An inverse analysis for determination of space-dependent heat flux in heat conduction problems in the presence of variable thermal conductivity

Farzad Mohebbi[1*], Ben Evans[1], Alexander Shaw[1], Mathieu Sellier[2]

[1] *Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Bay Campus, Fabian Way, Crymlyn Burrows, Swansea SA1 8EN, UK*

[2] *Department of Mechanical Engineering, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand*

Abstract. This paper presents an inverse problem of determination of a space-dependent heat flux in steady-state heat conduction problems. The thermal conductivity of a heat conducting body depends on the temperature distribution over the body. In this study the simulated measured temperature distribution on part of the boundary is related to the variable heat flux imposed on a different part of the boundary through incorporating the variable thermal conductivity components into the sensitivity coefficients. To do so, a body-fitted grid generation technique is used to mesh the two-dimensional irregular body and solve the direct heat conduction problem. An efficient, accurate, robust, and easy to implement method is presented to compute the sensitivity coefficients through derived expressions. The main novelty of the study lies in the sensitivity analysis in which all sensitivities can be obtained in only one direct solution at each iteration, irrespective of the number of unknown parameters. The conjugate gradient method along with the discrepancy principle is used in the inverse analysis to minimize the objective function and achieve the desired solution.

Keywords: Inverse Heat Transfer; Elliptic Grid Generation; Sensitivity Analysis; Finite Difference Method; Conjugate Gradient Method; Variable Thermal Conductivity; Variable Heat Flux

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[*] Corresponding author

Email address: farzad.mohebbi@swansea.ac.uk, farzadmohebbi@yahoo.com

NOMENCLATURE:

$d^{(\mathrm{k})}$	direction of descent at iteration k
$\dot{q}$	heat flux $(\frac{\mathrm{W}}{\mathrm{m}^2})$
h	heat transfer coefficient $(\frac{\mathrm{W}}{\mathrm{m}^2.^\circ\mathrm{C}})$
$\mathbf{Ja}$	Jacobian matrix
J	Jacobian of transformation
$\mathcal{J}$	objective function
k_T	thermal conductivity of the solid body $(\frac{\mathrm{W}}{\mathrm{m}.^\circ\mathrm{C}})$
$\mathbf{n}$	outward drawn unit vector
T	temperature $(^\circ\mathrm{C})$
T_m	measured outer surface temperature $(^\circ\mathrm{C})$
T_∞	ambient temperature $(^\circ\mathrm{C})$
x, y	Cartesian coordinates in the physical domain

Greek symbols

α, β, γ	metric coefficients in 2-D elliptic grid generation
$\beta^{(\mathrm{k})}$	search step size at iteration k
Γ	boundary
$\gamma^{(\mathrm{k})}$	conjugation coefficient at iteration k
Ω	domain
ξ, η	Cartesian coordinates in the computational domain

Subscripts

i	grid index in ξ- direction
j	grid index in η- direction
M	number of grid points in the ξ- direction
N	number of grid points in the η- direction

Superscript

k	iteration number

1. Introduction

Nowadays, due to the ever-increasing power of high-speed computers, the numerical treatment of inverse heat transfer problems (IHTP) has received much attention among mathematicians and engineers. IHTPs are ill-posed which makes them difficult to solve. The ill-posed problems are extremely unstable in that a small error in measurement can lead to a significant error in the estimated variable. There exist different methods to overcome the instabilities associated with the solution of IHTPs. Among such methods are the iterative regularization methods in which there is no need to modify the original objective function. In these gradient-based methods, the discrepancy principle may be used as a criterion to terminate the iteration and obtain a reasonably stable solution. In direct heat transfer problems, the known boundary conditions, the thermo-physical properties, the geometrical configuration of the heated body, and the applied heat flux on some part of the boundary are used to obtain the temperature distribution over the heated body. However, in inverse heat transfer problems, the measured temperature distribution on some part of the boundary of the heat conducting body is used to determine the boundary conditions, the thermo-physical properties, the geometrical configuration of the heated body, and the applied heat flux [1-3]. Inverse analysis has been employed to determine the thermo-physical properties such as the thermal conductivity and the convection heat transfer coefficient [4-35] , the heat flux [36-41], and the boundary shape of bodies [42-47].

In our recent works [16-19, 44-46], a novel sensitivity analysis scheme is proposed to compute the sensitivity coefficients in only one direct solution, without the need for the solution of the adjoint equations. In our studies, the determination of the geometrical configuration and the boundary conditions is performed by assuming a constant thermal conductivity. Only in [17], a variable (temperature-dependent) thermal conductivity is considered to estimate the variable thermal conductivity itself. The steady-state heat conduction equation is a nonlinear one when the thermal conductivity is not constant which makes the heat transfer equation difficult to solve. In this study, the applied heat flux is recovered using an inverse analysis. A novel method to obtain the sensitivity analysis expressions is presented in which the chain rule and differentiation are used to develop a relation between the measured temperatures and the imposed heat flux. As the measurement of temperature and application of the heat flux are not usually on the same parts of the boundary (for example, on reentry into the atmosphere, the temperature of the surface of the thermal shield of a space vehicle is so high that it cannot be measured directly with temperature sensors), the components of the thermal conductivity (not itself) can be used to establish a relation between the measured temperatures and the imposed heat flux. The reason for not being able to use the thermal conductivity itself is that it changes at each node. In other words, the magnitude of the thermal conductivity at the boundary surface where the temperature is measured is not equal to the one at the boundary surface where the heat flux is applied.

In the literature, there exist some limitations on numerical treatment of inverse heat conduction problems. Some of these limitations can be summarized as follows:

- the applicability of the direct solver to rectangular or circular heated bodies only (using traditional finite-difference method) and inability to consider a general 2D domain.

- the inability to handle a variety of boundary conditions. Most of the boundary conditions in the literature include a constant temperature (Dirichlet boundary condition) or insulated case.

- assuming a constant thermal conductivity in estimation of boundary conditions.

Thus in numerical treatment of the inverse heat transfer problems with a *general* 2D domain, a general methodology for accurate estimation of boundary conditions in the presence of a variable thermal conductivity is required. This paper deals with a two dimensional inverse steady-state heat conduction problem. The objective of this study is to estimate a variable (space-dependent) heat flux in an irregular body in the presence of a variable (temperature-dependent) thermal conductivity.

