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Dimensional Analysis

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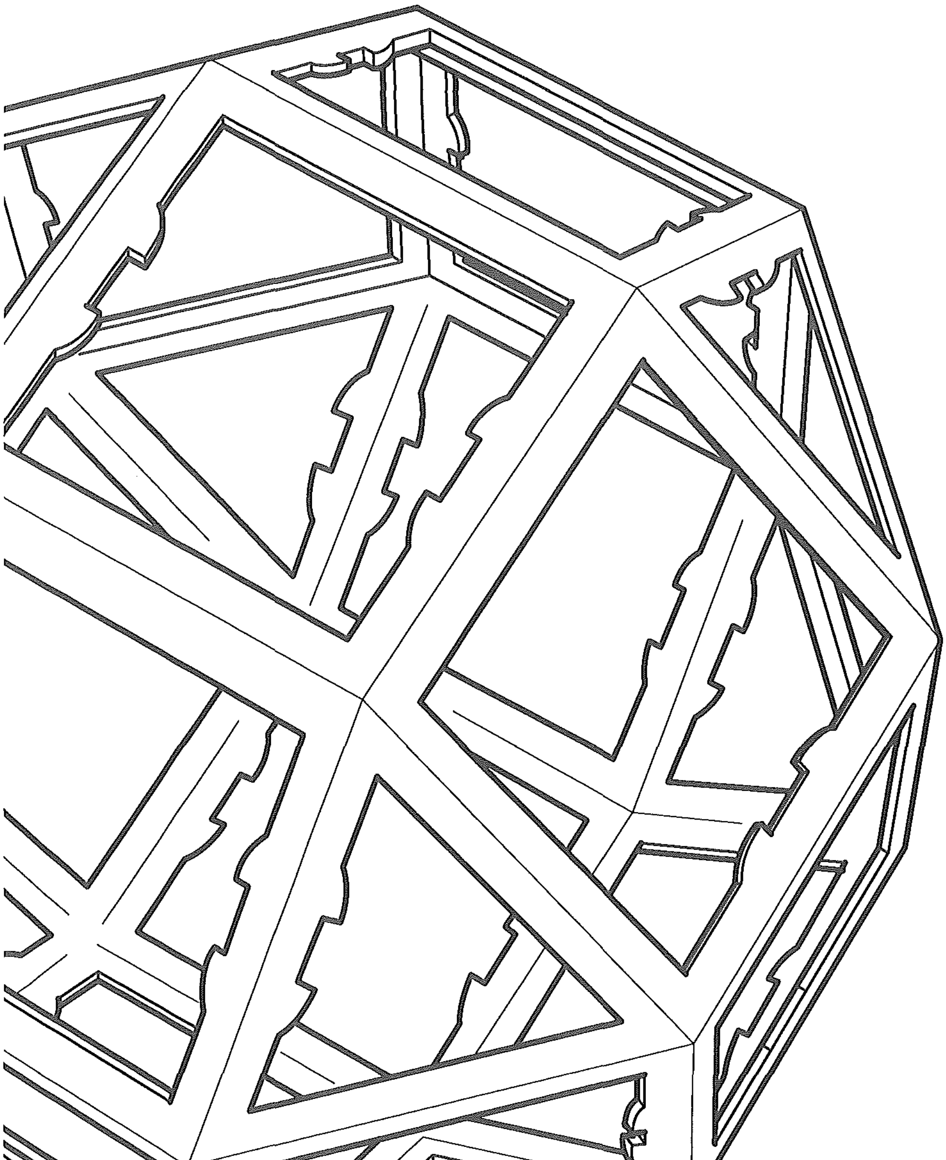

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Dimensional analysis

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Abstract

The Π-theorem was proved and dimensional analysis was used to derive basic formulae for various physical phenomena. Dimensional analysis was used to non-dimensionalize equations leading to the appearance of key dimensionless groups and the sometimes powerful extension due to Huntley was explored. The theory of modeling was explained and self-similar solutions were sought to problems. Types of similarity were classified, and it was shown that dimensional analysis can decrease the number of variables in a problem when formulating a solution under the assumption of complete or incomplete similarity and when dimensional analysis is used in conjunction with a transformation group. To illustrate the ideas we used real-world examples from a range of disciplines.

1 Introduction

Long before Dimensional Analysis was given its name and a proper mathematical formulation physicists were aware of its content. Newton knew that in his Law of Gravitation $F = \frac{Gm_1m_2}{r^2}$, expressions on the left and right of the equality both have the dimension of force. Fourier in 1822 used dimensional reasoning in his book "Theorie analytique de la chaleur" about a hundred years before Buckingham (1914) introduced and proved the $\Pi-$theorem on which dimensional analysis is based. In 1922 Bridgman wrote the first book on the subject titled "Dimensional Analysis". In his book Bridgman showed that dimensional analysis has applications in engineering, theoretical physics and in modeling experiments. In the 1940s when the first nuclear bombs were being tested by the United States, the energy released by the nuclear bomb was classified information. However, pictures of the explosions were publicly available. G.I Taylor estimated the energy released in such explosions by using these pictures and dimensional analysis alone (Barenblatt, 2003a).

Dimensional analysis can be used to establish the form for formulae of physical phenomena, from the simple formulae of the period of a pendulum or the range of a projectile in classical mechanics to the more sophisticated formula of Planck's radiation law in quantum physics. In the past two decades dimensional analysis has gained renewed importance because of its applications in fluid mechanics. On the more theoretical side of things, dimensional analysis is used in the formulation of self-similar solutions and explains the theory of model scaling. Today dimensional analysis is used in such vast fields as metrology (Esnault-Pelterie, 1950), astrophysics (R.Kurth, 1972), economics (Jong et al., 1976), biology (H.Brown and B.west, 2000), medications (Clement-O'Brien and Lawler, 1998) and fractal geometry (Barenblatt, 2003b).

To introduce some of the basic definitions and notations in dimensional analysis we begin by considering an example: Taylor's analysis of the basic intermediate stage of a nuclear explosion. He considered the ideal case in which a finite amount of energy is released in an infinitely concentrated form. This produces a spherical shock wave, with the pressure inside the shock wave thousands of times greater than the initial air pressure. In doing so Taylor was able to neglect the initial air pressure p_o and take the initial radius of the shock wave r_o as zero. This was a crucial step in his analysis of the explosion. In fact, when we are applying dimensional analysis to any phenomenon we have to decide on the variables or parameters that we think are important in describing that phenomenon. Too many superfluous variables can make our analysis difficult if not impossible. On the contrary, too few variables can lead to a false representation of the phenomenon.

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The questions that arise are: how does the radius of the shockwave front r_f grow? And, what are the quantities that it depends on? Having neglected r_o, p_o we make a list of the other quantities:

1. E, the total energy concentrated in the sphere.
2. ρ_o, the initial air density.
3. t, the time elapsed after the explosion.

For this problem we will work in the LMT class. The name of this class is informative as it means that the dimensions of all other physical quantities involved in this problem are products of L which is the dimension of length, M the dimension of mass and T the dimension of time. *The dimension of a physical quantity is the factor by which its numerical value changes when we pass from one system of units to another within the same class* (Barenblatt, 1987a). For example; if a is the length of something and it is measured to be 1 meter then a has a numerical value of 100 measured in centimeters. Here $L = 100$ which is the factor by which the numerical value of a changes. A *class* is a set of systems of units that differ only in magnitude of the fundamental units. Examples of systems of units are the SI system in which meters are the unit for length, seconds for time, and kilograms for mass and the CGS system in which centimeters are the unit for length, seconds for time, and grams for mass. Both the CGS and SI systems belong to the LMT class. Later it is shown that not all problems can be dealt with in the LMT class. If a problem involves temperature then we have to work in the $LMT\Theta$ class, where Θ describes the physical nature of temperature. For each problem we have to work in the appropriate class. Examples of other classes are the LFT class and the $LT\Theta$ class. One of the fundamental ideas in dimensional analysis is that any meaningful equation relating some physical quantities has to be dimensionally homogeneous. This means that for our example any expression containing E, ρ_o, t has to have the dimension of length because r_f has the dimension of length. The idea is formally referred to as *Dimensional Homogeneity.* We denote the dimension of a quantity x by $[x]$. Here the dimension function of radius r_f in the LMT class is $[r_f] = L$. The dimension function is a power law monomial which is proved as lemma 1. From this and the property of dimensional homogeneity we have

$$[r_f] = L = [E]^a[\rho_o]^b[t]^c = (ML^2T^{-2})^a(ML^{-3})^b(T)^c = M^{a+b}L^{2a-3b}T^{-2a+c}$$

where a, b and c are unknown constants. This gives us a system of equations which can be solved for a, b, c:

$$\begin{aligned} a + b &= 0 \\ 2a - 3b &= 1 \\ -2a + c &= 0 \end{aligned}$$

The solution is $a = \frac{1}{5}$, $b = -\frac{1}{5}$ and $c = \frac{2}{5}$. Therefore

$$r_f = CE^{\frac{1}{5}}\rho_o{}^{-\frac{1}{5}}t^{\frac{2}{5}} = C(E\rho_o t^2)^{\frac{1}{5}}.$$

Using the pictures taken of the first atomic explosion in New Mexico in July 1945 Taylor measured the radius as a function of time and was able to estimate the energy released in the explosion using the relation derived above. The constant C was measured to have a value close to unity (Barenblatt, 2003a). Although the equation is dimensionally homogeneous, the left and right hand sides could be numerically different which is why we included the dimensionless constant C.

2 Important results and Theorems

Lemma 1. *The dimension function is always a power-law monomial (Barenblatt, 1987b).*

Proof. Consider a physical quantity a in a general class $PQ...R$. The dimension of a is defined as $[a]$. It only depends on the quantities $PQ...R$. $[a] = \phi(P, Q, ..., R)$. We choose two new systems by decreasing the fundamental units by $P_1, Q_1..., R_1$ and $P_2, Q_2, .., R_2$ then the numerical value of a will increase by $\phi(P_1, Q_1..., R_1)$ and $\phi(P_2, Q_2, .., R_2)$ respectively. Let a_1 and a_2 be the physical quantity in these new systems. Then

$$a_1 = a\phi(P_1, Q_1..., R_1)$$
$$a_2 = a\phi(P_2, Q_2, .., R_2)$$
$$\implies \frac{a_2}{a_1} = \frac{\phi(P_2, Q_2, .., R_2)}{\phi(P_1, Q_1..., R_1)}$$

Here the second system can be obtained from the first by changing the units by a factor of $\frac{P_2}{P_1}\frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1}$

$$\text{Thus } a_2 = a_1\phi(\frac{P_2}{P_1}, \frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1}) \tag{1}$$
$$\implies \frac{\phi(P_2, Q_2, .., R_2)}{\phi(P_1, Q_1..., R_1)} = \phi(\frac{P_2}{P_1}, \frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1})$$

Assume the dimension function is smooth and differentiable. We differentiate both sides with respect to P_2 and set $P_2 = P_1 = P$, $Q_2 = Q_1 = Q, ..., R_2 = R_1 = R$.

$$\frac{1}{\phi(P_1, Q_1..., R_1)}\frac{\partial\phi(P_2, Q_2, .., R_2)}{\partial P_2} = \frac{1}{P_1}\frac{\partial\phi}{\partial P_2}(\frac{P_2}{P_1}\frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1})$$
$$\frac{1}{\phi(P, Q..., R)}\frac{\partial\phi(P, Q, .., R)}{\partial P} = \frac{1}{P}\frac{\partial\phi}{\partial P}(1, 1, ..., 1) = \frac{\alpha}{P}$$

where α is a constant.

$$\int\frac{1}{\phi(P, Q, .., R)}\partial\phi(P, Q, .., R) = \int\frac{\alpha}{P}\partial P$$
$$\ln\phi(P, Q, .., R) = \alpha\ln P + C(Q, ..., R)$$
$$\implies \phi(P, Q, .., R) = P^\alpha f(Q, ..., R)$$

Substituting in equation (1)

$$\frac{{P_2}^\alpha f(Q_2, ..., R_2)}{{P_1}^\alpha f(Q_1, ..., R_1)} = f(\frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1})\frac{{P_2}^\alpha}{{P_1}^\alpha}$$

