Unless $b_2 = 0$, $y_1 = b_1$ and $a_2 = z_2$, we have $|\tau_P(\mathcal{U}_{\mathrm{BT}}) \cap \ell| \leq 2$.*

Proof. From the description given in (2.20), we see that the points $P_{r,s,t} \in \tau_P(\mathcal{U}_{\mathrm{BT}})$ satisfy $s^{\sigma+2}+t^\sigma+st = y_1t+z_2$, and this equation has $q+1$ solutions. Substituting this into equation (2.22), the points $P_{r,s,t} \in \tau_P(\mathcal{U}_{\mathrm{BT}}) \cap \ell$ have s, t satisfying

$$s^2 + \delta t^2 + st + (y_1 + b_1)\, s + (y_1 + b_2\delta)\, t + a_1 = 0 \tag{2.24}$$

$$b_2 s + (y_1 + b_1 + b_2)\, t + a_2 + z_2 = 0 \tag{2.25}$$

$$s^{\sigma+2} + t^\sigma + st + y_1 t + z_2 = 0. \tag{2.26}$$

Recall that the points $(1, s, t, s^{\sigma+2} + t^\sigma + st)$, where $s, t \in \mathbb{F}_q$ are the q^2 affine points of a Tits ovoid. Hence, (2.26) represents an affine section of a Tits ovoid. Since it has $q + 1$ points, it is an oval projectively equivalent to the translation oval $\mathcal{D}_\sigma = \{(1, t, t^\sigma) \mid t \in \mathbb{F}_q\}$. Unless $b_2 = 0$ and $y_1 = b_1$, equation (2.25) represents a line in $\mathrm{AG}(2, q)$ which meets the oval (2.26) in at most two points, so we have at most two solutions to the system. If $b_2 = 0$, $y_1 = b_1$, and $a_2 \neq z_2$, then equation (2.25) has no solutions. □

Remark 2.2. *Lemma 2.5 is a refinement of [8, Theorem 4.33], where Barwick and Ebert rework Ebert's earlier proof in [20] that the feet of a point $P \notin (\ell_\infty \cup \mathcal{U}_{\mathrm{BT}})$ are not collinear. This reworked proof asserts that the feet cannot be collinear because the line given by equation* (2.25) *and the conic from equation* (2.24) *cannot have $q+1$ common solutions. However, we can see that this logic is not complete, and leaves an interesting case to examine when equation* (2.25) *vanishes. Ebert's original proof in [20] does not contain this error, instead arguing that equations* (2.24) *and* (2.26) *cannot have $q+1$ common solutions.*

It follows from Lemma 2.5 that the feet of a point $P \notin (\ell_\infty \cup \mathcal{U}_{\mathrm{BT}})$ is a set of $q+1$ points such that every line meets $\tau_P(\mathcal{U}_{\mathrm{BT}})$ in at most two points except for a set of q concurrent lines.

To this end, assume that $b_2 = 0$, $y_1 = b_1$ and $a_2 = z_2$. In this case, equation (2.25) vanishes. The system describing $\ell \cap \tau_P(\mathcal{U}_{\mathrm{BT}})$ is thus

$$s^2 + \delta t^2 + st = y_1 t + a_1 \tag{2.27}$$

$$s^{\sigma+2} + t^\sigma + st = y_1 t + z_2. \tag{2.28}$$

The lines that produce these cases are the lines with dual coordinates $[a_1 + z_2\epsilon, y_1, 1]$. These lines are concurrent at the point $(0, 1, y_1)$ which lies on ℓ_∞. We will show in Corollary 2.5 that these latter lines meet $\tau_P(\mathcal{U}_{\mathrm{BT}})$ in at most four points.

We require the following lemma, which adapts arguments found in [15, Lemma 2.1].

Lemma 2.6. *Let $\mathcal{O}$ be a translation oval in $\mathrm{PG}(2, q)$ projectively equivalent to $\mathcal{D}_\sigma$, and let $\mathcal{C}$ be a non-degenerate conic. If the nucleus of $\mathcal{O}$ is also the nucleus of $\mathcal{C}$, then $|\mathcal{O} \cap \mathcal{C}| \leq 4$.*

Proof. Without loss of generality we may take $\mathcal{O} = \mathcal{D}_\sigma$, so that the nucleus of $\mathcal{O}$ is $N = (0, 1, 0)$. If N is also the nucleus of $\mathcal{C}$, then $\mathcal{C}$ is a conic of the following form,

$$a_1x^2 + a_2y^2 + a_3z^2 + xz = 0, \tag{2.29}$$

for some $a_1, a_2, a_3 \in \mathbb{F}_q$ with $a_2 \neq 0$. Suppose that $(0, 0, 1) \notin \mathcal{C}$. Then $a_3 \neq 0$, and the point $(1, t, t^\sigma) \in \mathcal{O}$ if and only if t satisfies

$$a_1 + a_2t^2 + a_3t^{2\sigma} + t^\sigma = 0, \tag{2.30}$$

that is

$$0 = \left(a_1 + a_2t^2 + a_3t^{2\sigma} + t^\sigma\right)^{\sigma/2} = a_1^{\sigma/2} + a_2^{\sigma/2}t^\sigma + t^2 + t. \tag{2.31}$$

Therefore,

$$t^\sigma = \left(\frac{a_3}{a_2}\right)^{2^e} t^2 + \frac{1}{a_2^{2^e}}t + \left(\frac{a_1}{a_2}\right)^{2^e} \tag{2.32}$$

and substituting into equation (2.30), we find that this equation has at most four solutions. If instead $(0, 0, 1) \in \mathcal{C}$, then $a_3 = 0$ and arguing as above we find that equation (2.30) has at most two solutions, so $|\mathcal{O} \cap \mathcal{C}| \leq 3$. □

Corollary 2.5. *The feet of a point $P \notin (\ell_\infty \cup \mathcal{U}_{\mathrm{BT}})$ meet a line ℓ in at most four points.*

Proof. From Lemma 2.5, we know we can restrict ourselves to the case $b_2 = 0, y_1 = b_1, a_2 = z_2$ which means we are looking at the points $P_{r,s,t} \in \tau_P(\mathcal{U}_{\mathrm{BT}}) \cap \ell$ have s, t satisfying

$$s^2 + \delta t^2 + st = y_1t + a_1 \tag{2.33}$$

$$s^{\sigma+2} + t^\sigma + st = y_1t + z_2, \tag{2.34}$$

where equation (2.33) represents a conic $\mathcal{C}$, and equation (2.34) represents an oval $\mathcal{O}$ in $\mathrm{AG}(2, q)$. If the conic is degenerate, it's easy to see that the oval and conic have at most four points in common. So we may assume that the conic is non-degenerate. The nucleus of $\mathcal{C}$ is $N = (y_1, 0, 1)$. We now show that N is the nucleus of the oval $\mathcal{O}$. The line $t = 0$ goes through N and meets the oval (2.34) when $s^{\sigma+2} = z_2$, which has one solution as $\sigma + 2$ is a permutation of $\mathbb{F}_q$. The line $s + y_1 = 0$ through N meets the oval (2.34) when $t^\sigma = y^{\sigma+2} + z_2$ which has one solution for t. Therefore, N is the nucleus, as it is the intersection of two tangent lines to the oval. It now follows from Lemma 2.6 that equations (2.33) and (2.34) have at most four common solutions. □

We now show the existence of a point $P \notin (\mathcal{U}_{\mathrm{BT}} \cup \ell_\infty)$ and a line ℓ such that $|\ell \cap \tau_P(\mathcal{U}_{\mathrm{BT}})| = 3$, and demonstrate that our bound is sharp.

Lemma 2.7. *Let $y_1 = 0$, then the points of the oval given by equation* (2.34) *are*

$$\left\{ P_u = \left(\frac{z_2^{1-\sigma/2} u^\sigma}{1+u+u^\sigma}, \frac{z_2^{\sigma/2}(1+u^\sigma)}{1+u+u^\sigma} \right) \middle| u \in \mathbb{F}_q \right\} \cup \left\{ \left(z_2^{1-\sigma/2}, z_2^{\sigma/2} \right) \right\}. \tag{2.35}$$

Proof. If $y_1 = 0$, then equation (2.34) reduces to

$$s^{\sigma+2} + t^\sigma + st + z_2 = 0 \tag{2.36}$$

Using the properties of σ described in Section [[*Prior Results]], one can show the point $(z_2^{1-\sigma/2}, z_2^{\sigma/2})$ satisfies equation (2.36). Furthermore, the points $\overline{P_u} = (z_2^{1-\sigma/2}u^\sigma, z_2^{\sigma/2}(1 + u^\sigma), 1 + u + u^\sigma)$, where $u \in \mathbb{F}_q$, are projective points satisfying the following homogeneous equation

$$x^{\sigma+2} + y^\sigma z^2 + xyz^\sigma + z_2 z^{\sigma+2} = 0. \tag{2.37}$$

Because $\mathrm{Tr}(u + u^\sigma) = 0$, and $\mathrm{Tr}(1) = 1$ when $q = 2^{2e+1}$, we have $u^\sigma + u + 1 \neq 0$ for all $u \in \mathbb{F}_q$. Thus, normalising so $z = 1$, the points $\overline{P_u}$ have the form $(s, t, 1)$ where s and t satisfy equation (2.36). □

Corollary 2.6. *Let $y_1 = 0$ and consider the points P_u as described in Lemma 2.7. A point P_u lies on the conic given by equation* (2.33)*, if and only if u is a root of the following polynomial*

$$a_1^{\sigma/2}u^\sigma + (z_2^{\sigma-1} + \delta^{\sigma/2}z_2 + z_2^{\sigma/2} + a_1^{\sigma/2})u^2 + z_2^{\sigma/2}u + \delta^{\sigma/2}z_2 + a_1^{\sigma/2} \tag{2.38}$$

Proof. By directly substituting P_u into equation (2.33) we have

$$(z_2^{2-\sigma} + \delta z_2^\sigma + z_2 + a_1)u^{2\sigma} + z_2 u^\sigma + a_1 u^2 + (\delta z_2^\sigma + a_1) = 0 \tag{2.39}$$

Raising both sides of equation (2.39) to the power of $\sigma/2$ yields our result. □

Theorem 2.3. *Let U be a Buekenhout-Tits unital in* $\mathrm{PG}(2, q^2)$*. The feet of a point $P \notin (\ell_\infty \cup U)$ meet a line ℓ in at most four points. Moreover, there exists a line ℓ and point P such that $|\ell \cap \tau_P(U)| = k$ for each $k \in \{0, 1, 2, 3, 4\}$.*

Proof. By Theorem 2.1 we may assume that $U = \mathcal{U}_{\mathrm{BT}}$. The first part of the proof comes from Corollary 2.5. Let $P = (1, y_1, z_2\epsilon)$. All lines through P meet $\tau_P(U)$ in at most one point by definition, so it is clear that there exists lines ℓ such that $|\ell \cap \tau_P(U)|$ is zero or one. Because the points of $\tau_P(U)$ are not collinear, there exists a pair of points $Q, R \in \tau_P(U)$ such that the line QR does not contain $(0, 1, y_1)$. Hence, the line QR meets in precisely two points by Lemma 2.5.

Now consider a line ℓ with equation $(\delta + \epsilon)x + z = 0$ and let P be the point $(1, 0, \epsilon)$ (that is, $a_1 = \delta, a_2 = 1, b_1 = b_2 = y_1 = 0, z_2 = 1$). The number of points of $\ell \cap \tau_P(U)$ is the same as the number of solutions to equations (2.27) and (2.28). By Lemma 2.7 the points P_u satisfying equation (2.28) lie on the conic (2.27) when

$$\delta^{\sigma/2}u^{\sigma} + u = u(\delta^{\sigma/2}u^{\sigma-1} + 1) = 0, \tag{2.40}$$

which has two roots as $\sigma - 1$ is a permutation of $\mathbb{F}_q$. It can also be shown that $(z_2^{1-\sigma/2}, z_2^{\sigma/2}) = (1, 1)$ satisfies both equations. Hence, the intersection of the feet of the point $(1, 0, \epsilon)$ and ℓ has exactly three points.

Finally, consider the point $P(1, 0, \frac{1}{\delta^\sigma}\epsilon)$ and the line ℓ with dual coordinates $[\frac{1}{\delta} + \frac{1}{\delta^2}\epsilon, 0, 1]$. By Corollary 2.6, the number of feet of P on the line ℓ is the number of solutions to the equation (2.38), where $a_1 = \frac{1}{\delta}$ and $z_2 = \frac{1}{\delta^\sigma}$ which is

$$\frac{1}{\delta^{\sigma/2}}u^{\sigma} + (\frac{1}{\delta^{2-\sigma}} + \frac{1}{\delta})u^2 + \frac{1}{\delta}u = 0. \tag{2.41}$$

Since equation (2.41) is a $\mathbb{F}_2$-linearised polynomial, and there are at most 4 roots, we have that equation (2.38) has $1, 2$, or 4 roots. We will show that, under the condition $\mathrm{Tr}(\delta) = 1$, it has four roots. Multiplying equation (2.41) by δ yields $\delta^{1-\sigma/2}u^{\sigma} + (\delta^{\sigma-1} + 1)u^2 + u = 0$ and now substituting $a = \delta^{\sigma-1} + 1$ gives

$$(a^{\sigma/2} + 1)u^{\sigma} + au^2 + u = 0. \tag{2.42}$$

We find that $u = 0$ and $u = \frac{1}{a^{1+\sigma/2}}$ are solutions to equation (2.42). Now consider

$$u^{\sigma} + au^2 + 1 = 0. \tag{2.43}$$

Any solution to equation (2.43) also satisfies $(u^{\sigma} + au^2 + 1)^{\sigma/2} + u^{\sigma} + au^2 + 1 = 0$ which is precisely equation (2.42). Multiply equation (2.43) with $a^{\sigma+1}$, then we find $(a^{\sigma/2+1}u)^{\sigma} + (a^{\sigma/2+1}u)^2 + a^{\sigma+1} = 0$, and letting $z = (a^{\sigma/2+1}u)^2$,

$$z^{\sigma/2} + z + a^{\sigma+1} = 0, \tag{2.44}$$

which is known (see [32]) to have solutions if and only if $\mathrm{Tr}(a^{\sigma+1}) = 0$. As $z = 0$ and $z = 1$ are not solutions of equation (2.44), no solutions of equation (2.44) correspond to the solutions $u = 0$ or $u = \frac{1}{a^{1+\sigma/2}}$ of equation (2.41). Furthermore, recall that equation (2.41) has $1, 2$ or 4 solutions and that we have assumed that $\mathrm{Tr}(\delta) = 1$. Since $\delta^{\sigma-1} = a + 1$, it follows that $\delta = (a + 1)^{\sigma+1}$ and $\mathrm{Tr}(\delta) = \mathrm{Tr}(a^{\sigma+1} + a^{\sigma} + a + 1) = \mathrm{Tr}(a^{\sigma+1}) + \mathrm{Tr}(1) = \mathrm{Tr}(a^{\sigma+1}) + 1$. Hence, the conditions $\mathrm{Tr}(\delta) = 1$ and $\mathrm{Tr}(a^{\sigma+1}) = 0$ are equivalent, and we find exactly four roots to equation (2.41). $\square$

Chapter 3

Buekenhout-Metz Unitals

This chapter covers Buekenhout-Metz unitals in detail. The goals of the chapter are to:

1. Summarise our current understanding of Buekenhout-Metz unitals.
2. Examine recent work on the feet of Buekenhout-Metz unitals, and approach the problem with the same techniques we employed for Buekenhout-Tits unitals.
3. Show some new (albiet small) results on the intersection of Buekenhout-Metz unitals with ovoidal-Buekenhout-Metz unitals.

3.1 Prior Results

Baker and Ebert [4] find explicit coordinates of Buekenhout-Metz unitals in planes of odd order. They also establish that not all Buekenhout-Metz unitals are equivalent, and give a standard representation for the unitals in each projective equivalence class. We now present their work.

Lemma 3.1. *Let ζ be a primitive element of $\mathbb{F}_{q^2}$ for q odd and let $\epsilon = \zeta^{(q+1)/2}$. Then $\epsilon^q + \epsilon = 0$ and $\epsilon^2 = \omega$ is a primitive element of $\mathbb{F}_q$.*

Proof. Let ζ be a primitive element of $\mathbb{F}_{q^2}$ and $\epsilon = \zeta^{(q+1)/2}$. Then

$$x^2 - \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\zeta) \tag{3.1}$$

is a polynomial over $\mathbb{F}_q$ whose roots are precisely $\pm\epsilon$. Hence, $\epsilon^q = \pm\epsilon$. Because $q+1 \nmid (q+1)/2$, we know $\epsilon \notin \mathbb{F}_q$ so $\epsilon^q = -\epsilon$. Finally, because ζ is a primitive element of $\mathbb{F}_{q^2}$, we have $\epsilon^2 = \zeta^{q+1}$ is a primitive element of $\mathbb{F}_q$. $\square$

Result 3.1 ([4])**.** *Let* $U_{\alpha,\beta} = \{(x, \alpha x^2 + \beta x^{q+1} + r, 1) \mid x \in \mathbb{F}_{q^2}, r \in \mathbb{F}_q\} \cup \{(0,1,0)\}$*. Then* $U_{\alpha,\beta}$ *is a Buekenhout-Metz unital of* $\mathrm{PG}(2, q^2)$ *for* q *odd if and only if* $(\beta^q - \beta)^2 + 4\alpha^{q+1} \in \mathbb{F}_q$ *is non-square.*

Proof. Let $H_\infty : x_4 = 0$ be a fixed hyperplane of $\mathrm{PG}(4, q)$, and S to be the regular spread of H_∞ obtained by field reduction of $\mathrm{PG}(1, q^2)$. By Result 1.39, if $U_{\alpha,\beta}$ has a representation in the ABB construction as an elliptic cone tangent to H_∞, then $U_{\alpha,\beta}$ is a unital. Thus, we shall show that the ABB representation of $U_{\alpha,\beta}$ is an elliptic cone tangent to H_∞ if and only if $(\beta^q - \beta)^2 + 4\alpha^{q+1}$ is a non-square element of $\mathbb{F}_q$.