The proposed numerical approach takes advantage of the elliptic grid generation technique to generate a mesh over the irregular body and then solve for the nonlinear steady-state heat conduction equation using the finite-difference method, a nonlinear least square formulation to define the objective function, a novel, efficient, and accurate sensitivity analysis scheme to compute the sensitivity coefficients, and a gradient based optimization method.

The most innovative aspect of the numerical approach is its very efficient and accurate sensitivity analysis scheme. The sensitivity analysis scheme is formulated to compute the sensitivity of the temperatures to variation of the variable heat flux. The conjugate gradient method is employed to minimize the difference between the computed temperature on part of the boundary and the simulated measured temperature. As will be shown, this numerical methodology does not require the solution of an adjoint problem. Explicit expressions for the sensitivity coefficients are derived which allow for the computation of the sensitivity coefficients in one single solve only.

The proposed solution method introduced here is sufficiently general and can be employed for the estimation of a variable (space-dependent) heat flux applied on part of the boundary of a *general* two-dimensional region as long as the general two-dimensional region can be mapped onto a regular computational domain. Moreover, there is no limitation on the type of the boundary conditions. In other words, Dirichlet, Neumann, and Robin boundary conditions can be imposed on the domain boundary.

2. Governing Equation

The mathematical formulation for the steady state heat conduction problem with linearly temperature - dependent thermal conductivity is given by (see Fig. 1a)

$$\frac{\partial}{\partial x}(k_T \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_T \frac{\partial T}{\partial y}) = 0 \text{ in physical domain } \Omega \tag{1}$$

subject to the boundary conditions

$$\frac{\partial T}{\partial n} = \frac{\dot{q}}{k_T} \text{ on boundary surface } \Gamma_1 \tag{2}$$

$$\frac{\partial T}{\partial n} = -\frac{h_i}{k_T}(T_{\Gamma_i} - T_{\infty_i}) \text{ on boundary surface } \Gamma_i, i = 2, 3, 4 \tag{3}$$

where $k_T(T) = a + bT$ (a and b are constants and $b \neq 0$) and $\dot{q}(\Gamma_1) = a_1 + a_2 X_{\Gamma_1} + a_3 Y_{\Gamma_1}$ (a_1, a_2, and a_3 are constants).

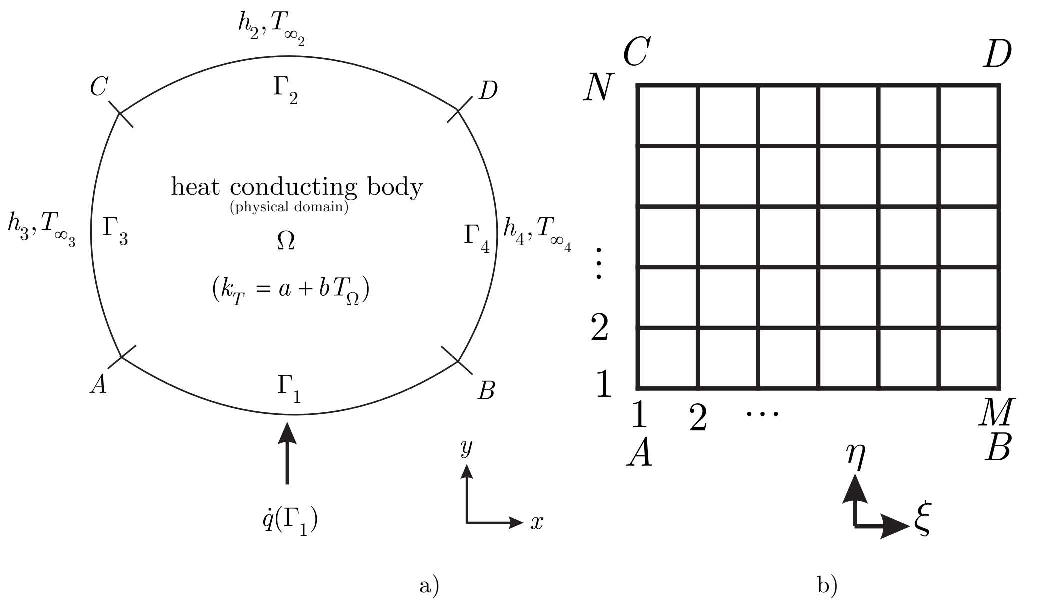


Fig. 1 Arbitrarily shaped two dimensional heat-conducting body (physical domain) subjected to convective heat transfer on surfaces $\Gamma_i, i = 2,3,4$ and a space-dependent heat flux $\dot{q}(\Gamma_1)$ on surface Γ_1 (a) and the corresponding computational domain (b). The thermal conductivity of the body, $k_T(T)$, is a temperature-dependent variable.

Here the discretization of the physical domain and approximation of the derivatives of the field variable (temperature) by algebraic ones are performed by using the elliptic grid generation method. In this method, the irregular physical domain is mapped from the x and y physical plane onto the ξ and η computational plane (Fig. 1b). Then the heat conduction equation and the boundary conditions (Eqs. (1) to (3)) are transformed from the x and y physical plane to the ξ and η computational plane. More details on the implementation of the elliptic grid generation technique and solution procedure for the steady-state heat conduction equation can be found in [48, 49]. Here since the thermal conductivity is not constant and is a linearly temperature-dependent variable, we can expand Eq. (1) as follows

$$\frac{\partial}{\partial x}((a+bT)\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}((a+bT)\frac{\partial T}{\partial y}) = 0$$

$$(a+bT)\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + b\left[(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\right] = 0 \tag{4}$$

we can substitute for T_x, T_y, T_{xx}, and T_{yy}, using the transformation relationships and finite difference expressions [49]:

$$(a+bT)\left[\frac{1}{J^2}(\alpha T_{\xi\xi}-2\beta T_{\xi\eta}+\gamma T_{\eta\eta})\right]+b\left[\frac{1}{J^2}(y_\eta T_\xi - y_\xi T_\eta)^2+\frac{1}{J^2}(-x_\eta T_\xi + x_\xi T_\eta)^2\right]=0$$

$$(a+bT)\left[(\alpha T_{\xi\xi}-2\beta T_{\xi\eta}+\gamma T_{\eta\eta})\right]+b\left[\alpha {T_\xi}^2-2\beta T_\xi T_\eta+\gamma {T_\eta}^2\right]=0$$

