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Implanting Soft Hairs on Kaluza–Klein Black Holes

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Abstract

The gravitational memory effect is the permanent alteration of the configuration of a system due to the passing of a gravitational shock wave. Research on this field has become increasingly intense in the past decade. We discuss the memory effect on a black hole due to a transient gravitational shock wave which leaves *soft hair* implanted on the horizon. To do so, we introduce the BMS group, an infinite dimensional group of symmetries at null infinity which contains the Poincaré group as a subgroup. We show that a transient shock wave leaves the black hole geometry deformed (supertranslated) and "turns on" non-vanishing superrotation charge (due to soft hair) at null infinity. Furthermore, we take the point of view of an observer in the near horizon limit to show that this shock wave is in fact a horizon supertranslation and a horizon superrotation. The horizon superrotation leads to charges that can be related to the change in entropy of the black hole. We then investigate the memory effect on black holes with the addition of a gauge field which is found to generate an infinite dimensional current algebra. We finally use the developed mathematical formalism to investigate the memory effect on black holes in the Kaluza–Klein theory. We find that the passage of this gravitational shock wave implants soft hair on the black hole horizon resulting in a horizon supertranslation, superrotation, and a finite gauge transformation.

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Chapter 1

Introduction

One of the most important discoveries in physics within the last ten years was the detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2015. These waves were predicted in 1916 as a solution to Einstein's equations of general relativity, nearly a hundred years before they were successfully detected. An additional effect has been discussed in the literature since 1972, known as the gravitational memory effect (or simply, the memory effect). This effect was first introduced by Zel'dovich and Polnarev in 1974 [1] and has been greatly developed in the past few decades by Christodoulou and others [2–6]. Figure 1.1 demonstrates the memory effect, where a set of particles are left in a permanently altered state after the passage of a gravitational wave. This is analogous to a wave striking the shore and leaving the formation of sand permanently altered. The issue with gravitational memory, however, is that our current detectors are not precise enough to detect it. Furthermore, the level of precision required to measure this effect would mean that seismic noise would interfere with the results [5]. The next step in gravitational wave —and potentially memory—detection is the Laser Interferometer Space Antenna (LISA). LISA is planned to be launched in 2034 and will consist of three spacecrafts which create a Michelson (like) interferometer with arms 2.5×10^9 m in length [7].

A natural extension to observing how particles would be altered by a gravitational wave is how different objects in the universe could also be altered, for example, black holes. Black holes are the subject of a no-hair conjecture, this conjecture states that black holes are solely characterised by Mass M, Charge Q, and Angular momentum J [8]. (Note that all of these characteristics are referred to as charges.) The no-hair theorem, however, leads to the black hole information paradox. The information paradox occurs because of quantum me-

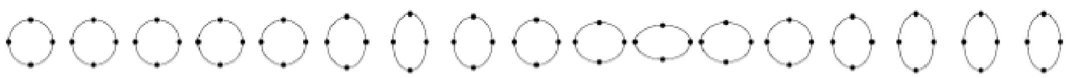

Figure 1.1: Particle configurations as a gravitational wave passes (time running left to right). The memory effect results in the final configuration differing from the initial. From [5]

chanics. In quantum theories, evolution of a wave function is determined by a unitary operator (i.e., probabilities are conserved). The paradox arises when considering Hawking radiation which causes a black hole to evaporate. If it evaporates entirely then any quantum information inside the black hole is lost and the wave function is now described by a non-unitary operator. Thus, moving forward, any theory of quantum gravity must either abandon what we know about evolution of wave functions and fields or overcome the information paradox. In a recent paper Hawking, Perry, and Strominger (HPS)[9] hypothesised that there exist "soft-hairs" (where soft refers to near zero energy) which encode the information of objects that have fallen past the event horizon on the boundary. These soft hairs are "implanted" on the horizon by a gravitational wave. Furthermore, these soft hairs would resolve the information paradox at null infinity ($\mathcal{I}$) — namely, the "region" where massless particles such as the photon and graviton propagate to after an infinitely long time. This idea is explored further in chapter 3.

The mathematical formalism used by HPS to describe soft hairs were the symmetries of asymptotically flat space at null infinity [9–11]. Generically, a physical system possesses symmetries if there is a set of physical transformations of the system which leave it unchanged [12, 13]. Furthermore, to be asymptotically flat,we must recover the Minkowski metric at large distances from the origin [11]. In flat space, we have isometries (transformations that are metric preserving) that form the Poincaré group. It was assumed for a long time that the only isometry group at *infinity* was the Poincaré group. (Note there are different notions of infinity.) However, in the 1960s a new symmetry group was discovered at null infinity by Bondi, van der Burg, Metzner and Sachs [14–16](BMS). The BMS group is generated by an infinite dimensional set of symmetries at null infinity which contains the Poincaré group as a subgroup [11, 17]. These symmetries were named *supertransformations* and consist of *supertranslations* and *superrotations*. As with any group, there are various methods to generate them. In chapter 2 we will discuss different definitions and generators of the BMS group.

It does not seem obvious why the memory effect and asymptotic symmetries are related. Figure 1.2 illustrates why a discussion of asymptotic symmetries takes place when discussing gravitational waves and the residual effect they may have. As we will illustrate heuristically inchapter 3, these two subjects (along with soft theorems) are actually part of what we call the infrared triangle. Furthermore, these subjects are actually the same, only expressed in different notations and with different starting points [11]. This allows two different ways of looking at the memory effect, one with a gravitational shock wave and one with asymptotic symmetries. Recent interest has been sparked by the fact that the gravitational memory effect is a physical realisation of abstractly formulated results. Furthermore, these results predict an infinite number of symmetries and conservation laws at null infinity.

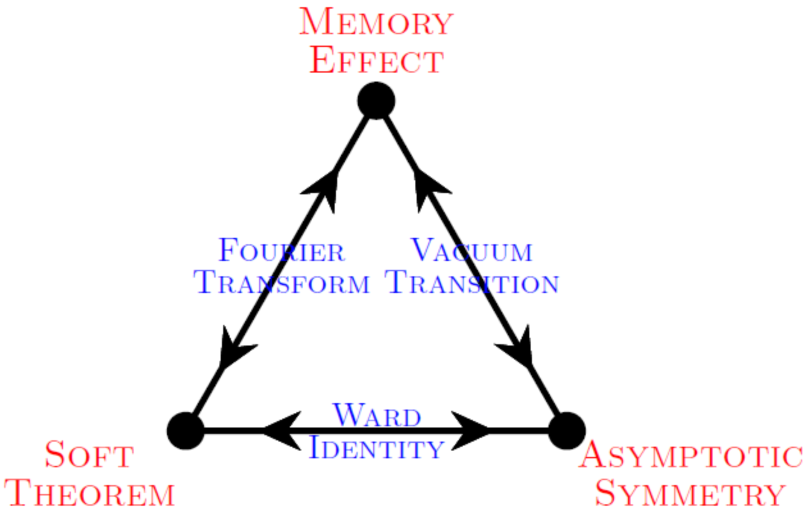

Figure 1.2: The infrared triangle from [11]: Three different areas of physics are realised as one in the infrared limit (at null infinity). The memory effect and asymptotic symmetries will be discussed further in chapter 2 and chapter 3. Soft theorems tell us that an infinite number of soft particles are produced in any physical process.

1.1 Outline of report

This report reviews recent results on gravitational memory, and then extends these to Kaluza–Klein black holes. In chapter 2 we introduce the generators of symmetries that form the BMS group. The generators of these symmetries are used throughout our analysis. In chapter 3 we demonstrate the perturbation induced in the spherically symmetric Schwarzschild black hole solution due to a gravitational wave by repeating the calculations of [18]. In chapter 4 we repeat the calculations in [18] which entail examination of the memory effect for gauge fields in addition to gravity. In chapter 5 these results are extended to the Kaluza–Klein theory [19] which combines gravity, electromagnetism and a scalar field in four dimensions from the dimensional reduction of a 5-dimensional Einstein gravity. My own contributions involved calculating details which are not in the literature in chapters 2-4 and then extending these calculations in chapter 5, Appendix C and Appendix D. Chapter 5, Appendix C, Appendix D are entirely original work.

Chapter 2

The BMS group

In 1962 Bondi, Metzner, van der Burg[1] and Sachs (BMS) [14, 16, 20] were attempting to find a group of diffeomorphisms — transformation which represent the same physical situation [13]— at null infinity which acted non-trivially on asymptotic data. To researchers at the time, including BMS, it seemed intuitive that any asymptotically flat space this data should be acted on non-trivially by the Poincaré group. However, BMS instead found an infinite dimensional group which contained the Poincaré group as a subgroup [11, 21–23]. This group has an infinite amount of generators known as supertransformations[9, 11, 23]. This was a surprising result as it meant that general relativity did not reduce to special relativity in the weak field limit and the symmetries found at null infinity were not those of the Poincaré group.

When investigating asymptotic symmetries we impose boundary conditions that reflect the nature of a spacetime at the boundary. These boundary conditions should be weak enough to allow all physically possible solutions to exist, but also strong enough that charges — i.e., globally conserved quantities — are finite and well defined [11, 21]. If we wish to discuss asymptotic symmetry groups, then we require the notions of gauge symmetries. These are related to a specific type of transformations that leave charges invariant. Allowed gauge symmetries are those that respect boundary conditions. But, we must factor our trivial gauge symmetries, namely transformations that only change spacetime coordinates [21].

We define asymptotic gauge symmetry groups (AGS) as the quotient group of *allowed gauge symmetries* and *trivial gauge symmetries* [11]. In general relativity, however, it is difficult to impose boundary conditions and therefore determine how a system *should* behave at the boundary. When considering spatial infinity, i^0, the AGS are the Poincaré group symmetries which consist of Lorentz transformations and spacetime translations. In terms of Noether's theorem, the ADM mass[2] is defined by symmetries at i^0 while the Bondi mass is defined by symmetries at futrue null infinity, $\mathcal{I}^+$.

[1]Sadly, even though van der Burg was an original founder, his name is left out of the acronym.

[2]Named after Arnowitt, Deser, and Misner (ADM). This mass is only defined for geometries that approach a well-defined metric at spatial infinity

2.1 Definitions of the BMS group

In general relativity, there is no global conservation of 4-momentum. This is because the extended form of energy–momentum conservation,

$$\nabla_\nu T^{\mu\nu} = 0,$$

is not integrable. However, if a spacetime possesses a Killing vector, K^μ then the current, $J^\mu = T^{\mu\nu} K_\nu$ gives a conserved charge. Therefore, we say that the existence of a Killing vector field results in the existence of continuous symmetries which imply a conserved charge via Noether's theorem. Furthermore, a Killing vector generates a Lie algebra which generate a Lie group. Thus, we say Killing vector fields generate a symmetry group. The BMS group, in this formulation, is an infinite dimensional symmetry group generated by the Lie algebra with an infinite number of asymptotic Killing vectors.