Let ζ be a primitive element of $\mathbb{F}_{q^2}$. Then, by Lemma 3.1, the element $\epsilon = \zeta^{(q+1)/2}$ satisfies $\epsilon^q = -\epsilon$ and $\epsilon^2 = \omega$ is a primitive element of $\mathbb{F}_q$. We can see that the point $(x, y, 1)$ lies on $U_{\alpha,\beta}$ if and only if $\alpha x^2 + \beta x^{q+1} - y \in \mathbb{F}_q$. Express x, y, α and β with respect to the basis $\{1, \epsilon\}$ of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ as $x_0 + x_1\epsilon$, $y_0 + y_1\epsilon$, $a_0 + a_1\epsilon$, and $b_0 + b_1\epsilon$. Then, let $H_\infty : x_4 = 0$ and S the spread of H_∞ obtained by field reduction from $\mathrm{PG}(1, q^2)$. A point $(x, y, 1)$ such that $\alpha x^2 + \beta x^{q+1} - y \in \mathbb{F}_q$ is therefore represented in the ABB construction as the point $(x_0, x_1, y_0, y_1, 1)$ of $\mathrm{PG}(4, q)$ such that:

$$a_1 x_0^2 + a_1 x_1^2 \omega + 2a_0 x_0 x_1 + b_1 x_0^2 - b_1 x_1^2 \omega + y_1 = 0. \tag{3.2}$$

We obtain condition (3.2) by expanding $\alpha x^2 + \beta x^{q+1} - y$ over the basis $\{1, \epsilon\}$ as $r_0 + r_1\epsilon$ and setting $r_1 = 0$. The line in H_∞ representing $(0, 1, 0)$ is $\langle (0, 0, 1, 0, 0), (0, 0, 0, 1, 0) \rangle$. Therefore, the points of $U_{\alpha,\beta}$ are represented in the ABB construction as a set of points (x_0, x_1, y_0, y_1, z) satisfying the homogeneous polynomial

$$a_1 x_0^2 + a_1 x_1^2 \omega + 2a_0 x_0 x_1 + b_1 x_0^2 - b_1 x_1^2 \omega + y_1 z = 0. \tag{3.3}$$

Based on equation (3.3), we can see these points form a cone with vertex $(0, 0, 1, 0, 0)$ having generator lines $\langle (0, 0, 1, 0, 0), (x_0, x_1, 0, y_1, z) \rangle$ where x_0, x_1, y_1, z satisfy equation (3.3). This cone is an elliptic cone if and only if the quadratic form

$$f(x_0, x_1) = a_1 x_0^2 + a_1 x_1^2 \omega + 2a_0 x_0 x_1 + b_1 x_0^2 - b_1 x_1^2 \omega \tag{3.4}$$

is irreducible. The discriminant of this form is $4a_0^2 - 4(a_1 + b_1)(a_1 - b_1)\omega = 4(a_0 - a_1^2\omega) + 4b_1^2\omega = (\beta^q - \beta)^2 + 4\alpha^{q+1}$ and the result follows. $\square$

For q even, we get a different discrimination condition from the same analysis. Let $\epsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ be a solution of $\epsilon^q + \epsilon + 1 = 0$, such that $\epsilon^2 + \epsilon + \delta = 0$ for some $\delta \neq 1$ with trace one. Then $(x + y\epsilon)^q = y$ and $(x + y\epsilon)^{q+1} = x^2 + xy + y^2\delta$. The following theorem (see [21]) describes when $U_{\alpha,\beta} = \{(x, \alpha x^2 + \beta x^{q+1} + r, 1) \mid x \in \mathbb{F}_{q^2},\ r \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2, q^2)$.

Result 3.2 ([21])**.** *Let $q \geq 4$ be an even prime power, then $U_{\alpha,\beta}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2,q^2)$ if and only if $\beta \notin \mathbb{F}_q$ and $d = \alpha^{q+1}/(\beta^q+\beta)^2$ has trace zero.*

Proof. By the same analysis as in Result 3.1, the point set $U_{\alpha,\beta}$ is represented as an elliptic cone in $\mathrm{PG}(4,q)$ if and only if

$$f(x_0,x_1) = b_1x_0^2 + b_1x_0x_1 + (a_0+a_1+a_1\delta+b_1\delta)x_1^2 = 0 \tag{3.5}$$

is an irreducible quadratic form. The discriminant condition for equation (3.5) to be irreducible is

$$\mathrm{Tr}((a_1+b_1)(a_0+a_1+a_1\delta+b_1\delta)/b_1^2) = 1. \tag{3.6}$$

Now, using the fact that $\mathrm{Tr}(x^2+x) = \mathrm{Tr}(x^2)+\mathrm{Tr}(x) = 0$ we see

$$\begin{aligned}
\mathrm{Tr}\left(\frac{(a_1+b_1)(a_0+a_1+a_1\delta+b_1\delta)}{b_1^2}\right) &= \mathrm{Tr}\left(\frac{a_0a_1+a_0b_1+a_1b_1+a_1^2+(a_1^2+b_1^2)\delta}{b_1^2}\right) && (3.7)\\
&= \mathrm{Tr}\left(\frac{a_0a_1+a_0b_1+a_1b_1+a_1^2+(a_1^2+b_1^2)\delta}{b_1^2}\right)\\
&\quad + \mathrm{Tr}\left(\frac{a_0+a_1}{b_1}+\left(\frac{a_0+a_1}{b_1}\right)^2\right) && (3.8)\\
&= \mathrm{Tr}\left(\frac{a_0^2+a_0a_1+(a_1^2+b_1^2)\delta}{b_1^2}\right) && (3.9)\\
&= \mathrm{Tr}\left(\frac{\alpha^{q+1}+(\beta^q+\beta)^2\delta}{(\beta^q+\beta^2)^2}\right) && (3.10)\\
&= \mathrm{Tr}\left(\frac{\alpha^{q+1}}{(\beta^q+\beta^2)^2}+\delta\right) && (3.11)\\
&= \mathrm{Tr}\left(\frac{\alpha^{q+1}}{(\beta^q+\beta^2)^2}\right)+1. && (3.12)
\end{aligned}$$

Hence equation (3.6) is equivalent to $\mathrm{Tr}\left(\frac{\alpha^{q+1}}{(\beta^q+\beta^2)^2}\right) = 0$. □

The stabiliser of Buekenhout-Metz unitals is computed separately for q odd [4] and q even [21]. We summarise their results here without proof.

Result 3.3 ([4])**.** *Let $q = p^e$ be an odd prime power, and G the $\mathrm{P\Gamma L}$-stabiliser of a Buekenhout-Metz unital $U_{\alpha,\beta}$ in $\mathrm{PG}(2,q^2)$. Let $d = (\beta^q-\beta)^2/4\alpha^{q+1}$, and $m = [\mathbb{F}_{q^2} : \mathbb{F}_p(d)]$. Then*

1. *The unital $U_{\alpha,\beta}$ is classical and $G \equiv \mathrm{P\Gamma U}(3,q^2)$ (and so $|G| = mq^3(q^2-1)$) or;*
2. *We have $\beta \in \mathbb{F}_q$ and $|G| = 2mq^3(q-1)$;*
3. *Otherwise, $|G| = mq^3(q-1)$.*

The homography subgroup $G_0 = G \cap \mathrm{PGL}(3, q^2)$ *has index* m *in* G*. The group* G *acts transitively on points of* $U_{\alpha,\beta} \setminus P_\infty$*, acts transitively on points of* $\ell_\infty \setminus \{P_\infty\}$*, and has either one or two orbits on* $\mathrm{PG}(2, q^2) \setminus (U_{\alpha,\beta} \cup \ell_\infty)$*, where* P_∞ *has coordinates* $(0, 1, 0)$ *and* ℓ_∞ *has dual coordinates* $[1, 0, 0]$*.*

For q even we have fewer cases.

Result 3.4 ([21])**.** *Let* $q \geq 4$ *be an even prime power, and* $U_{\alpha,\beta}$ *be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$*. Let* G *be the* $\mathrm{P\Gamma L}$*-stabiliser of* $U_{\alpha,\beta}$*. Then if* $d = \alpha^{q+1}/(\beta^q + \beta)^2$ *and* $m = [\mathbb{F}_{q^2} : \mathbb{F}_2(d)]$*, either*

1. *The unital* $U_{\alpha,\beta}$ *is classical and* $G \equiv \mathrm{P\Gamma U}(3, q^2)$ *(and so* $|G| = mq^3(q^2 - 1)$*) or;*
2. *The order of* G *is* $mq^3(q - 1)$*.*

The group $G_0 = G \cap \mathrm{PGL}(3, q^2)$ *has index* m *in* G*. The group* G *acts transitively on* $U_{\alpha,\beta} \setminus P_\infty$*, acts transitively on* $\ell_\infty \setminus \{P_\infty\}$*, and has one orbit on* $\mathrm{PG}(2, q^2) \setminus (U_{\alpha,\beta} \cup \ell_\infty)$*, where* P_∞ *has coordinates* $(0, 1, 0)$ *and* ℓ_∞ *has dual coordinates* $[1, 0, 0]$*.*

The difference in the possible group orders for q odd and q even motivates investigating $U_{\alpha,\beta}$ in $\mathrm{PG}(2, q^2)$ with q odd and $\beta \in \mathbb{F}_q$. In this case we can say something about the structure of the unital.

Result 3.5 ([8])**.** *Let* $U_{\alpha,\beta}$ *be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$ *with* q *odd. Then if* $\beta \in \mathbb{F}_q$*, the unital* $U_{\alpha,\beta}$ *is a union of irreducible conics.*

Proof. Ebert [4] demonstrates that all unitals $U_{\alpha,\beta}$ with $\beta \in \mathbb{F}_q$ are equivalent to $U_{\alpha,0}$, which is easily seen to be the union of the conics $\mathcal{C}_r : \alpha x^2 + rz^2 - yz$ for all $r \in \mathbb{F}_q$. These conics mutually intersect at the point $(0, 1, 0)$. $\square$

Note that a Buekenhout-Metz unital $U_{\alpha,\beta}$ in $\mathrm{PG}(2, q^2)$ with q even cannot contain any oval as the q^2+1 tangents to the oval would also be tangents to $U_{\alpha,\beta}$ and would be concurrent at a point which contradicts the fact that each point must lie on either one or $q + 1$ tangent lines.

3.2 Feet of Buekenhout-Metz Unitals

Ebert [8] presents a proof that the feet a point P of a non-classical Buekenhout-Metz unital $U_{\alpha,\beta}$ are collinear if and only if $P \notin U_{\alpha,\beta} \cup \ell_\infty$. We build on this work to describe how many points of $\tau_P(U_{\alpha,\beta})$ may lie on a line ℓ. Firstly, we need a description of the tangent lines to $U_{\alpha,\beta}$ — which we give without proof and refer to [8].

Lemma 3.2 ([8]). *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2,q^2)$ *for some prime power q (even or odd). Then, the tangent line to the point $P \in U_{\alpha,\beta}$ with homogeneous coordinates $(x, \alpha x^2 + \beta x^{q+1} + r, 1)$ has dual coordinates $[-2\alpha x + (\beta^q - \beta)x^q, 1, \alpha x^2 - \beta^q x^{q+1} - r]$.*

Result 3.6 ([8]). *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2,q^2)$*, where q is odd. Then the feet of a point $P \notin U_{\alpha,\beta} \cup \ell_\infty$ are not collinear unless $\alpha = 0$.*

Proof. Let ϵ be a primitive element of $\mathbb{F}_{q^2}$ such that $\epsilon^q = -\epsilon$ and $\epsilon^2 = \omega$. The PΓL-stabiliser of $U_{\alpha,\beta}$ has at most two orbits on $\mathrm{PG}(2,q^2) \setminus (U_{\alpha,\beta} \cup \ell_\infty)$, with representatives $(0, \epsilon, 1)$ and $(0, \omega\epsilon, 1)$. We consider only $P = (0, \epsilon, 1)$ as the other case is completely analogous. By Lemma 3.2, the $q+1$ tangent lines through P have dual coordinates of the form $[-2\alpha x + (\beta^q - \beta)x^q, 1, \alpha x^2 - \beta^q x^{q+1} - r]$ where $x \in \mathbb{F}_{q^2}$ and $r = \epsilon + \alpha x^2 - \beta^q x^{q+1} \in \mathbb{F}_q$, and the associated point on this tangent line is $(x, 2\alpha x^2 + (\beta^q - \beta)x^{q+1} + \epsilon, 1)$. Expressing $\alpha = a_0 + a_1\epsilon$, $\beta = b_0 + b_1\epsilon$ and $x = x_0 + x_1\epsilon$, and expanding each coordinate of $(x, 2\alpha x^2 + (\beta^q - \beta)x^{q+1} + \epsilon, 1)$ with respect to the basis $\{1, \epsilon\}$, we see that the feet of P are a subset of the points

$$(x_0 + x_1\epsilon, 2a_0x_0^2 + 2a_0\omega x_1^2 + 4a_1\omega x_0x_1 + \epsilon, 1) \tag{3.13}$$

where $x_0, x_1, a_0, a_1 \in \mathbb{F}_q$. Then by letting $r = \epsilon + \alpha x^2 - \beta^q x^{q+1} = r_0 + r_1\epsilon$, with $r_0, r_1 \in \mathbb{F}_q$, and setting $r_1 = 0$ so that $r \in \mathbb{F}_q$, we see that feet of P are points in the form of equation 3.13 such that

$$(a_1 + b_1)x_0^2 + (a_1 - b_1)\omega x_1^2 + 2a_0x_0x_1 + 1 = 0. \tag{3.14}$$

The condition on the feet, regarded as a equation in x_0 and x_1, represents an ellipse in $\mathrm{AG}(2,q)$ as it has $q+1$ solutions for each point of $\tau_P(U_{\alpha,\beta})$, and none of these points lie at infinity.

We shall proceed to show that no line can contain all the points of $\tau_P(U_{\alpha,\beta})$. Suppose that a line ℓ with dual coordinates $[s_0 + s_1\epsilon, 1, t_0 + t_1\epsilon]$ contained the set $\tau_P(U_{\alpha,\beta})$. Then a point with coordinates in the form of (3.13) meets ℓ if and only if $s_1x_0 + s_0x_1 + t_1 - 1 = 0$. Thus if ℓ contains $\tau_P(U_{\alpha,\beta})$, then

$$(a_1 + b_1)x_0^2 + (a_1 - b_1)\omega x_1^2 + 2a_0x_0x_1 + 1 = 0, \tag{3.15}$$

$$2a_0x_0^2 + 2a_0\omega x_1^2 + 4a_1\omega x_0x_1 + s_0x_0 + s_1\omega x_1t_0 = 0, \tag{3.16}$$

$$s_1x_0 + s_0x_1 + t_1 - 1 = 0, \tag{3.17}$$

for all feet of $(0, \epsilon, 1)$. If $\alpha = 0$, $s_0 = s_1 = 0$, $t_1 = 1$ and $t_0 = 0$, then equations (3.16) and (3.17) vanish. So if $\alpha = 0$ the feet of $(0, \epsilon, 1)$ lie on the line ℓ with dual coordinates $[0, 1, \epsilon]$. Now assume that $\alpha \neq 0$. Regarding equation (3.16) and equation (3.17) as equations in x_0 and x_1, we see that equation (3.16) represents a conic in $\mathrm{AG}(2,q)$, and equation (3.17) a line of $\mathrm{AG}(2,q)$. If ℓ contains $\tau_P(U_{\alpha,\beta})$, we need $q+1$ solutions to equations (3.16), (3.15) and

(3.17) simultaneously. If equation (3.17) does not vanish, it has at most two solutions in common with equation (3.15). Thus, $s_0 = s_1 = 0$ and $t_1 = 1$. Hence to have $q+1$ common solutions, equation (3.16) must represent the same ellipse as equation (3.15). Comparing equation (3.16) with equation (3.15) we have $a_0 t_0 = 2\omega a_1$ and $a_1 t_0 = 2\omega a_0$. Because $\alpha \neq 0$ we find $a_0 \neq 0$ and $a_1 \neq 0$ so $2a_0/a_1 = 2a_1\omega/a_0$ and therefore $a_0^2 - \omega a_1^2 = 0$. This last condition is equivalent to $\alpha^{q+1} = 0$ which contradicts $\alpha \neq 0$. So no such line contains the feet of P.

Now suppose that ℓ has the form $[1, 0, t_0 + t_1\epsilon]$. Then, we have $x_0 = -t_0$ and $x_1 = -t_1$. So lines of this form contain at most one point of the feet of P. Thus, no line ℓ may contain the feet of $\tau_P(U_{\alpha,\beta})$, unless $\alpha = 0$, and the result follows. □

We have the following immediate corollary.

Result 3.7. *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$*, with q odd. Then, $U_{\alpha,\beta}$ is classical if and only if $\alpha = 0$.*

In fact, similar analysis will show that this result holds for q even too.

Result 3.8. *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$*, with q even. Then, $U_{\alpha,\beta}$ is classical if and only if $\alpha = 0$.*

Abarzua, Pomareda and Vega [1] further improve Ebert's results, to provide the following bound on the feet of Buekenhout-Metz unitals, similar to the work in Section 2.4.

Result 3.9 ([1])**.** *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital, and $P \notin U_{\alpha,\beta} \cup \ell_\infty$, then a line m meets $\tau_P(U_{\alpha,\beta})$ in 0, 2, or 4 points.*

This result compares to Theorem 2.3 in the notable exclusion of a line that meets the feet of a point of $U_{\alpha,\beta}$ in at most three points.

Theorem 3.1. *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$ *with q odd and $\beta \in \mathbb{F}_q$. Then the feet of a point $P \notin U_{\alpha,\beta} \cup \ell_\infty$ lie on a conic, and in particular are an arc.*

Proof. In this case, the results of Theorem 3.3 tell us the points of $\mathrm{PG}(2, q^2) \setminus (U_{\alpha,\beta} \cup \ell_\infty)$ form a single orbit under the stabiliser group of $U_{\alpha,\beta}$, so without loss of generality we let $P = (0, \epsilon, 1)$. Consider the argument deriving equation 3.13 we find the feet form a subset of the points $(x, 2\alpha x^2 - \epsilon, 1)$ which lie on the conic $2\alpha x^2 + \epsilon z^2 - yz = 0$. The result now follows. □

The results in [1] on the feet of Buekenhout-Metz unitals are derived without reference to Ebert's original proof that the feet of affine points are not collinear. Using the same techniques as in Section 2.4 however, we can derive a part of the work in [1] in a straight forward fashion.

Theorem 3.2. *Let $U_{\alpha,\beta}$ be a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$ *with q odd. Then the feet of a point $P \notin U_{\alpha,\beta} \cup \ell_\infty$ meet any line m in at most four points. Moreover, all lines m that meet $\tau_P(U_{\alpha,\beta})$ in four points are concurrent.*

Proof. Consider again the system of equations that Ebert produces in Theorem 3.6.

$$(a_1 + b_1)x_0^2 + (a_1 - b_1)\omega x_1^2 + 2a_0x_0x_1 + 1 = 0, \tag{3.18}$$

$$2a_0x_0^2 + 2a_0\omega x_1^2 + 4a_1\omega x_0x_1 + s_0x_0 + s_1\omega x_1 + t_0 = 0, \tag{3.19}$$

$$s_1x_0 + s_0x_1 + t_1 - 1 = 0. \tag{3.20}$$

We can firstly see that because the equation (3.18) represents an ellipse and equation (3.20) a line in $\mathrm{AG}(2, q^2)$ unless it vanishes the whole system has no more that two solutions. If the line vanishes, we have $s_0 = s_1 = 0$ and $t_1 = 1$ so that the line $[s_0 + s_1\epsilon, t_0 + t_1\epsilon, 1]$ must contain $(1, 0, 0)$. It then follows that because equation (3.18) is an irreducible conic, if equation (3.19) is reducible the system has at most four points as the intersection of two lines with an irreducible conic contains at most four points. Lastly if equation (3.19) is irreducible then equations (3.18) and (3.19) have at most four common solutions. □

Recently Barwick, Jackson, and Wild [6], using extensions of the techniques we employ in Theorem 3.2 expands our understanding of the feet of Buekenhout-Metz unitals.