We follow the same technique,

$$\frac{1}{f(Q_1, ..., R_1)}\frac{\partial f(Q_2, ..., R_2)}{\partial Q_2} = \frac{1}{Q_1}\frac{\partial f}{\partial Q_2}(\frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1})$$

Setting $P_2 = P_1 = P$, $Q_2 = Q_1 = Q, ..., R_2 = R_1 = R$.

$$\int\frac{1}{f(Q, ..., R)}\frac{\partial f(Q, ..., R)}{\partial Q} = \int\frac{\beta}{Q}\partial Q$$

We get $f(Q, ..., R) = g(, ..., R)Q^\beta$

$$\phi(P, Q, .., R) = P^\alpha Q^\beta g(, ..., R)$$

Repeating the steps we find $\phi(P, Q, ..., R) = CP^\alpha Q^\beta ...R^\gamma$ as we had

$$\frac{\phi(P_2, Q_2, .., R_2)}{\phi(P_1, Q_1..., R_1)} = \phi(\frac{P_2}{P_1}\frac{Q_2}{Q_1}, ..., \frac{R_2}{R_1})$$
$$\implies \phi(1, 1, ..., 1) = 1 \text{ thus } C = 1 \text{ and } \phi(P, Q, ..., R) = P^\alpha Q^\beta ...R^\gamma$$

□

Theorem 1. *For a set of quantities $a_1, ..., a_k$ with independent dimensions in a given class of system of units PQ.... It is possible to pass from one system to another within the same class so that only one of the quantities changes its numerical value by a specified factor (Barenblatt, 1987c).*

Proof. From lemma 1 the dimensions of the quantities $a_1, ..., a_k$ have the form

$$[a_1] = P^{\alpha_1} Q^{\beta_1} \cdots$$
$$\vdots$$
$$[a_k] = P^{\alpha_k} Q^{\beta_k} \cdots$$

Without loss of generality assume a_1 changes by a factor A, $A \neq 1$

$$\begin{aligned} a'_1 &= a_1 A \\ a'_2 &= a_2 \\ &\vdots \\ a'_k &= a_k \end{aligned}$$

Here the terms with prime denote the new quantities.But

$$\begin{aligned} a'_1 &= a_1[a_1] = a_1 P^{\alpha_1} Q^{\beta_1} \cdots \\ a'_2 &= a_2[a_2] = a_2 P^{\alpha_2} Q^{\beta_2} \cdots \\ &\vdots \\ a'_k &= a_k[a_k] = a_k P^{\alpha_k} Q^{\beta_k} \cdots \end{aligned} \quad \Longrightarrow \quad \begin{aligned} P^{\alpha_1} Q^{\beta_1} \cdots &= A \\ P^{\alpha_2} Q^{\beta_2} \cdots &= 1 \\ &\vdots \\ P^{\alpha_k} Q^{\beta_k} \cdots &= 1 \end{aligned}$$

Taking logarithms we get

$$\begin{aligned} \alpha_1 \ln P + \beta_1 \ln Q \cdots &= \ln A \\ \alpha_2 \ln P + \beta_2 \ln Q \cdots &= 0 \\ &\vdots \\ \alpha_k \ln P + \beta_k \ln Q \cdots &= 0 \end{aligned}$$

Note that in each equation at least one of $\alpha_m, \beta_m, \cdots$ is non-zero or else the quantity a_m would be dimensionless.
Assume the system of equation is inconsistent. This means the left hand side of the first equation is a linear combination of the others. i.e

$$\begin{aligned} \alpha_1 \ln P + \beta_1 \ln Q \cdots &= c_2(\alpha_2 \ln P + \beta_2 \ln Q \cdots) + \cdots + c_k(\alpha_k \ln P + \beta_k \ln Q \cdots) \\ P^{\alpha_1} Q^{\beta_1} \cdots &= (P^{c_2\alpha_2} Q^{c_2\beta_2} \cdots) \cdots (P^{c_k\alpha_k} Q^{c_k\beta_k} \cdots) \\ [a_1] &= (P^{\alpha_2} Q^{\beta_2} \cdots)^{c_2} \cdots (P^{\alpha_k} Q^{\beta_k} \cdots)^{c_k} \\ &= [a_2]^{c_2} \cdots [a_k]^{c_k} \end{aligned}$$

We get a contradiction as the quantities $a_1, a_2, \cdots, a_k$ were defined to have independent dimensions. Thus the system of equations is consistent and the statement is proved □

Theorem 2 (Buckingham's Pi-theorem). *Let $a = f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m)$ be some function that defines a relation between the physical quantities $a, a_1, \cdots, a_k, b_1, \cdots, b_m$, where the parameters $a_1, \cdots, a_k$ have* *independent dimensions and dimensions of* $b_1, \cdots, b_m$ *are found from those of* $a_1, \cdots, a_k$*, then there exists* $\phi(\frac{b_1}{a_1{}^{p_1} \cdots a_k{}^{r_1}}, \cdots, \frac{b_m}{a_1{}^{p_m} \cdots a_k{}^{r_m}})$ *s.t*

$$\begin{aligned} f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m) &= a_1{}^p \cdots a_k{}^r \phi(\frac{b_1}{a_1{}^{p_1} \cdots a_k{}^{r_1}}, \cdots, \frac{b_m}{a_1{}^{p_m} \cdots a_k{}^{r_m}}) \\ &= a_1{}^p \cdots a_k{}^r \phi(\Pi_1, \cdots, \Pi_m) \end{aligned}$$

where the parameters $\Pi_1, \cdots, \Pi_m$ *are dimensionless (Barenblatt, 1987d).*

Proof. By definition

$$\begin{array}{lcr} \Pi_1 = \dfrac{b_1}{a_1{}^{p_1} \cdots a_k{}^{r_1}} & \text{Rearrange} & b_1 = \Pi_1(a_1{}^{p_1} \cdots a_k{}^{r_1}) \\ \vdots & \text{to get} & \vdots \\ \Pi_m = \dfrac{b_m}{a_1{}^{p_m} \cdots a_k{}^{r_m}} & & b_m = \Pi_m(a_1{}^{p_m} \cdots a_k{}^{r_m}) \end{array}$$

Let $\Pi = \frac{a}{a_1{}^p \cdots a_k{}^r}$ represent the dimensionless form of quantity a. Thus

$$\begin{aligned} \Pi &= \frac{f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m)}{a_1{}^p \cdots a_k{}^r} \\ &= \frac{1}{a_1{}^p \cdots a_k{}^r} f(a_1, \cdots a_k, \Pi_1 a_1{}^{p_1} \cdots a_k{}^{r_1}, \cdots, \Pi_m a_1{}^{p_m} \cdots a_k{}^{r_m}) \\ &= F(a_1, \cdots a_k, \Pi_1, \cdots \Pi_m) \end{aligned}$$

By theorem 1, we can pass onto another system so that a_1 changes by an arbitrary factor but the quantities $a_2, \cdots, a_k$ remain unchanged. The arguments and $\Pi, \Pi_1, \cdots, \Pi_m$ are dimensionless so they too remain unchanged. Therefore F is independent of the parameter a_1. By a similar argument F is also independent of $a_2 \cdots a_k$. Thus $F = \phi(\Pi_1, \cdots, \Pi_m)$

$$\Pi = \frac{f}{a_1{}^p \cdots a_k{}^r} \implies \frac{f}{a_1{}^p \cdots a_k{}^r} = \phi(\Pi_1, \cdots, \Pi_m)$$
$$f(a_1, \cdots, a_k, b_1, \cdots, b_m) = a_1{}^p \cdots a_k{}^r \phi(\Pi_1, \cdots, \Pi_m)$$

□

Corollary 3. *The number of dimensionless products* Π_j *is* m

Proof. Let n be the total number of parameters. $n = k + m$.

The number of dimensionless products is equal to the total number of parameters minus the number of parameters with independent dimensions $= n - k = k + m - k = m$ □

3 Examples

In this section we consider some basic examples and introduce the useful idea of the dimension matrix.

3.1 The period of a pendulum

Using dimensional analysis we derive the formula for the period τ of small oscillations of a pendulum, in the MLT class.

We make a list of the quantities that τ could depend on.

- Length of string l, $[l] = L$.

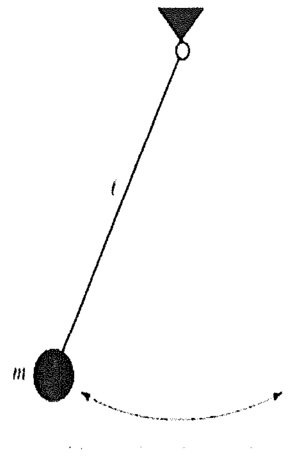

Figure 1: Period of a pendulum for small oscillations

- Mass of particle m, $[m] = M$.
- Acceleration due to gravity g, $[g] = LT^{-2}$.
- Amplitude of oscillation x, $[x] = L$.

Thus $\tau = f(m, l, g, x)$. We note that there are only three quantities with independent dimensions. Thus $n = 4$, $k = 3$, $m = 1$. Thus $\tau = f(m, l, g, x)$=$m^p l^q g^r \phi(\Pi_1)$. The left hand side has units of T, as the equation has to be dimensionally homogeneous it follows that $T = M^p L^q (LT^{-2})^r$. Thus

$$\begin{aligned} p &= 0 \\ q + r &= 0 \\ -2r &= 1 \\ \text{i.e } r &= -\frac{1}{2} \implies q = \frac{1}{2}. \end{aligned}$$

We have $\tau = \phi(\Pi_1) l^{\frac{1}{2}} g^{-\frac{1}{2}} = \phi(\Pi_1)\sqrt{\frac{l}{g}}$ where $\Pi_1 = \frac{a}{l}$. It turns out that $\phi(\Pi_1)$ is a constant, $\phi(\Pi_1) = 2\pi$.

In the above example l was taken to be the quantity with independent dimension and x to be the quantity with derived dimension, without any explanation. Since both l and x have dimensions of L one could argue as to why x was not taken as the quantity with independent dimension in which case the derived formula would be $\tau = \phi(\Pi_1)\sqrt{\frac{x}{g}}$. Mathematics and experimentation go hand in hand. If we had chosen x to be the independent quantity a simple experiment where we change the amplitude but not the length of the string would show otherwise.

3.2 Dimensional analysis of Poiseuille flow

Poiseuille flow is the flow of a Newtonian viscous fluid through a pipe of radius a and length l. Here we find a formula for the ΔP the pressure drop between the ends of the pipe. The other physical quantities we have to consider are viscosity μ, density ρ, velocity U of the fluid.

$$\left.\begin{aligned} [l] &= L \\ [a] &= L \end{aligned}\right\} \text{ Properties of pipe. } \left.\begin{aligned} [\mu] &= ML^{-1}T^{-1} \\ [\rho] &= ML^{-3} \end{aligned}\right\} \text{ Properties of the fluid.}$$

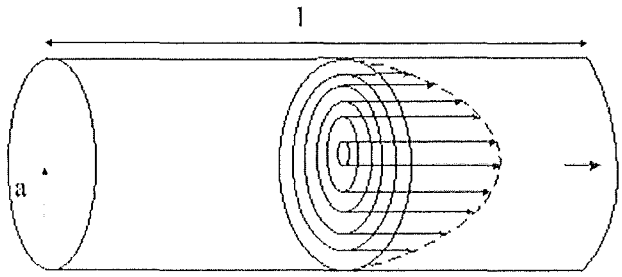

Figure 2: Velocity profile of viscous flow in a pipe

$[\Delta P] = ML^{-1}T^{-2}$	is pressure drop between the ends of pipe.
$[U] = LT^{-1}$	U is average velocity.