$$(a+bT_{i,j})\Bigg(\alpha(T_{i+1,j}-2T_{i,j}+T_{i-1,j})-2\beta(\frac{1}{4}T_{i+1,j+1}-\frac{1}{4}T_{i-1,j+1}-\frac{1}{4}T_{i+1,j-1}+\frac{1}{4}T_{i-1,j-1})$$

$$+\gamma(T_{i,j+1}-2T_{i,j}+T_{i,j-1})\Bigg)+b\Bigg(\alpha(\frac{1}{2}T_{i+1,j}-\frac{1}{2}T_{i-1,j})^2-2\beta(\frac{1}{2}T_{i+1,j}-\frac{1}{2}T_{i-1,j})$$

$$(\frac{1}{2}T_{i,j+1}-\frac{1}{2}T_{i,j-1})+\gamma(\frac{1}{2}T_{i,j+1}-\frac{1}{2}T_{i,j-1})^2\Bigg)=0 \tag{5}$$

where

$$\alpha = x_\eta^2 + y_\eta^2$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2 \tag{6}$$

are the coefficients obtained from the elliptic grid generation method. By having the values for a and b, Eq. (5) may be solved to obtain an expression for $T_{i,j}$. Eq. (5) is a quadratic one and an algebraic software such as Maple may be used to solve the equation in terms of $T_{i,j}$. The boundary condition equations also can be expanded and solved in a similar way. The variable heat flux imposed on the boundary surface Γ_1 is $\dot{q}=a_1+a_2x_{i,1}+a_3y_{i,1}$ ($i=1,\ldots,M$) which is known by having the constants a_1, a_2, and a_3 and $x_{i,1}$ and $y_{i,1}$ ($i=1,\ldots,M$) from the grid generation step. The direct heat conduction problem can be numerically solved to obtain the temperature distribution in the heat conducting body. By having the temperature values at any grid nodes as well as a and b, the thermal conductivity $k_T = a + bT_{i,j}$ can be calculated at any grid nodes (i,j).

3. The inverse analysis

3.1 Objective function

Here the inverse analysis is used to estimate the variable heat flux $\dot{q}(\Gamma_1) = a_1 + a_2 X_{\Gamma_1} + a_3 Y_{\Gamma_1}$ (using the estimation of a_1, a_2, and a_3) so that the square of the difference between the computed temperature of the outer surface Γ_2 and the measured temperature of the same surface is minimized. This can be mathematically expressed as

$$\min \left\{ \underset{\dot{q} \text{ on } \Gamma_1}{\mathcal{J}} := \left\| T_{\Gamma_2} - T_m \right\|^2 : \text{Eq.(1) in } \Omega,\ \text{BCs in Eqs.(2)-(3)} \right\} \tag{7}$$

where T_m is the measured temperature. The aim of the inverse analysis is to minimize the following objective function expression using optimization of the value of a_1, a_2, and a_3:

$$\mathcal{J} = \sum_{i=2}^{M-1} (T_{i,N} - T_{(i,N)_m})^2 \tag{8}$$

3.2 Sensitivity analysis

Since the proposed method is concerned with a gradient-based optimization method (here, conjugate gradient method), the required computation of derivative of the objective function with respect to the unknown variables (a_1, a_2, and a_3) is performed as follows

$$\frac{\partial \mathcal{J}}{\partial a_l} = 2\sum_{i=2}^{M-1} (T_{i,N} - T_{(i,N)_m}) \frac{\partial T_{i,N}}{\partial a_l} \tag{9}$$

where $l = 1,2,3$. In Eq. (9), $\dfrac{\partial T_{i,N}}{\partial a_l}$ ($l = 1,2,3$) are called the sensitivity coefficients. To obtain an algebraic expression for the sensitivity coefficients, we can use the chain rule to relate the temperature distribution on the surface Γ_2, $T_{i,N}$, to the variable heat flux (imposed on the surface Γ_1) components (a_1, a_2, and a_3) as follows

$$\frac{\partial T_{i,N}}{\partial a_l} = \frac{\dfrac{\partial T_{i,N}}{\partial a}}{\dfrac{\partial a_l}{\partial a}} \tag{10}$$

where $l = 1,2,3$ and a is the thermal conductivity constant ($k_T = a + bT$). To obtain an algebraic expression for the first term, $\dfrac{\partial T_{i,N}}{\partial a}$, from the boundary condition of the heat conduction equation at the surface Γ_2, Eq. (3), we have

$$\dot{q}_{\text{conduction}} \,|_{\Gamma_2} = \dot{q}_{\text{convection}} \,|_{\Gamma_2}$$

$$-k_T \frac{\partial T}{\partial n_2} = h_2 (T_{\Gamma_2} - T_{\infty_2})$$

$$-(a + bT_{i,N}) \frac{\partial T}{\partial n_2} = h_2 (T_{\Gamma_2} - T_{\infty_2}) \tag{11}$$

The term $\dfrac{\partial T}{\partial n}$ at a boundary surface in the physical domain is related to $\dfrac{\partial T}{\partial \xi}$ and/or $\dfrac{\partial T}{\partial \eta}$ at the corresponding transformed boundary surface in the computational domain. At surface Γ_2 we have

$$\frac{\partial T}{\partial n} \,|_{\Gamma_2} = \frac{\partial T}{\partial \eta} \,|_{\Gamma_2} = \frac{1}{J\sqrt{\gamma}} (\gamma T_\eta - \beta T_\xi) \,|_{\Gamma_2} \tag{12}$$

where the coefficients α, β, γ are defined in Eq. (6) and J is the Jacobian of transformation, $J = x_\xi y_\eta - x_\eta y_\xi$. Using the finite difference method, the T_η and T_ξ at every boundary surface with Neumann and Robin conditions can be discretized. By substituting Eq. (12) and the finite difference expressions for T_ξ and T_η into Eq. (11), we get

$$-(a+bT_{i,N})\left[\frac{1}{J\sqrt{\gamma}}\left(\gamma\frac{3T_{i,N}-4T_{i,N-1}+T_{i,N-2}}{2}-\beta\frac{T_{i+1,N}-T_{i-1,N}}{2}\right)\right]=h_2\left(T_{i,N}-T_{\infty_2}\right) \quad (13)$$