Result 3.10 ([6])**.** *Let $U_{\alpha,\beta}$ be a non-classical Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$*, with q odd. Let $P \notin U_{\alpha,\beta} \cup \ell_\infty$. Then, the feet $\tau_P(U_{\alpha,\beta})$ are contained in a Baer pencil with vertex a point on ℓ_∞. Moreover,*

1. *if α is non-square in $\mathbb{F}_{q^2}$, then $\tau_P(U_{\alpha,\beta})$ is an arc;*
2. *if α is a non-zero square in $\mathbb{F}_{q^2}$, then*
   - *(a) the feet $\tau_P(U_{\alpha,\beta})$ form a set of class* $(0, 1, 2, 4)$*,*
   - *(b) there are at least $(q - 3)/2$ lines meeting $\tau_P(U_{\alpha,\beta})$ in four points,*
   - *(c) every* 4*-secant of $\tau_P(U_{\alpha,\beta})$ contains the point vertex of the Baer pencil containing $\tau_P(U_{\alpha,\beta})$.*

Remark 3.1. *In Result 3.10, a set of* class $(a_1, a_2, \ldots, a_m)$ *is a set of points S in* $\mathrm{PG}(2, q)$ *such that for all lines ℓ, $|\ell \cap S| \in \{a_1, a_2, \ldots, a_m\}$ and for each a_i there exists a line ℓ such that $|\ell \cap S| = a_i$. For comparison, the feet of Buekenhout-Tits unitals either lie on a line or are a set of class* $(0, 1, 2, 3, 4)$*.*

Result 3.11 ([6])**.** *Let $U_{\alpha,\beta}$ be a non-classical orthogonal Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$*, with q even. Let $P \notin U_{\alpha,\beta} \cup \ell_\infty$. Then, the feet of P form an arc contained in a Baer subpencil with vertex a point on ℓ_∞.*

3.3 Intersections of Ovoidal-Buekenhout-Metz Unitals

The intersection of unitals is a mostly open problem. In this section we will slightly extend known results on the intersection of ovoidal-Buekenhout-Metz unitals.

The most general result on the intersection of unitals states that if $q = p^e$, then $|H \cap U| \equiv 1 \bmod p$ for all Hermitian unitals $\mathcal{H}$ and any unital U in $\mathrm{PG}(2, q^2)$.

Result 3.12 ([8])**.** *Let $\mathcal{H}$ be a Hermitian unital of $\mathrm{PG}(2, q^2)$ and U any unital, then $|\mathcal{H} \cap U| \equiv 1 \bmod p$.*

Proof. As $\mathcal{H}$ is a codeword in $\mathrm{PG}(2, q^2)$, we see that $\mathbf{v}^{\mathcal{H}} = \sum_{i=1}^{t} \mathbf{v}^{m_i}$ for some lines m_i. As $\mathbf{v}^{\mathcal{H}} \cdot \mathbf{v}^{\mathcal{H}} = |\mathcal{H}| = q^3 + 1$, we have

$$\mathbf{v}^{\mathcal{H}} \cdot \mathbf{v}^{\mathcal{H}} \equiv \mathbf{v}^{\mathcal{H}} \cdot \Big(\sum_{i=1}^{t} \mathbf{v}^{m_i}\Big) \mod p \tag{3.21}$$

$$\equiv \sum_{i=1}^{t} \mathbf{v}^{\mathcal{H}} \cdot \mathbf{v}^{m_i} \mod p \tag{3.22}$$

$$\equiv \sum_{i=1}^{t} 1 \mod p \tag{3.23}$$

$$\equiv t. \tag{3.24}$$

So $t \equiv 1 \bmod p$. Now,

$$\mathbf{v}^{U} \cdot \mathbf{v}^{\mathcal{H}} \equiv \mathbf{v}^{U} \cdot \Big(\sum_{i=1}^{t} \mathbf{v}^{m_i}\Big) \mod p \tag{3.25}$$

$$\equiv t \mod p \tag{3.26}$$

$$\equiv 1 \mod p. \tag{3.27}$$

□

To produce our novel result we require the following lemma concerning cones in $\mathrm{PG}(4, q)$.

Lemma 3.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be ovoidal cones with base an elliptic quadric in $\mathrm{PG}(4, q)$, both tangent to a hyperplane H_∞ at a shared generator line l. Then, there exists a collineation $\psi \in \mathrm{PGL}(5, q)$ mapping $\mathcal{C}$ to $\mathcal{C}'$ stabilising H_∞ and l.*

Proof. All ovoidal cones with base an elliptic quadric are projectively equivalent (this follows from the equivalence of elliptic quadrics in $\mathrm{PG}(3, q)$), so there exists a collineation $\psi \in \mathrm{PGL}(4, q)$ mapping $\mathcal{C}$ to $\mathcal{C}'$. If $\psi(l) = l$ then $\psi(H_\infty) = H_\infty$ as H_∞ is the unique tangent hyperplane at l for both $\mathcal{C}$ and $\mathcal{C}'$. Thus, assume that $\psi(l) \neq l$ and let H be a hyperplane such that $\mathcal{C}' \cap H = \mathcal{O}$ is an elliptic quadric. Then, as the stabiliser group of $\mathcal{O}$ is transitive on points of $\mathcal{O}$, there exists a collineation $\varphi \in \mathrm{PGL}(4, q)$ mapping $\psi(l) \cap \mathcal{O}$ to $l \cap \mathcal{O}$. There exists a collineation $\overline{\varphi} \in \mathrm{PGL}(5, q)$ fixing the vertex V' of the cone $\mathcal{C}'$ such that $\overline{\psi}\,|_H$ agrees with ψ. It then follows that $\overline{\varphi} \circ \psi$ is precisely a collineation of $\mathrm{PGL}(5, q)$ mapping $\mathcal{C}$ to $\mathcal{C}'$ stabilising l and H_∞. □

Recall from Section 1.9 that an ovoidal-Buekenhout-Metz unital is any unital constructed from the Buekenhout construction in $\mathrm{PG}(4, q)$ using an ovoidal cone tangent to a fixed hyperplane H_∞ with some ovoid of H_∞ as a base. The class of ovoidal-Buekenhout-Metz unitals includes all Buekenhout-Metz unitals as well as Buekenhout-Tits unitals. We make use of the behaviour of the collineation described in Lemma 3.3 to produce a novel result on the intersection of Buekenhout-Metz unitals with ovoidal-Buekenhout-Metz unitals, leveraging Result 3.12.

Theorem 3.3. *Let U be a Buekenhout-Metz unital, and U' an ovoidal-Buekenhout-Metz unital, both in* $\mathrm{PG}(2, q^2)$*. Suppose U meets U' in a point P such that:*

1. *the tangent line to U at P is also tangent to U';*

2. *secant lines of U containing P meet U in Baer sublines;*

3. *secant lines of U' containing P meet U' in Baer sublines.*

Then $|U \cap U'| \equiv 1 \bmod p$.

Proof. Without loss of generality, we may assume that U and U' meet at the point $P = (0, 1, 0)$ lying on $\ell_\infty : z = 0$, and that ℓ_∞ is tangent to both U and U'. Fix the hyperplane $H_\infty : x_4 = 0$, and the spread S obtained by field reduction from $\mathrm{PG}(1, q^2)$ into H_∞. Then under the ABB construction, the point P corresponds to the line $p = \langle (0, 0, 1, 0, 0), (0, 0, 0, 1, 0) \rangle$, and the unitals U and U' correspond to ovoidal cones $\mathcal{C}$ and $\mathcal{C}'$ tangent to H_∞ at p. By Lemma 3.3, there exists a collineation φ of $\mathrm{PG}(4, q)$ mapping $\mathcal{C}$ to the cone representing the Hermitian unital $U_{0,\beta}$ for some $\beta \in \mathbb{F}_{q^2}$, stabilising the line p and H_∞. The collineation φ induces a $\overline{\varphi}$ bijection of the points of $\mathrm{PG}(2, q^2) \setminus \ell_\infty$. Define the bijection ψ on points of $\mathrm{PG}(2, q^2)$

$$\psi(Q) = \begin{cases} Q & Q \in \ell_\infty, \\ \overline{\varphi}(Q) & \text{otherwise.} \end{cases} \tag{3.28}$$

Because ψ fixes P and agrees with $\overline{\varphi}$ at points $Q \notin \ell_\infty$, it follows that $\psi(U)$ is a set of points whose ABB representation is $\varphi(\mathcal{C})$ in $\mathrm{PG}(4, q)$. Likewise $\psi(U')$ is a set of points with ABB representation $\varphi(\mathcal{C}')$ in $\mathrm{PG}(4, q)$. Hence, $\psi(U)$ and $\psi(U')$ are both still unitals in $\mathrm{PG}(2, q^2)$

and, as $\varphi(U)$ is an ovoidal cone corresponding to a Hermitian unital, we also know that $\psi(U)$ is a Hermitian unital. It now follows from Theorem 3.12 that $|\psi(U) \cap \psi(U')| \equiv 1 \bmod p$ and so $|U \cap U'| \equiv 1 \bmod p$ as ψ is a bijection. $\square$

For the standard Buekenhout-Metz unitals $U_{\alpha,\beta}$ we have a stronger result. Note the condition on $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}|$ changes from $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}| \equiv 1 \bmod p$ in Theorem 3.3 to $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}| \equiv 1 \bmod q$.

Theorem 3.4. *For any pair of Buekenhout-Metz unitals $U_{\alpha,\beta}$ and $U_{\alpha',\beta'}$, we have $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}| \equiv 1 \bmod q$. Moreover for any $U_{\alpha,\beta}$ we have $|U_{\alpha,\beta} \cap U_{BT}| \equiv 1 \bmod q$, where U_{BT} is the Buekenhout-Tits unital given in Section 2.1.*

Proof. The unitals $U_{\alpha,\beta}$ and $U_{\alpha',\beta'}$ are unitals corresponding to ovoidal cones $\mathcal{C}$ and $\mathcal{C}'$ in $\mathrm{PG}(4,q)$ with a common vertex $V = (0,0,1,0,0)$. For each point $Q \in \mathcal{C} \cap \mathcal{C}'$ not in H_∞ : $x_4 = 0$ the line $QV \subseteq \mathcal{C} \cap \mathcal{C}'$. It now follows that $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}| = 1 + qk$ where k is the number of generator lines in common with $\mathcal{C}$ and $\mathcal{C}'$, so $|U_{\alpha,\beta} \cap U_{\alpha',\beta'}| \equiv 1 \bmod q$. The argument is similar for $U_{\alpha,\beta} \cap U_{BT}$. $\square$

Chapter 4

Kestenband Arcs

This chapter concerns *Kestenband arcs*, a family of complete $(q^2 - q + 1)$-arcs in $\mathrm{PG}(2, q^2)$ for $q > 2$. These arcs have a fascinating connection with Hermitian unitals, which we will discuss. In this chapter, we aim to:

1. Present a proof that all Kestenband arcs are equivalent. References to this result exist in the literature, we make the proof explicit.
2. Show that Hermitian unitals are codewords in $C_2(\mathrm{PG}(2, q^2))$ for q even using cyclic spreads and dual Kestenband arcs.
3. Show that a unital stabilised by a group of order $q^2 - q + 1$ is a Hermitian unital, using a clever method first presented by Giuzzi making use of Kestenband arcs.

4.1 Kestenband Arcs from Singer Groups

In this section we will construct Kestenband arcs as orbits of cyclic groups of order $q^2 - q + 1$. To do this, we first briefly introduce the *field model* of $\mathrm{PG}(2, q)$.

Let β be a generator of $\mathbb{F}_{q^3}^{\times}$. Then β^{q^2+q+1} is a generator for $\mathbb{F}_q^{\times}$. Let $\sim$ be an equivalence relation on $\mathbb{F}_{q^3}$, where $x \sim y$ if and only if $x = \beta^{i(q^2+q+1)} y$ for some i. Define $(x)_{\mathbb{F}_q}$ to be the equivalence class of $x \in \mathbb{F}_{q^3}$ under $\sim$.

We define the set Tr_α as the set of equivalence classes $(x)_{\mathbb{F}_{q^3}}$ such that $\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha x) = 0$. The set Tr_α is well-defined because if $(x)_{\mathbb{F}_{q^3}} \in \mathrm{Tr}_\alpha$, we have

$$\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta^{i(q^2+q+1)} \alpha x) = \beta^{i(q^2+q+1)} \mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha x) = 0,$$

so $\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha y) = 0$ for all $y \in (x)_{\mathbb{F}_q}$.

Result 4.1. *Let* $\mathcal{P} = \{(x)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^3}^{\times}\}$ *and* $\mathcal{L} = \{\mathrm{Tr}_\alpha \mid \alpha \in \mathbb{F}_{q^3}^{\times}\}$. *Then* $\Pi = (\mathcal{P}, \mathcal{L})$, *together with natural incidence, forms a Desarguesian projective plane of order* q.

Proof. To show that $\mathcal{P}$ and $\mathcal{L}$ form a Desarguesian projective plane, we find an incidence preserving bijection of points and lines to $\mathrm{PG}(2,q)$.

Let $\{1, \epsilon, \epsilon^2\}$ be a basis for $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$. Then, we can uniquely express $x \in \mathbb{F}_{q^3}$ as $x_0 + x_1\epsilon + x_2\epsilon^2$ for some $x_i \in \mathbb{F}_q$. Define $\varphi : \mathbb{F}_{q^3} \to (\mathbb{F}_q)^3$ to be the bijection

$$\varphi(x_0 + x_1\varepsilon + x_2\epsilon^2) = (x_0, x_2, x_3).$$

Clearly, φ is additive and $\mathbb{F}_q$-linear, and so it is an isomorphism of the $\mathbb{F}_q$-vector spaces $\mathbb{F}_{q^3}$ and $(\mathbb{F}_q)^3$. As $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \in \mathbb{F}_q$, the map $\varphi_\sim((x)_{\mathbb{F}_q}) = \langle\varphi(x)\rangle$ from $\mathcal{P}$ to points of $\mathrm{PG}(2,q)$ is well-defined. Moreover, as φ is bijective, the map $\varphi_\sim$ is also a bijection from $\mathcal{P}$ to $\mathrm{PG}(2,q)$.

If $\mathrm{Tr}_\alpha \in \mathcal{L}$, then define $\varphi_\sim(\mathrm{Tr}_\alpha) = \cup_{(x)_{\mathbb{F}_q} \in \mathrm{Tr}_\alpha} \varphi_\sim((x)_{\mathbb{F}_q})$. As the kernel of $\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha x)$ is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^3}$ with dimension two, the map φ sends $\ker(\mathrm{Tr}(\alpha x))$ to a hyperplane of $(\mathbb{F}_q)^3$. Hence $\varphi_\sim$ is a bijection from $\mathcal{L}$ to the lines of $\mathrm{PG}(2,q)$.

Let $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$. Then, by the definition of $\varphi_\sim$, we see that $P \in \ell$ if and only if $\varphi_\sim(P) \in \varphi_\sim(\ell)$. So $\varphi_\sim$ is an incidence preserving bijection, and Π is a Desarguesian projective plane of order q. □

The following corollary follows immediately from the description of $\varphi_\sim$.

Result 4.2. *The line in* $\mathrm{PG}(2,q)$ *spanning two points* $\varphi_\sim((x)_{\mathbb{F}_q})$ *and* $\varphi_\sim((y)_{\mathbb{F}_q})$ *contains all points* $\varphi_\sim((z)_{\mathbb{F}_q})$ *such that* $z = \delta x + \gamma y$ *where* $\delta, \gamma \in \mathbb{F}_q$.

A *Singer group* of a projective plane of order q is a cyclic group of collineations S of order q^2+q+1 acting sharply transitively on the points (and hence lines) of the plane. The generator β of $\mathbb{F}_{q^3}^\times$ induces the collineation $\theta : (x)_{\mathbb{F}_q} \mapsto (\beta x)_{\mathbb{F}_q}$. The action of θ is sharply transitive, so $\langle\theta\rangle$ is a Singer group.

Now consider $\mathrm{PG}(2,q^2)$ and let β be a generator for $\mathbb{F}_{q^6}^\times$. Then $\beta^{q^4+q^2+1}$ is a generator for $\mathbb{F}_{q^2}^\times$, and the relation $\sim$ is defined as before. Let θ be the collineation induced by β. Then the following set of points defines a Kestenband arc,

$$A = \left\{ (\beta^{i(q^2+q+1)})_{\mathbb{F}_{q^2}} \,\middle|\, i = 0, 1, \ldots, q^2 - q \right\}. \tag{4.1}$$

The set A is simply the orbit of the point $(1)_{\mathbb{F}_{q^2}}$ under $\langle\theta^{q^2+q+1}\rangle$.