A glance at the dimensions of the quantities reveals there are only 3 quantities with independent dimensions. To find the number of quantities with independent dimensions more formally we introduce the dimension matrix.

	l	a	μ	ρ	U
L	1	1	-1	-3	1
M	0	0	1	1	0
T	0	0	-1	0	-1

the dimension matrix is $\begin{bmatrix} 1 & 1 & -1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$

Using elementary row operations we reduce the matrix to $rref$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

It is easy to see that the number of quantities with independent dimensions must equal the rank of the dimension matrix. Here $rank = r = 3 \implies m = n - k = n - r = 5 - 3 = 2$ We have a choice as to which 3 quantities we pick. For example, any of the combinations l, μ, ρ, or a, ρ, U can be chosen and the dimensions of the other quantities derived from these. Choosing the quantities that best simplify a problem is not always easy. It requires intuition and deep understanding of the phenomena involved.

Here we take a, ρ, U as the quantities with independent dimensions. This choice of combination is not without reason. In fluid mechanics there are dimensionless groups which are commonly used. For example, $Re = \frac{\rho U a}{\mu}$ called Reynolds number is important in all types of fluid flow problems. Similarly, $\frac{P}{\rho U^2}$ $(P = pressure)$ called Euler number is used in problems in which pressure or pressure difference is important.

$$\text{From } \Pi\text{-theorem} \quad \Pi = \frac{\Delta P}{a^p \rho^q U^r}$$

$$ML^{-1}T^{-2} = L^p(ML^{-3})^q(LT^{-1})^r$$

$$\implies \quad q = 1,\ p - 3q + r = -1,\ \text{i.e } p + r = 2$$

$$\text{and } r = 2,\ \text{gives } p = 0$$

$$\Pi = \frac{\Delta P}{a^0 \rho U^2} = \frac{\Delta P}{\rho U^2}$$

$$\text{and we get } \Delta P = f(a, \rho, U, l, \mu) = \rho U^2 \phi(\Pi_1, \Pi_2)$$

where

$$\Pi_1 = \frac{l}{a^{p_1}\rho^{q_1}U^{r_1}}$$
$$L = L^{p_1}(ML^{-3})^{q_1}(LT^{-1})^{r_1}$$
$$\implies p_1 = 1,\ q_1 = r_1 = 0 \text{ we have } \Pi = \frac{l}{a}$$
$$\Pi_2 = \frac{\mu}{a^{p_2}\rho^{q_2}U^{r_2}}$$
$$ML^{-1}T^{-1} = L^{p_2}(ML^{-3})^{q_2}(LT^{-1})^{r_2}$$
$$q_2 = 1,\ p_2 - 3q_2 + r_2 = -1, \text{ i.e } p_2 + r_2 = 2$$
$$\text{and } r_2 = 1, \text{ gives } p_2 = 1$$
$$\Pi_2 = \frac{\mu}{a\rho U} = \frac{1}{Re}.$$

Thus $\Delta P = \rho U^2\Pi = \rho U^2\phi(\Pi_1, \Pi_2) = \rho U^2\phi(\frac{l}{a}, \frac{\mu}{a\rho U}) = \phi_1(\frac{l}{a}, Re)$.

Our initial choice of quantities with independent dimensions has played a role in establishing this particular form of the formula for ΔP.

3.3 Deriving Planck's Radiation Law using dimensional analysis

In Quantum Mechanics the intensity I radiated at a particular wavelength λ depends on temperature τ of the black body. A black body is a perfect absorber or reflector of electromagnetic radiation. The source of radiations are oscillators that have energy $k_B\tau$ per degree of freedom (Greiner, 1994) where k_B is boltzman constant. The medium of transfer of energy are photons that have energy $\frac{hc}{\lambda}$, where h is planck's constant and c is the speed of light in vacuum. Working in the $LMT\Theta$ class, the dimensions of these physical quantities are

$$[I] = ML^{-2}T^{-2} \qquad [\lambda] = L \qquad [\tau] = \Theta \qquad [k_B] = ML^2T^{-2}\Theta^{-1}$$
$$[h] = ML^2T^{-1} \qquad [c] = LT^{-1}.$$

Π-theorem assures us that I is some function of the rest. $I = f(\lambda, \tau, k_B, h, c)$

	λ	τ	k_B	h	c
L	1	0	2	2	1
M	0	0	1	1	0
T	0	1	-2	-1	-1
Θ	0	0	-1	0	0

gives the dimension matrix is $\begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

This has $rank = r = 4$. as $r = k$, we have $n = 5$, $k = 4$ and $m = n - k = 1$ so that $\Pi = \phi(\Pi_1)$. Takeing τ as the quantity with dependent dimension, we have

$$\Pi = \frac{I}{\lambda^p {k_B}^q h^r c^s}$$
$$ML^{-2}T^{-2} = L^p(ML^2T^{-2}\Theta^{-1})^q(ML^2T^{-1})^r(LT^{-1})^s$$
$$q = 0,\ q + r = 1 \implies r = 1,$$
$$p + 2r + s = -2,$$
$$-r - s = -2, \text{ i.e } s = 2 - r = 1 \text{ and } p = -5.$$
$$\Pi = \frac{I}{\lambda^{-5}hc},\ I = \frac{hc}{\lambda^5}\Pi$$

$$\Pi_1 = \frac{\tau}{\lambda^{p_1}k_B^{q_1}h^{r_1}c^{s_1}}$$
$$\Theta = L^{p_1}(ML^2T^{-2}\Theta^{-1})^{q_1}(ML^2T^{-1})^{r_1}(LT^{-1})^{s_1}$$
$$q_1 + r_1 = 0,\ p_1 + 2q_1 + 2r_1 + s_1 = 0,$$
$$-2q_1 - r_1 - s_1 = 0$$
$$q_1 = -1,\ r_1 = 1,\ s_1 = 1 \text{ and } p_1 = -1.$$
$$\Pi_1 = \frac{\tau}{\lambda^{-1}k_B^{-1}h^1c^1} = \frac{\lambda\tau k_B}{hc}$$

Thus we have

$$I = \frac{hc}{\lambda^5}\Pi = \frac{hc}{\lambda^5}\phi(\Pi_1) = \frac{hc}{\lambda^5}\phi(\frac{\lambda k_B T}{hc}) \equiv \frac{hc}{\lambda^5}F(\frac{hc}{\lambda k_B T}).$$

The exact expression for Planck's law is

$$I = 8\pi\frac{hc}{\lambda}\frac{1}{\exp(\frac{hc}{\lambda k_B T}) - 1}.$$

Using dimensional analysis we have achieved an approximate form of Planck's law. Dimensional analysis cannot tell us any thing about the function $\phi(\Pi_1)$.

4 Nondimensionalizing equations

Nondimensionalizing is a technique used to combine variables into dimensionless groups. In doing this, we also transfer to a scale which is intrinsic to the system. This enables us to compare the relative size or importance of the new dimensionless parameters. We can exploit this to drop terms in the derived equation(expression) further simplifying the problem or reducing it to a form which can be relatively easily solved. Here we nondimensionalize the Navier-Stokes (N-S) equations for incompressible Newtonian fluid flow. Very few exact solutions to the full N-S equations exist and in general solving N-S equations is either extremely hard or impossible. The N-S equations are derived from the conservation laws, namely conservation of momentum, energy and mass. The N-S equations are (Howison, 2005a):

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \mu\nabla^2\mathbf{u}, \qquad \nabla.\mathbf{u} = 0.$$

Here $\mathbf{u}$ is the fluid velocity and it describes the motion of the fluid as it is moving past the body. While p is the pressure, ρ is the density and μ the dynamic viscosity of the fluid. Suppose we have flow past a body of size L where the fluid has a constant horizontal velocity U as it approaches the body. We scale all distances with L, time with $\frac{L}{U}$ and velocities with U. It should be noted that $\mathbf{u}$ is function of the position vector $\mathbf{x} = (x, y, z)$ and time t.

$$\mathbf{x} = L\mathbf{x}', \qquad t = (\frac{L}{U})t', \qquad \mathbf{u} = U\mathbf{u}'$$

From the problem its not obvious as to how p should be scaled. We arbitrarily set $p = P_o p'$, and substitute these in N-S equations:

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = (\frac{\partial}{\partial x'}\frac{\partial x'}{\partial x}, \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y}, \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z}) = (\frac{1}{L}\frac{\partial}{\partial x'}, \frac{1}{L}\frac{\partial}{\partial y'}, \frac{1}{L}\frac{\partial}{\partial z'}) = \frac{1}{L}\nabla'$$

$$\nabla\cdot\mathbf{u} = \frac{1}{L}\nabla'\cdot(U\mathbf{u}') = \frac{U}{L}\nabla'\cdot\mathbf{u}' = 0 \implies \nabla'\cdot\mathbf{u}' = 0$$

$$\text{And } \rho(\frac{U^2}{L}\frac{\partial \mathbf{u}'}{\partial t'} + U\mathbf{u}'\cdot\frac{1}{L}\nabla'(U\mathbf{u}')) = -\frac{1}{L}\nabla'(P_o p') + \frac{\mu}{L^2}\nabla'^2(U\mathbf{u}')$$

$$\frac{\rho U^2}{L}(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}'\cdot\nabla'\mathbf{u}') = -\frac{P_o}{L}\nabla' p' + \frac{\mu U}{L^2}\nabla'^2\mathbf{u}'$$

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}'\cdot\nabla'\mathbf{u}' = -\frac{P_o}{\rho U^2}\nabla' p' + \frac{\mu}{\rho U L}\nabla'^2\mathbf{u}'.$$

This suggests we let $P_o = \rho U^2$. The dimensionless equation takes the form

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}'\cdot\nabla'\mathbf{u}' = -\nabla' p' + \frac{1}{Re}\nabla'^2\mathbf{u}'$$

$p' = \frac{P}{P_o} = \frac{P}{\rho U^2}$, Re are dimensionless quantities that we have encountered before using Π-theorem. If Re is large the equation reduces to

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' p'$$

called Euler's equation (Howison, 2005b). Re is the ratio of inertial to viscous forces (Howison, 2005c) and when it is large then viscous effects can be ignored. Euler's equation governs the flow of inviscid fluids. If we also assume the flow is irrotational then this equation has exact solutions (Howison, 2005d). If Re is small we resort to a different scaling of P. We had

$$\frac{\rho U^2}{L}\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}'\right) = -\frac{P_o}{L}\nabla' p' + \frac{\mu U}{L^2}\nabla'^2 \mathbf{u}'.$$

Multiply both sides with $\frac{L^2}{\mu U}$ and Let $P_o = \frac{\mu U}{L}$ to get

$$\frac{\rho U L}{\mu}\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}'\right) = -\frac{P_o L}{\mu U}\nabla' p' + \nabla'^2 \mathbf{u}'$$

$$Re\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}'\right) = -\nabla' p' + \nabla'^2 \mathbf{u}'.$$

Because Re is small the inertial terms on the left of the equation can be neglected (Howison, 2005e) to get $-\nabla' p' + \nabla'^2 \mathbf{u}' = 0$. This is Stokes model for slow flow, it is simpler than the N-S equations and it has no time dependence.

5 Huntley's Method

Consider the dimensions of density ρ, $[\rho] = ML^{-3}$ and velocity u, $[u] = LT^{-1}$. Density is a scalar while velocity is a vector, yet we have made no distinction between L in ML^{-3} and L in LT^{-1}. Similarly the dimension of torque $\tau = r \times F$ is $[\tau] = ML^2T^{-2}$. Here the dimension of length occurs twice. One L comes from the force and the other from the distance between the line of action of force and the pivot. The two are mutually perpendicular. Huntley proposes that, by making a distinction in such situations, dimensional analysis can reveal more than it would otherwise (Huntley, 1952).