Eq. (13) is a quadratic equation in terms of $T_{i,N}$. Using an algebraic software, one can obtain an expression for the term $\frac{\partial T_{i,N}}{\partial a}$ by differentiating the obtained expression for $T_{i,N}$ with respect to a. To obtain an explicit expression for the second term, $\frac{\partial a_l}{\partial a}$, from the boundary condition at the surface Γ_1, Eq. (2), we have

$$\dot{q}=-k_T\left.\frac{\partial T}{\partial n}\right|_{\text{at }\Gamma_1}$$

$$a_1+a_2x_{i,1}+a_3y_{i,1}=-(a+bT_{i,1})\left.\frac{\gamma T_\eta-\beta T_\xi}{J\sqrt{\gamma}}\right|_{\text{at }\Gamma_1} \quad (14)$$

At surface Γ_1, we can write, for example

$$T_\xi=\frac{T_{i+1,1}-T_{i-1,1}}{2}$$

$$T_\eta=\frac{-3T_{i,1}+4T_{i,2}-T_{i,3}}{2}$$

Using an algebraic software, we can solve Eq. (14) in terms of a_l ($l=1,2,3$):

$$a_1=-\frac{1}{2}\frac{1}{J\sqrt{\gamma}}(2a_2x_{i,1}J\sqrt{\gamma}+2a_3y_{i,1}J\sqrt{\gamma}-3bT_{i,1}^2\gamma+4bT_{i,1}T_{i,2}\gamma-bT_{i,1}T_{i,3}\gamma+bT_{i,1}T_{i-1,1}\beta-bT_{i,1}T_{i+1,1}\beta$$

$$-3aT_{i,1}\gamma+4aT_{i,2}\gamma-aT_{i,3}\gamma+aT_{i-1,1}\beta-aT_{i+1,1}\beta) \quad (15)$$

$$a_2 = -\frac{1}{2}\frac{1}{J\sqrt{\gamma}x_{i,1}}(2a_3 y_{i,1} J\sqrt{\gamma} - 3bT_{i,1}^2\gamma + 4bT_{i,1}T_{i,2}\gamma - bT_{i,1}T_{i,3}\gamma + bT_{i,1}T_{i-1,1}\beta - bT_{i,1}T_{i+1,1}\beta$$

$$+2a_1 J\sqrt{\gamma} - 3aT_{i,1}\gamma + 4aT_{i,2}\gamma - aT_{i,3}\gamma + aT_{i-1,1}\beta - aT_{i+1,1}\beta) \tag{16}$$

$$a_3 = -\frac{1}{2}\frac{1}{J\sqrt{\gamma}y_{i,1}}(2a_2 x_{i,1} J\sqrt{\gamma} - 3bT_{i,1}^2\gamma + 4bT_{i,1}T_{i,2}\gamma - bT_{i,1}T_{i,3}\gamma + bT_{i,1}T_{i-1,1}\beta - bT_{i,1}T_{i+1,1}\beta$$

$$+2a_1 J\sqrt{\gamma} - 3aT_{i,1}\gamma + 4aT_{i,2}\gamma - aT_{i,3}\gamma + aT_{i-1,1}\beta - aT_{i+1,1}\beta) \tag{17}$$

The terms $\frac{\partial a_l}{\partial a}$ ($l = 1,2,3$) can now be obtained by differentiating the obtained expressions for a_l ($l = 1,2,3$) with respect to a, as follows

$$\frac{\partial a_1}{\partial a} = \frac{3T_{i,1}\gamma - 4T_{i,2}\gamma + T_{i,3}\gamma - T_{i-1,1}\beta + T_{i+1,1}\beta}{2J\sqrt{\gamma}} \tag{18}$$

$$\frac{\partial a_2}{\partial a} = \frac{3T_{i,1}\gamma - 4T_{i,2}\gamma + T_{i,3}\gamma - T_{i-1,1}\beta + T_{i+1,1}\beta}{2x_{i,1}J\sqrt{\gamma}} \tag{19}$$

$$\frac{\partial a_3}{\partial a} = \frac{3T_{i,1}\gamma - 4T_{i,2}\gamma + T_{i,3}\gamma - T_{i-1,1}\beta + T_{i+1,1}\beta}{2y_{i,1}J\sqrt{\gamma}} \tag{20}$$

Thus the sensitivity coefficients, Eq. (10), can be computed in only one single direct problem solution without the need for solving the adjoint equation. The sensitivity matrix $\mathbf{Ja}$ can be explicitly written as

$$\mathbf{Ja}_{a_1} = \begin{bmatrix} \frac{\partial T_{2,N}}{\partial a_1} \\ \frac{\partial T_{3,N}}{\partial a_1} \\ \vdots \\ \frac{\partial T_{M-1,N}}{\partial a_1} \end{bmatrix}_{(M-2)\times 1}, \mathbf{Ja}_{a_2} = \begin{bmatrix} \frac{\partial T_{2,N}}{\partial a_2} \\ \frac{\partial T_{3,N}}{\partial a_2} \\ \vdots \\ \frac{\partial T_{M-1,N}}{\partial a_2} \end{bmatrix}_{(M-2)\times 1}, \mathbf{Ja}_{a_3} = \begin{bmatrix} \frac{\partial T_{2,N}}{\partial a_3} \\ \frac{\partial T_{3,N}}{\partial a_3} \\ \vdots \\ \frac{\partial T_{M-1,N}}{\partial a_3} \end{bmatrix}_{(M-2)\times 1} \tag{21}$$

3.3 The Conjugate Gradient Method (CGM)

In this study, the conjugate gradient optimization method, a powerful gradient-based optimization method, is used to solve the inverse heat transfer problem. The objective function given by Eq. (8) is minimized by searching along the direction of descent $d^{(\mathrm{k})}$ using a search step size $\beta^{(\mathrm{k})}$.

$$f^{(\mathrm{k}+1)} = f^{(\mathrm{k})} - \beta^{(\mathrm{k})} d^{(\mathrm{k})} \tag{22}$$

where $f \equiv a_1, a_2, a_3$. The direction of descent of the current iteration is obtained as a linear combination of the direction of descent of the previous iteration and the gradient direction $\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})}$. Therefore,

$$d^{(\mathrm{k})} = \nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})} + \gamma^{(\mathrm{k})} d^{(\mathrm{k}-1)} \tag{23}$$

The Polak-Ribiere formula [50] is employed to calculate the conjugation coefficient:

$$\gamma^{(\mathrm{k})} = \frac{\left[\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})}\right]^{\mathrm{T}} (\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})} - \nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k}-1)})}{\| \nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k}-1)} \|^2} = \frac{\left[\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})}\right]^{\mathrm{T}} (\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k})} - \nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k}-1)})}{\left[\nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k}-1)}\right]^{\mathrm{T}} \nabla \boldsymbol{\mathcal{J}}^{(\mathrm{k}-1)}} \tag{24}$$