Result 4.3 ([10])**.** *The set* $A = \left\{ (\beta^{i(q^2+q+1)})_{\mathbb{F}_{q^2}} \,\middle|\, i = 0, 1, \ldots, q^2 - q \right\}$ *is a* $(q^2 - q + 1)$*-arc.*

Proof. Suppose that A contains three distinct collinear points. By the transitivity of θ^{q^2+q+1}, assume these points to be $(1)_{\mathbb{F}_{q^2}}$, $(\beta^{i(q^2+q+1)})_{\mathbb{F}_{q^2}}$ and $(\beta^{j(q^2+q+1)})_{\mathbb{F}_{q^2}}$ with $i \neq j$. Then by Result 4.2, we have $1 + \alpha\beta^{i(q^2+q+1)} = \delta\beta^{j(q^2+q+1)}$ for some $\alpha, \delta \in \mathbb{F}_{q^2}$. Because $(\delta^{q^3+1})^{q^3} = \delta^{q^3+1}$, we obtain $\delta^{q^3+1} \in \mathbb{F}_{q^3} \cap \mathbb{F}_{q^2} = \mathbb{F}_q$. In addition, $\beta^{j(q^2+q+1)(q^3+1)} = \mathrm{N}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\beta^j) \in$

$\mathbb{F}_q$. Hence, $(1+\alpha\beta^{i(q^2+q+1)})^{q^3+1} = (\delta\beta^{j(q^2+q+1)})^{q^3+1} \in \mathbb{F}_q$. Likewise, $\alpha^{q+1}\beta^{i(q^2+q+1)(q^3+1)} = N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha)N_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\beta^i) \in \mathbb{F}_q$. Therefore,

$$\begin{aligned}
(1+\alpha\beta^{i(q^2+q+1)})^{q^3+1} - 1 - \alpha^{q+1}\beta^{i(q^2+q+1)q^3} &= (1+\alpha\beta^{i(q^2+q+1)})^{q^3}(1+\alpha\beta^{i(q^2+q+1)}) && (4.2)\\
&= (1+\alpha^q\beta^{i(q^2+q+1)q^3})(1+\alpha\beta^{i(q^2+q+1)}) \\
&\quad - 1 - \alpha^{q+1}\beta^{i(q^2+q+1)q^3} && (4.3)\\
&= 1+\alpha\beta^{i(q^2+q+1)} + \alpha^q\beta^{i(q^2+q+1)q^3} + \\
&\quad \alpha^{q+1}\beta^{i(q^2+q+1)q^3} - 1 - \alpha^{q+1}\beta^{i(q^2+q+1)q^3} && (4.4)\\
&= \alpha\beta^{i(q^2+q+1)} + \alpha^q\beta^{i(q^2+q+1)q^3}, && (4.5)
\end{aligned}$$

is a member of $\mathbb{F}_q$. Hence $f(x) = (x-\alpha\beta^{i(q^2+q+1)})(x-\alpha^q\beta^{i(q^2+q+1)q^3})$ is a quadratic polynomial over $\mathbb{F}_q$, with $\alpha\beta^{i(q^2+q+1)}$ as a root. This implies $\alpha\beta^{i(q^2+q+1)} \in \mathbb{F}_{q^2}$, so $(\alpha\beta^{i(q^2+q+1)})_{\mathbb{F}_{q^2}} = (1)_{\mathbb{F}_{q^2}}$, which is a contradiction. Thus the points $(1)_{\mathbb{F}_{q^2}}$, $(\beta^{i(q^2+q+1)})_{\mathbb{F}_{q^2}}$, and $(\beta^{j(q^2+q+1)})_{\mathbb{F}_{q^2}}$ are not collinear, and A is an arc. □

From Result 4.3, it follows that at each point $P \in A$ there are precisely q^2-q secants and $q+1$ tangent lines to A.

Kestenband arcs are examples of complete arcs for q even and q odd, as the following result demonstrates.

Result 4.4 ([10])**.** *Let θ be a Singer cycle of order q^2-q+1, and $A = \mathrm{orb}_{\langle\theta\rangle}(P)$ for some point P in $\mathrm{PG}(2,q^2)$. If $q \geq 4$, then A is a complete (q^2-q+1)-arc.*

Proof. Suppose there exists a point Q such that $A \cup \{Q\}$ is an arc. Then, every line joining Q to a point of A is tangent to A. As θ stabilises A, every line joining Q^{θ^i} to a point of A is tangent to A, for any $i = 0,1,\ldots,q^2-q$. Because $A' = \mathrm{orb}_{\langle\theta\rangle}(Q)$ is also an arc, each tangent of A contains at most two points of A'. Thus, counting the lines joining each point of A' to P, there are at least $(q^2-q+1)/2$ tangent lines to A at P. This is a contradiction as there should be $q+1$ tangents through any point of A, and $(q^2-q+1)/2 > q+1$ if $q \geq 4$. Hence, A is complete. □

4.2 Kestenband Arcs from Classical Unitals

Kestenband [30] describes Kestenband arcs as the intersection of Hermitian unitals. In this section, we describe his work.

Recall that a matrix H, with entries in $\mathbb{F}_{q^2}$, is *Hermitian* if $H^q = H^T$. For some non-singular Hermitian matrix H, we will denote U_H to be the unital induced by H. A corollary of Kestenband's work is the following:

Result 4.5 ([30]). *Suppose $\mathcal{H}$ is a Hermitian unital in $\mathrm{PG}(2, q^2)$ consisting of points with homogeneous $\mathbf{x}$ such that $\mathbf{x}H\mathbf{x}^T = \mathbf{0}$ for some Hermitian matrix H with entries in $\mathbb{F}_{q^2}$. Then $\mathcal{H}$ meets the Hermitian unital U_I in a complete (q^2-q+1)-arc if and only if the characteristic polynomial of H is irreducible over $\mathbb{F}_{q^2}$.*

The proof of Result 4.5 relies on a couple of lemmas.

Result 4.6 ([30]). *Let H_1 and H_2, be two non-singular Hermitian matrices such that $H_1 \neq cH_2$ for all $c \in \mathbb{F}_{q^2}$. Then H_1 and H_2 induce Hermitian unitals $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose $\mathcal{H}_3$ and $\mathcal{H}_4$ are unitals induced by some distinct pair of non-zero Hermitian matrices $r_1H_1 + r_2H_2$, $s_1H_1 + s_2H_2$ such that*

$$\det \begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix} \neq 0.$$

Then, $\mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{H}_3 \cap \mathcal{H}_4$. Moreover, for all $P \in \mathrm{PG}(2, q)$ there exists $r, s \in \mathbb{F}_q$ such that the Hermitian variety induced by $rH_1 + sH_2$ contains P.

Proof. Consider the system of equations describing the points of $\mathcal{H}_3 \cap \mathcal{H}_4$

$$\mathbf{x}^T(s_1H_1 + s_2H_2)\mathbf{x}^{(q)} = 0, \tag{4.6}$$

$$\mathbf{x}^T(r_1H_1 + r_2H_2)\mathbf{x}^{(q)} = 0. \tag{4.7}$$

Some rearranging shows,

$$\begin{bmatrix} s_1 & s_2 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}^T H_1 \mathbf{x}^{(q)} \\ \mathbf{x}^T H_2 \mathbf{x}^{(q)} \end{bmatrix} = 0, \tag{4.8}$$

and by assumption this equation has only the trivial solution. Thus,

$$\begin{bmatrix} \mathbf{x}H_1\mathbf{x}^T \\ \mathbf{x}^T H_2 \mathbf{x}^{(q)} \end{bmatrix} = 0, \tag{4.9}$$

so $\mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{H}_3 \cap \mathcal{H}_4$.

Let $P \in \mathrm{PG}(2, q)$ be a point with homogeneous coordinates $\mathbf{x}$. If $\mathbf{x}^T H_1 \mathbf{x}^{(q)} = m$ and $\mathbf{x}^T H_2 \mathbf{x}^{(q)} = n$, and $(m, n) \neq (0, 0)$, then the variety induced by the matrix $nH_1 - mH_2$ contains P. □

Remark 4.1. *The Hermitian matrices $nH_1 - mH_2$ may not be full rank, depending on H_1 and H_2, and hence may not determine Hermitian unitals in general. In particular if the characteristic polynomial of a Hermitian matrix H has a root $\lambda \in \mathbb{F}_q$, then $H - \lambda I$ is not full rank.*

We can now discuss the intersection of a particular family of Hermitian unitals.

Result 4.7 ([30])**.** *Suppose H is a Hermitian matrix with characteristic polynomial $p(x)$ that is irreducible over $\mathbb{F}_{q^2}$. Let $S = \{U_{H-\lambda I} \mid \lambda \in \mathbb{F}_q\} \cup \{U_I\}$. Then, $|\cap_{\mathcal{H}\in S}\mathcal{H}| = q^2 - q + 1$.*

Proof. Because $p(x)$ is irreducible over $\mathbb{F}_{q^2}$, the matrices $H - \lambda I$ are full rank for all $\lambda \in \mathbb{F}_q$ and hence induce Hermitian unitals. By Result 4.6, the set $\cup_{\mathcal{H}\in S}\mathcal{H}$ contains all points of $\mathrm{PG}(2, q)$ and the unitals in S mutually intersect in the same k points. Thus, $(q + 1)(q^3 + 1) - qk = q^4 + q^2 + 1$, which implies that $k = q^2 - q + 1$. □

Another useful lemma shows that if two Hermitian unitals share three collinear points, they share a Baer subline.

Result 4.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian unitals such that $\mathcal{H}_1 \cap \mathcal{H}_2$ contains three points on a line ℓ. Then $\ell \cap \mathcal{H}_1 = \ell \cap \mathcal{H}_2$.*

Proof. The secant line ℓ meets $\mathcal{H}_1$ and $\mathcal{H}_2$ in Baer sublines b_1 and b_2 respectively. If b_1 and b_2 share three common points, then $b_1 = b_2$, as Baer sublines are uniquely determined by three collinear points. Thus $\ell \cap \mathcal{H}_1 = \ell \cap \mathcal{H}_2$. □

We can now show that the unitals in S as defined in Result 4.6 meet in a $(q^2 - q + 1)$-arc.

Result 4.9 ([30])**.** *Suppose H is a Hermitian matrix with characteristic polynomial $p(x)$ that is irreducible over $\mathbb{F}_{q^2}$. Let $S = \{U_{H-\lambda I} \mid \lambda \in \mathbb{F}_q\} \cup \{U_I\}$. Then, $\cap_{\mathcal{H}\in S}\mathcal{H}$ is a $(q^2 - q + 1)$-arc.*

Proof. Let $A = \cap_{\mathcal{H}\in S}\mathcal{H}$. It follows from Result 4.7 that $|A| = q^2 - q + 1$. Suppose that a line ℓ meets A in three points. Then, by Lemma 4.8, for each $\mathcal{H}, \mathcal{H}' \in S$ we have $\ell \cap \mathcal{H} = \ell \cap \mathcal{H}'$. Hence, the set $\cup_{\mathcal{H}\in S}\mathcal{H}$ does not contain ℓ, and so does not contain all points of $\mathrm{PG}(2, q^2)$. This contradicts Result 4.6, so ℓ cannot meet A in more than two points. □

4.3 Equivalence of Kestenband Arcs

In Section 4.1, we saw that we could construct Kestenband arcs as the orbit of cyclic groups of order $q^2 - q + 1$. It is then natural to ask if different choices of such groups lead to projectively inequivalent arcs. The answer to this question is no, and this section will demonstrate this.

We first show that all cyclic subgroups of $\mathrm{P\Gamma L}(3, q^2)$ of order $q^2 - q + 1$ are conjugate. As a consequence, every point-orbit of these cyclic subgroups is equivalent.

To begin, we require the following well-known theorem from number theory. We give the result here without proof.

Result 4.10 (Zsigmondy's Theorem [41])**.** *Let $a > b > 0$ be coprime integers. Then, for any integer $n \geq 1$, there exists a prime p such that $p \mid a^n - b^n$ and $p \nmid a^k - b^k$ for all $k < n$ with the following exceptions:*

- *If* $n = 1$ *and* $a - b = 1$.
- *If* $n = 2$ *and* $a + b$ *is a power of two.*
- *If* $n = 6$, $a = 2$ *and* $b = 1$.

The same result holds when replacing $a^n - b^n$ *with* $a^n + b^n$, *with the sole exception of* $2^3 + 1^3$.

We also require some group representation theory. A *linear representation* of a group G is a pair (V, π), where V is a vector space and $\pi : G \to \mathrm{GL}(V)$ a group homomorphism. A linear representation (V, π) is *reducible* if there exists a subspace $W \subset V$ such that for all $g \in G$ and $\mathbf{w} \in W$, $\pi(g)\mathbf{w} \in W$.

Let (V_1, π_1) and (V_2, π_2) be two representations of a group G. A *G-intertwining map* $f : V_1 \to V_2$ (or just *intertwining map* if G is clear from context) is a map such that $\pi_2(g) \circ f = f \circ \pi_1(g)$ for all $g \in G$. Schur's Lemma constrains f when the representations are both irreducible.

Result 4.11 (Schur's Lemma)**.** *Let* (V_1, π_1), *and* (V_2, π_2) *be two irreducible* F*-linear representations of a group* G. *If* f *is an intertwining map between* π_1 *and* π_2, *then either* f *is zero or* f *is an isomorphism from* V_1 *to* V_2.

Proof. Let $\mathbf{x} \in \ker(f) \subset V_1$, then

$$
\begin{aligned}
(f \circ \pi_1(g))(\mathbf{x}) &= (\pi_2(g) \circ f)(\mathbf{x}) && (4.10)\\
&= \pi_2(g)\mathbf{0} && (4.11)\\
&= \mathbf{0}. && (4.12)
\end{aligned}
$$

So $\ker(f)$ is an invariant subspace of V_1. As (V_1, π_1) is irreducible we have either $\ker(f) = V_1$ (i.e. f is the zero map) or $\ker(f) = \emptyset$. In the latter case, let $\mathbf{y} \in f(V_1) \subset V_2$, then

$$
\begin{aligned}
\pi_2(g)\mathbf{y} &= (\pi_2(g) \circ f)(\mathbf{x}) && (4.13)\\
&= (f \circ \pi_1(g))(\mathbf{x}). && (4.14)
\end{aligned}
$$

So $f(V_1)$ is an invariant subspace of V_2. Therefore, $f(V_1) = V_2$ as $\ker(f) = \emptyset$. $\square$

We then obtain the following immediate corollary.

Lemma 4.1. *Suppose that* G *is a subgroup of* $\mathrm{GL}(k, q)$ *acting irreducibly on* $(\mathbb{F}_q)^k$. *Then, the set* $C(G) \cup \{0\}$ *is isomorphic to a subfield of* $\mathbb{F}_{q^k}$ *under the usual operations of addition and multiplication.*

Proof. Let G be a subgroup of $\mathrm{GL}(k,q)$ acting irreducibly on $(\mathbb{F}_q)^k$. Let $A, B \in C(G)$ with $A + B \neq 0$. We now have

$$(A + B)X = X(A + B)$$

for all $X \in G$. Therefore, $f(\mathbf{x}) = (A + B)\mathbf{x}$ is an intertwining map from $(\mathbb{F}_q)^k$ to itself. By Result 4.11 $f(\mathbf{x})$ is invertible, and so $A + B \in \mathrm{GL}(k,q)$. As $A + B \in \mathrm{GL}(k,q)$, and $A + B$ commutes with all $X \in G$, we have $A + B \in C(G)$. Hence $C(G)$ is closed under addition and multiplication, and $F = C(G) \cup \{0\}$ forms a finite division algebra. As all finite division algebras are fields, the division algebra F is a field. All finite fields are isomorphic, so $F = C(G) \cup \{0\}$ is isomorphic to a finite field $\mathbb{F}_{q^i}$ (since it must contain $\mathrm{Z}(\mathrm{GL}(k,q)) \simeq \mathbb{F}_q^\times$).

Let φ be an isomorphism from $\mathbb{F}_{q^i}$ to F, and let $\alpha \in \mathbb{F}_{q^i}$ be a generator. Then, $\varphi(\alpha) \in \mathrm{GL}(k,q)$ is a $k \times k$ matrix that, by the Cayley-Hamilton Theorem, is a root of its own characteristic polynomial $f \in \mathbb{F}_q[x]$ of degree at most k. Moreover f is irreducible as α is a generator. It now follows that $f(\alpha) = 0$, and so $\alpha \in \mathbb{F}_q^k$. Thus F is isomorphic to a subfield of $\mathbb{F}_{q^k}$. □

We can now show that Singer groups are conjugate.

Result 4.12. *Let $q = p^e$, then any two Singer subgroups of $\mathrm{P\Gamma L}(k,q)$ are conjugate, except if $k = 2$ and $q = 8$.*

Proof.* We first show that any two Singer cyclic groups are conjugate in $\mathrm{PGL}(k,q)$. Let S be a Singer cyclic group of $\mathrm{PG}(k,q)$. Then, barring a few trivial cases we won't consider here (see Remark 4.2), we have by Result 4.10 a prime r that divides $q^k - 1$ but not $q^i - 1$ for all $i < k$. Therefore, r divides $\frac{q^k-1}{q-1}$ and so there exists a Sylow r-subgroup in S, let R be such a subgroup. Because $[\mathrm{PGL}(k,q) : S] = \prod_{i=0}^{k-1} q^i(q^{k-i} - 1)$, we see that $r \nmid [\mathrm{PGL}(k,q) : S]$. Thus, R is also a Sylow r-subgroup in $\mathrm{PGL}(k,q)$.

Consider now the centraliser $C(R)$. As S is cyclic (and therefore abelian) it follows that $S \subset C(R)$. Let π be the natural homomorphism from $\mathrm{GL}(k,q)$ to $\mathrm{PGL}(k,q)$. We then have $\pi^{-1}(R) = \hat{R} \times Z$, where Z is the group of invertible of diagonal matrices and $\hat{R}$ is a Sylow r-subgroup of $\mathrm{GL}(k,q)$. Note that $\pi^{-1}(C(R)) = C(\hat{R})$.

The subgroup $\pi^{-1}(S)$ acts sharply transitively on $(\mathbb{F}_q)^k$, and hence $\hat{R}$ acts freely on $(\mathbb{F}_q)^k$. Suppose that $U \subset (\mathbb{F}_q)^k$ is a proper invariant subspace of $\hat{R}$. Then $\hat{R}$ partitions U into orbits of length $|\hat{R}|$ because $\hat{R}$ acts freely on U. Thus, $|\hat{R}| \mid |U|$ and in particular $r \mid |U|$. However, $|U| = q^i$ for some $i < k$ and $r \nmid q^i$ because $\gcd(q^i, q^k - 1) = 1$ and $r \mid q^k - 1$. So R must act irreducibly on $(\mathbb{F}_q)^k$.

By Result 4.1, the centraliser $C(\hat{R})$ is isomorphic to the multiplicative group of $\mathbb{F}_{q^k}$. Because $\pi^{-1}(S) \subset C(\hat{R})$ is isomorphic to $\mathbb{F}_{q^k}^\times$, we conclude $C(\hat{R}) = \pi^{-1}(S)$. It now follows that $C(R) = S$ in $\mathrm{PGL}(k,q)$. Because Sylow r-subgroups are conjugate, their centralisers are also conjugate. Hence all Singer groups in $\mathrm{PGL}(3,q)$ are conjugate.