5.1 Huntley's Method applied to motion of a projectile

A projectile is fired horizontally from a height h with a velocity u. What is its horizontal range R? The

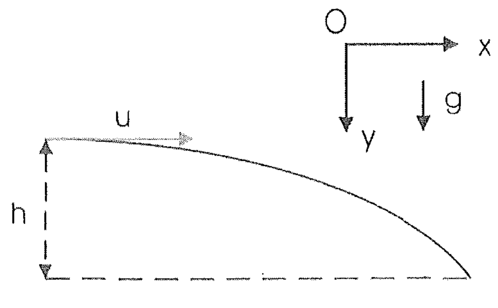

Figure 3: The motion of a projectile fired horizontally from a height

dimensions of quantities involved are

$$[h] = L, \qquad [u] = LT^{-1}, \qquad [R] = L, \qquad [g] = LT^{-2}$$

	h	u	g
L	1	1	1
M	0	0	0
T	0	-1	-2

gives the dimension matrix is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$

This has $rank = r = 2$. Here $n = 3$ and as $r = k, m = n - k = 3 - 2 = 1$ and $\Pi = \phi(\Pi_1)$. Clearly

$$\Pi = \frac{R}{h}, \qquad \Pi_1 = \frac{u}{h^{p_1} g^{q_1}} = \frac{u}{h^{\frac{1}{2}} g^{\frac{1}{2}}} = \frac{u}{\sqrt{hg}}$$

Thus
$$R = h\phi(\frac{u}{\sqrt{hg}})$$

Here we introduce more fundamental dimensions of length L_x, L_y and L_z. The dimensions of the quantities are now

$$[R] = L_x, \qquad [u] = L_x T^{-1}, \qquad [h] = L_z, \qquad [g] = L_z T^{-2}$$

	h	u	g
L_x	0	1	0
L_y	0	0	0
L_z	1	0	1
M	0	0	0
T	0	-1	-2

gives the dimension matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$

This has $rank = r = 3$. So $n = k = 3$, $m = 0$. Thus $\Pi = \phi$ is a constant.

$$\Pi = \frac{R}{h^p g^q u^r}$$
$$L_x = (L_z)^p (L_z T^{-2})^q (L_x T^{-1})^r$$
$$p + q = 0, \quad -2q - r = 0, \quad r = 1 \implies q = -\frac{1}{2}, \; p = \frac{1}{2}$$
$$\Pi = \frac{R}{h^{\frac{1}{2}} g^{-\frac{1}{2}} u}. \text{ Thus } R = u\sqrt{\frac{h}{g}}\phi$$

We have made remarkable progress compared to the first formula for R. Dimensional analysis has now revealed the complete formula except a constant, ϕ being the constant in this case. Where as previously ϕ was some unknown function of the dimensionless parameter $\frac{u}{\sqrt{hg}}$. We know that $R = ut$ and from the equations of motion in mechanics $h = \frac{1}{2}gt^2$. Rearrange to get $t = \sqrt{\frac{2h}{g}}$. Therefore $R = u\sqrt{\frac{2h}{g}}$ and the constant $\phi = \sqrt{2}$.

5.2 Drawback of Huntley's Method

The draw back of Huntley's method is that it is not universally applicable. For example, the Navier-Stokes equation for incompressible newtonian viscous flow is (Howison, 2005a):

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u}, \qquad \nabla . \mathbf{u} = 0$$

If we just consider the x-component of this vector equation we have

$$\rho(\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u) = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u,$$

where we have taken $\mathbf{u} = \mathbf{u}(u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$.

The term
$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
where $\mathbf{u} = \mathbf{u}(u, v, w)$ is the horizontal velocity component. Huntley's method gives
$$[u] = L_x T^{-1}, \qquad [x] = L_x, \qquad [y] = L_y, \qquad [z] = L_z$$
Thus
$$[\nabla \cdot \mathbf{u}] = \frac{L_x T^{-1}}{L_x} + \frac{L_x T^{-1}}{L_y^2} + \frac{L_x T^{-1}}{L_z^2}.$$
We see that application of Huntley's method has rendered the well established N-S equation meaningless as adding different dimensions makes no sense.

6 Self-similarity, Self-similar solutions and dimensional analysis

Self-similarity is the property of a phenomenon remaining geometrically similar to itself as time passes. For example, if we make a single ripple by dipping a finger in a large bowl containing water, the ripple starts from a point and moves outwards. The ripple grows bigger as it travels but it remains a circular wave at all times until it hits the boundary of the container. The phenomenon has the property of self-similarity and any solution that describes it must be self-similar. For a self-similar solution we do not need to trace back the origin in time of the phenomenon, and once a solution for a particular instant in time is found, solutions at other times can be obtained from this by simple similarity transformations. The general form of a self-similar solution of a phenomenon $a = f(a_1, a_2 \cdots a_k, b_1)$ which has only one governing parameter with dependent dimension is
$$a = f(a_1, a_2 \cdots a_k, b_1) = g(t)\phi(\frac{b_1}{h(t)})$$
Here one of $a_j = t$ and $\frac{b_1}{h(t)}$ is called the similarity variable. In the scales $\frac{a}{g(t)}$ and $\frac{b_1}{h(t)}$ the phenomenon is time invariant. Often physical phenomena are both time and space dependent which give rise to PDE models. Establishing self-similarity eliminates this time dependence in the appropriate scales and the PDE model reduces to an ODE model. Dimensional analysis can be used to reveal self-similarity and find the similarity variables.This is demonstrated in the problem of heat flow in an infinite long bar. Heat flow is governed by the heat equation. The one-dimensional heat equation is (Barenblatt, 1987e):
$$\frac{\partial \tau(x,t)}{\partial t} = D\frac{\partial^2 \tau(x,t)}{\partial x^2} \tag{2}$$
where, $D = \frac{k}{dc_\rho}$ is called thermal diffusivity and $\tau(x,t)$ is the temperature. In $D = \frac{k}{dc_\rho}$, k is thermal conductivity, $[k] = MLT^{-3}\theta^{-1}$, its dimension can be derived from the Fourier heat law where it appears as the constant of proportionality (Halliday et al., 2004a), c_ρ is specific heat(the energy required to raise the temperature of one kilogram of a substance by one degree) (Halliday et al., 2004b), $[c_\rho] = L^2T^{-2}\theta^{-1}$ and d is density, $[d] = ML^{-3}$.
At an initial time $t = 0$, we add an amount of heat Q_o at some point of the bar, which we arbitrarily call $x = 0$. Let the rod have a cross-sectional area A. The quantity of heat contained in the volume element $A\delta x$ is $dc_\rho \tau A\delta x$ (Barenblatt, 1987f). We assume that, apart from the heat we added no heat is gained or lost to the environment. Therefore conservation of heat gives
$$dc_\rho A\int_{-\infty}^{\infty} \tau(x,t)dx = Q_o \text{ or equally } \int_{-\infty}^{\infty} \tau(x,t)dx = \frac{Q_o}{dc_\rho A} \tag{3}$$
How does this heat diffuse away from $x = 0$ as a function of time t i.e what is $\tau(x,t)$? We identify the parameters. The temperature depends on x, t, D and from equation (3) $Q \equiv \frac{Q_o}{dc_\rho A}$ because the variables d,

c_ρ, A, and Q_o only appear in the problem in their group form Q and not separately.
Here $[x] = L, [t] = T, [D] = L^2T^{-1}, [Q] = L\theta$. We take t, D, Q as quantities with independent dimensions. We have $n = 4, k = 3, m = 1$.

$$\tau(t, D, Q, x) = t^p D^q Q^r \phi(\Pi_1) \qquad \text{from } \Pi\text{-theorem.}$$

Here,

$$\Pi_1 = \frac{x}{t^{p_1} D^{q_1} Q^{r_1}}$$
$$L = T^{p_1}(L^2T^{-1})^{q_1}(L\theta)^{r_1}$$
$$p_1 - q_1 = 0, \; 2q_1 + r_1 = 1,$$
$$r_1 = 0 \implies q_1 = \frac{1}{2} \text{ and } p_1 = \frac{1}{2}$$
$$\Pi_1 = \frac{x}{(Dt)^{\frac{1}{2}}}$$

And

$$\Pi = \frac{\tau}{t^p D^q Q^r}$$
$$\theta = T^p(L^2T^{-1})^q(L\theta)^r$$
$$p - q = 0, \; 2q + r = 0,$$
$$r = 1 \implies q = -\frac{1}{2} \text{ and } p = -\frac{1}{2}$$
$$\Pi = \frac{\tau}{(Dt)^{-\frac{1}{2}}Q}$$

We have

$$\tau(t, D, Q, x) = \frac{Q}{(tD)^{\frac{1}{2}}}\phi(\frac{x}{(tD)^{\frac{1}{2}}})$$

We see that ϕ is only a function of the combination $\frac{x}{(tD)^{\frac{1}{2}}}$, and not x, t separately. To determine ϕ, we let $z = \frac{x}{(tD)^{\frac{1}{2}}}$. Using the chain rule we calculate various derivatives:

$$\begin{aligned}
\frac{\partial \tau}{\partial x} &= \frac{Q}{(tD)^{\frac{1}{2}}}\frac{\partial z}{\partial x}\frac{d\phi(z)}{dz} = \frac{Q}{Dt}\frac{d\phi(z)}{dz} \\
\frac{\partial^2 \tau}{\partial x^2} &= \frac{Q}{(Dt)^{\frac{3}{2}}}\frac{d^2\phi(z)}{dz^2} \\
\frac{\partial \tau}{\partial t} &= -\frac{1}{2}\frac{Q}{D^{\frac{1}{2}}t^{\frac{3}{2}}}\phi(z) + \frac{Q}{(Dt)^{\frac{1}{2}}}\frac{\partial z}{\partial t}\frac{d\phi(z)}{dz} \\
&= -\frac{1}{2}\frac{Q}{D^{\frac{1}{2}}t^{\frac{3}{2}}}[\phi(z) + z\frac{d\phi(z)}{dz}].
\end{aligned}$$

Substituting in equation (2)

$$\begin{aligned}
-\frac{1}{2}\frac{Q}{D^{\frac{1}{2}}t^{\frac{3}{2}}}[\phi(z) + z\frac{d\phi(z)}{dz}] &= D\frac{Q}{(Dt)^{\frac{3}{2}}}\frac{d^2\phi(z)}{dz^2} \\
\frac{d^2\phi(z)}{dz^2} + \frac{z}{2}\frac{d\phi(z)}{dz} + \frac{1}{2}\phi(z) &= 0.
\end{aligned}$$

Dimensional analysis has reduced the problem from the solution of a partial differential equation in two variables to the solution of an ordinary differential equation in one variable. Here a reasonable solution should have $\phi(z) \to 0$ as $x \to \pm\infty$ and $\frac{d\phi}{dx} \to 0$ as $x \to \pm\infty$. With these conditions the differential equation is easily solved to give $\phi(z) = C\exp(-\frac{z^2}{4})$. To find C we proceed by putting $\tau(t, D, Q, x) = \frac{Q}{(tD)^{\frac{1}{2}}}\phi(z)$ in equation (3), we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{Q}{(tD)^{\frac{1}{2}}}\phi(z)dz(Dt)^{\frac{1}{2}} &= Q\int_{-\infty}^{\infty}\phi(z)dz = Q \\
\implies \int_{-\infty}^{\infty}\phi(z)dz &= 1, \text{ which gives}
\end{aligned}$$

$$C\int_{-\infty}^{\infty}\exp(-\frac{z^2}{4})dz = C(4\pi)^{\frac{1}{2}} = 1.$$

We get
$$C = \frac{1}{(4\pi)^{\frac{1}{2}}}.$$

Finally
$$\tau(x,t) = \frac{Q}{(4\pi Dt)^{\frac{1}{2}}}\exp(-\frac{x^2}{4Dt}).$$

In figure 4 the temperature distributions at various instants of time are shown. The figure shows that the solutions are self similar in the appropriate scales.

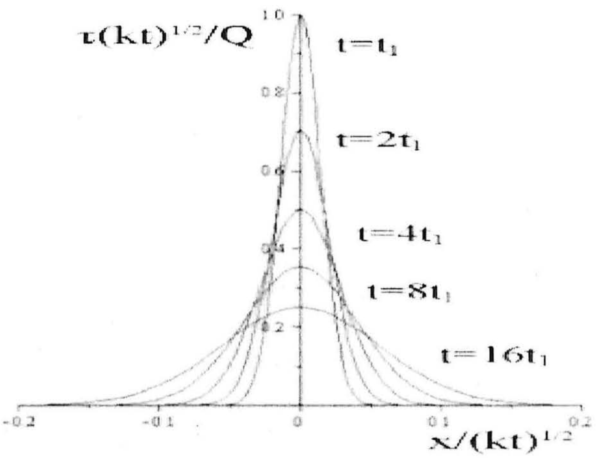

Figure 4: The temperature distributions at various instants of time.