The search step size is given as follows [3]

$$\beta^{(\mathrm{k})} = \frac{[\mathbf{Ja}^{(\mathrm{k})} d^{(\mathrm{k})}]^{\mathrm{T}} [T_{i,N} - T_{(i,N)_m}]}{[\mathbf{Ja}^{(\mathrm{k})} d^{(\mathrm{k})}]^{\mathrm{T}} [\mathbf{Ja}^{(\mathrm{k})} d^{(\mathrm{k})}]} \tag{25}$$

3.3.1 Optimization algorithm

The following algorithm represents the direct and inverse analysis steps used to estimate the space-dependent heat flux in steady-state heat conduction problems in the presence of the temperature-dependent thermal conductivity:

1. Specify the physical domain, the boundary conditions, and the measured outer surface temperature.

2. Generate the boundary-fitted grid using the elliptic grid generation method.

3. Solve the direct problem of finding the temperature values at any grid points of the physical domain using an initial variable heat flux (initial guess for a_1, a_2, a_3).

4. Using Eq. (8), compute the objective function ($\mathcal{J}^{(\mathrm{k})}$).

5. If value of the objective function obtained in step 4 is less than the specified stopping criterion, the optimization is finished. Otherwise, go to step 6.

6. Compute the sensitivity matrices $\mathbf{Ja}_{a_1}$, $\mathbf{Ja}_{a_2}$, and $\mathbf{Ja}_{a_3}$ from Eq. (21)

7. Compute the gradient directions $\nabla \mathcal{J}_{a_l}^{(\mathrm{k})}$ ($l = 1,2,3$) from Eq. (9), respectively.

8. Compute the conjugation coefficients $\gamma_{a_l}^{(\mathrm{k})}$ ($l = 1,2,3$) from Eq. (24). For $\mathrm{k} = 0$, set $\gamma^{(0)} = 0$.

9. Compute the directions of descent $d_{a_l}^{(\mathrm{k})}$ ($l = 1,2,3$) from Eq. (23).

10. Compute the search step sizes $\beta_{a_l}^{(\mathrm{k})}$ ($l = 1,2,3$) from Eq. (25).

11. From Eq. (22), evaluate the new values for a_l ($l = 1,2,3$) separately, namely $a_1^{(\mathrm{k}+1)}$, $a_2^{(\mathrm{k}+1)}$, $a_3^{(\mathrm{k}+1)}$.

12. Set the next iteration ($\mathrm{k} = \mathrm{k} + 1$) and return to the step 2.

3.4 stopping criterion

If the problem involves no measurement errors, the traditional check condition is specified as

$$\mathcal{J}^{(\mathrm{k})} < \varepsilon \tag{26}$$

where ε is a small specified number. In this study, for the case of no measurement error, $\varepsilon = 10^{-12}$. However, the measured temperatures will contain errors. In this case, the objective function value will not be zero at the end of the iterative process. As the computed temperatures approach the measured temperatures containing errors, during the minimization of the objective function (Eq. (8)), large oscillations may appear in the inverse problem solution resulting in an ill-posed character for the inverse problem. However, the conjugate gradient method may become well-posed if the *Discrepancy Principle* is used to stop the iterative procedure. In the Discrepancy Principle, the solution is assumed to be sufficiently accurate when the difference between computed and measured temperatures is of the order of magnitude of the measurement errors, that is,

$$\left| T_{\text{computed}} - T_{\text{measured}} \right| \approx \sigma \tag{27}$$

where σ is the standard deviation of the measurement errors, which is assumed constant in the present analysis. We can obtain the following value for ε by substituting Eq. (27) into Eq. (8) (objective function definition)

$$\varepsilon = (M-2)\sigma^2 \tag{28}$$

Then the iterative procedure is stopped when the following criterion is satisfied [3]

$$\mathcal{J}^{(\mathrm{k})} < \varepsilon \tag{29}$$

4. Results

The following test case is given to demonstrate the accuracy and efficiency of the proposed inverse analysis in the numerical treatment of inverse heat conduction problems involving variable (space-dependent) heat flux and variable (temperature-dependent) thermal conductivity. It is first assumed that the variable heat flux $\dot{q}$ is known, the heat conduction problem is solved using the given boundary conditions to obtain the temperature distribution on the surface Γ_2. Then the computed temperature distribution $T_{i,N}$ $(i = 2,\ldots,M-1)$ is used as the simulated measured temperatures for inverse analysis to recover the initially used variable heat flux $\dot{q}$. To do so, the steady-state heat conduction problem is initially solved using the known values for the linearly temperature-dependent thermal conductivity of the body $k_T = 0.15 + 0.03T_{i,j}$ $(\frac{\mathrm{W}}{\mathrm{m.C}})$, the constant heat transfer coefficients $h_i = 5$ $(\frac{\mathrm{W}}{\mathrm{m^2.C}})$ imposed on the surfaces Γ_i ($i = 2,3,4$), and the space-dependent heat flux $\dot{q} = 1000.0 + 1.5x_{i,1} + 0.5y_{i,1}$ $(\frac{\mathrm{W}}{\mathrm{m^2}})$ applied on the surface Γ_1 to obtain the temperature distribution on the outer surface Γ_2 ($T_{i,N}$, $i = 2,\ldots,M-1$). To facilitate the computation of the sensitivity matrix coefficients using the central finite-difference relations, the grid nodes $(1,N)$ and (M,N) on corners of the outer surface Γ_2 are excluded from computing the temperature distribution. Then the resulting outer surface temperature distribution is used as the simulated measured temperatures in the inverse analysis to recover the initially used values for three parameters $a_1 = 1000.0$, $a_2 = 1.5$, and $a_3 = 0.5$. To do so, the square of the difference between the temperature distribution of the outer surface Γ_2 (obtained from the solution the direct problem at each iteration) and the simulated measured temperature distribution of the same surface (Γ_2) is to be minimized. It is worth noting that the measured temperatures and applied heat flux are on two different surfaces Γ_2 and Γ_1, respectively, and the relation between them is made through thermal conductivity component a (or b) in sensitivity analysis scheme.