To extend the result to show that all Singer cyclic groups are conjugate in $\mathrm{P\Gamma L}(k,q)$, let r be again a prime that divides q^k-1 but not q^i-1 for all $0 \le i < k$ and let S be a Singer group of $\mathrm{P\Gamma L}(k,q)$. Then $r \nmid e$, as otherwise if $r = es$,

$$q^k \equiv 1 \mod r \tag{4.15}$$

$$p^{ke} \equiv 1 \mod r \tag{4.16}$$

$$p^{krs} \equiv 1 \mod r \tag{4.17}$$

$$p^{ks} \equiv 1 \mod r \tag{4.18}$$

so $r|p^{ks}-1$, which is a contradiction as $ks < ke$. Thus, the Sylow r-subgroup R in S is still contained in $\mathrm{PGL}(k,q)$. Let π be the natural homomorphism from $\Gamma L(k,p^e)$ to $\mathrm{P\Gamma L}(k,p^e)$, and let $\hat{R} = \pi^{-1}(R)$, and $\hat{S} = \pi^{-1}(S)$. Then, as $\Gamma\mathrm{L}(k,p^e)$ embeds in $\mathrm{GL}(ke,p)$, we may regard $\hat{R}$ as a subgroup of $\mathrm{GL}(ke,p)$. As before, $\hat{R}$ acts irreducibly and $C(\hat{R}) \equiv \hat{S}$. So the conjugacy of Singer groups in $\mathrm{P\Gamma L}(k,q)$ follows. $\square$

Remark 4.2. *There are exceptional cases to Result 4.12, where Result 4.10 does not apply. The first is* $\mathrm{PGL}(6,2)$*, and this can be dealt with computationally. The second is* $k=2$ *and* $q+1$ *a power of two. Let* G *be a Singer cyclic group of* $\mathrm{PGL}(2,q)$*, where* $q+1$ *is a power of two. In this case,* $|\mathrm{PGL}(2,q)| = (q-1)q(q+1)$*. Because* $q+1$ *is a power of two, there exists a Sylow* 2*-subgroup* R *of* $\mathrm{PGL}(2,q)$ *containing* G*. Because* $2 \mid (q+1)$ *but not* q *or* $q-1$*, the Sylow* 2*-subgroup* R *has order* $2(q+1)$ *and* $G \triangleleft R$*. We thus obtain* $R \equiv D_{q+1}$ *and so* G *is the unique subgroup of* R *having order* $q+1$*. The result now follows from the conjugacy of Sylow* 2*-subgroups. In extending the results to* $\mathrm{P\Gamma L}(k,q)$ *we have two additional exceptions to Theorem 4.10, namely* $q=4$ *and* $k=3$ *as well as* $q=8$ *and* $k=2$*. The conjugacy of Singer groups in* $\mathrm{P\Gamma L}(3,4)$ *is easily checked by computer. In* $\mathrm{P\Gamma L}(2,8)$ *there are two conjugacy classes of Singer groups.*

Result 4.13. *Let* $q=p^e$*, any two cyclic subgroups of order* q^2-q+1 *in* $\mathrm{P\Gamma L}(3,q^2)$ *are conjugate, except if* $q=2$*.*

Proof.* By Zsigmondy's Theorem, there exists a prime divisor r of q^6-1 that does not divide q^k-1 for any $k<6$. Thus, as $q^6-1=(q^3-1)(q^3+1)$, we see that $r \mid q^3+1$. As $q^2-1=(q-1)(q+1)$, we find that $r \mid q+1$. It follows that $r \mid (q^3+1)/(q+1) = q^2-q+1$ and $r \mid (q^2-q+1)(q^2+q+1) = (q^6-1)/(q^2-1)$. As in Result 4.12, r does not divide e or q^i for all $1 \le i < 6$.

Now let S be a Singer group, and K the unique subgroup of order q^2-q+1 contained in S. Then, because $r \mid q^2-q+1$, there exists a Sylow r-subgroup R contained in K. Because

$$[\mathrm{P\Gamma L}(3,q^2):K] = eq^6(q^4-1)(q^2-1)(q^2+q+1),$$

it follows that r does not divide $[\mathrm{P\Gamma L}(3,q^2):K]$. Hence, R is also a Sylow r-subgroup in $\mathrm{P\Gamma L}(3,q^2)$. By the same arguments of Lemma 4.12, we obtain $C(R)=S$.

Finally let K' be an arbitrary cyclic subgroup of order q^2-q+1. It contains a Sylow r-subgroup R' that must be conjugate to R. Let $\varphi\in\mathrm{P\Gamma L}(3,q^2)$ be the map such that $\varphi R\varphi^{-1}=R'$. Then, $\varphi C(R)\varphi^{-1}=C(R')$. As both $\varphi K\varphi^{-1}$ and K' are subgroups of order q^2-q+1 in $C(R')$, and $C(R')$ is cyclic, we need $\varphi K\varphi^{-1}=K'$. Hence, all cyclic subgroups of order q^2-q+1 in $\mathrm{P\Gamma L}(3,q^2)$ are conjugate. □

Remark 4.3. *Result 4.13 does not hold for* $\mathrm{P\Gamma L}(3,4)$*, where there are three conjugacy classes of groups of order* $q^2-q+1=3$.

As a consequence, we have also shown the following

Result 4.14. *All Kestenband arcs are* $\mathrm{P\Gamma L}$*-equivalent.*

The natural extension of this question is whether all complete (q^2-q+1)-arcs in $\mathrm{PG}(2,q^2)$ are projectively equivalent. Exhaustive searches of complete arcs in $\mathrm{PG}(2,q)$ for $q\le 29$ have been made by computer (see for instance [17]) and have thus verified that all (q^2-q+1)-arcs in $\mathrm{PG}(2,q^2)$ are projectively equivalent for $q\le 5$. The next largest case $\mathrm{PG}(2,36)$ seems out of reach for the time being. These arcs appear to be unique for even q, which is remarkable considering that there are inequivalent ovals in $\mathrm{PG}(2,q^2)$ for even q. In Section 4.4 we show some conditions on the structure of an arbitrary complete (q^2-q+1)-arc.

4.4 Kestenband Arcs and Cyclic Spreads of Hermitian Unitals

The relationship between Kestenband arcs and Hermitian unitals has been explored in Section 4.2. In this section, we show that the tangent lines to a Kestenband arc dualise to a Hermitian unital when q is even, and that a Kestenband arc is a dual cyclic spread of some Hermitian unital. These two facts are well known. For instance, the tangent line result was first mentioned in [29].

A *spread* of a unital U is a set of q^2-q+1 lines that partition the points of U. To introduce this concept we give an example of a trivial spread of a Hermitian unital.

Result 4.15. *Let $\mathcal{H}$ be a Hermitian unital with associated polarity ρ of* $\mathrm{PG}(2,q^2)$*, and $P\notin\mathcal{H}$. Let $\ell_1,\ell_2,\dots,\ell_{q^2-q}$ be the secant lines of $\mathcal{H}$ through P. Then, the set of q^2-q+1 lines $S=\{\ell_1,\ell_2,\dots,\ell_{q^2-q},P^\rho\}$ form a spread of $\mathcal{H}$.*

Proof. To show that S is a spread, we show the lines of S do not meet in H. As $\ell_i\cap\ell_j=P\notin H$, we need only consider $\ell_i\cap P^\rho$. Because $(\ell_i\cap P^\rho)^\rho=\langle P,\ell_i^\rho\rangle$, and $P\notin\ell_i\cap H$, $(\ell_i\cap P^\rho)^\rho$ is a secant line of U and hence $\ell_i\cap P^\rho$ is not a point of H. Thus, S is a spread of H. □

Hermitian unitals admit another kind of spread, the *cyclic spreads*. These spreads are constructed by taking the orbit of a particular secant line ℓ of H under a cyclic group of order q^2-q+1 stabilising H. The motivation for looking at cyclic spreads is the following theorem, which gives an explicit description for $\mathbf{v}^H$ as the sum of lines in a cyclic spread for q even.

Result 4.16 ([19]). *Let $\mathcal{H}$ be a Hermitian unital of* $\mathrm{PG}(2,q^2)$ *with q even admitting a cyclic spread $S=\{\ell_i \mid i=1..q^2-q+1\}$. Then, $\mathbf{v}^{\mathcal{H}}=\sum_{i=1}^{q^2-q+1}\mathbf{v}^{\ell_i}$.*

Proof. We aim to show that $\mathbf{v}^{\mathcal{H}}=\sum_{i=1}^{q^2-q+1}\mathbf{v}^{\ell_i}$. Let $P\in\mathcal{H}$, then P lies on precisely one line of S. It is clear that S is a dual Kestenband arc, being the orbit of a line under a group of order q^2-q+1. So if $P\notin\mathcal{H}$, P either lies on zero or two lines of S. Thus, if $\mathbf{u}=\sum_{i=1}^{q^2-q+1}\mathbf{v}^{\ell_i}$, $\mathbf{u}_P=1 \mod 2$ and $\mathbf{u}_P=0 \mod 2$ if $P\in\mathcal{H}$ and $P\notin\mathcal{H}$ respectively. Hence, $\mathbf{u}=\mathbf{v}^{\mathcal{H}}$ and the result follows. □

We now establish the existence of cyclic spreads of Hermitian unitals for q even. Let β be a generator for $\mathbb{F}_{q^6}$, and let θ be the collineation of $\mathrm{PG}(2,q^2)$ induced by β, when viewing $\mathrm{PG}(2,q^2)$ in the field model. We may partition $\mathrm{PG}(2,q^2)$ into $\mathcal{B}=\{B_i \mid i=0\dots q^2-q\}$ where

$$B_i=\{(\beta^{i+t(q^2-q+1)})_{\mathbb{F}_{q^2}} \mid t=0\dots q^2+q+1\}. \tag{4.19}$$

Note that B_0 is the Baer subplane obtained by canonically embedding $\mathrm{PG}(2,q^3)$. The set $\mathcal{B}$ is the orbit of B_0 under $\langle\beta\rangle$.

A second partition of $\mathrm{PG}(2,q^2)$ is obtained by considering $\mathcal{A}=\{A_i \mid i=0\dots q^2+q\}$, where

$$A_i=\{(\beta^{i+t(q^2+q+1)})_{\mathbb{F}_{q^2}} \mid t=0\dots q^2-q+1\}. \tag{4.20}$$

The set A_0 is a Kestenband arc, and $\mathcal{A}$ is the orbit of A_0 under $\langle\beta\rangle$. To show there exists a unique cyclic spread of H stabilised by θ we use the following lemmas dealing with the intersection of lines ℓ with arcs in $\mathcal{A}$ and Baer subplanes in $\mathcal{B}$.

Lemma 4.2. *Let $\mathcal{A}=\{A_i\}$ and $\mathcal{B}=\{B_j\}$ be as defined in equation* (4.20) *and Equation 4.19 respectively. Then, $|A_i\cap B_j|=1$ for all i and j.*

Proof. As $\mathcal{B}$ and $\mathcal{A}$ are orbits under β, assume that $B_j=B_0$. Suppose that $|A_i\cap B_0|\geq 0$. Then $(\beta^{i+t(q^2+q+1)})_{\mathbb{F}_{q^2}}\in B_0$ for some $0\leq t\leq q^2-q$, and so $\beta^{i+t(q^2+q+1)}\in\mathbb{F}_{q^3}$. If $0\leq k<q^2-q+1$ and $(\beta^{j+(t+k)(q^2+q+1)})_{\mathbb{F}_{q^2}}\in B_0$, then $\beta^{k(q^2+q+1)}=\beta^{j+(t+k)(q^2+q+1)}/\beta^{t(q^2+q+1)}\in\mathbb{F}_{q^3}$. However, this is true if and only if $q^3+1\mid k(q^2+q+1)$ and because $\gcd(q^3+1,q^2+q+1)=1$ this implies $q^3+1\mid k$. So $k=0$ as $k<q^3+1$. Therefore, $|A_i\cap B_0|\leq 1$ and hence $|A_i\cap B_j|\leq 1$ for all i,j. There are q^2-q+1 points of A_i and q^2-q+1 Baer subplanes in $\mathcal{B}$ partitioning the plane, so $|A_i\cap B_j|=1$ for all i,j. □

We may also prove that secant lines of the Baer subplanes are tangent lines of the arcs.

Lemma 4.3. *Let $\mathcal{B} = \{B_i \,|\, i = 0 \dots q^2 - q + 1\}$ be defined as in equation* (4.19). *Let $P \in B_i$, and let A_j be the arc meeting B_i at P. Then the $q+1$ secant lines of B_i are the tangent lines of $p \in A_j$.*

Proof. Again assume that $B_i = B_0$. We further assume that $P = (1)_{\mathbb{F}_q}$ because the collineation induced by β^{q^2-q+1} is transitive on B_0, and stabilises both $\mathcal{B}$ and $\mathcal{A}$. The unique arc through P is A_0. Suppose that some secant line of B_0 through p and $(x)_{\mathbb{F}_q} \in B_0$ meets A_0 in a point $(\beta^{i(q^2+q+1)})_{\mathbb{F}_q^2}$. Then, we have $1 + \delta x = \gamma\beta^{i(q^2+q+1)}$ for some $\delta, \gamma \in \mathbb{F}_{q^2}$. Because $\left(\gamma\beta^{i(q^2+q+1)}\right)^{q^3+1} \in \mathbb{F}_q$,

$$
\begin{aligned}
(1+\delta x)^{q^3+1} &= (1+\alpha x)(1+\alpha x)^{q^3} && (4.21)\\
&= (1+\alpha x)(1+\alpha^q x) && (4.22)\\
&= 1 + (\alpha^q + \alpha)x + \alpha^{q+1}x^2 && (4.23)\\
\Rightarrow 0 &= (1+\delta x)^{q^3+1} + 1 + (\alpha^q + \alpha)x + \alpha^{q+1}x^2. && (4.24)
\end{aligned}
$$

Since $\alpha^q + \alpha = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha) \in \mathbb{F}_q$ and $\alpha^{q+1} = \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha) \in \mathbb{F}_q$, it follows that x satisfies a quadratic equation over $\mathbb{F}_q$ and is hence in $\mathbb{F}_{q^2}$. Therefore, we have $(x)_{\mathbb{F}_q^2} = (1)_{\mathbb{F}_q^2}$. This is a contradiction of the assumption that $(x)_{\mathbb{F}_q^2}$ was a distinct point to $(1)_{\mathbb{F}_q^2}$, so all lines of B_0 through P are tangent to the arc A_0. $\square$

The following characterisation of lines tangent to arcs in $\mathcal{A}$ is immediate.

Corollary 4.1. *Let $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ be as defined in equation* (4.20) *and Equation 4.19 respectively. Then a line ℓ is tangent to some A_i if and only if it is the secant line of some B_j.*

We can then prove the following result,

Lemma 4.4. *Let $\mathcal{B} = \{B_i\}$ be as defined in equation* (4.19). *Then a line ℓ is secant to precisely one Baer subplane B_i.*

Proof. There are $q^2 + 1$ points on ℓ, and the Baer subplanes of $\mathcal{B}$ partition the points of ℓ into subsets of size 1 or $q+1$. Let t be the number of Baer subplanes secant to ℓ, then

$$q^2 + 1 = t(q+1) - (q^2 - q + 1 - t),$$

whence $t = 1$ follows. $\square$

We have all the pieces now to show our main theorem,

Corollary 4.2. *Let $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ be the arcs and Baer subplanes as defined in equation* (4.20) *and Equation 4.19 respectively. Then any line $\ell \in \mathrm{PG}(2, q^2)$ is tangent to exactly $q + 1$ arcs in $\mathcal{A}$, forming a Baer subline of* $\mathrm{PG}(2, q^2)$.

Proof. By Lemma 4.3 and Lemma 4.4, a line ℓ is a secant line to exactly one Baer subplane of $\mathcal{B}$, and therefore tangent to exactly $q + 1$ arcs in $\mathcal{A}$. □

Theorem 4.1. *Let $\mathcal{H}$ be a classical unital in* $\mathrm{PG}(2, q^2)$, *with q even, having a cyclic stabiliser K of order $q^2 - q + 1$, partitioning $\mathcal{H}$ into $q + 1$ Kestenband arcs $\mathcal{A}_{\mathcal{H}}$. Then every secant line ℓ of $\mathcal{H}$ is tangent to all arcs of $\mathcal{A}_{\mathcal{H}}$ or precisely one arc of $\mathcal{A}_{\mathcal{H}}$. Moreover, there are exactly $q^2 - q + 1$ lines of the former kind, forming a cyclic spread of $\mathcal{H}$.*

Proof. We may view $\mathcal{A}_H$ as a subset of a partition of the whole plane $\mathrm{PG}(2, q^2)$ into Kestenband arcs $\mathcal{A}$. As K is a cyclic group of order $q^2 - q + 1$, it is contained in a Singer group of order S, inducing a partition of $\mathrm{PG}(2, q^2)$ into Baer subplanes $\mathcal{B}$ as in equation (4.19). Suppose that ℓ is a secant line meeting two arcs in $\mathcal{A}_H$. Then because $|\ell \cap H| = 1 \mod p$, we must have ℓ tangent to at least three arcs in $\mathcal{A}_H$. At the same time, ℓ is tangent to precisely $q + 1$ arcs in $\mathcal{A}$, and the points of tangency form a Baer subline. As a Baer subline is uniquely determined by three points this Baer subline must be $\ell \cap \mathcal{H}$, and so ℓ is tangent to all arcs in $\mathcal{A}_H$. Let S be the set of all lines tangent to all arcs in $\mathcal{A}_H$. The Hermitian unital meets each Baer subplane in exactly one Baer subline (see Theorem 6.21 of [8]), and each such line must be tangent to all arcs in $\mathcal{A}_H$. So $|S| = |\mathcal{B}| = q^2 - q + 1$. The other lines are tangent to at least (and hence exactly) one arc as $|\ell \cap H| = q + 1 \equiv 1 \mod p$ and $|\ell \cap A_i| \leq 2$. The lines tangent to all arcs $\mathcal{A}_H$ then must be a single orbit under K, as if ℓ is tangent to all arcs in $\mathcal{A}_H$ then ℓ^θ is tangent to all arcs in $\mathcal{A}_H$ for all $\theta \in K$. □

We will now show that given any complete $(q^2 - q + 1)$-arc A, the dual to its tangent lines forms a Hermitian unital. To show this, we need the following result about arcs (given without proof, see [28] for details).