6.1 Complete and incomplete similarity

Dimensional analysis reduces the number of variables by grouping them into dimensionless parameters. For

$$a = f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m) = a_1{}^p \cdots a_k{}^r \phi(\Pi_1, \cdots, \Pi_m)$$

it may be possible to further reduce the number of parameters of ϕ. If any of the dimensionless parameter Π_i has a value too large or small then we can omit it from ϕ based on the assumption that for intermediate values Π_i has no effect on ϕ. More generally if $\Pi_{l+1}, \cdots, \Pi_m$ go to zero or infinity while $\Pi_1, \cdots, \Pi_l$ remain constant and ϕ converges to a finite non-zero limit, then for large or small $\Pi_{l+1}, \cdots, \Pi_m$ we replace ϕ by a function $\phi_1(\Pi_1, \cdots, \Pi_l)$ of fewer parameters. This is called complete similarity in the parameters $\Pi_{l+1}, \cdots, \Pi_m$ (Barenblatt, 2003c). Consider the problem of heat conduction, this time in a finite bar of length l. Since the bar is of finite length we have to take into account some other physical quantities that we didn't consider previously. In addition to x, t, D and Q we must consider the length of the bar l, the length of the cross section where the heat was initially released h and the coordinate of the central point of initial heat release, which we had arbitrarily assigned $x_o = 0$. And so we are just left with two new arguments l, h. Thus

$$\tau(t, D, Q, x, l, h) = \frac{Q}{(Dt)^{\frac{1}{2}}}\phi(\Pi_1, \Pi_2, \Pi_3).$$

Here $\Pi_1 = \frac{x}{(tD)^{\frac{1}{2}}}$ and because $[x][= [l] = [h] = L$ we must have $\Pi_2 = \frac{l}{(tD)^{\frac{1}{2}}}$, and $\Pi_3 = \frac{h}{(tD)^{\frac{1}{2}}}$. We should note that there are three time scales, in each of which heat flow is different. (1) The initial time when the heat released is still confined to a region of about the size of h, (2) A time when the heat has reached the ends of the bar and the bar is reaching a constant temperature, (3) the intermediate time between these two events. It is this intermediate time that we are interested in the heat flow in the bar. In this interval heat would have flowed a distance much larger compared to h, so $\Pi_3 = \frac{h}{(tD)^{\frac{1}{2}}} \ll 1$, but it hasn't reached the ends of the bar, and so $\Pi_2 = \frac{l}{(tD)^{\frac{1}{2}}} \gg 1$. Assume that Π_1 remains constant as $\Pi_2 \to \infty$ and $\Pi_3 \to 0$, and ϕ approaches a finite non-zero limit. Then this problem reduces to the problem of heat conduction in an infinite bar, and we have $\tau = \frac{Q}{(Dt)^{\frac{1}{2}}}\phi(\Pi_1)$. This is an example of complete similarity in the parameters Π_2 and Π_3. Going back to our discussion of self-similar solutions, they are always solutions to idealized problems. This however, does not mean that self-similar solutions have no real life applications, because as seen here, within the appropriate time interval, these self-similar solutions can serve to approximate the solution to real life problems. In the above example, we were lucky that ϕ did have a non-zero limit. In general there is no guarantee that a non-zero limit for ϕ exists. So our conduct should be to assume complete similarity and then check our solution against available data. If there are discrepancies, then we may assume incomplete similarity to formulate a solution, and if there are discrepancies again, then we can conclude that there is no self-similar solution to the problem.

If $\Pi_{l+1}, \cdots, \Pi_m$ go to zero or infinity and ϕ doesn't converge to a limit, then under the assumption of incomplete similarity, we may still be able to obtain a self-similar solution. A formal mathematical definition of incomplete similarity is as follows.

Let $\Pi_{l+1}, \cdots, \Pi_m$ go to zero or infinity and assume ϕ does not converge to a limit, then for $a = f(a_1, \cdots, a_k, b_1, \cdots, b_m) = a_1^p \cdots a_k^r \phi(\Pi_1, \cdots, \Pi_l, \Pi_{l+1}, \cdots, \Pi_m)$, we replace ϕ by ϕ_1, where

$$\phi = \Pi_{l+1}^{\alpha_{l+1}} \cdots \Pi_m^{\alpha_m} \phi_1\left(\frac{\Pi_1}{\Pi_{l+1}^{\beta_1} \cdots \Pi_m^{\delta_1}}, \cdots, \frac{\Pi_l}{\Pi_{l+1}^{\beta_l} \cdots \Pi_m^{\delta_l}}\right).$$

This is called incomplete similarity in the variables $\Pi_{l+1}, \cdots, \Pi_m$ (Barenblatt, 2003c). Here dimensional analysis can not reveal the values of the constants $\alpha_{l+1}, \cdots, \delta_l$, because $\Pi_{l+1}, \cdots, \Pi_m$ are dimensionless and we can not manipulate the dimensions of the parameters to establish the unknown constants. Moreover, in this case the parameters $\Pi_{l+1}, \cdots, \Pi_m$ haven't disappeared from the scene, consequently the corresponding dimensional parameters $b_{l+1}, \cdots, b_m$ remain essential in describing the phenomena. Consider the flow of an ideal fluid past a wedge of length l. The flow is described by the continuity equation (Barenblatt, 1987g):

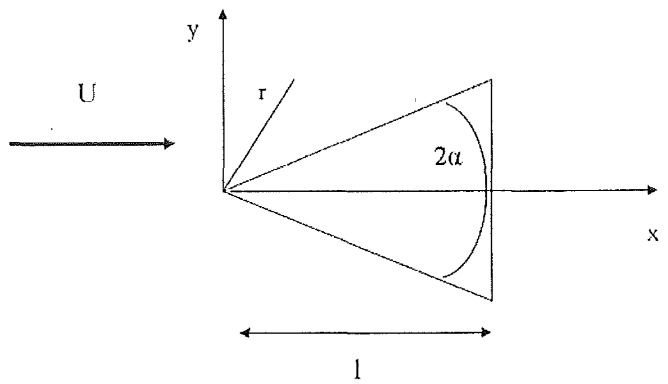

Figure 5: Flow of an ideal incompressible fluid past a wedge

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} = 0.$$

Here $u(x, y)$, $v(x, y)$ are the horizontal and vertical components of velocity respectively. For ideal flow with no internal resistance it is known that there exists some potential function (Barenblatt, 1987g) $\Phi(x, y)$ so that

$$u(x, y) = \frac{\partial \Phi}{\partial x}, \qquad v(x, y) = \frac{\partial \Phi}{\partial y}.$$

Substituting this in the continuity equation we get laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

In polar coordinates this becomes (see appendix A)

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 \tag{4}$$

The solution to this problem is well-known in hydrodynamics. It is of the form

$$\Phi = \text{Real part of } U\zeta,$$

where $\zeta = F(z)$ is a function of the complex variable $z = x + iy$, which carries out a conformal mapping of the exterior of the upper half of the wedge onto the segment $0 \leq x \leq a$ of the x-axis (Barenblatt, 1987h). The Schwartz-Christoffel $(S - C)$ transformation (see Appendix B) can be used to obtain this conformal

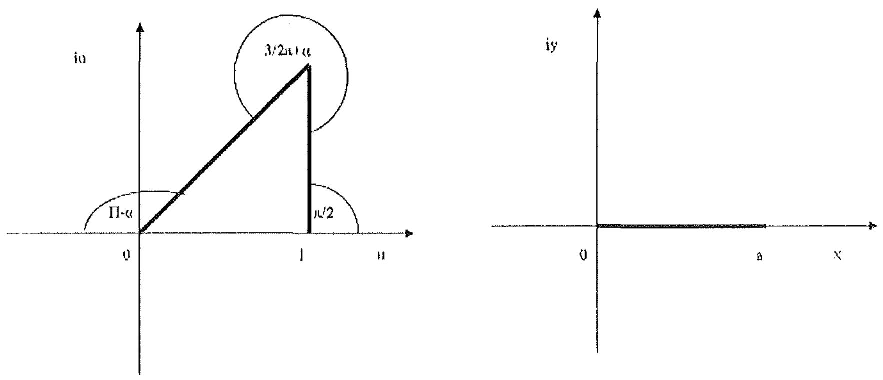

Figure 6: The upper half of the wedge mapped to the line segment $0 \leq x \leq a$

mapping. From the $S - C$ formula we have

$$z = \int_0^{\zeta} (t - 0)^{\omega_1}(t - b)^{\omega_2}(t - a)^{\omega_3} dt$$

where $0 = F(0)$, $b = F(l + il \tan \alpha)$; the point b lies on the segment $0 \leq x \leq a$, $a = F(l)$ and $\omega_k = \frac{\sigma_k}{\pi} - 1$. The constant σ_k is the internal angle. From Figure 6 we have $\sigma_1 = \pi - \alpha$, $\sigma_2 = \frac{3}{2}\pi + \alpha$ and $\sigma_3 = \frac{\pi}{2}$.

Therefore
$$\omega_1 = \frac{\sigma_1}{\pi} - 1 = \frac{\pi - \alpha}{\pi} - 1 = -\frac{\alpha}{\pi}$$
$$\omega_2 = \frac{\sigma_2}{\pi} - 1 = \frac{\frac{3}{2}\pi + \alpha}{\pi} - 1 = \frac{1}{2} + \frac{\alpha}{\pi}$$
$$\omega_3 = \frac{\sigma_3}{\pi} - 1 = \frac{\frac{\pi}{2}}{\pi} - 1 = -\frac{1}{2}$$
$$\Longrightarrow z = \int t^{-\frac{\alpha}{\pi}}(t - b)^{\frac{1}{2}+\frac{\alpha}{\pi}}(t - a)^{-\frac{1}{2}} dt.$$

Similar to the problem of heat flow in a bar of finite length, where we were considered heat flow during intermediate times; here we are interested in the fluid flow near the tip of the wedge, where the distance r from the edge of the wedge O is much smaller then the length of the wedge l, so that $r \ll l$. Setting $a = l\delta$, $b = l\mu$ and $t = l\tau$ (where $\tau \ll 1$) we expand the terms in the integrand as an infinite series using the generalized binomial theorem to get the simplified integral below (See Appendix C).

$$z = l\beta e^{i\alpha} \int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}} d\tau = l\beta e^{i\alpha} (\frac{\zeta}{l})^{\frac{\pi-\alpha}{\pi}} (\frac{\pi}{\pi - \alpha}),$$

where β is a constant. We rearrange to get

$$(\frac{\zeta}{l})^{\frac{\pi-\alpha}{\pi}} = \frac{z}{l} e^{-i\alpha} \beta^{-1} (\frac{\pi - \alpha}{\pi})$$
$$\zeta = l(\frac{z}{l})^{\frac{\pi}{\pi-\alpha}} e^{-i\frac{\pi\alpha}{\pi-\alpha}} \gamma$$

as $z = re^{i\theta}$ we have

$$\zeta = l(\frac{r}{l})^{\frac{\pi}{\pi-\alpha}} e^{i(\frac{\theta\pi}{\pi-\alpha} - \frac{\alpha\pi}{\pi-\alpha})} \gamma$$

so that

$$\Phi = Ur\cos[(\lambda + 1)\theta + \chi]\gamma(\frac{r}{l})^{\lambda}$$

where

$$\lambda = \frac{\alpha}{\pi - \alpha}, \qquad \chi = -\frac{\pi\alpha}{\pi - \alpha},$$

and γ is a dimensionless constant.