Test Case: Numerical values of the coefficients involved in this test case are listed in Table 1. The temperature distribution in the body (using a grid size of 40×30) and the simulated measured temperature distribution on the outer surface Γ_2, $T_{(i,N)_m}$, are demonstrated in Fig. 2a and Fig. 2b, respectively. The distribution of the space-dependent heat flux imposed on the surface Γ_1 is shown in Fig. 3. $T_{(i,N)_m}$ will be used in the inverse analysis to recover the initial values of a_1, a_2, a_3. Using an inverse analysis, the known (desired) values of $a_{1_d} = 1000.0$, $a_{2_d} = 1.5$, and $a_{3_d} = 0.5$ are to be recovered by utilizing three different initial guesses:

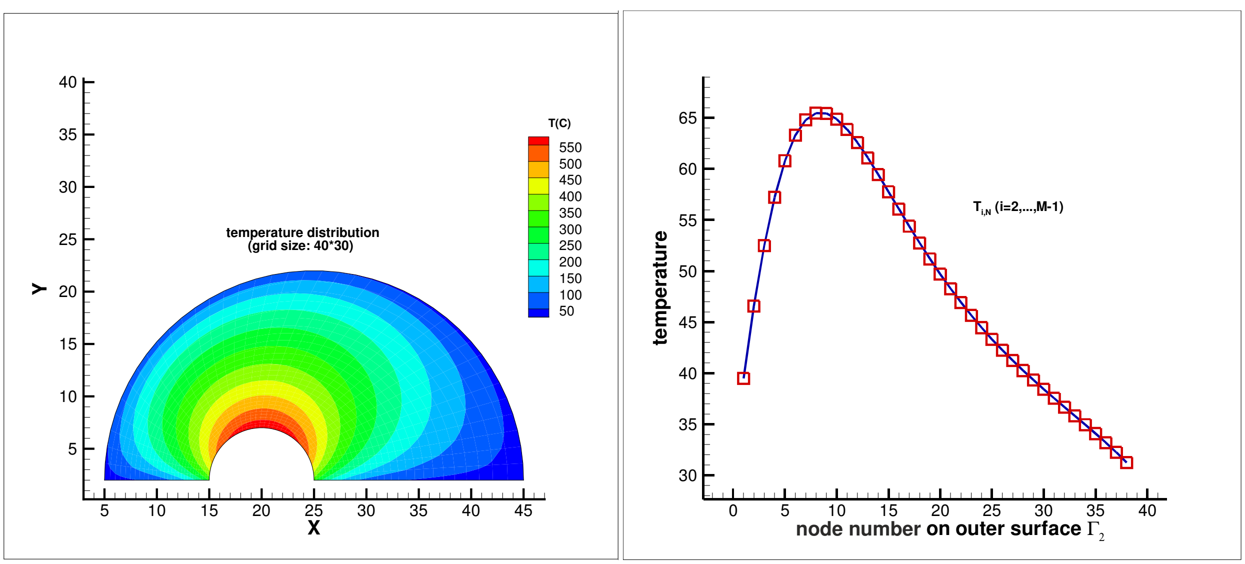
$$a_{1_{\text{initial}_1}} = 200.0, \quad a_{2_{\text{initial}_1}} = \text{-}1.004, \quad a_{3_{\text{initial}_1}} = \text{-}5.08$$

$$a_{1_{\text{initial}_2}} = 750.0, \quad a_{2_{\text{initial}_2}} = 10.7, \quad a_{3_{\text{initial}_2}} = 5.2$$

$$a_{1_{\text{initial}_3}} = 500.0, \quad a_{2_{\text{initial}_3}} = 0.04, \quad a_{3_{\text{initial}_3}} = 0.001$$

$\dot{q}(\frac{\mathrm{W}}{\mathrm{m}^2})$	$k_T(\frac{\mathrm{W}}{\mathrm{m.C}})$	$h_i(\frac{\mathrm{W}}{\mathrm{m}^2.\mathrm{C}}), i = 2,3,4$	$T_{\infty_i}(^{\circ}\mathrm{C}), i = 2,3,4$
$1000.0 + 1.5x_{i,1} + 0.5y_{i,1}$	$0.15 + 0.03T_{i,j}$	5	30

Table 1 Data used for Test Case 1.



a) b)

Fig. 2 Temperature distribution in irregular physical domain (a) and on outer surface Γ_2 (used as T_m for inverse analysis) (b).

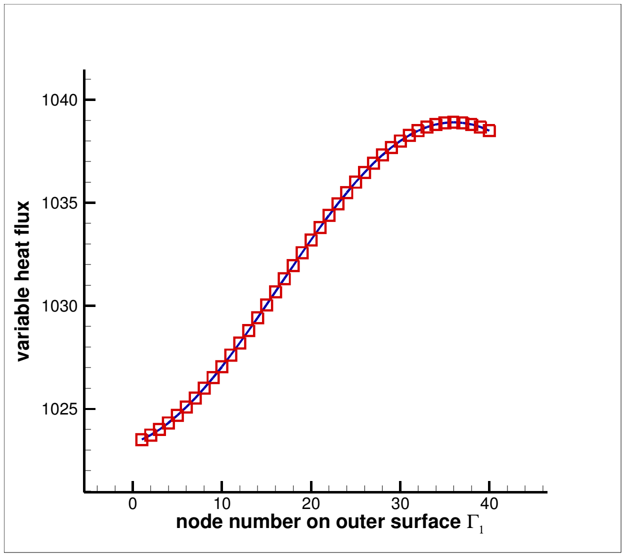
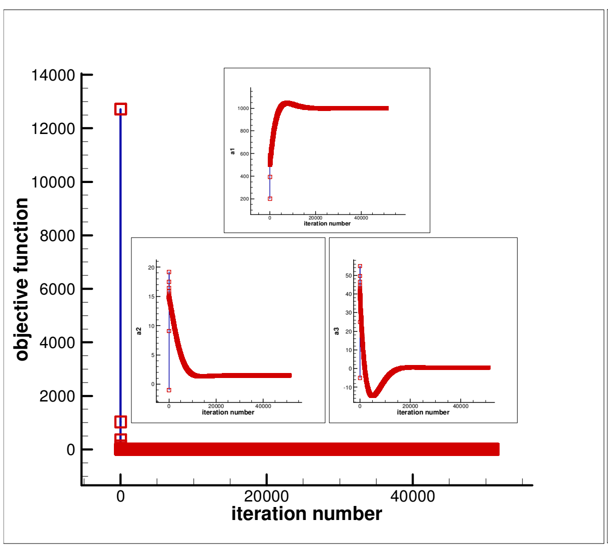
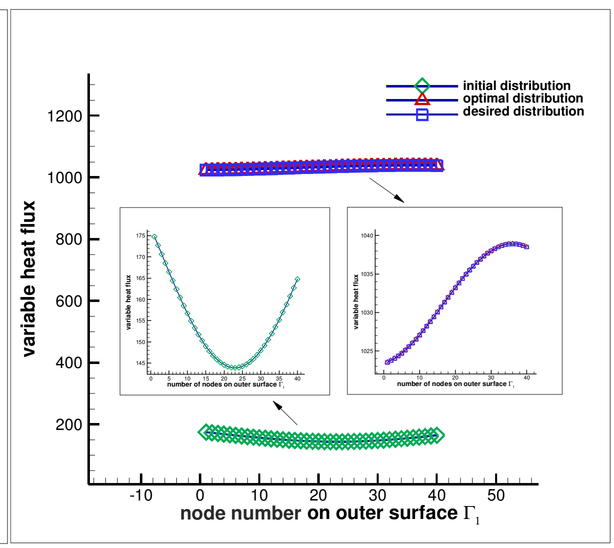