We denote the BMS group in abstract mathematical notation [11, 24],

$$\mathrm{BMS} = S \rtimes \mathrm{SO}(1,3).$$

Note that $\mathrm{SO}(1,3)$ is a representation of the Lorentz group and S is the group of supertranslations. Here $\rtimes$ is the semi-direct product which simply means that elements of the BMS group are pairs of elements in the Lorentz transformation group, and the group of supertranslations. To compare this definition with something more familiar, we can define the Poincaré group as [12, 23]

$$\text{Poincaré} = T_4 \rtimes \mathrm{SO}(1,3).$$

Here T_4 is the group of spacetime translations. The notion we will use, however, is one with asymptotic Killing vectors.

2.2 Generators

Consider the following example to distinguish the difference between supertranslations and regular Minkowski spacetime translations. In Cartesian coordinates (t, x, y, z) a translation in the z-direction is generated by the vector field

$$\mathbf{T_z} = \partial_z,$$

in Bondi coordinates (u,r,θ,ϕ) this becomes:

$$\mathbf{T_z} = -\cos\theta\partial_u + \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta.$$

A general supertranslation is found by replacing the coefficient of ∂_u (in this case $\cos\theta$) with a general spherical harmonic function, $Y_l^m(\theta,\phi)$ [25].

We will attempt to find Killing vectors at null infinity to find symmetries. We can do this by applying boundary conditions and finding what are called asymptotic Killing vectors. In general we find Killing vectors by solving Killing's equation,

$$\mathcal{L}_\xi g_{\mu\nu} = 0, \tag{1}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\boldsymbol{\xi}$ and $g_{\mu\nu}$ is a general metric tensor. When deriving asymptotic Killing vectors, we solve the "asymptotic" version of Killing's equation,

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(r^{-n}), \quad n \in \mathbb{Z}^+. \tag{2}$$

Here the right hand side simply denotes that Killing's equations will admit a Killing vector as r becomes large.

Before deriving the generator of symmetries at future null infinity we need to define our boundary conditions. In the Bondi Gauge, the most general four-dimensional metric that is not flat but asymptotically flat [20] is given by

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + g_{AB}\Big(d\Theta^A + \frac{1}{2}U^A du\Big)\Big(d\Theta^B + \frac{1}{2}U^B du\Big) \tag{3}$$

with

$$\partial_r\Big(det(\frac{g_{AB}}{r^2})\Big) = 0. \tag{4}$$

Here V, β and U^A are arbitrary functions that can be used to describe any particular background and $\Theta^A \in (\theta, \phi)$. We can expand the metric further, as done by BMS in [14, 16, 20] to account for terms smaller than $\frac{1}{r}$ which is then given by

$$\begin{aligned}
ds^2 &= -du^2 - 2dudr + r^2\gamma_{AB}d\Theta^A d\Theta^B \\
&+\frac{2m_B}{r}du^2 + rC_{AB}d\Theta^A d\Theta^B - D^B C_{AB}dud\Theta^A \\
&+ \frac{1}{16r^2}C^{AB}C_{AB}dudr \\
&+ \frac{1}{r}\Big(\frac{4}{3}N_A + \frac{4u}{3}\partial_A m_B - \frac{1}{8}\partial_A(C_{AB}C^{AB})\Big)dud\Theta^A \\
&+ \frac{1}{4}\gamma_{AB}C_{FD}C^{FD}d\Theta^A d\Theta^B + ...
\end{aligned} \tag{5}$$

The first line is the usual Minkowski background and the remaining are perturbation terms. Furthermore, $C_{AB}, N_A,$ and m_B all depend on $\{u, \Theta^A\}$ at future null infinity but not on r, and D^B is the covariant derivative on the two-sphere. In fact C_{AB} is related to the Bondi news tensor, which characterises the gravitational radiation that passes through $\mathcal{I}^+$. m_B and N_A are called the Bondi mass aspect and the angular momentum aspect respectively, which will be explored in section 2.3. We are now ready to define the boundary conditions which can be read off from (5)

$$\begin{aligned}
&g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{uA} = \mathcal{O}(r^{-2}), \\
&g_{AB} = r^2\gamma_{AB} + \mathcal{O}(1), \quad g_{rr} = g_{rA} = 0.
\end{aligned} \tag{6}$$

Looking at the g_{uA} condition, it is obvious that it does not follow from the metric expansion (5). This is because while we can use metric expansions to identify "suitable" boundary conditions, there is no consensus in the literature on how to specify fall offs [11] . The form of this fall off is chosen to match a subset of the literature [11, 12, 18].

With these boundary conditions, we are now in a position to derive a general asymptotic Killing vector. We first assume the general form of the Killing vector (field) to be

$$\xi^\mu = \xi^u \partial_u + \xi^r \partial_r + \xi^A \partial_A,$$

where again, $u = t - r$ and $A \in (\theta, \phi)$. Next we set up Killing's equation and thus take the Lie derivative of metric components along the vector field, ξ:

$$\begin{aligned} \mathcal{L}_\xi g_{rA} &= \xi^\mu \partial_\mu g_{rA} + g_{\mu r} \partial_A \xi^\mu + g_{A\mu} \partial_r \xi^\mu = 0, \\ \mathcal{L}_\xi g_{rr} &= \xi^\mu \partial_\mu g_{rr} + 2 g_{\mu r} \partial_r \xi^\mu = 0, \\ g^{AB} \mathcal{L}_\xi g_{AB} &= g^{AB} \Big(\xi^\mu \partial_\mu g_{AB} + g_{\mu B} \partial_A \xi^\mu + g_{A\mu} \partial_B \xi^\mu \Big) = 0. \end{aligned} \tag{7}$$

After expanding these equations with the sums over μ and substituting in the non-zero metric components, the set of equations (7) become

$$\begin{aligned} \mathcal{L}_\xi g_{rA} &= r^2 \gamma_{AB} \partial_r \xi^B - \partial_A \xi^u = 0, \\ \mathcal{L}_\xi g_{rr} &= -2 \partial_r \xi^u = 0, \\ \frac{r}{2} g^{AB} \mathcal{L}_\xi g_{AB} &= 2\xi^r + r D_A \xi^A = 0, \end{aligned} \tag{8}$$

where we have used $g_{AB} = r^2 \gamma_{AB} + \mathcal{O}(1)$. Note that we will often switch between the usual derivative, ∂_μ and the covariant derivative, D_μ following the conventions of [18]. One can now see that the three equations in (8) form a linear system of differential equations which we will now solve. The second equation in (8) can be solved directly and implies that

$$\xi^u = f(u, A). \tag{9}$$

Now the first equation in (8) becomes

$$r^2 \gamma_{AB} \partial_r \xi^B - \partial_A f(u, A) = 0,$$

which can now be solved in terms of the arbitrary function, $f(u, A)$ resulting in

$$\xi^B = Y^B(u, A) + D^B f(u, A) \int^r \frac{1}{r'^2} dr'. \tag{10}$$

Here we have introduced another arbitrary function of integration, $Y^B(u, A)$. We can now use (10) to solve the final equation in (8), to find

$$\xi^r = -\frac{r}{2} \Big(D_A Y^A(u, A) + D^2 f(u, A) \int^r \frac{1}{r'^2} dr' \Big). \tag{11}$$

Note we have used $D_A D^A = D^2$, which is the Laplacian on the two-sphere. We can now put (9)–(11) together to retrieve the general asymptotic Killing vector:

$$\xi^\mu \partial_u = f\partial_u - \Big[\frac{1}{2}\Big(rD_A Y^A - D^2 f(u,A)\Big)\Big]\partial_r + \Big(Y^A - \frac{1}{r}D^A f\Big)\partial_A. \tag{12}$$

Here we have carried out the integrals in (10)–(11). Recall that we have not actually used the boundary/fall off conditions in our derivation of the Killing vector field. These are instead used to restrict our arbitrary functions further. Note firstly that we have

$$\mathcal{L}_\xi g_{ur} = \partial_u \xi^u + \partial_r \xi^r = \mathcal{O}(r^{-2}).$$

If we ignore all terms that are $\mathcal{O}(r^{-2})$ and substitute (9),(11) into the above equation it is clear that we have

$$\partial_u f = \partial_A Y^A. \tag{13}$$

Secondly we note that

$$\mathcal{L}_\xi g_{uA} = g_{AB}\partial_u \xi^B + g_{uu}\partial_A \xi^u + g_{ur}\partial_A \xi^r = \mathcal{O}(r^{-2}).$$

Since the only term that is $\mathcal{O}(r^2)$ is the $g_{AB}\partial_u \xi^B$ we can set it to zero. Using (12) and ignoring terms with second partial derivatives[3], we obtain

$$\partial_u Y^A = 0. \tag{14}$$

With (13)–(14) we find two conditions, the first being

$$\partial_u^2 f = 0 \tag{15}$$

and the second being

$$f = s(A) + u\partial_A Y^A. \tag{16}$$

The term $s(A)$ in (16) is referred to as the supertranslation symmetry term and the term $u\partial_A Y^A$ is known as the superrotation symmetry term. [9, 12]

The set of Killing vectors found in the above analysis are, as stated before, asymptotic Killing vectors. These Killing vectors generate diffeomorphisms at null infinity. When we speak of (as in chapter 3) supertranslating a spacetime we will find the spacetime is mapped to a physically inequivalent one. This is not of concern as we only require the form of the metric (5) to be preserved at null infinity. Thus, these Killing vectors do in fact generate diffeomorphisms [11, 13].