Now we look for a solution under the assumption of incomplete similarity. In polar coordinates we expect the potential function Φ to depend on the velocity of the incident fluid U, the radius r of the point where we are considering the flow, the corresponding polar angle θ, the opening angle of the wedge 2α and the length of the wedge l.

$$\Phi = f(U, r, \theta, \alpha, l).$$

$$\text{Here } [\Phi] = L^2T^{-1}, \qquad [U] = LT^{-1}, \qquad [r] = L, \qquad [\theta] = [\alpha] = 1, \qquad [l] = L$$

$$\Phi = f(U, r, \theta, \alpha, l) = Ur\phi(\theta, \alpha, \frac{l}{r}).$$

We are interested in distances from the sharp edge of the wedge such that $r \ll l$. Thus the parameter $\frac{l}{r}$ is large and under the assumption of incomplete similarity we have

$$\phi = (\frac{l}{r})^{\kappa}\phi_1(\theta, \alpha)$$
$$\text{i.e } \Phi = Ur(\frac{l}{r})^{\kappa}\phi_1(\theta, \alpha)$$

where κ is a constant. We calculate the partial derivatives and substitute in equation (4).

$$(\kappa - 2)(\kappa - 1)Ulr^{\kappa-3}\phi_1(\theta, \alpha) + (\kappa - 1)Ulr^{\kappa-3}\phi_1(\theta, \alpha) + Ulr^{\kappa-3}\frac{\partial^2\phi_1}{\partial\theta^2} = 0.$$

Since the parameter α is a constant and ϕ_1 is a function of the variable θ only we have

$$(\kappa - 1)^2\phi_1 + \frac{d^2\phi_1}{d\theta^2} = 0.$$

This is the differential equation for a harmonic oscillator. Its solution is of the form

$$\phi_1 = A\cos[(\kappa - 1)\theta].$$

The flow along the faces of the wedge and along the line of symmetry must be zero. Therefore the boundary conditions are $\frac{d\phi_1}{d\theta} = 0$ for $\theta = \alpha$ and $\theta = \pi$.

$$\begin{aligned} \frac{d\phi_1}{d\theta} &= -A(\kappa - 1)\sin\left[(\kappa - 1)\theta\right] = 0 \\ \Longrightarrow\ (\kappa - 1)\pi &= m\pi \\ (\kappa - 1)\alpha &= n\pi \text{ where } n \text{ and } m \text{ are integers.} \\ \text{thus } \kappa &= \frac{(m-n)\pi}{\pi - \alpha} + 1 = \frac{\pi(m-n+1) - \alpha}{\pi - \alpha} \\ \text{setting } m - n + 1 = 0 \text{ and } \kappa &= -\frac{\alpha}{\pi - \alpha} = \lambda^{-1}. \end{aligned}$$

Finally

$$\Phi = Ur(\frac{r}{l})^{\lambda} A\cos[-(\lambda + 1)\theta] = Ur(\frac{r}{l})^{\lambda} A\cos[(\lambda + 1)\theta],$$

which has the same form as the solution obtained using the $C - S$ transformation. If we had looked for a solution under the assumption of complete similarity we would have had

$$\Phi = Ur\phi(\theta, \alpha).$$

Substituting this in equation (4) and solving for Φ, we get $\Phi = UrA\cos(\theta)$. Here $\frac{d\Phi(\alpha)}{d\theta} \neq 0$ as $0 \leq \alpha \leq \frac{\pi}{2}$ and so Φ does not satisfy the required boundary conditions.

7 The theory of modeling and dimensional analysis

In modeling, tests are carried out on models to understand behavior of full-size prototypes. For example, it would be impossible to carry out experiments on drag and lift properties of full-sized aircrafts, in a closed setup such as a laboratory. Instead wind tunnels are used to do tests on small scale models of aeroplanes. But how do we take the results done on such small scale models and apply them to the prototype. The answer lies in the concept of similarity and similar phenomena that we encountered in Section 6. Consider the relationship between some physical quantities $a, a_1, \cdots, a_k, b_1, \cdots, b_m$;

$$a = f(a_1, \cdots, a_k, b_1, \cdots, b_m).$$

To make distinction we denote the relationship for the model as

$$a^M = f(a_1^M, \cdots, a_k^M, b_1^M, \cdots, b_m^M),$$

the superscript M referring to the model, and let

$$a^P = f(a_1^P, \cdots, a_k^P, b_1^P, \cdots, b_m^P)$$

denote the relationship for the prototype. Here the function f is the same for both the model and prototype, as we require them to be similar. However, the magnitudes of the quantities $a, a_1, \cdots, a_k, b_1, \cdots, b_m$ can differ. We can apply dimensional analysis and obtain

$$\Pi^M = \phi(\Pi_1^M, \cdots, \Pi_m^M) \text{ and } \Pi^P = \phi(\Pi_1^P, \cdots, \Pi_m^P).$$

In order to be able to translate results from the model to the prototype similarity requires that $\Pi^M = \Pi^P$. In fact, similarity requires that all the dimensionless quantities of the model be equal to their corresponding quantities of the prototype, i.e $\Pi_1^M = \Pi_1^P, \cdots, \Pi_m^M = \Pi_m^P$.

$$\frac{b_1^M}{a_1^{M^{p_1}} \cdots a_k^{M^{r_1}}} = \frac{b_1^P}{a_1^{P^{p_1}} \cdots a_k^{P^{r_1}}} \quad \Longrightarrow \quad b_1^M = b_1^P (\frac{a_1^M}{a_1^P})^{p_1} \cdots (\frac{a_k^M}{a_k^P})^{r_1}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{b_m^M}{a_1^{M^{p_m}} \cdots a_k^{M^{r_m}}} = \frac{b_m^P}{a_1^{P^{p_m}} \cdots a_k^{P^{r_m}}} \quad \Longrightarrow \quad b_m^M = b_m^P (\frac{a_1^M}{a_1^P})^{p_m} \cdots (\frac{a_k^M}{a_k^P})^{r_m}$$

This is all that is required for the results of the model to be applicable to the prototype (Barenblatt, 2003d). Here of course, we are taking for granted, the condition of geometrical similarity between the model and the prototype. It is remarkable as to how simple the idea of similarity is. G.I Barenblatt in his book *Scaling* emphasizes that this is the entire theory of similarity and that there is nothing more to it.

Assume we have to test the effects of drag on a prototype racing car. To save resources, we decide to first build a model small-size car so that we can do the necessary tests on the model. Below we demonstrate how we can use our knowledge of dimensional analysis to scale the results so that they are applicable to the prototype. But first we derive a simple relation for the drag force. The governing parameters are the length(size) of car l, the viscosity of fluid μ, density of fluid ρ, and the relative speed of the car and the fluid U. These have dimensions

$$[l] = L, \qquad [\mu] = ML^{-1}T^{-1}, \qquad [\rho] = ML^{-3}, \qquad [U] = LT^{-1}$$

	l	μ	ρ	U
L	1	-1	-3	1
M	0	1	1	0
T	0	-1	0	-1

the dimension matrix is $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

This has rank 3. We let l, ρ, U be the 3 quantities with independent dimensions.

$$\text{Now} \qquad \Pi = \frac{F_{drag}}{l^p \rho^q U^r} = C_D(\text{drag coefficient}) \text{ from } \Pi\text{-theorem}$$

$$MLT^{-2} = L^p(ML^{-3})^q(LT^{-1})^r$$

$$q = 1,\ p - 3q + r = 1,\ p + r = 4,\ r = 2,\ \Longrightarrow\ p = 2$$

$$\Pi = \frac{F_{drag}}{\rho l^2 U^2},\ \Pi = \phi(\Pi_1),\ \text{where } \Pi_1 = Re.$$

We must have, $\Pi_1^M = \Pi_1^P$ and $\Pi^M = \Pi^P$, i.e

$$\frac{l^M \rho^M U^M}{\mu^M} = \frac{l^P \rho^P U^P}{\mu^P}.\ \text{Testing in a windtunnel gives } \rho^M = \rho^P,\ \mu^M = \mu^P,\ \text{therefore,}$$

$$U^M = U^P(\frac{l^P}{l^M}) \implies F_{drag}^M = F_{drag}^P.$$

As $l^M < l^P$, U^M is greater than U^P by the ration $\frac{L^P}{L^M}$. The general drag equation is $F_{drag} = \frac{1}{2}\rho U^2 A C_d$ (Halliday et al., 2004c), where A is the reference area and C_d the drag coefficient. For a sports car, C_d is between 0.27 to 0.38 (Palmer, 2005). Taking $C_d = 0.3$, $A = 3m^2$, $\rho_{air} = 1.293 kgm^{-3}$. At $150km/h(40.67ms^{-1})$ the drag force is

$$F_{drag} = \frac{1}{2} \times 1.293 \times (41.67)^2 \times 3 \times 0.3 = 1010.3N.$$

The drag force of $136.2N$ is the same for both the model and prototype. Had we chosen to test the model under different conditions; for example, under different pressure settings, then the drag force would have been different from that on the prototype.

8 Transformation groups

A transformation groups as the name suggest is a set of transformations that also form a group, i.e

- There exists the identity transformation.
- For each transformation in the set there exists the inverse transformation that also belongs to the set.
- For transformations A, B there exists C which belongs to the set and C is obtained from successive application of transformations A and B.

Given a function $a = f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m)$ among some physical quantities, once we have established that it possesses the property of generalized homogeneity, i.e

$$a = f(a_1, a_2 \cdots a_k, b_1, \cdots, b_m) = a_1{}^p \cdots a_k{}^r \phi(\Pi_1, \cdots, \Pi_m),$$

it may be possible to find a transformation group that further reduces the number of arguments of ϕ by the number of parameters of the transformation group (Barenblatt, 2003e). This idea is illustrated below in the example of fluid flow past a flat plate.

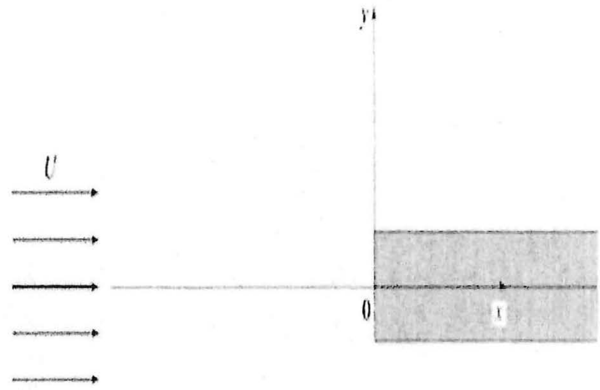

Figure 7: Viscous flow past a thin semi-infinite plate

Before tackling the problem we briefly discuss fluid mechanics, which should help highlight the importance of dimensional analysis as a tool that can often establish important results, with little effort. The Navier-Stokes equations and the equation of continuity can in theory be used to explain the whole of fluid mechanics. However, they are extremely difficult to deal with. Even for the viscous flow past a thin flat plate, despite its seeming simplicity, to this day no analytical solution has been found. In 1904 Ludwig Prandtl proposed the idea of boundary layer, which is regarded as having revolutionized the study of fluid mechanics. The boundary layer concept is that for viscous flow past the flat plate the fluid flow can be divided into two regions. A boundary layer close to the surface where the flow is viscous and a region where the flow can essentially be considered nonviscous. For sufficiently high Reynolds number the boundary layer is very thin. Reynolds number which we have encountered before is a dimensionless number. It is the ratio of inertial to viscous forces in a fluid. The Reynolds number is a natural product of the application of dimensional analysis to fluid flow problems. When non-dimensionalizing the $N-S$ equations Re crystallizes out in the process making it inherently important in the study of fluid mechanics. With the introduction of boundary layer it became possible to reduce the $N-S$ equations to relatively simpler boundary layer equations using asymptotic analysis. We now apply dimensional analysis to reveal an important property of the fluid flow problem past the flat plate. The boundary equations for this problem are (Barenblatt, 2003e):

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

with boundary conditions at $x>0$, $y>0$. $\quad u(0,y)=U,\ u(x,\infty)=U,\ u(x,0)=v(x,0)=0.$

Here x and y are the cartesian coordinate variables, $u(x,y)$ the horizontal velocity, $v(x,y)$ the vertical velocity component, ν is the kinematic viscosity, and U is the constant speed of the uniform external flow of the fluid. The governing parameters are ν, x, U, y. Dimensional analysis suggests u and v are some functions of these parameters.

$$u=f_u(\nu,x,U,y) \text{ and } v=f_v(\nu,x,U,y).$$

Here $\qquad [u]=[v]=[U]=LT^{-1}, \qquad [x]=[y]=L, \qquad [\nu]=[\frac{\mu}{\rho}]=L^2T^{-1}$