Fig. 3 Distribution of variable heat flux on the outer surface Γ_1.

Initial guess 1: $\dot{q}_{\text{initial}_1} = 200.0 - 1.004x_{i,1} - 5.08y_{i,1}(\frac{\mathrm{W}}{\mathrm{m}^2})$

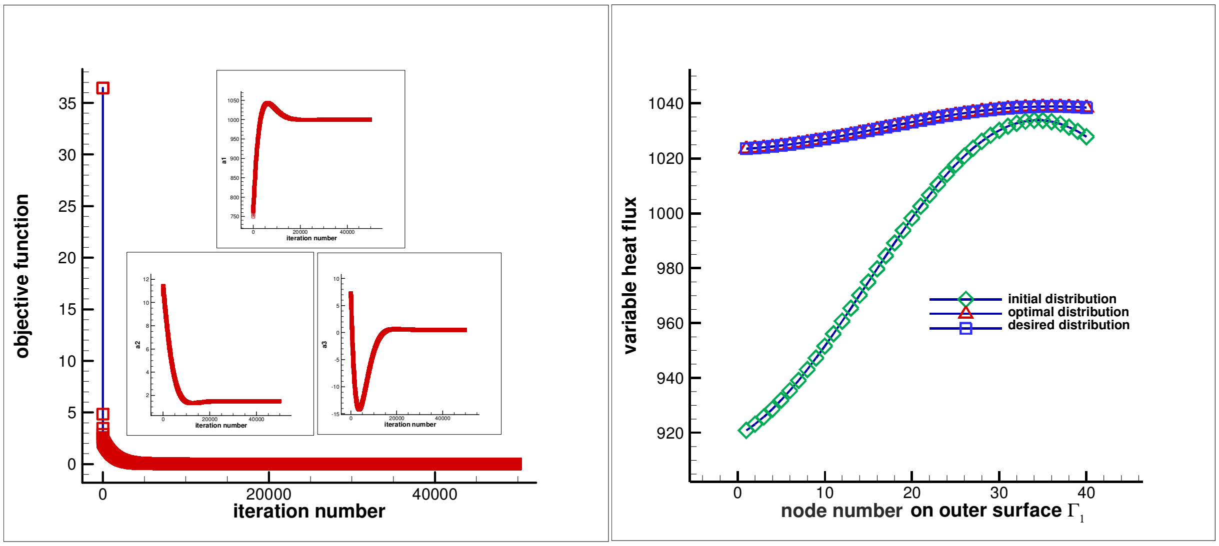




a) b)

Fig. 4 Estimation of a_1, a_2, a_3 ($\dot{q} = a_1 + a_2 x_{i,1} + a_3 y_{i,1}$) and objective function versus iteration number for initial heat flux $\dot{q}_{\text{initial}_1} = 200.0 - 1.004x_{i,1} - 5.08y_{i,1}(\mathrm{W/\,m}^2)$ (a), and comparison of initial, optimal, and desired (simulated measured) heat flux distributions (b).

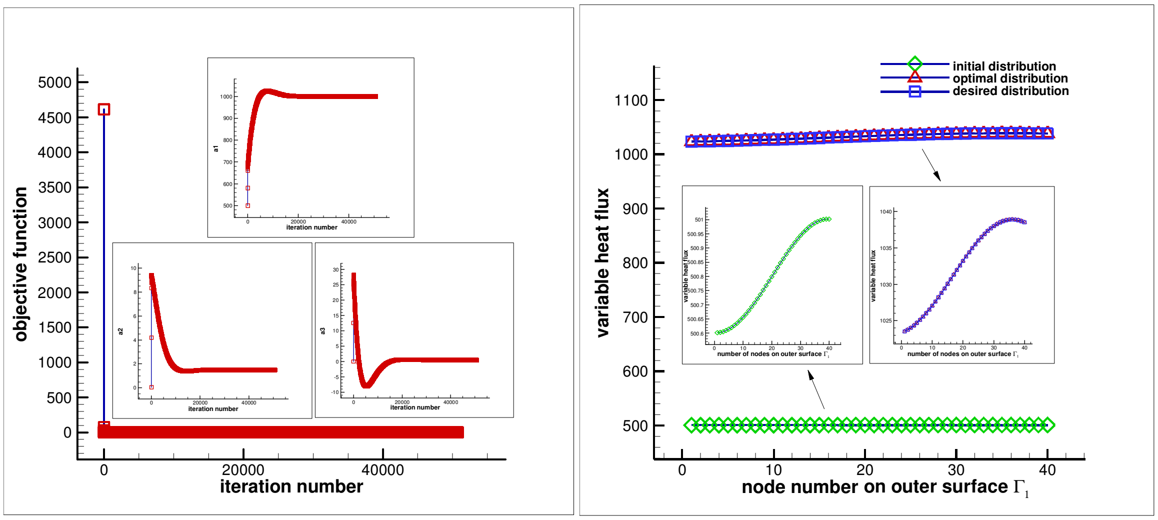
Initial guess 2: $\dot{q}_{\mathrm{initial}_2} = 750.0 + 10.7x_{i,1} + 5.2y_{i,1}(\dfrac{\mathrm{W}}{\mathrm{m}^2})$



a) b)

Fig. 5 Estimation of a_1, a_2, a_3 ($\dot{q} = a_1 + a_2 x_{i,1} + a_3 y_{i,1}$) and objective function versus iteration number for initial heat flux $\dot{q}_{\mathrm{initial}_2} = 750.0 + 10.7x_{i,1} + 5.2y_{i,1}(\mathrm{W/\,m}^2)$ (a), and comparison of initial, optimal, and desired (simulated measured) heat flux distributions (b).