2.3 The scattering problem and conserved charges

In classical general relativity, the scattering problem refers to finding a map of Cauchy data on $\mathcal{I}^-$ to Cauchy data on $\mathcal{I}^+$ [11]. For this, a prescription

[3] We follow the calculations of [12]. While it is not clear why derivatives of the form $\partial_v \partial_A$ are neglected, we remain consistent with this approach.

is required to attach $\mathcal{I}^+$ to the boundary of the set containing development of Cauchy data from $\mathcal{I}^-$. This is done by specifying a BMS+ frame (an asymptotic inertial frame at $\mathcal{I}^+$) and then determining initial values for integrating the Bondi mass aspect, m_B and angular momentum aspect,N_A along $\mathcal{I}^+$. When integrated over the sphere at null infinity, m_B determines the Bondi mass, whole the contraction of N_A and a vector on the 2-sphere at null infinity similarly gives the total angular momentum. It was proposed in [9, 11] that BMS+ frames should be determined by the matching conditions

$$m_B(\Theta)|_{\mathcal{I}^+_-} = m_B(\Theta)|_{\mathcal{I}^-_+} \quad \text{and} \quad N_A(\Theta)|_{\mathcal{I}^+_-} = N_A(\Theta)|_{\mathcal{I}^-_+}. \tag{17}$$

Thus far we have discussed generators of symmetries at null infinity. Supertranslations and superrotations are continuous symmetries, and via Noether's theorem have associated conserved charges referred to as supertranslation charge and superrotation charge respectively [9, 11]. These charges are derived in Appendix A of [26]. The conservation law for supertranslation charge is given by

$$Q_f^+ = \frac{1}{4\pi}\int_{\mathcal{I}^+} d^2\Theta\,\sqrt{\gamma}\,fm_B = \frac{1}{4\pi}\int_{\mathcal{I}^-} d^2\Theta\,\sqrt{\gamma}\,fm_B = Q_f^- \tag{18}$$

where the two charges at past and future null infinity are equated due to the matching conditions in (17). Conservation here means that the charges at past null infinity, $\mathcal{I}^-$ are the same as charges are future null infinity, $\mathcal{I}^+$. In the case where $f = 1$ the conserved charge implies conservation of total energy while the case where f is a $l = 1$ spherical harmonic function we have ADM momentum conservation. The conservation law for superrotation charge is given by

$$Q_Y^+ = \frac{1}{8\pi}\int_{\mathcal{I}^+_-} d^2\Theta\,\sqrt{\gamma}\,Y^A N_A = \frac{1}{8\pi}\int_{\mathcal{I}^-_+} d^2\Theta\,\sqrt{\gamma}\,Y^A N_A = Q_Y^-, \tag{19}$$

in the case that Y^A is one of the 6 global conformal Killing vectors on the 2-sphere. Equation (19) expresses conservation of ADM angular momentum and boost charges [9]. One can see how there are an infinite number of charges generated with these definitions at null infinity. The existence of these conserved charges is, at least in principle, verifiable with the gravitational memory effect, whic in future may become possible to measure.

Chapter 3

Supertransformations on Schwarzschild Black holes

3.1 Shock wave diffeomorphism

We begin our discussion of BMS hair and black holes with a Schwarzschild black hole. Consider the Schwarzschild line element in advanced Bondi coordinates given by

$$ds_0^2 = g^0_{\mu\nu}dx^\mu dx^\nu = -\Big(1 - \frac{2M}{r}\Big)dv^2 + 2dvdr + r^2\gamma_{AB}dz^A dz^B, \tag{20}$$

where $v = t+r^*$ is the advanced time coordinate, where $r^* = r+2M\,\ln\left|\frac{r}{2M} - 1\right|$, γ_{AB} is the unit 2-sphere metric, and $z^A \in (\theta, \phi)$. The event horizon is located at $r_+ = 2M$ (found by solving for when $g_{vv} = 0$). We say this represents a *bald* Schwarzschild black hole because the only non-vanishing charge at null infinity is the mass, M. We wish to show how a gravitational shock wave at the black hole horizon can be viewed as a supertranslation from null infinity. In [9, 18] a linearised gravitational shock wave was constructed at advanced time $v = v_0$ with an energy density given by the energy-momentum tensor,

$$T_{vv} = \frac{\mu + T(z)}{4\pi r^2}\delta(v - v_0), \tag{21}$$

where μ is the monopole contribution of the shock wave and $T(z)$ characterises the angular profile of the wave. The $\delta(v-v_0)$ term simply denotes the fact that the shock wave is prepared at the advanced time $v = v_0$. If we then solve the conservation of energy momentum equation ($\nabla_\mu T^{\mu\nu} = 0$) in the background (20) we find subleading contributions to (21), namely

$$T_{vv} = \Big(\frac{\mu + T(z)}{4\pi r^2} + \frac{D_A T^A(z)}{4\pi r^3}\Big)\delta(v - v_0), \quad T_{vA} = \frac{T^A(z)}{4\pi r^2}\delta(v - v_0). \tag{22}$$

Here $T^A(z)$ obeys $(D^2 + 2)D_A T^A = -6MT$. To find the metric perturbations we must solve the linearised Einstein equations. This was done in [9, 11] by using a Green's function which is defined as

$$D^2(D^2 + 2)G(z, \omega) = \frac{4}{\sqrt{\gamma}}\delta^{(2)}(z - \omega). \tag{23}$$

This Green's function connects two different angular positions, z and ω. After defining the Goldstone boson[1],

$$C(z) = \int d^2\omega G\left(z,\omega\right) T(\omega) \tag{24}$$

we can rewrite (22) as

$$\begin{aligned} T_{vv} &= \frac{1}{4\pi r}\Big(\mu + \frac{1}{4}D^2(D^2+2)C - \frac{3M}{2r}D^2C\Big)\delta(v-v_0),\\ T_{vA} &= -\frac{3M}{8\pi r^2}D_A C\,\delta(v-v_0). \end{aligned} \tag{25}$$

We then find the metric perturbations, $h_{\mu\nu}$ to be

$$\begin{aligned} h_{vv} &= \Big(\frac{2\mu}{r} - \frac{M}{r^2}D^2C\Big)\Theta(v-v_0),\\ h_{vA} &= D_A\Big(\frac{r-2M}{r}C + \frac{1}{2}D^2C\Big)\Theta(v-v_0),\\ h_{AB} &= -2r\Big(D_AD_BC - \frac{1}{2}\gamma_{AB}D^2C\Big)\Theta(v-v_0), \end{aligned} \tag{26}$$

where Θ is the Heaviside function.

3.2 Null infinity analysis

We will now examine the action of the asymptotic Killing vector on the background (20). Note, in the background (20) we find that Y^A vanishes. The asymptotic Killing vector (12) becomes $\xi^\mu\partial_\mu = f\partial_v + \frac{1}{r}D^Af\partial_A - \frac{1}{2}D^2f\partial_r$. After solving $h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ we find

$$\begin{aligned} \bar{g}_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} = \Big(\frac{2M}{r} - 1 + \frac{M}{r^2}D^2f\Big)dv^2 + 2dvdr - D_A\Big(2f - \frac{4M}{r}f + D^2f\Big)dvdz^A +\\ \Big(r^2\gamma_{AB} + 2rD_AD_Bf - r\gamma_{AB}D^2f\Big)dz^Adz^B. \end{aligned} \tag{27}$$

To find the location of the new event horizon, we in general solve the equation $g_{vv} = 0$. We proceed by solving this equation in a general sense. Let us begin with $r = r_+ + \delta$, where r_+ is the location of the unperturbed event horizon. Furthermore, let (in general) $\bar{g}_{vv} = g_{vv} + h_{vv} = -\Delta + \frac{1}{2}D^2f\partial_r\Delta$, where $\Delta = g_{vv}$ and so,

$$\begin{aligned} &-\Delta|_{r=r_++\delta} = -\Delta|_{r_+} - \delta\Delta'|_{r_+} + ... + \frac{1}{2}D^2f\Big[\Delta'|_{r_+} + \delta\Delta''|_{r+} + ...\Big] = 0\\ &= \delta\Big(\Delta'|_{r_+} + \frac{1}{2}D^2f\Delta''|_{r_+}\Big) = \frac{1}{2}D^2f\Delta'|_{r_+}\\ &\implies \delta = \frac{1}{2}D^2f\Big(1 - \frac{\frac{1}{2}D^2f\Delta''|_{r_+}}{\Delta'|_{r_+}}\Big)^{-1}. \end{aligned} \tag{28}$$

[1] The Goldstone bosons are bosons that appear in models which have a spontaneous breaking of continuous symmetries. In this case they parameterise the classically inequivalent gravitational vacua after the passage of a gravitational wave which supertranslates vacua at null infinity [11]

Note that here, we have used a Taylor expansion. From (28) we find that if $(1/2)D^2 f$ is small, then, $\delta = (1/2)D^2 f + \mathcal{O}((D^2 f)^2)$. Therefore, to linear order in $D^2 f$ we find the location of the perturbed event horizon in all cases is

$$(r_+)_f = r_+ + \frac{1}{2} D^2 f. \tag{29}$$

Now, setting $f = -C(z)\Theta(v - v_0)$ (and ignoring the monopole contribution, μ) one finds the perturbation metric $h_{\mu\nu}$ in (27) is equivalent to that in (26). The line element given by (27) is known as the supertranslated (hairy) Schwarzschild black hole. Thus, as claimed, we can see that the action of a linearised gravitational shock wave on the Schwarzschild black hole can be viewed as a BMS transformation at null infinity. In particular, we see that the action of this diffeomorphism turns on a superrotation charge at null infinity and so the supertranslated (hairy) Schwarzschild black hole is a physically different configuration to the bald Schwarzschild black hole. Figure 3.1 demonstrates the process discussed, where at advanced time $v > v_0$ the geometry is described by (27) and when $v < v_0$ the geometry is described by (20). The superrotation charge can be found by using (19). First note the angular momentum aspect is found in the g_{vA} (see (5)) component of the metric and in the case of (27)

$$N_A = -3 D_A M f. \tag{30}$$

Therefore, the conserved superrotation charge is given by

$$Q_Y = \frac{1}{8\pi} \int_{\mathcal{I}^-_+} d^2 z \sqrt{\gamma}\, Y^A N_A = -\frac{3M}{8\pi} \int_{\mathcal{I}^-_+} d^2 z \sqrt{\gamma}\, Y^A \partial_A f. \tag{31}$$

Here Y^A is an arbitrary vector field on the 2-sphere. One may wonder why there is no supertranslation charge found at null infinity when the solution (27) is called the supertranslated (hairy) black hole. This is because *supertranslations* do not impart *supertranslation charge* just as a spatial translation would not impart momentum to the black hole.

As mentioned previously, the background (27) is referred to as a "hairy" black hole and the process is also referred to as "implanting supertranslation hair". This may be puzzling as we stated there is a "no-hair theorem". The no-hair theorem and Hawking radiation lead to the information paradox. Supertranslation hair, is in fact an attempt to reconcile the information paradox at null infinity. This stems from the discussion in section 2.3, where it was stated that we require supertranslation and superrotation charge conservation. As an infinite amount of superrotation and supertranslation charges should be conserved whether or not the spacetime contains a black hole. Black holes require an infinite number of "soft hairs" (or an infinite number of "hairdos" [11]).

These soft hairs correspond to supertranslation and superrotation charges which are radiated to future null infinity in the process of Hawking evaporation. Therefore, due to Hawking radiation of these charges the total charge at future null infinity will be equal to that at past null infinity (and thus, overcoming the information paradox at null infinity) [9, 11]. Something not obvious is that the argument above makes an assertion about quantum gravity, and the significance of the much sought after graviton.