Choose ν and U to be the two quantities with independent dimensions, therefore $k=2$ and $m=2$. Standard dimensional analysis gives:

$$\Pi_u=\frac{f_u}{\nu^p x^q U^r y^s}=\frac{u}{U} \qquad \Pi_v=\frac{f_u}{\nu^p x^q U^r y^s}=\frac{v}{U}$$

as $m=2$, $\qquad \Pi_u=\Pi_u(\Pi_1,\Pi_2) \qquad \Pi_v=\Pi_v(\Pi_1,\Pi_2).$

As $\nu=\frac{\mu}{\rho}$, with out performing any calculations, from our previous examples we know that both Π_1, and Π_2 must be *Reynolds* numbers.

$$\Pi_1=\xi=\frac{Ux}{\nu}, \qquad \Pi_2=\eta=\frac{Uy}{\nu}.$$

We rewrite the equations and boundary conditions in variables ξ and η.

$$\frac{\partial\phi_u}{\partial x}=\frac{\partial\phi_u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\phi_u}{\partial\eta}\frac{\partial\eta}{\partial x}, \qquad \frac{\partial\phi_u}{\partial y}=\frac{\partial\phi_u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\phi_u}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$\frac{\partial\phi_u}{\partial x}=\frac{U}{\nu}\frac{\partial\phi_u}{\partial\xi}, \qquad \frac{\partial\phi_u}{\partial y}=\frac{U}{\nu}\frac{\partial\phi_u}{\partial\eta}$$

$$u=U\phi_u \implies \frac{\partial u}{\partial x}=U\frac{\partial\phi_u}{\partial x}, \qquad \frac{\partial u}{\partial y}=U\frac{\partial\phi_u}{\partial y}$$

so

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= U^2\phi_u\frac{\partial\phi_u}{\partial x}+U^2\phi_v\frac{\partial u}{\partial y} \text{ as } v=U\phi_v \\ &= \frac{U^3}{\nu}\phi_u\frac{\partial\phi_u}{\partial\xi}+\frac{U^3}{\nu}\phi_v\frac{\partial\phi_u}{\partial\eta} \end{aligned}$$

$$\begin{aligned} u=U\phi_u \implies \frac{\partial u}{\partial y}=U\frac{\partial\phi_u}{\partial y},\ \frac{\partial^2 u}{\partial y^2} &= U\frac{\partial}{\partial y}(\frac{\partial\phi_u}{\partial y})=U\frac{\partial}{\partial y}(\frac{U}{\nu}\frac{\partial\phi_u}{\partial\eta})=\frac{U^2}{\nu}\frac{\partial}{\partial y}(\frac{\partial\phi_u}{\partial\eta}) \\ &= \frac{U^2}{\nu}\frac{\partial^2\phi_u}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial y}+\frac{U^2}{\nu}\frac{\partial^2\phi_u}{\partial\eta^2}\frac{\partial\eta}{\partial y}=\frac{U^3}{\nu^2}\frac{\partial^2\phi_u}{\partial\eta^2}=\frac{\partial^2 u}{\partial y^2} \end{aligned}$$

and we have,

$$\phi_u\frac{\partial\phi_u}{\partial\xi}+\phi_v\frac{\partial\phi_u}{\partial\eta}=\frac{\partial^2\phi_u}{\partial\eta^2}, \quad \frac{\partial\phi_u}{\partial\xi}+\frac{\partial\phi_v}{\partial\eta}=0 \tag{6}$$

$$u(0,y)=u(x,\infty)=U \text{ gives } \phi_u(0,\eta)=\phi_u(\xi,\infty)=1$$

$$u(x,0)=v(x,0)=0 \text{ gives } \phi_u(\xi,0)=\phi_v(\xi,0)=0.$$

We can now show that the system is invariant under an additional transformation group. We let $\phi_u(\xi,\eta)$, $\phi_v(\xi,\eta)$ be unique solution to our derived equations and consider the one-parameter transformation group

$$\xi' = \alpha^a \xi, \qquad \eta' = \alpha^b \eta, \qquad \phi_u' = \alpha^c \phi_u, \qquad \phi_v' = \alpha^d \phi_v$$

$$\phi_u' = \alpha^c \phi_u \implies \frac{\partial \phi_u'}{\partial \xi'} = \frac{\partial(\alpha^c \phi_u)}{\partial \xi'} = \alpha^c \frac{\partial \phi_u}{\partial \xi'}$$

$$\frac{\partial \phi_u}{\partial \xi} = \frac{\partial \phi_u}{\partial \xi'}\frac{\partial \xi'}{\partial \xi} = \alpha^a \frac{\partial \phi_u}{\partial \xi'} = \alpha^{a-c}\frac{\partial \phi_u'}{\partial \xi'}$$

Similarly $$\frac{\partial \phi_u}{\partial \eta} = \alpha^{b-c}\frac{\partial \phi_u'}{\partial \eta'} \text{ and } \frac{\partial \phi_v}{\partial \eta} = \alpha^{b-d}\frac{\partial \phi_v'}{\partial \eta'}$$

Substituting in equation (6)

$$\alpha^{a-2c}\phi_u'\frac{\partial \phi_u'}{\partial \xi'} + \alpha^{b-c-d}\phi_v'\frac{\partial \phi_u'}{\partial \eta'} = \alpha^{2b-c}\frac{\partial^2 \phi_u'}{\partial \eta'^2}, \quad \alpha^{a-c}\frac{\partial \phi_u'}{\partial \xi'} + \alpha^{b-d}\frac{\partial \phi_v'}{\partial \eta'} = 0.$$

To get the same equation and boundary conditions in the new variables we must have

$$a - 2c = b - c - d, \qquad a - 2c = 2b - c, \qquad a - c = b - d$$

from which we have

$$a - 2b - c = 0, \qquad a - c - b + d = 0.$$

Subtracting gives

$$b + d = 0. \text{ For simplicity choose } b = 1 \implies d = -1.$$

To get the same boundary conditions we must have $c = 0$ which gives $a = 2$. Finally,

$$\xi' = \alpha^2 \xi, \qquad \eta' = \alpha\eta$$

$$\phi_u'(\xi',\eta') = \phi_u(\xi,\eta), \qquad \phi_v'(\xi',\eta') = \alpha^{-1}\phi_v(\xi,\eta).$$

Here ϕ_u', ϕ_v' are also unique. This set of transformations forms a group. Setting $\alpha = 1$ gives the identity transformation. $\beta = \frac{1}{\alpha}$ gives the inverse transformation to α and $\gamma = \alpha\beta$ can give the product of transformations α and β applied successively. This group property is important because we want the inverse transformation to exist so that we can change any solutions to the original variables in the original coordinates. We therefore have

$$\phi_u(\xi,\eta) = \phi_u'(\xi',\eta') = \phi_u(\alpha^2\xi, \alpha\eta)$$
$$\phi_v(\xi,\eta) = \alpha\phi_v'(\xi',\eta') = \alpha\phi_v(\alpha^2\xi, \alpha\eta).$$

We now let $\alpha = \frac{1}{\sqrt{\xi}}$ and get

$$\phi_u(\xi,\eta) = \phi_u(1, \frac{\eta}{\sqrt{\xi}}) = f_u(\frac{\eta}{\sqrt{\xi}}) = f_u(\frac{y}{\sqrt{\frac{\nu x}{U}}})$$

$$\phi_v(\xi,\eta) = \frac{1}{\sqrt{\xi}}\phi_v(1, \frac{\eta}{\sqrt{\xi}}) = \frac{1}{\sqrt{\xi}} f_v(\frac{\eta}{\sqrt{\xi}}) = \sqrt{\frac{\nu}{Ux}} f_v(\frac{y}{\sqrt{\frac{\nu x}{U}}})$$

In doing so we have found what Ludwig Prandtl showed; the flow past a flate plate doesn't depend on the two space coordinates x and y separately, but on their combination $\frac{y}{\sqrt{\frac{\nu x}{U}}}$

8.1 An alternative approach

It is possible to apply Huntley's method to this boundary-layer equation and obtain the same result with less effort. The dimensions of the quantities become

$$[u] = [U] = L_x T^{-1}, \qquad [v] = L_y T^{-1}, \qquad [y] = L_y, [x] = L_x, \qquad [\nu] = L_y^2 T^{-1}.$$

The dimensions of the other quantities are obvious except ν. The dimension for ν is obtained as follows.

As we move vertically from the surface of the flat plate, where the fluid velocity is taken to be zero, the fluid velocity increases. Thus a velocity gradient $\frac{du}{dy}$ is setup. It is assumed that the shearing stress $\frac{F}{A}$ is directly proportional to this velocity gradient (Streeter et al., 1997). By the definition of shearing stress, F is parallel to the flat plate, and A is the area of the flat plate.

$$\frac{F}{A} \propto \frac{du}{dy}, \qquad \text{i.e} \qquad \frac{F}{A} = \mu \frac{du}{dy}$$

This constant of proportionality is called the dynamic viscosity. We can now find the dimension of μ from this equation.

$$[F] = ML_x T^{-2}, \qquad [A] = L_x L_z, \qquad [u] = L_x T^{-1}, \qquad [y] = L_y$$

Substituting the dimensions in the formula, we have

$$\frac{ML_x T^{-2}}{L_x L_z} = [\mu] \frac{L_x T^{-1}}{L_y}$$

$$[\mu] = L_x^{-1} L_y L_z^{-1} M T^{-1}$$

$$\text{therefore } [\nu] = [\frac{\mu}{\rho}] = \frac{Lx^{-1} L_y L_z^{-1} M T^{-1}}{L_x^{-1} L_y^{-1} L_z^{-1} M} = L_y^2 T^{-1}.$$

Now we have three quantities with independent dimensions. We arbitrarily let ν, U and x be these quantities, so $k = 3$ and as $n = 4$, we have $m = n - k = 1$. Thus

$$\Pi'_u = \phi'_u(\Pi'_1) = \frac{u}{U}$$

$$\Pi'_1 = \frac{y}{\nu^{p_1} U^{q_1} x_1^r}$$

$$L_y = (L_y^2 T^{-1})^{p_1} (L_x T^{-1})^{q_1} (L_x)^{r_1}$$

$$p_1 = \frac{1}{2}, p_1 + q + 1 = 0, q_1 + r_1 = 0$$

$$\implies q_1 = -\frac{1}{2}, r_1 = \frac{1}{2}$$

$$\Pi'_1 = \frac{y}{\nu^{\frac{1}{2}} U^{\frac{-1}{2}} x^{\frac{1}{2}}} = \frac{y}{\sqrt{\frac{\nu x}{U}}}.$$

and

$$\Pi'_v = \phi'_v(\Pi'_1) = \frac{v}{\nu^p U^q x^r}$$

$$L_y T^{-1} = (L_y^2 T^{-1})^p (L_x T^{-1})^q (L_x)^r$$

$$p = \frac{1}{2}, p + q = 1, q + r = 0$$

$$\implies q = \frac{1}{2}, r = -\frac{1}{2}$$

$$\Pi'_v = \frac{v}{\nu^{\frac{1}{2}} U^{\frac{1}{2}} x^{\frac{-1}{2}}} = \frac{v}{\sqrt{\frac{\nu U}{x}}}$$

In this instance Huntley's method was applied to equation (5) and it has given us the correct solution. We have already shown that it is not applicable to the full N-S equations.

Conclusion

Our choice of length, mass, time and temperature as fundamental quantities and their dimensions as basic dimensions means that the dimensions for other physical quantities can be derived from these basic dimensions. Therefore all other physical quantities can be expressed as some sort of combination of length, mass, time and temperature; for example, velocity as change in length over time, energy as the product of mass and the velocity of light squared, and so on. Dimensional analysis is the use of this idea in establishing the forms for other physical quantities. This is similar to the ancient Greek's belief that every thing is made up of fire, wind, water and earth.