Result 4.17. *Let K be a k-arc in* $\mathrm{PG}(2, q)$ *with q even. Then the tk tangents of K where $t = q + 2 - k$ belong to an algebraic envelope Γ_t of class t with the properties:*

1. *Γ_t is unique if $k > t$, that is, $k > q/2 + 1$;*
2. *Γ_t contains no secant of K and so no pencil with vertex P in K;*
3. *each tangent of K is counted exactly once in Γ_t.*

Remark 4.4. *In this context, an* algebraic envelope *Γ_t of class t is the dual of an algebraic curve with degree t. In particular, a point lies on at most t lines of Γ_t.*

Result 4.18 ([29])**.** *Let A be a complete (q^2-q+1)-arc in* $\mathrm{PG}(2,q^2)$*, and let Γ be the q^3+1 tangents to A. Then Γ is a dual Hermitian unital.*

Proof. Label the points of $\mathrm{PG}(2,q^2)\setminus A$ in some order as $\{x_i \mid i=1,2,\ldots,q^4+q\}$. For each $x_i \notin A$, let s_i be the number of tangents to A incident to x_i. We first double count ordered pairs (x_i,ℓ), with $x_i \in \ell$ and $\ell \in \Gamma$. On the one hand each point $x_i \in \mathrm{PG}(2,q^2)\setminus A$ is incident to s_i tangents, so there are $S=\sum_i s_i$ ordered pairs (x_i,L). For each $\ell \in \Gamma$, there are q^2 points on $\ell \setminus A$ and so

$$\sum_{i=1}^{q^4+q} s_i = S = |\Gamma| q^2 = (q+1)(q^2-q+1)q^2.$$

We now count ordered triples (x_i,ℓ,ℓ'), where $x_i=\ell\cap\ell' \notin A$ for distinct $\ell,\ell' \in \Gamma$. On the one hand for a given $x_i \in \mathrm{PG}(2,q^2)\setminus A$, there are $s_i(s_i-1)$ ordered pairs (ℓ,ℓ') such that $x_i=\ell\cap\ell'$. Hence, there are $S'=\sum_i s_i(s_i-1)$ ordered triples. On the other hand there are $|\Gamma|(|\Gamma|-(q+1))$ ordered pairs (ℓ,ℓ') such that $\ell\cap\ell' \notin A$, so

$$\sum_{i=1}^{q^4+q} s_i(s_i-1) = S' = (q+1)(q^2-q+1)((q+1)(q^2-q+1)-(q+1)).$$

It now follows that

$$\sum_{i=1}^{q^4+q} (s_i-1)(s_i-(q+1)) \tag{4.25}$$

$$= \sum_{i=1}^{q^4+q} s_i(s_i-1) - (q+1)\sum_{i=1}^{q^4+q} s_i + (q+1)(q^4+q) \tag{4.26}$$

$$\begin{aligned} &= (q+1)^2(q^2-q+1)(q^2-q) \\ &\quad - (q+1)^2(q^2-q+1)q^2 + (q+1)(q^4+q) \end{aligned} \tag{4.27}$$

$$= 0. \tag{4.28}$$

As Γ is an algebraic envelope of class $q+1$ and q^2-q+1 is odd, $1 \le s_i \le q+1$ and we have $(s_i-1)(s_i-(q+1)) \le 0$ for all i. Because $\sum_{i=1}^{q^4+q}(s_i-1)(s_i-(q+1))=0$, we have $(s_i-1)(s_i-(q+1))=0$ for all i. So each point $x_i \notin A$ lies on 1 or $q+1$ tangents lines to A. Denote Γ^* to be the dual of Γ. Then as every point lies on either one or 1 or $q+1$ tangents to A, every line meets Γ^* in 1 or $q+1$ points. Hence, Γ^* is a unital, and because Result 4.16 gives an expression for Γ^* as a codeword in $\mathrm{PG}(2,q^2)$, we see that Γ^* is a Hermitian unital. □

Theorem 4.18 has some implications for an arbitrary complete (q^2-q+1)-arc A when q even is even, namely:

1. the tangents to A form a dual Hermitian unital;
2. the stabiliser group of A is a subgroup of $\mathrm{PGU}(3, q^2)$ (the stabiliser group of any Hermitian unital);
3. the tangents to any point $P \in A$ are a *Baer subpencil*, a set of lines forming a pencil in a Baer subplane;
4. the arc A is a dual spread of a dual Hermitian unital $\mathcal{H}$.

These facts, combined with the computational evidence, suggest that A is likely not some arbitrary arc, and further motivates Conjecture 4.1.

Conjecture 4.1. *All complete $(q^2 - q + 1)$-arcs in $\mathrm{PG}(2, q^2)$, with q even, are projectively equivalent to a Kestenband arc.*

4.5 The Arc and Unital Plane Model

In this section we aim to relay a succinct proof of the following theorem, due to [25],

Result 4.19 ([25])**.** *A unital U is classical if and only if it is stabilised by a cyclic group of order $q^2 - q + 1$.*

Originally, Theorem 4.19 was shown in [18] using cyclic partitions of the projective plane into Baer subplanes, and a careful embedding of $\mathrm{PG}(2, q^2)$ inside $\mathrm{PG}(2, q^6)$. The method of Giuzzi, which we now illustrate, employs a construction of the Desarguesian projective plane $\mathrm{PG}(2, q)$ using Kestenband arcs and Hermitian unitals.

Consider the field model for the projective plane $\mathrm{PG}(2, q^2)$ with β as the generator for $\mathbb{F}_{q^6}$, and θ the corresponding collineation. Let $A_i = \{(\beta^{i+t(q^2+q+1)})_{\mathbb{F}_{q^2}} \,|\, t = 0, 1, \ldots, q^2 - q\}$. For each $\alpha \in \mathbb{F}_{q^3}$, we define the form $s_\alpha(x, y) = \mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(\alpha x y^{q^3})$.

Result 4.20 ([25])**.** *For each $\alpha \in \mathbb{F}_{q^3}$, the form s_α is a reflexive non-degenerate Hermitian form.*

Proof. We first show s_α is non-degenerate. As $y \mapsto \alpha y^{q^3}$ is a permutation of $\mathbb{F}_{q^6}$, the form s_α is degenerate if and only if $\mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(xz) = 0$ is degenerate. Because trace pairing is a non-degenerate form, so too is $s_\alpha(x, y)$. Because $\mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(\alpha x y^{q^3})^{q^3} = \mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(\alpha x^{q^3} y)$ we see that $s_\alpha(x, y)$ is both reflexive and Hermitian. $\square$

The Hermitian polarity associated with $s_\alpha(x, y)$ is the map $T_\alpha(x) = \mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(\alpha x^{q^3+1})$. The absolute points $\mathcal{H}_\alpha$ of T_α form a Hermitian unital by definition. Let $\mathcal{P} = \{A_i \,|\, i = 0, 1, \ldots, q^2 - q\}$, and $\mathcal{L} = \{\mathcal{H}_\alpha \,|\, \alpha \in \mathbb{F}_{q^3}\}$. Then, together with natural incidence, we will show $(\mathcal{P}, \mathcal{L})$ forms a Desarguesian projective plane of order q.

Result 4.21 ([25]). *Let $\mathcal{L} = \{\mathcal{H}_\alpha \,|\, \alpha \in \mathbb{F}_{q^3}\}$. Then $|\mathcal{L}| = q^2 + q + 1$.*

Proof. Because $\mathbb{F}_{q^3} \cap \mathbb{F}_{q^2} = \mathbb{F}_q$, it follows that $\mathcal{H}_{\alpha_1} = \mathcal{H}_{\alpha_2}$ if and only if $\alpha_1/\alpha_2 \in \mathbb{F}_q$. Therefore, $|\mathcal{L}| = (q^3 + 1)/(q - 1) = q^2 + q + 1$. □

Result 4.22 ([25]). *Let $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha'}$ be two distinct unitals, then $\mathcal{H}_\alpha \cap \mathcal{H}_{\alpha'}$ is a Kestenband arc stabilised by θ.*

Proof. As $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha'}$ are two unitals stabilised by θ, their intersection is stabilised by θ, so $\mathcal{H}_\alpha \cap \mathcal{H}_{\alpha'}$ is stabilised by θ. Because $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha'}$ are codewords, they meet in $1 \bmod p$ points (Result 3.12), and hence their intersection is the union of at least one Kestenband arc stabilised by θ. However, $2(q^2 - q + 1) > (q + 1)^2$ as $q \geq 4$ so $\mathcal{H}_\alpha \cap \mathcal{H}_{\alpha'}$ is precisely one Kestenband arc stabilised by θ. □

The following is an immediate corollary of Result 4.21 and 4.22.

Result 4.23. *[25] Let $\mathcal{P} = \{A_i \,|\, i = 1, \dots, q^2 - q\}$ and $\mathcal{L} = \{\mathcal{H}_\alpha \,|\, \alpha \in \mathbb{F}_{q^3}\}$. Then, the set $\mathcal{P}$ and $\mathcal{L}$ together with natural incidence form a projective plane of order q.*

The projective plane constructed with Result 4.23 is Desarguesian.

Result 4.24 ([25]). *Let $\mathcal{P} = \{A_i \,|\, i = 0, 1, \dots, q^2 - q\}$ and $\mathcal{L} = \{\mathcal{H}_\alpha \,|\, \alpha \in \mathbb{F}_{q^3}\}$. The projective plane with points $\mathcal{P}$, lines $\mathcal{L}$ and natural incidence is Desarguesian.*

Proof. If A_i is contained in $\mathcal{H}_\alpha$, then $\mathrm{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^2}}(\beta^{i(q^3+1)}\alpha) = \mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta^{i(q^3+1)}\alpha)$. On the other hand if $\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta^{i(q^3+1)}\alpha) = 0$ then since $\beta^{(q^2-q+1)(q^3+1)}$ is a generator for $\mathbb{F}_q$,

$$\begin{aligned} \mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta^{i(q^3+1)+t(q^2-q+1)(q^3+1)}\alpha) &= \beta^{t(q^2-q+1)(q^3+1)}\,\mathrm{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta^{i(q^3+1)}\alpha) && (4.29)\\ &= 0. && (4.30)\end{aligned}$$

The incidence described is identical to incidence in the field model of $\mathrm{PG}(2, q)$, and so the projective plane with points $\mathcal{P}$, lines $\mathcal{L}$ and natural incidence is Desarguesian. □

As an application of what we have just shown, we prove that any unital stabilised by a cyclic group of order $q^2 - q + 1$ is classical. This proof is the work of [25], and makes use of the following trivial result about blocking sets.

Result 4.25. *Let S be a blocking set of $\mathrm{PG}(2, q)$ with $|S| = q + 1$. Then, S is a line of $\mathrm{PG}(2, q)$.*

Proof. Let P, Q be two points of S, and suppose that there exists a point $R \in \langle P, Q\rangle$ not in S. Then, as there are exactly $q + 1$ points of S and $q + 1$ lines through R, if S is a blocking set each line through R contains exactly one point of S. However, the line $\langle P, Q\rangle$ contains R and at least two points of S, so by contradiction R cannot exist and $S = \langle P, Q\rangle$. □

Result 4.26 ([25])**.** *Let U be a unital of* $\mathrm{PG}(2, q^2)$*. Then, U is classical if and only if it is stabilised by a cyclic group of order $q^2 - q + 1$.*

Proof. As the classical unital is stabilised by $\mathrm{PGU}(3, q^2)$, which does contain a cyclic group of order $q^2 - q + 1$, we only need to show the converse. Let U be a unital stabilised by a cyclic group K of order $q^2 - q + 1$. By Result 4.13, there exists a collineation φ such that $\langle \theta \rangle = \varphi K \varphi^{-1}$. Therefore, the cyclic group $\langle \theta \rangle$ stabilises $U' = \varphi(U)$. Partition the plane $\mathrm{PG}(2, q^2)$ into $q^2 + q + 1$ Kestenband arcs $\mathcal{P} = \{A_i \mid 0, 1, \ldots, q^2 - q\}$, and $q^2 + q + 1$ classical unitals $\mathcal{L} = \{\mathcal{H}_\alpha \mid \alpha \in \mathbb{F}_{q^3}^\times\}$ to form a projective plane of order q as in Result 4.23. Because $\langle \theta \rangle$ stabilises U', the unital U' is the disjoint union of $q + 1$ Kestenband arcs A_i. By Result 3.12, U' meets each $\mathcal{H}_\alpha$ in at least one point and so at least one Kestenband arc — as θ stabilises both U' and $\mathcal{H}_\alpha$. Hence, U' is a blocking set of order $q + 1$ in the projective plane formed by $\mathcal{P}$ and $\mathcal{L}$. By Lemma 4.25, $U' \in \mathcal{L}$ and therefore $U = \mathcal{H}_\alpha$ for some α. Because U' is classical, and $U = \varphi^{-1}(U')$, it follows that U is classical. $\square$

Chapter 5

Unitals and Ovals in Figueroa Planes

This chapter looks at unitals and ovals in Figueroa planes, a family of non-translation projective planes. In this chapter we will:

1. Provide a detailed background of Figueroa planes.

2. Introduce the known ovals and unitals in Figueroa planes, and understand the similarities in their construction.

3. Attempt to generalise the construction of ovals and unitals, outlining some difficulties in doing so.

5.1 Background

The *Figueroa plane* of order q^3, for some prime power q, is a non-translation plane first described by Figueroa [24]. Figueroa constructs Figueroa planes for $q \not\equiv 1 \bmod 3$ and describes incidence in an algebraic fashion. Grundhöfer [26] later describes Figueroa planes of order q^3 for all prime powers q using a synthetic description of incidence. We present Grundhöfer's work.

Let α be an order 3 collineation of $\mathrm{PG}(2, q^3)$. We classify points into three types with respect to α.

Type-I points Points fixed by α.

Type-II points Points P such that P, P^α, P^{α^2} are collinear.

Type-III points Points P such that P, P^α, P^{α^2} are not collinear.

Lines may be dually classified as type-I, type-II or type-III. If α is clear from context, we may simply say P is type-I, type-II or type-III as opposed to type-I, type-II or type-III with respect to α.

Result 5.1 ([26]). *Suppose α is a collineation of order 3. Let μ be the function defined as $\mu(P) = P^{\alpha}P^{\alpha^2}$ for any type-III point P and $\mu(l) = l^{\alpha} \cap l^{\alpha^2}$ for any type-III line l. Then μ is an involution mapping type-III points to type-III lines.*

Proof. Suppose that P is type-III. Then since the points $P, P^{\alpha}, P^{\alpha^2}$ are not collinear, the line $l = P^{\alpha}P^{\alpha^2}$ is a type-III line. Dually, if l is a type-III line, then $\mu(l)$ is a type-III point. As $\mu(\mu(P)) = \mu(P^{\alpha}P^{\alpha^2}) = (PP^{\alpha^2}) \cap (PP^{\alpha}) = P$, μ is an involution of type-III points and lines. $\square$

The following result constrains the structure of a collineation α of order three.

Result 5.2 ([26]). *Let α be an order three collineation of* $\mathrm{PG}(2, q^3)$. *Then either α is a perspectivity of* $\mathrm{PG}(2, q)$ *or α fixes a subplane of order q point-wise.*

The Figueroa plane will be constructed using μ to modify incidence for type-III points and lines. We require the following important lemma that describes when a type-II line may contain two type-III points.

Result 5.3 ([26]). *Let α be an order three collineation of* $\mathrm{PG}(2, q^3)$. *Suppose that P and Q are two type-III points of* $\mathrm{PG}(2, q^3)$ *with respect to α. Then PQ is type-II if and only if $P^{\mu} \cap Q^{\mu}$ is type-II.*

Proof. Let P and Q be two type-III points. Let $R = P^{\alpha}P^{\alpha^2} \cap Q^{\alpha}Q^{\alpha^2} = P^{\mu} \cap Q^{\mu}$. By Desargues' Theorem, the triangles $\{P, P^{\alpha}, P^{\alpha^2}\}$ and $\{Q, Q^{\alpha}, Q^{\alpha^2}\}$ are in perspective from a point $V = PQ \cap P^{\alpha}Q^{\alpha}$ if and only if R, R^{α} and R^{α^2} are collinear. If PQ, $P^{\alpha}Q^{\alpha}$ and $P^{\alpha^2}Q^{\alpha^2}$ are concurrent at V, then $V = V^{\alpha} = V^{\alpha^2}$ and so V is type-I. Hence PQ is type-II if and only if $P^{\mu} \cap Q^{\mu}$ is type-II. $\square$

We also require the following useful corollary of Desargues' Theorem.

Result 5.4 ([26]). *Let l be a fixed line containing three distinct points P_0, P_1 and P_2, and consider two further distinct lines m and n meeting at a point Z. Let $\mathcal{R}$ be the set of points R_0 for which there exists $R_1 \in m$ and $R_2 \in n$ forming a triangle, such that $P_0 \in R_1R_2$, $P_1 \in R_0R_2$ and $P_2 \in R_0R_1$. Then, the points of $\mathcal{R}$ are collinear on a line s containing Z.*

Proof. Let R_0, R_1, R_2 and R_0', R_1', R_2' be a pair of triples as described in the statement of the result. Then, the triangles $\{R_0, R_1, R_2\}$ and $\{R_0', R_1', R_2'\}$ are in perspective from the line l in the points P_0, P_1, P_2. So they must also be in perspective from a line s containing $R_1R_1' \cap R_2R_2' = m \cap n = Z$. If R_0'', R_1'' and R_2'' is a further triple of points, then because the triangle $\{R_0'', R_1'', R_2''\}$ is in perspective with both $\{R_0, R_1, R_2\}$ and $\{R_0', R_1', R_2'\}$ from l, $R_0'' \in R_0R_0' = s$. $\square$

Result 5.5 ([26])**.** *Let α be a collineation of $\mathrm{PG}(2,q^3)$ of order 3 contained in $\mathrm{P\Gamma L}(3,q^3) \setminus \mathrm{PGL}(3,q^3)$, and let μ be the involution defined in Result 5.1. The set of points $\mathcal{P}$, and lines $\mathcal{L}$ form a projective plane $\mathrm{Fig}_\alpha(q^3)$ of order q^3 where:*

1. *the set of points $\mathcal{P}$ is the set of points in $\mathrm{PG}(2,q^3)$;*
2. *for each type-I or type-II line ℓ of $\mathrm{PG}(2,q^3)$ there is a corresponding Figueroa line $\ell_{\mathcal{F}_\alpha} \in \mathcal{L}$ containing all points of ℓ;*
3. *for each type-III line ℓ of $\mathrm{PG}(2,q^3)$ there is a corresponding Figueroa line $\ell_{\mathcal{F}_\alpha} \in \mathcal{L}$ containing all type-II points of ℓ, and all type-III points P such that $P^\mu \ni \ell^\mu$.*

Proof. To show that $\mathrm{Fig}_\alpha(q^3)$ is a projective plane, we must show the axioms of a projective plane hold.

Let P and Q be two points of $\mathrm{Fig}(q^3)$. If both P and Q have type-I or type-II then $\ell = PQ$ is the unique line of $\mathrm{PG}(2,q^3)$ such that $\ell_{\mathcal{F}_\alpha}$ contains both P and Q. If P is type-I and Q is type-III, then any line containing both P and Q is type-II. So the line $\ell = PQ$ is again the unique type-II line containing both P and Q. Now assume that P and Q have type-III. By Result 5.3, the line $\ell = PQ$ is type-II if and only if $P^\mu \cap Q^\mu$ is type-II. The point $R = P^\mu \cap Q^\mu$ is the unique point such that $R \in P^\mu$ and $R \in Q^\mu$. Hence the unique line containing P and Q is $(\ell)_{\mathcal{F}_\alpha}$ if ℓ is type-II, and $((P^\mu \cap Q^\mu)^\mu)_{\mathcal{F}_\alpha}$ otherwise.

Suppose that P is type-II and Q is type-III. Consider all triangles with vertices (R_0, R_1, R_2) where $R_1 \in QQ^{\alpha^2}$ and $R_2 \in QQ^\alpha$, such that $P^{\alpha^2} \in R_0R_1$, $P^\alpha \in R_0R_2$ and $P \in R_1R_2$. By Result 5.4, the points R_0 lie on a unique line m containing Q. Let $X = m \cap Q^\mu$. Then the points $X' = XP^{\alpha^2} \cap QQ^{\alpha^2}$ and $X'' = XP^\alpha \cap QQ^\alpha$ are the unique points such that $\{X, X', X''\}$ is a triple of the form (R_0, R_1, R_2). This implies that $\{X, X', X''\}$ is an orbit under α. Hence, $X = m \cap Q^\mu$ is the unique point such that $X \in Q^\mu$ and $P \in X^\mu$. If X is type-III, then $(X^\mu)_{\mathcal{F}_\alpha}$ contains both Q and P. We will show that X is type-III if and only if PQ has type-III, thus ensuring that X^μ is the unique line such that $P, Q \in (X^\mu)_{\mathcal{F}_\alpha}$.