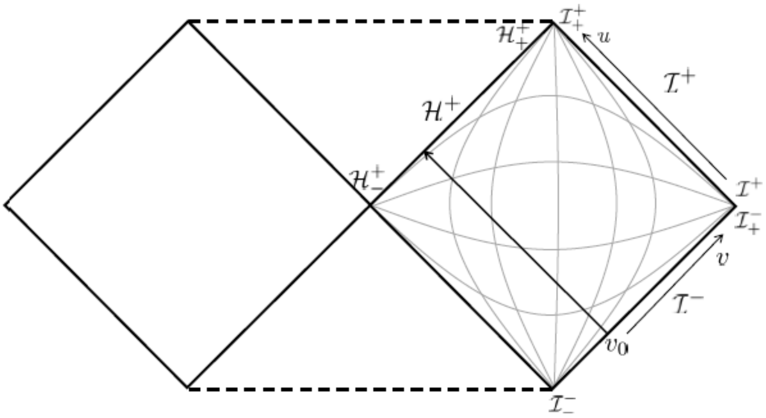

Figure 3.1: Carter Penrose diagram from [18] of a Schwarzschild black hole. The shock wave is given by the arrow from $v = v_0$ to the horizon. This shock wave divides the exterior geometry. This can be seen from the asymptotic horizon charges being different.

3.3 Near horizon analysis

In this section we will discuss the action of the shock wave described by (21) as seen by an observer who is located near the horizon. In [27] it was shown that the general form of a near horizon metric is given by

$$ds^2 = -2\rho\kappa dv^2 + 2dvd\rho + 2\rho\theta_A dvdz^A + \Omega_{AB}dz^A dz^B + ..., \tag{32}$$

where κ is the surface gravity[2], θ_A, and Ω_{AB} $(= \Omega\gamma_{AB})$ are in principle arbitrary metric functions of v and z^A. Note that the horizon is now located at $\rho = 0$ and the ellipsis are to denote terms that are $\mathcal{O}(\rho^2)$. It is clear we have:

$$g_{\rho\rho} = 0, \quad g_{v\rho} = 1, \quad g_{A\rho} = 0. \tag{33}$$

We can find a set of asymptotic Killing vectors which preserve (33), generating algebra consisting of *both* supertranslations and superrotations. We reproduce the calculations (8)–(12) and find the Killing vector to be

$$\xi^\mu\partial_\mu = f\partial_v + \Big(Y^A - \partial_B f\int^\rho d\rho' g^{AB}\Big)\partial_A + \Big(Z(v,z^A) - \rho\partial_v f + \partial_A f\int^\rho d\rho' g^{AB}g_{vB}\Big)\partial_\rho. \tag{34}$$

The boundary conditions we use are similar to those found in [29] and are,

$$\begin{aligned} g_{vv} &= -2\rho\kappa + \mathcal{O}(\rho^2), \\ g_{vA} &= \rho\theta_A + \mathcal{O}(\rho^2), \\ g_{AB} &= \Omega_{AB} + \mathcal{O}(\rho), \end{aligned} \tag{35}$$

which provide constraints on Y^A, Z, and f, namely,

$$\partial_v Y^A = 0, \quad \kappa\partial_v f + \partial_v^2 f = 0, \quad Z = 0. \tag{36}$$

[2]Surface gravity is defined as the acceleration exerted at infinity to keep an object at the horizon [28].

As mentioned, the generic form of the metric itself is preserved. However, we find the functions κ, θ_A, and Ω_{AB} are changed (or supertranslated). The supertranslated metric functions subject to conditions (35) are

$$\begin{aligned} \delta_\xi \kappa &= \mathcal{L}_\xi \kappa = 0, \\ \delta_\xi \theta_A &= \mathcal{L}_Y \theta_A + f\partial_v \theta_A - 2\kappa \partial_A f - 2\partial_v \partial_A f + \Omega^{BC} \partial_v \Omega_{AB} D_C f, \\ \delta_\xi \Omega_{AB} &= f\partial_v \Omega_{AB} + \mathcal{L}_Y \Omega_{AB}. \end{aligned} \tag{37}$$

We will now bring the perturbed black hole solution (27) to the near horizon limit to investigate horizon supertranslations and horizon superrotations. The perturbed black hole (27) in the near horizon limit ($r = r_+$) takes the form

$$ds^2 = -\frac{1}{r_+}\rho dv^2 + 2dvd\rho - \frac{2}{r_+}\rho D_A f dz^A dv + (r_+^2 \gamma_{AB} + 2r_+ D_A D_B f) dz^A dz^B + ..., \tag{38}$$

where $\rho = r - (r_+)_f$ and $d\rho = dr - (1/2)D_A D^2 f dz^A$. From (38) one can determine the metric functions of the perturbed black hole, namely[3],

$$\begin{aligned} \kappa + \delta_\xi \kappa &= \frac{1}{2r_+}, \\ \theta_A + \delta_\xi \theta_A &= -2\kappa D_A f, \\ \Omega_{AB} + \delta_\xi \Omega_{AB} &= r_+^2 \gamma_{AB} + 2r_+ D_A D_B f. \end{aligned} \tag{39}$$

By noting that $\delta_\xi \Omega_{AB} = 2r_+ D_A D_B f$, $\Omega_{AB} = r_+^2 \gamma_{AB}$, and then using the third equation of (37) we find

$$Y_A = \frac{1}{r_+} D_A f. \tag{40}$$

This means we have a horizon superrotation as well as a horizon supertranslation given by f. Note that f is the supertranslation function which was responsible for the hairy black hole solution (27)[4]. We have, therefore, shown that the disturbance of the shock wave (21) that is seen as a pure BMS supertranslation at null infinity is seen by a near-horizon observer as supertranslation f *and* an induced superrotation, Y^A. Once again, we find a superrotation charge which is turned on from the horizon supertranslation which is given by

$$Q_Y = \frac{1}{8\pi} \int d^2 z \sqrt{\gamma} Y^A \theta_A \Omega, \tag{41}$$

where we have replaced N_A with the the corresponding component in the near horizon limit.

Previously it was stated that supertranslations do not "turn on" supertranslation charge. Superrotations on the other hand, do. The fact that there is a superrotation on the horizon now, leads to supertranslation charge being turned

[3] The notation used here, is to distinguish the bald Schwarzschild black hole metric functions and the hairy Schwarzschild black hole metric functions.

[4] If $f = 0$ then (27) is just the bald Schwarzschild black hole (20). Therefore we say f *is* the supertranslation.

on at the horizon. This charge is, however, absent at $\mathcal{I}^-$. We note that in this form of the metric, the Bondi mass aspect is given by

$$m_B = M\kappa, \tag{42}$$

M being the mass of the black hole. We thus find the supertranslation charge is given by

$$\delta Q_T = \frac{\kappa}{8\pi}\int d^2z f\delta\sqrt{\det\Omega_{AB}}, \tag{43}$$

where we recover the Bondi mass aspect by noting $\Omega_{AB} = \Omega\gamma_{AB} = 4M^2\gamma_{AB}$. As explained in [18] the zero-mode (i.e. when $f = 1$) of this charge is

$$\delta Q_T = \frac{\kappa}{2\pi}\frac{\delta A}{4}, \tag{44}$$

where δA is the variation in the surface area of the black hole and integration has been conducted over the 2-sphere. We can express this as

$$\delta Q_T = T_H\delta S_{BH}, \tag{45}$$

where T_H is the Hawking temperature and S_{BH} is the Bekenstein-Hawking entropy equation for black holes. Thus the zero-mode of this supertranslation charge relates to the variation in the entropy of this black hole. This concludes the discussion of Schwarzschild black holes.

Chapter 4

Horizon symmetries with gauge fields

4.1 Diffeomorphisms and Gauge field Transformations

We will now investigate the horizon symmetries with the addition of a gauge field in the Einstein-Maxwell theory. The restrictions we place on the metric are the same as those given by (33) and (35). The Taylor expansion of a U(1) electromagnetic gauge field near $\rho = 0$ (in the coordinates $\rho = r - r_+$) is given by (see Appendix D for details)

$$\begin{aligned} A_v &= A_v^{(0)} + \rho A_v^{(1)}(v, z^A) + \mathcal{O}(\rho^2), \\ A_B &= A_B^{(0)}(z^A) + \rho A_B^{(1)}(v, z^A) + \mathcal{O}(\rho^2), \\ A_\rho &= 0. \end{aligned} \tag{46}$$

Here $A_v^{(0)}$ is the Coulomb potential and is considered to be fixed at the horizon. An analysis of horizon symmetries can now be undertaken on metrics obeying (35),(36) and gauge field obeying (46). Similar to the previous horizon analysis, we say a set of field transformations in the near horizon limit can be defined by

$$\delta_{(\xi,\epsilon)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad \delta_{(\xi,\epsilon)} A_\mu = \mathcal{L}_\xi A_\mu + \partial_\mu \epsilon. \tag{47}$$

The vector field $\xi = \xi^\mu \partial_\mu$ here generates diffeomorphisms and ϵ is the U(1) electromagnetic gauge parameter. This approach is undertaken in [18] and is similar to that in [30]. With the gauge fixing *and* boundary conditions we find

$$\begin{aligned} \xi^v &= f(v, z^A), \\ \xi^\rho &= -\partial_v f(v, z^A)\rho + \frac{\rho^2}{2\Omega}\theta_A(z^B)\partial^A f(v, z^A) + \mathcal{O}(\rho^3), \\ \xi^A &= Y^A(z^A) - \frac{\rho}{\Omega}\partial^A f(v, z^A) + \mathcal{O}(\rho^2), \end{aligned} \tag{48}$$

where instead of leaving the components in integral form, as in (34), we have evaluated them. For the gauge parameter the third equation in (46) implies

$$\mathcal{L}_\xi A_\rho + \partial_\rho \epsilon = 0, \tag{49}$$

from which we find

$$\epsilon = \epsilon^{(0)}(v, z^A) - \int_0^\rho d\rho' A_B \partial_\rho \xi^B = \epsilon^{(0)} + \frac{\rho}{\Omega}\partial^B f A_B^{(0)} + \mathcal{O}(\rho^2). \tag{50}$$

To find $\epsilon^{(0)}$ we use (46) to find suitable boundary conditions, which are

$$\mathcal{L}_\xi A_v + \partial_v \epsilon = \mathcal{O}(\rho), \quad \mathcal{L}_\xi A_B + \partial_B \epsilon = \mathcal{O}(1). \tag{51}$$

Solving these equations, we find

$$\epsilon^{(0)}(v, z^A) = U(z^A) - f(v, z^A) A_v^{(0)}. \tag{52}$$

With (51) we rewrite (49) as

$$\epsilon(v, z^A) = U(z^A) - f(v, z^A) A_v^{(0)} + \frac{\rho}{\Omega}\partial^B f(v, z^A) A_B^{(0)}(z^A) + \mathcal{O}(\rho^2). \tag{53}$$

With (48) and (53) we can now define the perturbations in the near horizon metric functions (κ, θ_A, and Ω_{AB}) and the gauge field (A_v and A_B),

$$\begin{aligned}
&\delta_{(\xi,\epsilon)}\kappa = 0 = \kappa \partial_v f + \partial_v^2 f, \\
&\delta_{(\xi,\epsilon)}\theta_A = \mathcal{L}_Y \theta_A + f \partial_v \theta_A - 2\kappa \partial_A f - 2\partial_v \partial_A f + \Omega^{BC}\partial_v \Omega_{AB} D_C f, \\
&\delta_{(\xi,\epsilon)}\Omega_{AB} = f \partial_v \Omega_{AB} + \mathcal{L}_Y \Omega_{AB}, \\
&\delta_{(\xi,\epsilon)} A_v = 0, \\
&\delta_{(\xi,\epsilon)} A_B = Y^C \partial_C A_B^{(0)} + A_C^{(0)} \partial_B Y^C + \partial_B U.
\end{aligned} \tag{54}$$

The perturbations in (54) generate a Lie algebra, whose properties we now discuss.