We have shown that dimensional analysis can give the form for formulae of physical quantities . It also gives a method for forming the dimensionless variables $\Pi, \Pi_1, \cdots, \Pi_m$ however; it cannot tell us anything about the function $\phi(\Pi_1, \cdots, \Pi_m)$. We used dimensional analysis to nondimensionalize the N-S equations and in the process we were able to establish two important equations used in fluid mechanics: the Euler equations for incompressible fluids and the Stokes equation for slow flow. In Section 6 we used complete similarity to derive the solution for heat flow in a bar of finite length, and incomplete similarity to establish the form for the solution of fluid flow past a wedge. These examples show that dimensional analysis can reveal self-similarity and gives the similarity variables. We have also shown that dimensional analysis gives the rules for scaling the results for a physically similar model up to a prototype. In this case we formulated the rules for scaling the velocity and drag force of a model racing car up to the prototype. The example of fluid flow past a flat plate shows that dimensional analysis can be instructive in revealing that a problem is invariant under a transformation group, which can then be used to formulate a solution.

Whatever dimensional analysis reveals about a phenomenon is the result of our choice of fundamental units. On one hand Huntley's introduction of even more fundamental units L_x, L_y and L_z of length in some cases, such as the example of range of a projectile, takes us a step further but it can also be at odds with well established equations like the N-S equations. On the other hand Relativity introduces the concept of space-time: that space and time are inseparable. We have yet to study the effects of a new dimension of space-time. No matter what fundamental units we use, our choice of quantities with independent dimensions from a list of quantities is important. As seen in the examples, we chose the quantities with independent dimensions carefully to obtain a relation that has the form of the formula already known for that phenomenon. In practice we would have to work by trial and error to establish a formula that best fits available data.

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Appendices

A Derivation of Laplace's equation in polar coordinates

We have

$$x = r\cos(\theta), \qquad y = r\sin(\theta)$$

$$\begin{aligned}\frac{\partial\Phi}{\partial r} &= \frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial r}\\ &= \cos\theta\frac{\partial\Phi}{\partial x} + \sin\theta\frac{\partial\Phi}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\Phi}{\partial r^2} &= \cos\theta[\frac{\partial^2\Phi}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2\Phi}{\partial y\partial x}\frac{\partial y}{\partial r}] + \sin\theta[\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2\Phi}{\partial y^2}\frac{\partial y}{\partial r}]\\ &= \cos^2\theta\frac{\partial^2\Phi}{\partial x^2} + \cos\theta\sin\theta\frac{\partial^2\Phi}{\partial y\partial x} + \cos\theta\sin\theta\frac{\partial^2\Phi}{\partial x\partial y} + \sin^2\theta\frac{\partial^2\Phi}{\partial y^2}\\ &= \cos^2\theta\frac{\partial^2\Phi}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2\Phi}{\partial y\partial x} + \sin^2\theta\frac{\partial^2\Phi}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial\Phi}{\partial\theta} &= \frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial\theta}\\ &= -r\sin\theta\frac{\partial\Phi}{\partial x} + r\cos\theta\frac{\partial\Phi}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\theta^2} &= \frac{\partial}{\partial\theta}(-r\sin\theta\frac{\partial\Phi}{\partial x}) + \frac{\partial}{\partial\theta}(-r\cos\theta\frac{\partial\Phi}{\partial y})\\ &= -r\cos\theta\frac{\partial\Phi}{\partial x} - r\sin\theta\frac{\partial}{\partial\theta}(\frac{\partial\Phi}{\partial x}) - r\sin\theta\frac{\partial\Phi}{\partial y} + r\cos\theta\frac{\partial}{\partial\theta}(\frac{\partial\Phi}{\partial y})\\ &= -r\cos\theta\frac{\partial\Phi}{\partial x} - r\sin\theta\frac{\partial\Phi}{\partial y} + [-r\sin\theta(\frac{\partial^2\Phi}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2\Phi}{\partial y\partial x}\frac{\partial y}{\partial\theta}) + r\cos\theta(\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2\Phi}{\partial y^2}\frac{\partial y}{\partial\theta})]\\ &= -r\frac{\partial\Phi}{\partial r} + [-r\sin\theta(-r\sin\theta\frac{\partial^2\Phi}{\partial x^2} + r\cos\theta\frac{\partial^2\Phi}{\partial y\partial x}) + r\cos\theta(-r\sin\theta\frac{\partial^2\Phi}{\partial x\partial y} + r\cos\theta\frac{\partial^2\Phi}{\partial y^2})]\end{aligned}$$

we have

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = -\frac{1}{r}\frac{\partial\Phi}{\partial r} + \sin^2\theta\frac{\partial^2\Phi}{\partial x^2} + \cos^2\theta\frac{\partial^2\Phi}{\partial y^2} - 2\cos\theta\sin\theta\frac{\partial^2\Phi}{\partial y\partial x}$$

adding the partial derivatives together we have

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{1}{r}\frac{\partial\Phi}{\partial r}$$

Rearrange to get the desired result

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}$$

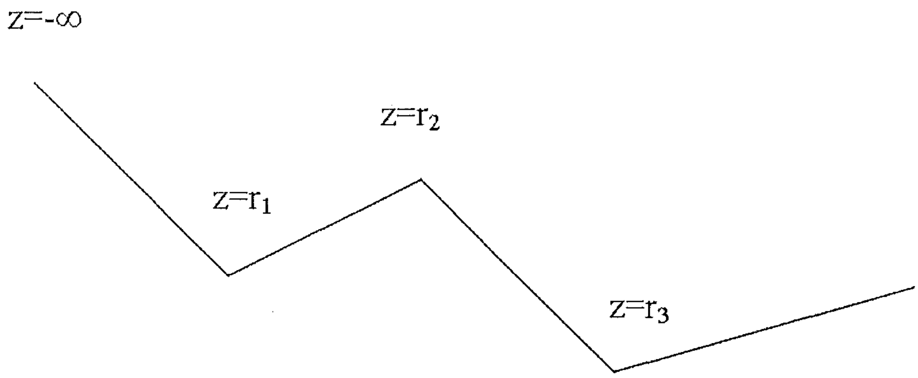
B Schwarz-Christoffel (S-C) Transformation

The S-C transformation maps top half of z-plane into a w-plane region defined by a series of straight line segments, straight lines and/or rays. Consider $dw = A(z-r_1)^{\phi_1}(z-r_2)^{\phi_2}\cdots(z-r_3)^{\phi_3}dz$ where A is a complex constant, $r_1 \cdots, r_n$ are all real such that $r_1 \leq r_2 \leq \cdots \leq r_n$ and $\phi_1 \cdots \phi_n$ are also real.

We want to map boundary of upper half of z-plane (i.e real z-plane axis) into w-plane. Let z trace along the real axis from $-\infty$ to $+\infty$. Then

$$arg(dw) = arg(A) + arg((z-r_1)^{\phi_1}) + \cdots + arg((z-r_n)^{\phi_n}) + arg(dz)$$
$$arg(dw) = arg(A) + \phi_1 arg(z-r_1) + \cdots + \phi_n arg(z-r_n) + arg(dz)$$

Now $z - r_k$ is real so $arg(z-r_k)$ equals zero when $z \geq r_k$ and π when $z \leq r_k$. And $arg(dz) = 0$ as z is increasing along the real axis. So $arg(dw)$ is constant for all z values between a pair of consecutive r values



(say $r_1 \leq z \leq r_2$) and is also constant for all $z \leq r_1$ and $z \geq r_n$. But when z crosses r_k the $arg(z-r_k)$ drops from π to 0. Therefore $arg(dw)$ changes by $-\phi_k\pi$. Hence $w(z)$ looks like figure 8.

In practice we usually use internal angles α_1 and α_2 where $\alpha_k - \phi_k\pi = 180\deg = \pi$.

$$\text{i.e } \frac{\alpha_k}{\pi} - 1 = \phi_k$$

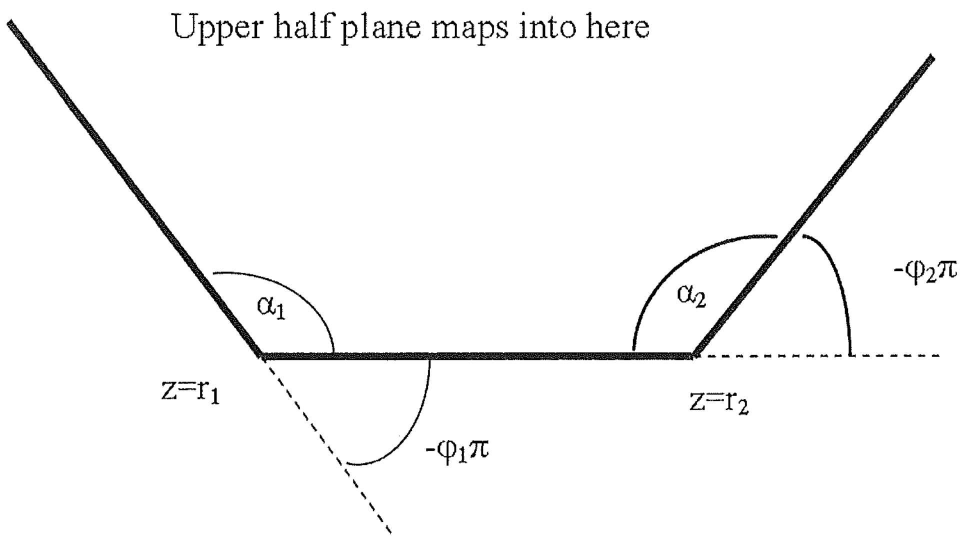


Figure 8: Scwwarz-Christoffel Transformation

C Simplifying the S-C integral

$$z = l\int_0^{\frac{\zeta}{l}} (l\tau)^{-\frac{\alpha}{\pi}}(l\tau - l\mu)^{\frac{1}{2}+\frac{\alpha}{\pi}}(l\tau - l\delta)^{-\frac{1}{2}}d\tau$$
$$z = l\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}(\tau - \mu)^{\frac{1}{2}+\frac{\alpha}{\pi}}(\tau - \delta)^{-\frac{1}{2}}d\tau$$
$$z = l\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}(\frac{\tau}{\mu} - 1)^{\frac{1}{2}+\frac{\alpha}{\pi}}(\frac{\tau}{\delta} - 1)^{-\frac{1}{2}}\mu^{\frac{1}{2}+\frac{\alpha}{\pi}}\delta^{-\frac{1}{2}}d\tau$$
$$z = l\mu^{\frac{1}{2}+\frac{\alpha}{\pi}}\delta^{-\frac{1}{2}}e^{i\alpha}\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}(1 - \frac{\tau}{\mu})^{\frac{1}{2}+\frac{\alpha}{\pi}}(1 - \frac{\tau}{\delta})^{-\frac{1}{2}}d\tau$$
$$z = l\mu^{\frac{1}{2}+\frac{\alpha}{\pi}}\delta^{-\frac{1}{2}}e^{i\alpha}\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}(1 - (\frac{1}{2} + \frac{\alpha}{\pi})\frac{\tau}{\mu})(1 + \frac{\tau}{2\delta})d\tau$$
$$z = l\beta e^{i\alpha}\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}(1 + \frac{\tau}{2\delta} - (\frac{1}{2} + \frac{\alpha}{\pi})\frac{\tau}{\mu} - (\frac{1}{2} + \frac{\alpha}{\pi})\frac{\tau^2}{2\mu\delta})d\tau$$
$$z = l\beta e^{i\alpha}\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}} + (\frac{1}{2\delta} - \frac{1}{2\mu} - \frac{\alpha}{\pi\mu})\tau^{\frac{\pi-\alpha}{\pi}} - \frac{1}{2\mu\delta}(\frac{1}{2} + \frac{\alpha}{\pi})\tau^{\frac{2\pi-\alpha}{\pi}}d\tau$$

as $\tau \ll 1$, $\tau^{\frac{\pi-\alpha}{\pi}}$ and $\tau^{\frac{2\pi-\alpha}{\pi}}$ are insignificant compared to $\tau^{-\frac{\alpha}{\pi}}$, because $0 \le \alpha \le \frac{\pi}{2}$. We have

$$z = l\beta e^{i\alpha}\int_0^{\frac{\zeta}{l}} \tau^{-\frac{\alpha}{\pi}}d\tau = l\beta e^{i\alpha}(\frac{\zeta}{l})^{\frac{\pi-\alpha}{\pi}}(\frac{\pi}{\pi-\alpha}).$$