Assume that X is type-II. Then $P \in X^\alpha X^{\alpha^2} = XX^\alpha$ and $P^\alpha \in X^{\alpha^2}X = XX^\alpha$. So $XX^\alpha = PP^\alpha$ and thus $X \in PP^\alpha$. Therefore, the lines m, PP^α and Q^μ are concurrent in X. If $P \in QQ^\alpha$ or $P \in QQ^{\alpha^2}$ then the unique Figueroa line spanning P and Q is just $((Q^\alpha)^\mu)_{\mathcal{F}_\alpha}$ or $((Q^{\alpha^2})^\mu)_{\mathcal{F}_\alpha}$ respectively. So we may assume $P \notin QQ^\alpha$ and $P \notin QQ^{\alpha^2}$. Now let $l = P^{\alpha^2}(PQ^\alpha \cap P^{\alpha^2}Q^{\alpha^2})$ (see Figure 5.1) — note that this line exists as P does not lie on QQ^α or QQ^{α^2}. The point $R_0 = l \cap P^\alpha Q^\alpha$, together $R_1 = Q^{\alpha^2}Q \cap l$ and $R_2 = Q^\alpha$ are the vertices of a triangle (R_0, R_1, R_2) with $R_1 \in QQ^{\alpha^2}$, $R_2 \in QQ^\alpha$, $P^{\alpha^2} \in R_0R_1$, $P^\alpha \in R_0R_2$ and $P \in R_1R_2$. So by Result 5.4, we must have $R_0 \in m$ and so $m \cap P^\alpha Q^\alpha = l \cap P^\alpha Q^\alpha$. Hence, the line l contains the points P^{α^2}, $PQ^\alpha \cap QQ^{\alpha^2}$, and $m \cap P^\alpha Q^\alpha$. By Result 1.11 applied to the tuple $(Q^\alpha, Q^{\alpha^2}, X; P^{\alpha^2}, m \cap P^\alpha Q^\alpha, PQ^\alpha \cap QQ^{\alpha^2})$, we obtain that P, Q, and $P^\alpha Q^\alpha \cap P^{\alpha^2}Q^{\alpha^2}$ are collinear. Hence, PQ is type-II.

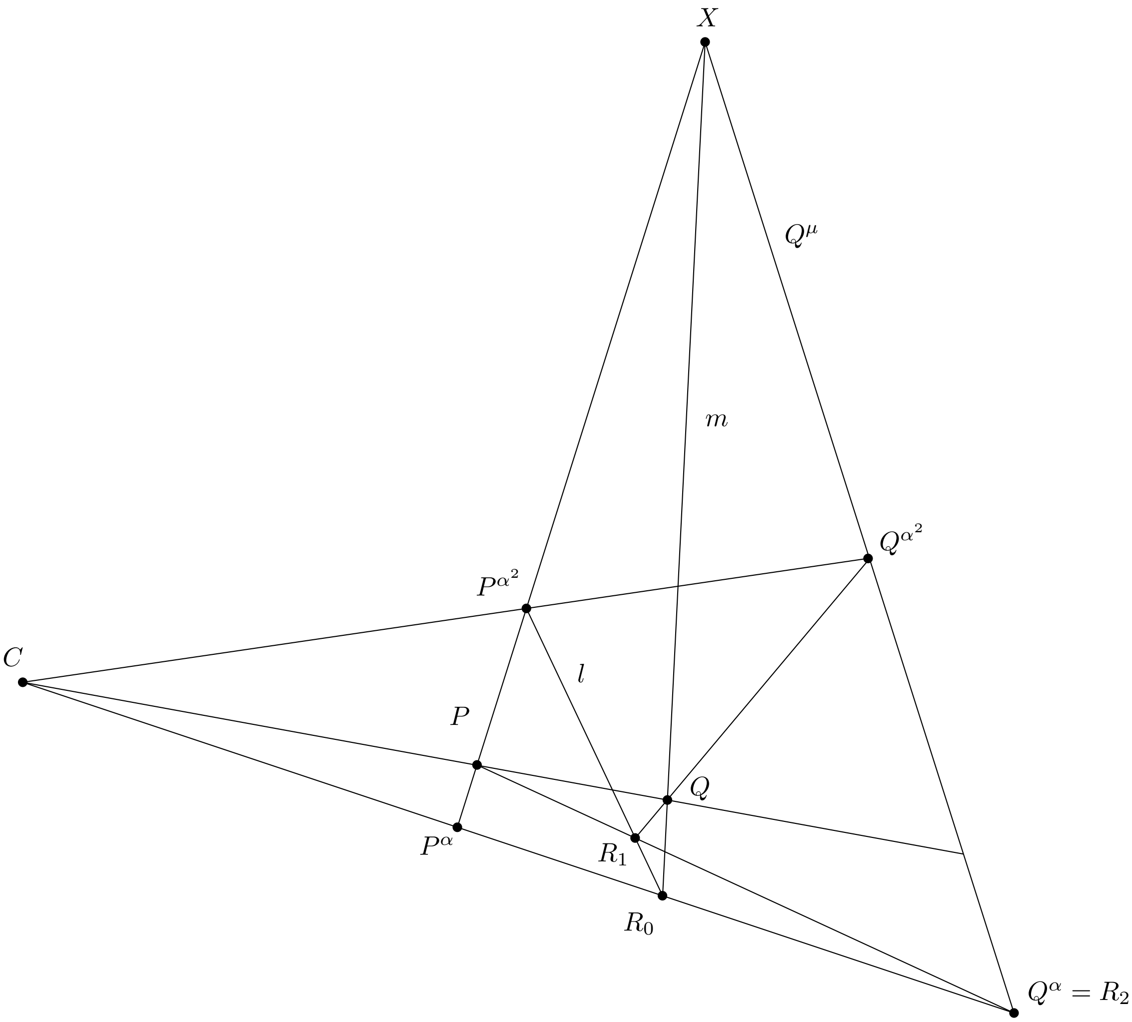

Figure 5.1: Illustration of the configuration in Result 5.5.

Now suppose that PQ is type-II. Then, PQ, $(PQ)^\alpha$ and $(PQ)^{\alpha^2}$ are concurrent at a point C. Then, applying Result 1.11 again to the hexagon $\{C, P, Q; P^{\alpha^2}, PQ^\alpha \cap QQ^{\alpha^2}, m \cap P^\alpha Q^\alpha\}$, we find that the points Q^α, Q^{α^2}, and $PP^\alpha \cap m$ are collinear. So $X = Q^\mu \cap m = PP^\alpha \cap m$ must have type-II as it lies on the type-I line PP^α. Thus, the unique line $\ell_{\mathcal{F}_\alpha} \in \mathcal{L}$ spanning P and Q is $(PQ)_{\mathcal{F}_\alpha}$ if PQ is type-II and $(X^\mu)_{\mathcal{F}}$ otherwise. We conclude that there is a unique line of $\mathcal{L}$ spanning any two distinct points P and Q of $\mathrm{Fig}_\alpha(q^3)$.

We shall show that two lines in $\mathcal{L}$ meet in a unique point dually. Suppose that P is a point of $\mathrm{PG}(2, q^3)$ with homogeneous coordinates (x, y, z). Then recall (Section 1.2) that the dual of P is a line P^ρ with dual coordinates $[x, y, z]$. It therefore follows that $(P^\rho)^\alpha = (P^\alpha)^\rho$ and, if P and ℓ are type-III, $P^\mu \in \ell^\mu$ if and only if $(\ell^\rho)^\mu \in (P^\rho)^\mu$. Thus, if $l_{\mathcal{F}_\alpha}, m_{\mathcal{F}_\alpha} \in \mathcal{L}$, we have $P \in l_{\mathcal{F}_\alpha} \cap m_{\mathcal{F}_\alpha}$ if and only if $(P^\rho)_{\mathcal{F}_\alpha} = (l^\rho m^\rho)_{\mathcal{F}_\alpha}$ and so P exists and is unique as there is a unique Figueroa line spanning two points.

Lastly we show there exists four points of $\mathrm{Fig}(q^3)$, no three lying on a line of $\mathcal{L}$. As

$\alpha \in \mathrm{P\Gamma L}(3, q^3) \setminus \mathrm{PGL}(3, q^3)$, the type-I points of α are an order three subplane of $\mathrm{PG}(2, q^3)$ by Result 5.2. Because this subplane contains four points no three collinear, there exists four type-I points of $\mathrm{Fig}_\alpha(q^3)$, no three lying on a line $\ell_{\mathcal{F}_\alpha} \in \mathcal{L}$. $\square$

The usual choice for α is the collineation mapping the point P is the collineation induced by the automorphism $x \to x^q$ of $\mathbb{F}_{q^3}$. The fixed points of α are precisely the points of the order q subplane $\mathrm{PG}(2, q)$ canonically embedded into $\mathrm{PG}(2, q^3)$.

The following results are corollaries of Result 5.8.

Result 5.6. *Let* Π *be a subplane of* $\mathrm{PG}(2, q^3)$ *of order* q*. The group of collineations fixing* Π *point-wise is a group of order three.*

Proof. Let Π_0 be the subplane obtained by canonically embedding $\mathrm{PG}(2, q)$ in $\mathrm{PG}(2, q^3)$. Let α be the collineation induced by the automorphism $x \to x^q$ of $\mathbb{F}_{q^3}$. The group of collineations fixing $\mathrm{PG}(2, q)$ point-wise is clearly the group of order three generated by the collineation α. By Result 1.22, the subplane Π is projectively equivalent to Π_0. Hence the point-wise stabiliser of Π is isomorphic to the point-wise stabiliser of Π_0. $\square$

Result 5.7. *Let* α *and* β *be two order three collineations in* $\mathrm{P\Gamma L}(3, q^3) \setminus \mathrm{PGL}(3, q^3)$*. Then* $\mathrm{Fig}_\alpha(q^3)$ *is isomorphic to* $\mathrm{Fig}_\beta(q^3)$*.*

Proof. By Result 5.2, the collineations α and β both fix subplanes Π and Π' of order q respectively. As in Result 5.1, let μ_α and μ_β be the involutions induced by α and β respectively. By Result 1.22, there exists a collineation ψ mapping Π to Π'. Then, $\psi \circ \alpha \circ \psi^{-1}$ is a collineation whose fixed points are precisely the points of Π'. By Result 5.6, $\psi \circ \alpha \circ \psi^{-1} = \beta$ or $\psi \circ \alpha \circ \psi^{-1} = \beta^2$. We will assume that $\psi \circ \alpha \circ \psi^{-1} = \beta$ as the other case is similar. Suppose that P is type-I with respect to α, then $\psi(P) = \psi(P^\alpha) = \psi(P)^\beta$, so $\psi(P)$ is type-I with respect to β. Likewise P, P^α and P^{α^2} are collinear if and only if $\psi(P)$, $\psi(P)^\beta$, and $\psi(P)^{\beta^2}$ are collinear; the collineation ψ is a type-preserving bijection of points and lines. We now see

$$\begin{aligned}
\psi(P^{\mu_\alpha}) &= \psi(P^\alpha P^{\alpha^2}) && (5.1)\\
&= \psi(P^\alpha)\psi(P^{\alpha^2}) && (5.2)\\
&= \psi(P)^\beta \psi(P)^{\beta^2} && (5.3)\\
&= \psi(P)^{\mu_\beta} && (5.4)
\end{aligned}$$

for any point P that is type-III with respect to α. Because ψ is a type-preserving collineation of $\mathrm{PG}(2, q^3)$, if P is a point and ℓ a line, and both are not type-III with respect to α then

$P \in \ell_{\mathcal{F}_\alpha}$ if and only if $\psi(P) \in \psi(\ell)_{\mathcal{F}_\beta}$. If P and ℓ have type-III,

$$\begin{aligned} & P \in \ell_{\mathcal{F}_\alpha} && (5.5)\\ & \Leftrightarrow \ell^{\mu_\alpha} \in P^{\mu_\alpha} && (5.6)\\ & \Leftrightarrow \psi(\ell^{\mu_\alpha}) \in \psi(P^{\mu_\alpha}) && (5.7)\\ & \Leftrightarrow \psi(\ell)^{\mu_\beta} \in \psi(P)^{\mu_\beta} && (5.8)\\ & \Leftrightarrow \psi(P) \in \psi(\ell)_{\mathcal{F}_\beta}. && (5.9) \end{aligned}$$

Hence ψ is an incidence preserving bijection of points and lines between the points and lines of $\mathrm{Fig}_\alpha(q^3)$ and $\mathrm{Fig}_\beta(q^3)$ and the planes are isomorphic. □

Result 5.7 justifies the notation $\mathrm{Fig}(q^3)$ to refer to the Figueroa plane of order q^3.

The collineation group of the Figueroa plane is determined by Hering and Schaeffer [27].

Result 5.8 ([27])**.** *Let $q = p^e$. Then the collineation group of $\mathrm{Fig}(q^3)$ is isomorphic to $\mathrm{PGL}(3, q) \ltimes \langle\tau\rangle$ where τ is the automorphism $x \to x^p$.*

It is useful to know the number of type-I, type-II and type-III points, as well as the number of points on lines of $\mathrm{Fig}(q^3)$ of each type.

Result 5.9. *Let $\alpha \in \mathrm{P\Gamma L}(3, q^3) \setminus \mathrm{PGL}(3, q^3)$ be a collineation of order 3. Then, $\mathrm{PG}(2, q^3)$ has q^2+q+1 type-I points, $(q^2+q+1)(q^3-q)$ type-II points and $q^6-q^5-q^4+q^3+q^2+q+1 = (q^3-q)(q^3-q^2)$ type-III points with respect to α.*

Proof. By Result 5.2, the fixed points of α form a subplane of order q. It follows that there are $q^2 + q + 1$ type-I points and lines. For each type-II point P, the collineation α fixes the line PP^α. So each type-II point lies on a type-I line. Any two type-I lines meet in a type-I point because the fixed points of α are a subplane of $\mathrm{PG}(2, q^3)$. Hence, the type-I lines partition the type-II points. As each of the $q^2 + q + 1$ type-I lines contains $q + 1$ type-I points, there are $q^3 - q$ type-II points on each type-I line. Thus, there are $(q^2 + q + 1)(q^3 - q)$ type-II points in $\mathrm{PG}(2, q^3)$. The number of type-III points is then $q^6 + q^3 + 1 - (q^2 + q + 1)(q^3 - q) = (q^3 - q)(q^3 - q^2)$. □

Result 5.10. *Let $\ell_{\mathcal{F}}$ be a line in $\mathrm{Fig}(q^3)$, where ℓ is a line in $\mathrm{PG}(2, q^3)$.*

1. *If ℓ is type-I, then there are $q + 1$ type-I and $q^3 - q$ type-II points incident with $\ell_{\mathcal{F}}$.*
2. *If ℓ is type-II, then there is one type-I, q^2 type-II and $q^3 - q^2$ type-III points incident with $\ell_{\mathcal{F}}$.*
3. *If ℓ is type-III, then there are $q^2 + q + 1$ type-II and $q^3 - q^2 - q$ type-III points incident with $\ell_{\mathcal{F}}$.*

Proof.

1. If ℓ is type-I, then $\ell_{\mathcal{F}} = \ell$. We have already seen in Result 5.9 that the number of type-II points on ℓ is $q^3 - q$. The remaining $q+1$ points are all type-I as ℓ is a secant line of a subplane of order q.

2. If ℓ is type-II, then again $\ell_{\mathcal{F}} = \ell$. The point $\ell^\alpha \cap \ell$ is the unique type-I point lying on ℓ. Then, it follows that each type-I line not through $\ell^\alpha \cap \ell$ meets ℓ in a distinct type-II point. Thus there is one type-I, q^2 type-II and $q^3 - q^2$ type-III points on $\ell = \ell_{\mathcal{F}}$.

3. If ℓ is type-III, then because $\ell \cap \ell_{\mathcal{F}}$ is the set of type-II points of ℓ, then ℓ contains the same number of type-II (and hence also type-III) points as ℓ. The line ℓ has q^2+q+1 type-II points corresponding to its intersection with each type-I line. The remaining $q^3 - q^2 - q$ points have type-III as no type-III line ℓ may contain a type-I point, since type-I points lie only on type-I and type-II lines. So $\ell_{\mathcal{F}}$ contains q^2+q+1 type-II and $q^3 - q^2 - q$ type-III points.

□

Result 5.11. *Let G be the collineation group of* $\mathrm{Fig}(q^3)$*. Then all points of the same type are equivalent under G.*

Proof. By Result 5.7, we may take α to be the collineation induced by $x \to x^q$. As $\mathrm{PGL}(3,q) \lhd G$, and $\mathrm{PGL}(3,q)$ acts transitively on the canonically embedded $\mathrm{PG}(2,q)$, the action of G is transitive on type-I points.

Let P be a type-II point. Then, any element of $\mathrm{PGL}(3,q)_P$ fixes P, P^α and P^{α^2} and hence fixes the type-I line $\ell = PP^\alpha$ point-wise. Therefore, the group $\mathrm{PGL}(3,q)_P$ is precisely the group of perspectivities with axis ℓ and has order $q(q-1)(q+1)$. Thus, by the orbit-stabiliser theorem, we have $|\,\mathrm{PGL}(3,q)^P| = |\,\mathrm{PGL}(3,q)|/q(q-1)(q+1) = (q^2+q+1)(q^3-1)$ and so $\mathrm{PGL}(3,q)$ is transitive on type-II points.

Likewise, if P is a type-III point, the point-wise $\mathrm{PGL}(3,q)$ stabiliser of P fixes the points P^α and P^{α^2}. As $\mathrm{PGL}(3,q)$ is sharply transitive on frames, an element $\sigma \in \mathrm{PGL}(3,q)_P$ is uniquely determined by its image on a single type-I point. Hence, we obtain $|\,\mathrm{PGL}(3,q)_P| \leq q^2+q+1$. Again the orbit-stabiliser theorem implies $|\,\mathrm{PGL}(3,q)^P| \geq \frac{|\,\mathrm{PGL}(3,q)|}{q^2+q+1} = (q^3-q)(q^3-q^2)$, which is equal to the number of type-III points. It now follows that $|\,\mathrm{PGL}(3,q)^P| = (q^3-q)(q^3-q^2)$ and so all type-III points are equivalent under $\mathrm{PGL}(3,q)$. □

5.2 Ovali di Roma

Ovali di Roma are the only known ovals in $\mathrm{Fig}(2,q^3)$ with q odd. An Ovale di Roma is constructed using a polarity ρ of $\mathrm{PG}(2,q^3)$ that commutes with α.