4.2 Non-extremal horizons and symmetry algebra

The framework we use to analyse charges is conformal field theory (CFT). Conformal field theory is a quantum field theory on a Euclidean space which is invariant under conformal (angle preserving) transformations. In general, a conformal field theory will have an infinite dimensional symmetry (Lie) algebra. Or in other words, the symmetries of a conformal field theory will form a vector space that is infinite dimensional with a commutator. In conformal field theory, an important algebra is the Witt algebra which is spanned by the basis, $l_n = -z^{n+1}\partial_z$ and the fundamental commutator is

$$[l_m, l_n] = (m - n) l_{m+n} \quad (n, m \in \mathbb{Z}). \tag{55}$$

The Witt algebra is important as two copies of the Witt algebra form the algebra of conformal transformations. Another important concept that is needed for symmetry discussions is that of a current algebra. Current algebras first appeared as the algebra of current density operators ($Q_i(x)$) of a quantum field theory [31]. They obey the commutation relations

$$[Q_i(x), Q_j(y)] = i\delta(x-y)C^k_{ij}Q_k(y), \tag{56}$$

where C^k_{ij} are the structure constants which determine the Lie bracket for the entire algebra. With this information in mind, we will continue with our discussion of horizon symmetries for the case where the surface gravity does not vanish.

For isolated non-extremal horizons surface gravity is constant, and nonzero, $\kappa \neq 0$. This means we can use the first equation in (54) to obtain

$$f(v, z^A) = T(z^A) + e^{-\kappa v}X(z^A). \tag{57}$$

The supertranslation generator, f, has two components. We say that $T(z^A)$ generates supertranslations at the horizon. The second term resembles the exponential decay in retarded time of the energy of a photon moving tangential to the event horizon [18]. We can now proceed to define Fourier modes of the supertranslation, superrotation, and electromagnetic charge generators (54) as done in [18, 23, 30]

$$\begin{aligned} &T_{(m,n)} = z^m\bar{z}^n, \quad X_{(m,n)} = e^{-\kappa v}z^m\bar{z}^n, \\ &Y_n = -z^{n+1}\partial_z, \quad \bar{Y}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \\ &U_{(m,n)} = z^m\bar{z}^n. \end{aligned} \tag{58}$$

Note here that we have made a change to using z and $\bar{z}$ with $z^n = \cot\left(\frac{\theta}{2}\right)e^{in\phi}$. Furthermore the supertranslations are associated with transformations on the null coordinate, v. With these modes, we can determine the entire algebra (see Appendix B for a derviation),

$$\begin{aligned} &[Y_m, Y_n] = (m-n)Y_{m+n}, \\ &[\bar{Y}_m, \bar{Y}_n] = (m-n)\bar{Y}_{m+n}, \\ &[Y_k, T_{(m,n)}] = -mT_{(m+k,n)}, \\ &[\bar{Y}_k, T_{m,n)}] = -nT_{(m,n+k)}, \\ &[Y_k, X_{(m,n)}] = -mX_{(m+k,n)}, \\ &[\bar{Y}_k, X_{(m,n)}] = -nX_{(m,n+k)}, \\ &[X_{(k,l)}, T_{(m,n)}] = \kappa X_{(m+k,n+l)}, \\ &[Y_k, U_{(m,n)}] = -mU_{(m+k,n)}, \\ &[\bar{Y}_k, U_{(m,n)}] = -nU_{(m,n+k)}, \end{aligned} \tag{59}$$

where all other commutators are zero (all other components commute). If we now compare the forms of (59) with (55)–(56) we see clearly that the algebra contains two sets of Witt algebra (generated by Y_n and $\bar{Y}_n$). Furthermore, the algebra also contains supertranslation (current) algebras generated by $U_{(m,n)}$, $X_{(m,n)}$, and $T_{(m,n)}$ (up to structure functions that complete the last 7 commutators in (59)). Once again, we can compare this algebra with something more familiar, namely, the Lorentz algebra which is spanned by Y_{-1}, Y_0, Y_1, $\bar{Y}_{-1}$, $\bar{Y}_0$, and $\bar{Y}_1$ [17, 23, 24]. The supertranslation generator $T_{(m,n)}$ has a zero-mode, $T_{(0,0)}$ which is the Killing vector associated with translations in advanced time, v. Therefore, $T_{(0,0)}$ is associated with the notion of energy[1]. The generators Y_n,$\bar{Y}_n$, $U_{(m,n)}$, and $T_{(m,n)}$ commute with this zero-mode. Thus we say these are candidates for *soft horizon hairs* as these hairs are zero-energy [18, 32].

The diffeomorphisms and gauge field transformations (59) all have associated conserved charges. These charges have a similar form to those discussed in Section 2.3. After including the gauge field, it was shown in [18]

$$Q[T, Y^A, U] = \frac{1}{16\pi} \int d^2z \, \sqrt{\gamma}\, \Omega \Big(2T\kappa - Y^A \theta_A - 4U A_v^{(1)} - 4A_B^{(0)} Y^B A_v^{(1)} \Big). \quad (60)$$

Note that the first two terms are those found in (41) and (43) which relate to supertranslation and superrotation horizon charges. The third term is due to the electromagnetic generator and the last term mixes the superrotation vector field with the gauge field. With (61), we can now define the zero-modes of the conserved charges related to Y_n, $U_{(m,n)}$, and $T_{(m,n)}$ and then discuss the physical meaning of these zero-modes for specific black holes. Let $\mathcal{Y}_n$, $\bar{\mathcal{Y}}_n$, $\mathcal{T}_{(m,n)}$, and $\mathcal{U}_{(m,n)}$ denote the charges associated to Y_n, $\bar{Y}_n$, $T_{(m,n)}$, and $U_{(m,n)}$ respectively. Then the zero-modes of the conserved charges are

$$\begin{aligned} \mathcal{T}_{(0,0)} &= Q[1,0,0] = \frac{2}{16\pi} \int d^2z \, \sqrt{\gamma}\, \Omega \kappa, \\ \mathcal{U}_{(0,0)} &= Q[0,0,1] = \frac{-4}{16\pi} \int d^2z \, \sqrt{\gamma}\, \Omega A_v^{(1)}, \\ \mathcal{Y}_{(0,0)} &= Q[0,1,0] = \frac{1}{16\pi} \int d^2z \, \sqrt{\gamma}\, \Omega \Big(\theta_\theta + \theta_\phi - 4(A_\theta^{(0)} + A_\phi^{(0)}) A_v^{(0)} \Big). \end{aligned} \quad (61)$$

We will now take this information and apply it to black holes in the Kaluza–Klein theory.

[1] Recall that we stated that a Killing vector is associated with conserved charges, and a timelike Killing vector is associated with a conserved energy.

Chapter 5

Soft Hair on Kaluza–Klein black holes

The Kaluza–Klein (KK) theory was an early classical attempt at unifying gravitation and electromagnetism using a small compact fifth spatial dimension [33]. It took the form of general relativity, with the 15 components of the 5-dimensional spacetime metric split as 10 for the 4-dimensional gravity metric and 4 for the electromagnetic vector potential and one component for a scalar field. We consider a spherically symmetric, time independent solution of vacuum Einstein equations which has a symmetry in the compact 5th spatial dimension generated by a spacelike Killing vector as in [19]. Due to this Killing vector, we can use dimensional reduction and obtain a solution not involving the fifth dimension. The line element we use for our analysis is from [19]

$$g_{\mu\nu}dx^\mu dx^\nu = -\frac{N^2}{\sqrt{BC}}dv^2 + 2dvdr + \sqrt{BC}(d\theta^2 + \sin^2\theta d\phi^2), \tag{62}$$

where

$$N^2 = (r - GM)^2 - \frac{G}{4\pi}(k^2M^2 + 16\pi^2\Sigma - Q^2 - P^2), \tag{63}$$

$$B = \left(r + \frac{k\Sigma}{\sqrt{3}}\right)^2 - \frac{2k^2Q^2\Sigma}{4\pi(4\pi\Sigma + 3kM)}, \tag{64}$$

$$C = \left(r - \frac{k\Sigma}{\sqrt{3}}\right)^2 - \frac{2k^2P^2\Sigma}{4\pi(4\pi\Sigma - 3kM)}. \tag{65}$$

Here $k^2 = 4\pi G$. M, Σ, Q, and P are the ADM mass, scalar charge, total electric charge, and total magnetic charge respectively. The gauge field is given by

$$A_\mu dx^\mu = -\frac{Q_0}{B}\left(r - \frac{k\Sigma}{\sqrt{3}}\right)dv - P_0\cos\theta d\phi. \tag{66}$$

Where we have set $Q_0 = Q/4\pi$ and $P_0 = P/4\pi$. The line element (62) becomes singular when $A = 0$ or $B = 0$. The event horizon, however, is found by setting $N^2 = 0$. The event horizon(s) are, therefore, at

$$r_\pm = GM \pm \sqrt{\frac{G}{4\pi}(k^2M^2 + 16\pi^2\Sigma - Q^2 - P^2)}, \tag{67}$$

5.1 Non-extremal horizon charge analysis

We will now expand the metric (62) in the near horizon limit using $\rho = r + r_+$ to be able to conduct a near horizon symmetry analysis. Near $\rho = 0$ we find the metric functions to be

$$\kappa = \frac{1}{2}\frac{r_+ - r_-}{\sqrt{BC(r_+)}}, \quad \theta_A = 0, \quad \Omega_{\theta\theta} = \sqrt{BC(r_+)}, \quad \Omega_{\phi\phi} = \sqrt{BC(r_+)}\sin^2\theta. \tag{68}$$

We also expand the gauge field near $\rho = 0$ to find

$$\begin{aligned} A_v^{(0)} &= -\frac{Q_0}{B(r_+)}\Big(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\Big), \\ A_v^{(1)} &= -\frac{Q_0}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right), \\ A_B^{(0)} &= -P_0\cos\theta\delta_B^\phi. \end{aligned} \tag{69}$$

Here r_{B+} and r_{B-} are the roots of the B, the location of the singularities. We can now define our general horizon charge expression using (60) [1]:

$$\begin{aligned} Q[T, Y^A, U] = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{\pi} d\phi\, d\theta \sin\theta \Bigg(&(r_+ - r_-)T \\ &+ 4U\frac{Q_0\sqrt{BC(r_+)}}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right) \\ -4\frac{Q_0\sqrt{BC(r_+)}}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)&\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right)P_0\cos\theta Y^\phi\Bigg). \end{aligned} \tag{70}$$

The charges associated to the zero-modes are

$$\begin{aligned} \mathcal{T}_{(0,0)} &= Q[1,0,0] = \frac{1}{2}\kappa\sqrt{BC(r_+)}, \\ \mathcal{U}_{(0,0)} &= Q[0,0,1] = \frac{Q_0}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right), \\ \mathcal{Y}_{(0,0)} &= Q[0,1,0] = 0. \end{aligned} \tag{71}$$

The physical interpretation of the zero-mode of the supertranslation charge is the same as it was for the Schwarzschild black hole. Is the product of the Hawking temperature, $T_H = k/2\pi$ and the Bekenstein-Hawking entropy, $S_{BH} = \pi\sqrt{BC(r_+)}$. The zero-mode charge corresponding to the electromagnetic charge generator is related to the total electric charge of the black hole. The final charge gives the angular momentum of the black hole which is zero. This is what one would expect for a non-rotating black hole solution.

[1] The determinant here is $\sin^2\theta$ and $\Omega = \sqrt{BC(r_+)}$

5.2 Soft hair analysis

To obtain expressions for soft hair on Kaluza–Klein black holes, we must examine these charges on the perturbed horizon. To find an expression for the perturbed horizon and therefore, the perturbed metric functions, the calculations in section 3.2 and 3.3 must be repeated. First, we use the asymptotic Killing vector at null infinity, $\xi = f\partial_v + \frac{1}{r}D^A f\partial_A - \frac{1}{2}D^2 f\partial_r$. (See Appendix C for a derivation.) We find the perturbed metric components by computing

$$h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \tag{72}$$

to obtain the perturbed metric,

$$\begin{aligned}\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = -\Delta + \frac{1}{2}D^2 f\Big[\frac{\partial_r(N^2)}{\sqrt{AB}} - \frac{N^2\partial_r(AB)}{AB^{\frac{3}{2}}}\Big] + 2dudr \\ - D_A\Big(\Delta f + \frac{1}{2}D^2 f\Big)dvdz^A + (\sqrt{AB}\gamma_{AB} + 2rD_A D_B f - r\gamma_{AB}D^2 f)dz^A dz^B,\end{aligned} \tag{73}$$

where $\Delta = N^2/\sqrt{AB}$. Using the argument in Section 3.2 the location of the perturbed black hole must be at

$$(r_+)_f = r_+ + \frac{1}{2}D^2 f. \tag{74}$$

The perturbed metric (73) indicates, as in the case of the Schwarzschild black hole, a superrotation charge at null infinity that has been turned on by the gravitational wave. This superrotation charge is given by coefficients of the $1/r$ terms in Δ.

We also examine horizon charges that are turned on by this gravitational wave, using the analysis of chapter 4. First we will compute transformations in the gauge field using (47). First note

$$\epsilon = -\frac{1}{r}A_B D^B f, \tag{75}$$

from the condition $\mathcal{L}_\xi A_r + \partial_r\epsilon = 0$. To find the gauge field transformations we compute

$$\begin{aligned}\mathcal{L}_\xi A_v + \partial_v\epsilon = \mathcal{O}(\rho), \\ \mathcal{L}_\xi A_B + \partial_B\epsilon = \mathcal{O}(1).\end{aligned} \tag{76}$$

Carrying out the the Lie derivatives in (76) we find

$$\begin{aligned}\delta_{(\xi,\epsilon)}A_v &= -\partial_r\Big(\frac{Q}{4\pi B}\Big)\Big(r - \frac{k\Sigma}{\sqrt{3}}\Big) + \frac{k\Sigma}{\sqrt{3}}\frac{Q}{4\pi B}, \\ \delta_{(\xi,\epsilon)}A_B &= -\frac{Q}{4\pi B}\Big(r - \frac{k\Sigma}{\sqrt{3}}\Big)D_B f + \frac{1}{r}\frac{P}{4\pi}\sin\theta D^A f.\end{aligned} \tag{77}$$

Next we bring the perturbed Kaluza–Klein solution (73) into the near-horizon limit by using

$$\rho = r - (r_+)_f, \quad d\rho = dr - \frac{1}{2}D_A D^2 f dz^A.$$

At order $\mathcal{O}(f)$ we find the perturbed metric functions to be

$$\kappa = \frac{1}{2}\frac{r_+ - r_-}{\sqrt{BC(r_+)}}, \quad \theta_A = -2\kappa D_A f, \quad \Omega_{AB} = \sqrt{BC(r_+)}\gamma_{AB} + 2r_+ D_A D_B f. \tag{78}$$

Note that κ is unchanged from the bald Kaluza–Klein black hole. This is due to the fact that we can find an expression for surface gravity that applies for all spherically symmetric black holes. First, let us find a general expression for the surface gravity from the general near horizon metric (32). From (32) we note that the surface gravity, κ is the coefficient of ρ in the g_{vv} component. Therefore, we assume the general form of κ may be found by expanding Δ near $r = r_+ + \rho$, thus by using a Taylor series,

$$-2\kappa\rho = \Delta|_{r=r_+ + \rho} = \Delta|_{r_+} + \rho\Delta'|_{r_+} + ... \tag{79}$$

The first term in this expansion is zero by definition and thus to order $\mathcal{O}(\rho^2)$ we find

$$\kappa = -\frac{1}{2}\Delta|_{r_+}. \tag{80}$$

Secondly, we want to evaluate the perturbed g_{vv} component, $\bar{\Delta} = \Delta - (1/2)D^2 f \partial_r \Delta$ at $r = r_+ + (1/2)D^2 f$. Using a Taylor series we find

$$\begin{aligned}
\bar{\Delta}|_{r_+ + \frac{1}{2}D^2 f} &= -\frac{1}{2}\Delta'|_{r_+ + \frac{1}{2}D^2 f} + \frac{1}{4}D^2 f \Delta''|_{r_+ + \frac{1}{2}D^2 f} \\
&= -\frac{1}{2}\Big(\Delta'|_{r_+} + \frac{1}{2}D^2 f \Delta''|_{r_+} + ...\Big) + \frac{1}{4}D^2 f \Big(\Delta''|_{r_+} + \frac{1}{2}D^2 f \Delta'''|_{r_+} + ...\Big) \\
&= -\frac{1}{2}\Delta'|_{r_+} + \mathcal{O}((D^2 f)^2),
\end{aligned} \tag{81}$$

which shows that for a small perturbation, in this case, the one due to the supertransformation, the surface gravity remains unchanged at order $\mathcal{O}((D^2 f)^2)$.

To find a horizon superrotation as we did in the case of Schwarzschild black holes, we use the third equation in (37) and obtain

$$Y_A = \frac{r_+}{\sqrt{BC(r_+)}} D_A f. \tag{82}$$

Therefore, it is evident there is a horizon superrotation *and* supertranslation as in the case of the Schwarzschild black hole. This is to be expected since intuitively the same gravitational wave should not produce wildly different results when acting on different black holes.

Let us now consider the transformed gauge field (77) near the perturbed horizon $(r = r_+ + (1/2)D^2 f)$,

$$\begin{aligned}
\delta_{(\xi,\epsilon)} A_v^{(0)} &= -\partial_r\Big(\frac{Q}{4\pi B}\Big)\Big|_{r=r_+}\Big(r_+ - \frac{k\Sigma}{\sqrt{3}}\Big) + \frac{k\Sigma}{\sqrt{3}}\frac{Q}{4\pi B(r_+)}, \\
\delta_{(\xi,\epsilon)} A_B^{(0)} &= -\frac{Q}{4\pi B(r_+)}\Big(r_+ - \frac{k\Sigma}{\sqrt{3}}\Big) D_B f + \frac{1}{r_+}\frac{P}{4\pi}\sin\theta D^A f.
\end{aligned} \tag{83}$$

We can now use the last equation in (54) to find the electromagnetic charge implanted on the horizon from the transient shock wave,

$$U = -\frac{Q}{4\pi B(r_+)}\Big(r_+ - \frac{k\Sigma}{\sqrt{3}}\Big)f - \frac{r_+}{\sqrt{BC(r_+)}}A^{(0)}_B D^B f + \\ \int \Big(\frac{1}{r_+}\frac{P}{4\pi}\sin\theta D^A f - \frac{Pr_+}{4\pi\sqrt{BC(r_+)}}\sin\theta D^A f\Big)dB. \quad (84)$$

Given an integrable $D^A f$ we can find an exact analytic result for (84). It is evident, that as in the case of the perturbed metric (73), a non-trivial charge is generated from the action of horizon symmetry transformation on the bald Kaluza–Klein solution. Furthermore, as discussed in chapter 4, the generator, U is a potential candidate for what we call *soft hairs* on the horizon. Notice from (54) that if there was no superrotation at the horizon due to the shock wave, we would still have a gauge transformation, and this transformation would depend on the gauge parameter alone. This concludes the analysis and discussion on the memory effect due to a transient shock wave striking a black hole in the Kaluza–Klein theory.

While it is a useful exercise to examine the memory effect for a charged black hole, one should note that the 5-dimension Kaluza–Klein is a "toy" model. It is usually treated as a precursor to string theory which contains the idea of unification in higher dimensions. The unification, however, is classical.The status of the Kaluza–Klein theory in 1984 can be found in [34]. An interesting solution to explore in future would be rotating black holes found in the Kerr solution.

Chapter 6

Conclusion

This report explored the connection between the gravitational memory effect and the BMS group. We introduced the BMS group which is an infinite dimensional group of symmetries at null infinity containing the Poincaré group. We derived the generators of this group, using an asymptotic version of Killing's equations and discussed the associated charges, namely, supertranslation and superrotation charges. The action of a gravitational wave striking a Schwarzschild black hole was shown to be equivalent to a pure BMS supertranslation on the Schwarzschild geometry at null infinity. This process turned on a non-vanishing superrotation charge at null infinity, which has a zero mode associated to angular momentum, a measurable quantity. The process was then examined from the point of view of an observer near the horizon. Here the shock wave was found to produce a horizon supertranslation *and* superrotation. The horizon superrotation yields a supertranslation charge that was absent at null infinity.