Result 5.12. *Let ρ be a polarity in $\mathrm{PG}(2,q^3)$ commuting with a collineation $\alpha \in \mathrm{P\Gamma L}(3,q^3) \setminus \mathrm{PGL}(3,q^3)$ of order 3. Let Ω_0 denote the absolute points of ρ of type-I and type-II, and Ω_1 the absolute points of ρ of type-III. Define the involution $\rho_{\mathcal{F}_\alpha}$ of points and lines to map $P^{\rho_{\mathcal{F}_\alpha}} = (P^\rho)_{\mathcal{F}_\alpha}$ for all points P and $\ell^{\rho_{\mathcal{F}_\alpha}} = (\ell^\rho)_{\mathcal{F}_\alpha}$. Then $\rho_{\mathcal{F}_\alpha}$ is a polarity in $\mathrm{Fig}_\alpha(q^3)$ with absolute points $\Omega_0 \cup \{P^{\rho\mu} \,|\, P \in \Omega_1\}$.*

Proof. Because ρ commutes with α, we can see that $\rho_{\mathcal{F}_\alpha}$ is a type-preserving bijection of points and lines of $\mathrm{Fig}_\alpha(q^3)$. In particular $P^\alpha = P$ if and only if $(P^\rho)^\alpha = P^\rho$ and $P, P^\alpha, P^{\alpha^2}$ is collinear if and only if $P^\rho, (P^\rho)^\alpha, (P^\rho)^{\alpha^2}$ are concurrent. It also follows that $P^{\mu\rho} = (P^\alpha P^{\alpha^2})^\rho = (P^\rho)^\alpha (P^\rho)^{\alpha^2} = P^{\rho\mu}$. Hence, if P and Q do not both have type-III, $P \in Q^{\rho_{\mathcal{F}_\alpha}}$ if and only if $Q \in P^{\rho_{\mathcal{F}_\alpha}}$. For type-III points P and Q, $P \in Q^{\rho_{\mathcal{F}_\alpha}}$ if and only if $Q^{\rho\mu} \in P^\mu$ but as μ commutes with α,

$$\begin{aligned} & P \in Q^{\rho_{\mathcal{F}_\alpha}} && (5.10)\\ & \Leftrightarrow Q^{\rho\mu} \in P^\mu && (5.11)\\ & \Leftrightarrow P^{\mu\rho} \in Q^{\rho\mu\rho} && (5.12)\\ & \Leftrightarrow P^{\rho\mu} \in Q^\mu && (5.13)\\ & \Leftrightarrow Q \in P^{\rho_{\mathcal{F}_\alpha}}. && (5.14) \end{aligned}$$

Thus, $\rho_{\mathcal{F}_\alpha}$ is a polarity of $\mathrm{Fig}_\alpha(q^3)$, whose absolute points are $\Omega_0 \cup \{P^{\rho\mu} \,|\, P \in \Omega_1\}$. □

We now require the following result of Seib [37] and Baer [3].

Result 5.13 ([37]). *Let ρ be a polarity of a projective plane of order n. Then,*

1. *If ρ has $n+1$ absolute points, then the absolute points of ρ form a line if n is even, and an oval if n is odd.*
2. *If $n = m^2$ and ρ has m^3+1 absolute points, then the absolute points of ρ form a unital.*

The following theorem is now a corollary of Results 5.12 and 5.13. It establishes that Ovali di Roma are ovals of $\mathrm{Fig}(q^3)$.

Result 5.14 ([16]). *Let ρ be a polarity in $\mathrm{PG}(2,q^3)$ commuting with a collineation $\alpha \in \mathrm{P\Gamma L}(3,q^3) \setminus \mathrm{PGL}(3,q^3)$ of order 3. Let Ω_0 denote the absolute points of ρ of type-I and type-II, and Ω_1 the absolute points of ρ of type-III. Then ρ induces a polarity $\rho_{\mathcal{F}_\alpha}$ in $\mathrm{Fig}_\alpha(q^3)$, whose absolute points $\Omega_0 \cup \{P^{\mu\rho} \,|\, P \in \Omega_1\}$ form an oval of $\mathrm{Fig}_\alpha(q^3)$.*

We know little of Ovali di Roma aside from their existence. Here are some simple results.

Result 5.15. *Let α be the order 3 collineation induced by the automorphism $x \to x^q$, and suppose $\mathcal{O}$ is an Ovale di Roma of $\mathrm{Fig}_\alpha(q^3)$ constructed from a conic $\mathcal{C}$ in $\mathrm{PG}(2, q^3)$. If G is the collineation group of $\mathrm{Fig}_\alpha(q^3)$, then $G_{\mathcal{O}} = (\mathrm{PGL}(2, q^3)_C \cap \mathrm{PGL}(2, q)) \ltimes \langle\alpha\rangle$.*

Proof. This follows from the characterisation of the collineation group of the Figueroa plane in Result 5.8 and the construction of the Ovale di Roma. $\square$

Lemma 5.1. *Let α be the order three collineation induced by the automorphism $x \to x^q$ of $\mathbb{F}_{q^3}$. A polarity ρ of $\mathrm{PG}(2, q^3)$ commutes with α if and only if ρ is induced by a non-singular matrix M such that $M^q = \lambda M$.*

Proof. Suppose that ρ is a polarity commuting with α. Because ρ is a polarity of $\mathrm{PG}(2, q^3)$, there exists a matrix M inducing ρ of the form

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}. \tag{5.15}$$

Consider the points P_0, P_1, P_2, P_3 with homogeneous coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ respectively. Then the condition that $P^{\rho\alpha} = P^{\alpha\rho}$ for all points P ensures

$$M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^q = \lambda_1 \left(M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)^q \tag{5.16}$$

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^q = \lambda_2 \left(M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)^q \tag{5.17}$$

$$M \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^q = \lambda_3 \left(M \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^q \tag{5.18}$$

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^q = \lambda_4 \left(M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)^q. \tag{5.19}$$

These conditions are then equivalent to

$$(a, d, g) = \lambda_1(a^q, d^q, g^q) \tag{5.20}$$

$$(b, e, h) = \lambda_2(b^q, e^q, h^q) \tag{5.21}$$

$$(c, f, i) = \lambda_3(c^q, f^q, i^q) \tag{5.22}$$

$$(a + b + c, d + e + f, g + h + i) = \lambda_4(a^q + b^q + c^q, d^q + e^q + f^q, g^q + h^q + i^q). \tag{5.23}$$

Combining these equations yields

$$\begin{bmatrix} a^q & b^q & c^q \\ d^q & e^q & f^q \\ g^q & h^q & i^q \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a^q & b^q & c^q \\ d^q & e^q & f^q \\ g^q & h^q & i^q \end{bmatrix} \begin{bmatrix} \lambda_4 \\ \lambda_4 \\ \lambda_4 \end{bmatrix}, \tag{5.24}$$

whence $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ follows. So $M^q = \lambda M$. □

All our discussion has been limited to ovals in $\mathrm{Fig}(q^3)$ for q odd, however it is worth considering the case where q is even. Here, the work of de Resmini and Hamilton [35] shows us that hyperovals in q even can actually be inherited into the Figueroa plane.

Result 5.16 ([35])**.** *Let $\alpha \in \mathrm{P\Gamma L}(3, q^3) \setminus \mathrm{PGL}(3, q^3)$ be an order three collineation and suppose $\mathcal{H}$ is a regular hyperoval in $\mathrm{PG}(2, q^3)$ for $q > 2$ even. If $\mathcal{H}$ is stabilised by α, then $\mathcal{H}$ is also a hyperoval in $\mathrm{Fig}_\alpha(q^3)$.*

5.3 Unitals in the Figueroa Plane

Unitals have been shown to exist in a number of non-Desarguesian planes. In the Figueroa plane know of exactly one class of unitals, called the *Figueroa unitals.* Their construction is similar to the Oval di Roma.

Result 5.17 ([35])**.** *Let $\mathcal{H}$ be a Hermitian unital in $\mathrm{PG}(2, q^6)$ with a Hermitian polarity ρ commuting with a collineation $\alpha \in \mathrm{P\Gamma L}(3, q^6) \setminus \mathrm{PGL}(3, q^6)$. If $\mathcal{H}_0$ is the set of type-I and type-II points of $\mathcal{H}$ and $\mathcal{H}_1$ the set of type-III points of $\mathcal{H}$, then $\mathcal{H}_* = \mathcal{H}_0 \cup \{P^{\rho\mu} \mid P \in \mathcal{H}_1\}$ is unital in $\mathrm{Fig}_\alpha(q^6)$.*

Proof. The proof follows from Results 5.12 and 5.13. □

Tai and Wong [38] demonstrate that Figueroa unitals contain *O'Nan configurations.* An O'Nan configuration is a collection of four distinct secant lines, meeting each other in six distinct points. The presence of an O'Nan configuration in the Figueroa unital is significant as it provides evidence towards a conjecture of Piper [33] that the Hermitian unital is the only unital that does not contain an O'Nan configuration. Let ρ be the Hermitian polarity of $\mathrm{PG}(2, q^6)$ induced by the automorphism $x \to x^{q^3}$ of $\mathbb{F}_{q^6}$ and let α be the order three collineation induced by the automorphism $x \to x^{q^2}$. Then, ρ commutes with α and so induces a polarity $\rho_{\mathcal{F}_\alpha}$ of $\mathrm{Fig}_\alpha(q^6)$, whose absolute points are a unital $\mathcal{H}_*$.

Result 5.18 ([38])**.** *The Figueroa unital $\mathcal{H}_*$ of $\mathrm{Fig}_\alpha(q^6)$ contains an O'Nan configuration for all q.*

Recall that the Hermitian unital is characterised as the only unital that is a codeword of $\mathrm{PG}(2, q^2)$. A natural question to then ask is if the Figueroa unital is also a codeword of the code of the Figueroa plane. Unfortunately establishing this in general would require knowing more about the structure of the code of $\mathrm{Fig}_\alpha(q^6)$. However, for $q = 2$ we can simply check and see that the Figueroa unital $\mathcal{H}_*$ does indeed lie in the code of $\mathrm{Fig}_\alpha(64)$. We can also see that $\mathcal{C}_2(\mathrm{Fig}_\alpha(64)) \cap \mathcal{C}_2(\mathrm{PG}(2, 64))$ is precisely the code generated by the type-I and type-II lines of $\mathrm{PG}(2, 64)$. Extending these computational results much further than the smallest case, to $q = 3$, requires computing a matrix of dimensions roughly 500000×500000, which is unfortunately is too big for available hardware without a more sophisticated approach.

5.4 Semiovals in the Figueroa Plane

The Figueroa unital and Ovali di Roma are both constructed using a polarity commuting with α. The natural question to then ask is if such a construction can be generalised.

A *semi-oval* is a set of points $\mathcal{O}$ such that each point $P \in \mathcal{O}$ has a unique tangent t_P to $\mathcal{O}$ at P. Both ovals and unitals are examples of semi-ovals. A natural generalisation of the Figueroa construction would then be to start with a semi-oval $\mathcal{O}$ stabilised by some order 3 collineation $\alpha \in \mathrm{P\Gamma L}(3, q^3) \setminus \mathrm{PGL}(3, q^3)$, keep all type-I and type-II points, and replace the type-III points P of $\mathcal{O}$ with t_P^μ.

Unfortunately there are problems with this generalisation, chiefly the fact that a type-III point may not have a type-III tangent, and μ is a only a bijection on the type-III points and lines of $\mathrm{PG}(2, q^6)$. A suitable semi-oval $\mathcal{O}$ would have the property that if $P \in \mathcal{O}$, then P, P^α, and P^{α^2} are collinear if and only if t_P, t_{P^α} and $t_{P^{\alpha^2}}$ are concurrent. A Hermitian unital $\mathcal{H}$ stabilised by α satisfies this property because the feet of any point $P \notin \mathcal{H}$ are collinear. However, we have seen that this is not true of Buekenhout-Tits and Buekenhout-Metz unitals. So it is unlikely that the generalised semi-oval construction we outline makes sense for such unitals.

One important class of semi-ovals are the *vertex-less triangles*. A vertex-less triangle in a projective plane Π is the union of three distinct, non-concurrent lines l, m and n, without the points $l \cap m$, $m \cap n$ and $l \cap n$. We will show that if q is even and ℓ is a type-III line, then the generalised semi-oval construction does produce semi-ovals in the Figueroa plane $Fig_\alpha(q^3)$ when applied to a vertex-less triangle with sides ℓ, ℓ^α and ℓ^{α^2}.

Lemma 5.2. *Let l, m, and n be distinct non-concurrent lines in a projective plane Π of order $q > 2$. Then $S = (l \cup m \cup n) \setminus \{l \cap m, m \cap n, l \cap n\}$ is a semi-oval.*

Proof. Let $P \in S$, we will assume that $P \in l$ as the argument is the same for $P \in m$ and $P \in n$. Each line through P is uniquely determined by its intersection with m. The line

$P(m \cap n)$ is tangent to S at the point P. Each point $Q \in m \setminus \{l \cap m, m \cap n\}$ determines a unique line PQ through P that is not tangent to S. Lastly, the line $(l \cap m)P = l$ is not tangent to S as $S \cap l = l \setminus \{l \cap m, l \cap n\}$ has $q - 1 > 1$ points. Hence, the line $P(m \cap n)$ is the unique tangent line to S at P. □

Let α be the order three collineation of $\mathrm{PG}(2, q^3)$ mapping the point P with homogeneous coordinates (x, y, z) to the point P^α with homogeneous coordinates (z^q, x^q, y^q). If ℓ is a type-III line with respect to α, then $S = (\ell \cup \ell^\alpha \cup \ell^{\alpha^2}) \setminus \{\ell \cap \ell^\alpha, \ell \cap \ell^{\alpha^2}, \ell^\alpha \cap \ell^{\alpha^2}\}$ is a semi-oval of $\mathrm{PG}(2, q^3)$. It turns out that the generalised Figueroa semi-oval construction does work for S if q is even.

Lemma 5.3. *Let α be the order three collineation of $\mathrm{PG}(2, q^3)$ mapping the point P with homogeneous coordinates (x, y, z) to the point P^α with homogeneous coordinates (z^q, x^q, y^q). Let ℓ be a type-III line with respect to α and $S = (\ell \cup \ell^\alpha \cup \ell^{\alpha^2}) \setminus \{\ell \cap \ell^\alpha, \ell \cap \ell^{\alpha^2}, \ell^\alpha \cap \ell^{\alpha^2}\}$. Then if q is even, the tangent line to $P \in S$ is the same type as P. Otherwise, if q is odd, the tangent line to $P \in S$ is type-III if P is type-II, and type-II or type-III if P is type-III.*

Proof. By Result 5.11, type-III lines are equivalent under a group of type-preserving collineations commuting with α. Hence, we may assume that ℓ has equation $x = 0$. So $S = (\ell \cup \ell^\alpha \cup \ell^{\alpha^2}) \setminus \{\ell \cap \ell^\alpha, \ell \cap \ell^{\alpha^2}, \ell^\alpha \cap \ell^{\alpha^2}\}$ is the set of points

$$\{(1, a, 0) \mid a \in \mathbb{F}_q^\times\} \cup \{(1, 0, b) \mid b \in \mathbb{F}_q^\times\} \cup \{(0, 1, c) \mid c \in \mathbb{F}_q^\times\}. \tag{5.25}$$

A point $P = (1, a, 0)$ is type-II if and only if

$$\det \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a^q \\ a^{q^2} & 0 & 1 \end{bmatrix} = 0. \tag{5.26}$$

Equation (5.26) is equivalent to $a^{q^2+q+1} = -1$. The line tangent to S at $(1, a, 0)$ is $\langle (1, a, 0), (0, 0, 1) \rangle$ and has equation $ax - y = 0$. This line is type-II if and only if

$$\det \begin{bmatrix} a & -1 & 0 \\ 0 & a^q & -1 \\ -1 & 0 & a^{q^2} \end{bmatrix} = 0. \tag{5.27}$$

Equation (5.27) is equivalent to $a^{q^2+q+1} = 1$. If q is even, then $-1 = 1$ and so a point with coordinates $(1, a, 0)$ is type-II if and only if it's tangent line is type-II. On the other hand if q is odd, a type-II point $P \in S$ with coordinates $(1, a, 0)$ has a type-III tangent to S at P. If P has type-III and q is odd, then P has type-II if $a^{q^2+q+1} \neq 1$ and type-III otherwise. □

Theorem 5.1. *Let q be even, and let α be the order 3 collineation as described in Lemma 5.3. Suppose ℓ is a type-III line, so that $S = (\ell \cup \ell^\alpha \cup \ell^{\alpha^2}) \setminus \{\ell \cap \ell^\alpha, \ell \cap \ell^{\alpha^2}, \ell^\alpha \cap \ell^{\alpha^2}\}$ is a semi-oval in* $\mathrm{PG}(2, q^3)$*. Let S_0 be the type-II points of S, and S_1 the type-III points of S. Then, the set $S^* = S_0 \cup \{t_P^\mu \,|\, P \in S_1\}$ is a semi-oval of $Fig_\alpha(q^3)$. Moreover, $S^* = (\ell_{\mathcal{F}_\alpha} \cup \ell_{\mathcal{F}_\alpha}^\alpha \cup \ell_{\mathcal{F}_\alpha}^{\alpha^2}) \setminus \{\ell_{\mathcal{F}_\alpha} \cap \ell_{\mathcal{F}_\alpha}^\alpha, \ell_{\mathcal{F}_\alpha} \cap \ell_{\mathcal{F}_\alpha}^{\alpha^2}, \ell_{\mathcal{F}_\alpha}^\alpha \cap \ell_{\mathcal{F}_\alpha}^{\alpha^2}\}$.*

Proof. As in Result 5.11, we assume that ℓ has equation $x = 0$. Let P be a type-III point of S with coordinates $(1, a, 0)$. By Lemma 5.3, the tangent line t_P to S at P is type-III. Because $\ell^\mu = (0, 0, 1)$, and $t_P = \langle (0, 0, 1), (1, a, 0) \rangle$ contains ℓ^μ, by definition $t_P^\mu \in \ell_{\mathcal{F}}$. Similarly $t_Q^\mu \in \ell_{\mathcal{F}}^\alpha$ and $t_R^\mu \in \ell_{\mathcal{F}}^{\alpha^2}$ for all type-III points $Q \in \ell^\alpha$ and $R \in \ell^{\alpha^2}$. Hence,

$$S^* = (\ell_{\mathcal{F}} \cup \ell_{\mathcal{F}}^\alpha \cup \ell_{\mathcal{F}}^{\alpha^2}) \setminus \{\ell_{\mathcal{F}} \cap \ell_{\mathcal{F}}^\alpha, \ell_{\mathcal{F}} \cap \ell_{\mathcal{F}}^{\alpha^2}, \ell_{\mathcal{F}}^\alpha \cap \ell_{\mathcal{F}}^{\alpha^2}\}. \tag{5.28}$$

By Lemma 5.2, the set S^* is a semi-oval in $\mathrm{Fig}(q^3)$. $\square$

Chapter 6

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