We proceeded to turn on a gauge field at the horizon which resulted in an extra set of supertranslations that acted non-trivially on the horizon superrotations. It was found in [18] that these supertransformations and gauge transformations generated an algebra of transformations which generated charges. With these tools, we examined the action of a gravitational shock wave on black holes in the Kaluza–Klein theory from [19]. This examination showed two things: Firstly, there was BMS supertranslation as seen from null infinity. Secondly, there was a non-trivial gauge transformation on the black hole horizon due to the shock wave. The generators of these transformations are referred to as a *soft* hair.

Perhaps the most important question regarding the memory effect and the BMS group is that of potential observations. The BMS group contains, as mentioned, an infinite amount of generators, supertranslations and superrotations. While the action of these generators is understood mathematically, how one measures higher modes of these transformations and their charges still faces conceptual hurdles. The memory effect may be detectable with the launch of LISA in the 2030s. With the mathematical formalism of asymptotic symmetries, a deeper physical understanding of supertransformation charges, in order to predict the observational signatures.

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Bibliography

[1] Y. B. Zel'Dovich and A. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars", Soviet Astronomy **18**, 17 (1974).

[2] D. Christodoulou, "Nonlinear nature of gravitation and gravitational-wave experiments", Phys. Rev. Lett. **67**, 1486–1489 (1991).

[3] A. Tolish and R. M. Wald, "Retarded fields of null particles and the memory effect", Phys. Rev. D **89**, 064008 (2014).

[4] M. Favata, "Nonlinear gravitational-wave memory from binary black hole mergers", AP. J. Lett **696**, L159 (2009).

[5] A. Tolish, "The Gravitational Wave Memory Effect and Classical Scattering Problems", PhD thesis (University of Chicago, 2017).

[6] A. Tolish, L. Bieri, D. Garfinkle, and R. M. Wald, "Examination of a simple example of gravitational wave memory", Phys. Rev. D **90**, 044060 (2014).

[7] P. Amaro-Seoane, H. Audley, S. Babak, J. Baker, E. Barausse, P. Bender, E. Berti, P. Binetruy, M. Born, D. Bortoluzzi, et al., "Laser interferometer space antenna", arXiv:1702.00786 (2017).

[8] R. Ruffini and J. A. Wheeler, "Introducing the Black Hole.", Physics Today **24**, 30 (1971).

[9] S. W. Hawking, M. J. Perry, and A. Strominger, "Superrotation charge and supertranslation hair on black holes", JHEP **05**, 161 (2017).

[10] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft Hair on Black Holes", Phys. Rev. Lett **116**, 231301 (2016).

[11] A. Strominger, *Lectures on the infrared structure of gravity and gauge theory* (Princeton University Press, 2018).

[12] Z. L. Lim, "BMS Superrotation and supertranslation (A review of Strominger's work)", National University of Singapore, undergraduate report. Available at URL: https://www.researchgate.net/publication/318429419_BMS_Superrotation_and_supertranslation_A_review_of_Strominger's_work (2017).

[13] S. M. Carroll, *Spacetime and geometry* (Cambridge University Press, 2019).

[14] R. Sachs, "Asymptotic Symmetries in Gravitational Theory", Phys. Rev. **128**, 2851–2864 (1962).

[15] R. K. Sachs, "Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time", Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **270**, 103–126 (1962).

[16] H. Bondi, M. G. J. Van der Burg, and A. Metzner, "Gravitational waves in general relativity, VII. Waves from axi-symmetric isolated system", Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **269**, 21–52 (1962).

[17] G. Barnich and C. Troessaert, "Supertranslations call for superrotations", arXiv e-prints (2011).

[18] L. Donnay, G. Giribet, H. A. González, and A. Puhm, "Black hole memory effect", Phys. Rev. **98**, 124016 (2018).

[19] G. W. Gibbons and D. L. Wiltshire, "Black holes in Kaluza-Klein theory", Annals of Physics **167**, 201–223 (1986).

[20] H. Bondi, "Gravitational waves in general relativity", Nature **186**, 535 (1960).

[21] G. Compère, *Advanced lectures on general relativity*, Vol. 952 (Springer, 2019).

[22] M. Boyle, "Transformations of asymptotic gravitational-wave data", Phys. Rev. D **93**, 084031 (2016).

[23] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence", JHEP **05**, 62 (2010).

[24] H. van der Ven, "The BMS algebra and black hole information", PhD thesis (Utrecht University, 2016).

[25] E. Teo, *Supertranslated vaccum and black-hole spacetimes*, Talk given by Edward Teo, from National University of Singapore, 2017.

[26] G. Barnich and C. Troessaert, "BMS charge algebra", JHEP **12**, 105 (2011).

[27] P. Chrusciel, "The geometry of black holes", BlackHolesViennaJanuary2015.pdf, Erwin Schrödinger Institute and Faculty of Phys, 2015.

[28] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics* (Cambridge university press, 2004).

[29] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon", Phys. Rev. Lett **116**, 091101 (2016).

[30] G. Barnich and P.-H. Lambert, "Einstein-Yang-Mills theory: Asymptotic symmetries", Phys. Rev. D **88**, 103006 (2013).

[31] J. A. de Azcárraga and J. M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics* (Cambridge University Press, 1998).

[32] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Extended symmetries at the black hole horizon", JHEP **09**, 100 (2016).

[33] T. Kaluza, "Zum Unitätsproblem der Physik", Sitzungsber. Preuss. Akad. Wiss. Phys. Math. **1921**, 966–972 (1921).

[34] A. Chodos, "Kaluza-Klein theories: an overview", Comments Nucl. Part. Phys. **13**, 171–181 (1984).

Appendix A

Perturbed Schwarzschild Solution

A.1 Null infinity limit

Here will determine the perturbed metric components. These are found by taking Lie derivatives of metric components. We will consider the following perturbations:

$$\begin{aligned} h_{vv} &= \mathcal{L}_\xi g_{vv} = \xi^\mu \partial_\mu g_{vv} + 2g_{\mu v}\partial_v \xi^\mu, \\ h_{vA} &= \mathcal{L}_\xi g_{vA} = \xi^\mu \partial_\mu g_{vA} + g_{\mu A}\partial_v \xi^\mu + g_{v\mu}\partial_A \xi^\mu, \\ h_{AB} &= \mathcal{L}_\xi g_{AB} = \xi^\mu \partial_\mu g_{AB} + 2g_{C(A}\partial_{B)}\xi^C. \end{aligned} \tag{85}$$

Here, we use the asymptotic Killing vector,

$$\xi^\mu \partial_\mu = f\partial_v - \frac{1}{2}D^2 f \partial_r + \frac{1}{r}D^A f \partial_A.$$

We compute the first component by expanding the sums in this equation and then substituting in the non-zero components and find

$$h_{vv} = -\frac{1}{2}D^2 f \partial_r(g_{vv}) + 0 = -\frac{1}{2}D^2 f\Big(\frac{M}{r^2}D^2 f\Big). \tag{86}$$

This follows the fact that partial derivatives commute and $\partial_r f = 0$. The equation for the second component becomes

$$h_{vA} = 0 + g_{AB}\partial_v \xi^B + 2\Big(g_{vv}\partial_A f + \partial_A\Big(\frac{1}{2}D^2 f\Big)\Big). \tag{87}$$

We now use the condition on f we find from the boundary conditions, namely, $\partial_v f = 0$ and so we find,

$$h_{vA} = -2D_A\Big(f - \frac{2M}{r} + D^2 f\Big). \tag{88}$$

Finally, after computing the final equation in (85) we find,

$$h_{AB} = 2rD_A D_B f - r\gamma_{AB}D^2 f. \tag{89}$$

Collecting (85)–(89) together, we find the metric perturbations that are given in chapter 3.

A.2 Near horizon limit

In the case of the Schwarzschild solution, we are able to bring the perturbed metric (27) to the near horizon limit simply. The first term, $\kappa\rho$ was derived in chapter 3. The third term in (27), however, will be derived here. Recall the term we are trying to expand near the horizon is,

$$2f - \frac{4M}{r}f + D^2 f, \tag{90}$$

this is the term that is also in (88). We wish switch coordinates to $\rho = r - r_{(f+)}$ where $r_{(f+)}$ is the perturbed horizon. We proceed by substituting this into (90), and then Taylor expand about the horizon ($\rho = 0$),

$$2f - \frac{4M}{\rho + r_{(f+)}}f + D^2 f = 2f - f\frac{4M}{r_+ + (1/2)D^2 f} + f\frac{\rho\, 4M}{(r_+ + (1/2)D^2 f)^2} + D^2 f + ... \tag{91}$$

The ellipsis here denote terms that are $\mathcal{O}(\rho^2)$. Using an similar to that in chapter 3 we show that to linear order, in f, we are left with

$$2f - f\frac{4M}{r_+} + \frac{\rho\, 4M}{(r_+)^2} + D^2 f. \tag{92}$$

Now using the definition for the Schwarzschild radius, $r = 2M$, we find the perturbation, h_{vA} near the horizon is

$$h_{vA}(\text{horizon}) = \frac{2}{r_+}\rho + D^2 f. \tag{93}$$

Note the $D^2 f$ becomes absorbed into the second term in (27) now with the change in the 1-form, $d\rho$. The final term in (27) follows directly from the boundary condition, i.e., Ω_{AB} is order $\mathcal{O}(\rho)$.

Appendix B

Symmetry algebra

It was previously stated that Killing vectors generate a Lie algebra and the commutation relations of one such algebra were given in chapter 4. Here we explain how one would derive these commutation relations. Let us begin with the definition of the Modified Lie bracket given in [23],

$$[\xi_1, \xi_2]_M = [\xi_1, \xi_2] + \delta_{\xi_1}\xi_2 - \delta_{\xi_2}\xi_1. \tag{94}$$

Here ξ_1 and ξ_2 are Killing vectors. This definition is implemented to "subtract off" the dependence on any fields. The bracket takes the original Lie bracket and subtracts the Killing vector action on the field dependent components in the second Killing vector. In the near horizon limit considered in chapter 4, the Killing vector does not actually depend on fields at order $\mathcal{O}(\rho)$ and so the definition in (94) does not need to be used. We can compute the normal commutator to find

$$\begin{aligned}[\xi_1, \xi_2]_M = (f_1\partial_v f_2 - f_2\partial_v f_1 + Y_1\partial_A f_2 - Y_2\partial_A f_1)\partial_v + \\ (f_1\partial_v Y_2 - f_2\partial_v Y_1 + Y_1\partial_A Y_2 - Y_2\partial_A Y_1)\partial_A.\end{aligned} \tag{95}$$

Here we have taken $\xi^\mu\partial_\mu$ to be order $\mathcal{O}(\rho)$. Furthermore, it is obvious that ξ^r does not play a role in this algebra since $\partial_r f = 0$ and every term component of the Killing vector contains constants with respect to r. The gauge field, however, does require the use of (94) as at order $\mathcal{O}(\rho)$ we have dependence on fields. By this we mean (53) is

$$\epsilon(v, z^A) = U(z^A) - f(v, z^A)A_v^{(0)}. \tag{96}$$

It is apparent that this is still dependent on the gauge field through the $A_v^{(0)}$ term. For the gauge field, the Lie bracket we consider is given by,

$$[\epsilon_1, \epsilon_2]_M = \xi_1^\mu\partial_\mu\epsilon_2 - \xi_2^\mu\partial_\mu\epsilon_1 + \delta_{\epsilon_1}\epsilon_2 - \delta_{\epsilon_2}\epsilon_1. \tag{97}$$

The above equation results in,

$$\begin{aligned}[\epsilon_1, \epsilon_2] = Y_1^A\partial_A U_2 - Y_2^A\partial_A U_1 + f_1\partial_v(f_2 A_v^{(0)}) - f_2\partial_u(f_1 A_v^{(0)}) \\ + Y_1^A\partial_A(f_2 A_v^{(0)}) - Y_2^A\partial_A(f_1 A_v^{(0)}) + \delta_{\epsilon_1}\epsilon_2 - \delta_{\epsilon_2}\epsilon_1.\end{aligned} \tag{98}$$

We now note that the final two terms only consider the field contributions, and are identical to the middle 4 terms in (98). We are, therefore, only left with the first two terms. We say the algebra closes with the brackets, (95) and (98). We can now derive the commutation relations found in chapter 4. Let us first note that $f = T(z^A) + e^{-\kappa v} X(z^A)$ in this scenario. Note that (95) specifies that the algebra closes with the commutator of f_1 and f_2, let us label $T = f_1$ and $e^{-\kappa v} X = f_2$. We now see the commutator of these must be

$$[z^k \bar{z}^l \partial_v, e^{-\kappa v} z^m \bar{z}^n \partial_v] = -\kappa X_{(m+k,l+n)}. \tag{99}$$

Here we have used the expanded forms of T and X given in chapter 4. What is not obvious in chapter 4 is that the commutator of $T_{(m,n)}$ and $U_{(m,n)}$ is 0. However, from (98) we now see that this is because the commutator of supertranslations (T and U) and the electromagnetic charge generator, $U_{(m,n)}$ is not in the algebra, thus is trivially zero. The other commutators can now be determined by the process above. First, we note which components are in the closure of the algebra, and then compute the commutators using the appropriate derivative operators, ∂_μ.

Appendix C

Asymptotic Killing vector in the Kaluza–Klein theory

The asymptotic Killing vector, as shown in chapter 2, is derived from taking Lie derivatives of the metric. In most cases, such as the Schwarzschild, this is straightforward. In the case of the Kaluza–Klein solution (62), a few extra steps must be taken as the "r" coordinate is shifted. We must take the limit as r becomes large (at null infinity) to find the asymptotic Killing vector. The Lie derivatives we consider are

$$\begin{aligned}
&\mathcal{L}_\xi g_{rA} = g_{AB}\partial_r\xi^B + \partial_A\xi^v = 0, \\
&\mathcal{L}_\xi g_{rr} = 2\partial_r\xi^v = 0, \\
&g^{AB}\mathcal{L}_\xi g_{AB} = g^{AB}(\xi^r\partial_r g_{AB} + 2g_{C(A}\partial_{B)}\xi^C) = 0.
\end{aligned} \tag{100}$$

Here, the third equation is a condition we use that is found in [23]. Furthermore, this condition holds when r becomes large. From the second equation in (100) we find

$$\xi^v = f(v, A), \tag{101}$$

therefore, the first equation in (100) becomes

$$\xi^B = -\int^r \frac{1}{\sqrt{BC}} D^A f dr'. \tag{102}$$

Instead of attempting to integrate this equation directly, we recall that there are prescribed boundary conditions, which are in general $\mathcal{O}(r^{-1})$ or $\mathcal{O}(r^{-2})$. Therefore, we proceed by expanding $1/\sqrt{BC}$ near $r = \infty$. Let us first rewrite B and C from (64) and (65) respectively as,

$$\begin{aligned}
B &= (r-\alpha)^2 - \beta, \\
C &= (r+\alpha)^2 - \zeta,
\end{aligned} \tag{103}$$

where

$$\alpha = \frac{k\Sigma}{\sqrt{3}}, \quad \beta = \frac{2k^2Q^2\Sigma}{4\pi(4\pi\Sigma + 3kM)}, \quad \text{and} \quad \zeta = \frac{2k^2P^2\Sigma}{4\pi(4\pi\Sigma - 3kM)}.$$

Using (103) we expand BC to obtain

$$BC = r^4 - \zeta r^2 - 2\alpha^2 r^2 - \beta r^2 + 2\alpha\zeta r - 2\beta\alpha r - \alpha^2\zeta + \alpha^4 + \beta\zeta - \beta\alpha^2. \quad (104)$$

We now rewrite $1/\sqrt{BC}$ as

$$\frac{1}{\sqrt{BC}} = \frac{1}{r^2}\Big[1 - \frac{2\alpha^2}{r^2} - \frac{\beta}{r^2} - \frac{\zeta}{r^2} + \mathcal{O}(r^{-3)}\Big]^{-\frac{1}{2}}. \quad (105)$$

Now note, that as $r \to \infty$ we can use a binomial expansion of the square root to obtain

$$\frac{1}{\sqrt{BC}} = \frac{1}{r^2} + \frac{2\alpha^2 + \beta + \zeta}{r^4} + \mathcal{O}\Big(\frac{1}{r^5}\Big). \quad (106)$$

Now, noting that the integral of any term beyond the first, that is, beyond $1/r^2$ we will result in $\mathcal{O}(r^{-3})$ terms, we can neglect these terms in (102) by understanding ξ^B to be order $\mathcal{O}(r^{-2})$. Therefore, we find

$$\xi^B = \frac{1}{r}D^A f. \quad (107)$$

We are now able to solve the third equation in (100) to find ξ^r. First note that

$$\sqrt{BC} = r^2 - \alpha^2 - \beta - \zeta + \mathcal{O}(r^{-1}) \quad (108)$$

from the same steps used in (104)–(106). The third equation in (100) becomes

$$2r\gamma_{AB}\xi^r + \sqrt{BC}\,\gamma_{CB}\,D_A\Big(\frac{1}{r}D^C f\Big) = 0, \quad (109)$$

where we have used $g_{AB} = \sqrt{BC}\,\gamma_{AB}$, γ_{AB} being the unit 2−sphere metric. Note that we have also neglected terms order $\mathcal{O}(r^{-1})$ in $\sqrt{BC}$ when taking computing $\partial_r\sqrt{BC}\gamma_{AB}$. This is because these terms would become order $\mathcal{O}(r^{-2})$ and thus are not considered. Rearranging (109) we find

$$\xi^r = -\frac{1}{2}\frac{\sqrt{BC}}{r^2}D^2 f, \quad (110)$$

by considering (108), and neglecting terms order $\mathcal{O}(r^{-2})$ then we find

$$\xi^r = -\frac{1}{2}D^2 f. \quad (111)$$

Therefore, as in the case of the Schwarzschild metric black holes (found in [18]) we find the asymptotic Killing vector to be

$$\xi^\mu\partial_\mu = f\partial_v - \frac{1}{2}D^2 f\partial_r + \frac{1}{r}D^A f\partial_A. \quad (112)$$

Appendix D

Gauge field expansion

We refer to the gauge expansion terms as $A_v^{(0)}$ and $A_v^{(1)}$. Consider the following example: let

$$A_v = -\frac{q}{r}. \tag{113}$$

Then changing coordinates to the near horizon coordinates we use, $\rho = r - r_+$, (113) becomes

$$A_v = -\frac{q}{r_+ + \rho}. \tag{114}$$

If we now perform a Taylor expansion of (114) about $\rho = 0$ then we find,

$$\begin{aligned} A_v &= -\frac{q}{r}\bigg|_{r_+} + \rho\frac{q}{r^2}\bigg|_{r_+} + \mathcal{O}(\rho^2) \\ &= -\frac{q}{r_+} + \rho\frac{q}{r_+^2} + \mathcal{O}(\rho^2). \end{aligned} \tag{115}$$

We now say that $A_v^{(0)} = -q/r_+$ and $A_v^{(1)} = q/r_+^2$. We will now derive the $A_v^{(0)}$ and the $A_v^{(1)}$ terms for the Kaluza–Klein gauge field. Consider the v component of the gauge field in the coordinates $\rho = r - r_+$,

$$A_v = \frac{Q_0}{(r_+ + \rho - r_{B+})(r_+ + \rho - r_{B-})}\Big(\rho + r_+ - \frac{k\Sigma}{\sqrt{3}}\Big). \tag{116}$$

We can rewrite (116) in a form that makes the expansion step clear,

$$\begin{aligned} A_v = \frac{Q_0}{(r_+ - r_{B+})(r_+ - r_{B-})}\left(r_+ - \frac{k\Sigma}{\sqrt{3}}\right)\Bigg[&\Big(1 + \frac{\rho}{r_+ + \frac{k\Sigma}{\sqrt{3}}}\Big) \\ &\Big(1 + \frac{\rho}{(r_+ - r_{B+})}\Big)^{-1}\Big(1 + \frac{\rho}{(r_+ - r_{B-})}\Big)^{-1}\Bigg]. \end{aligned} \tag{117}$$

Now note that as $\rho \to 0$ we can use a binomial expansion of the last two terms in (117). To order $\mathcal{O}(\rho^2)$ we find,

$$A_v = -\frac{Q_0}{B(r_+)}\Big(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\Big) - \frac{Q_0\rho}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right). \tag{118}$$

Therefore, after making comparisons with the explanations in (113)–(115), we find

$$\begin{aligned} A_v^{(0)} &= -\frac{Q_0}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right), \\ A_v^{(1)} &= -\frac{Q_0}{B(r_+)}\left(r_+ - \frac{\kappa\Sigma}{\sqrt{3}}\right)\left(\frac{1}{r_+ - \frac{k\Sigma}{\sqrt{3}}} - \frac{1}{r_+ - r_{B+}} - \frac{1}{r_+ - r_{B-}}\right). \end{aligned} \tag{119}$$