Initial guess 3: $\dot{q}_{\text{initial}_3} = 500.0 + 0.04x_{i,1} + 0.001y_{i,1}(\dfrac{\text{W}}{\text{m}^2})$



a) b)

Fig. 6 Estimation of a_1, a_2, a_3 ($\dot{q} = a_1 + a_2x_{i,1} + a_3y_{i,1}$) and objective function versus iteration number for initial heat flux $\dot{q}_{\text{initial}_3} = 500.0 + 0.04x_{i,1} + 0.001y_{i,1}(\text{W/ m}^2)$ (a), and comparison of initial, optimal, and desired (simulated measured) heat flux distributions (b).

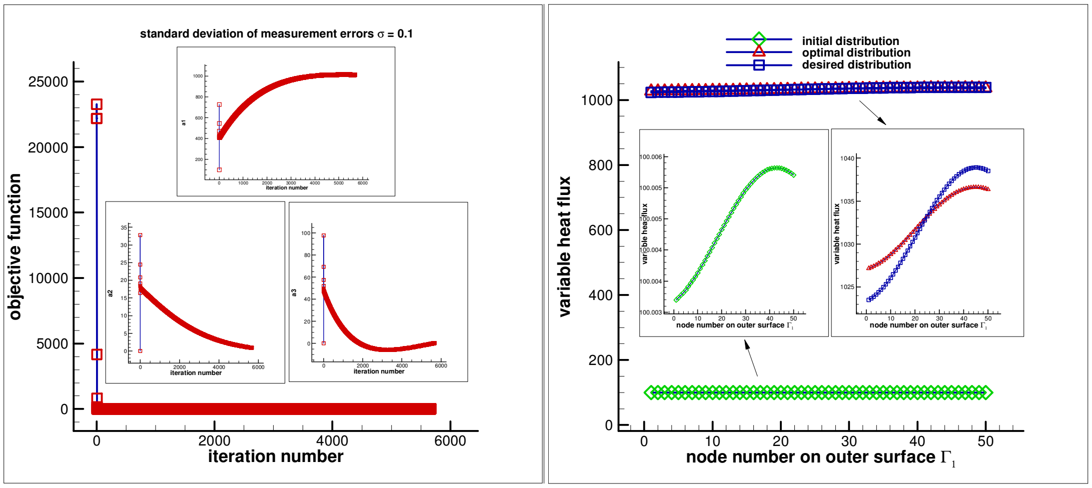
Initial guess 4 (with measurement error): $\dot{q}_{\text{initial}_4} = 100.0 + 0.0002x_{i,1} + 0.0001y_{i,1}(\dfrac{\text{W}}{\text{m}^2})$

In this study, the measured temperature containing random errors, $T_{i,N}^{\text{meas}}$ $(i = 2,\ldots, M-1)$, is generated by adding an error term $\omega\sigma$ to the exact temperature $T_{i,N}^{\text{exact}}$ to give

$$T_{i,N}^{\text{meas}} = T_{i,N}^{\text{exact}} + \omega\sigma \tag{30}$$

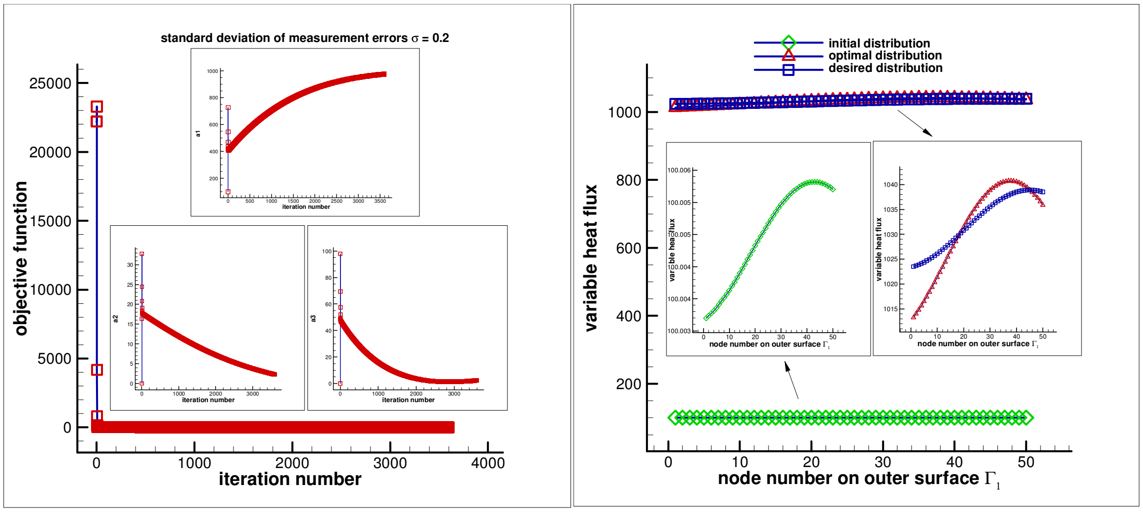
where ω is a random variable with normal distribution, zero mean, and unitary standard deviation. Assuming 99% confidence for the measured temperature, ω lies in the range $-2.576 \leq \omega \leq 2.576$ and it is randomly generated by using MATLAB. σ is the standard deviation of the measurement errors. In this study, $\sigma = 0.1$ and $\sigma = 0.2$, and a finer mesh (50×50) is used. The initial guess

$\dot{q}_{\mathrm{initial}_4} = 100.0 + 0.0002x_{i,1} + 0.0001y_{i,1}(\dfrac{\mathrm{W}}{\mathrm{m}^2})$ is considered to initiate the optimization process.



a) b)

Fig. 7 Estimation of a_1, a_2, a_3 ($\dot{q} = a_1 + a_2x_{i,1} + a_3y_{i,1}$) and objective function versus iteration number for initial heat flux $\dot{q}_{\mathrm{initial}_4} = 100.0 + 0.0002x_{i,1} + 0.0001y_{i,1}(\mathrm{W/m}^2)$ by considering measurement error ($\sigma = 0.1$) (a), and comparison of initial, optimal, and desired (simulated measured) heat flux distributions (b).



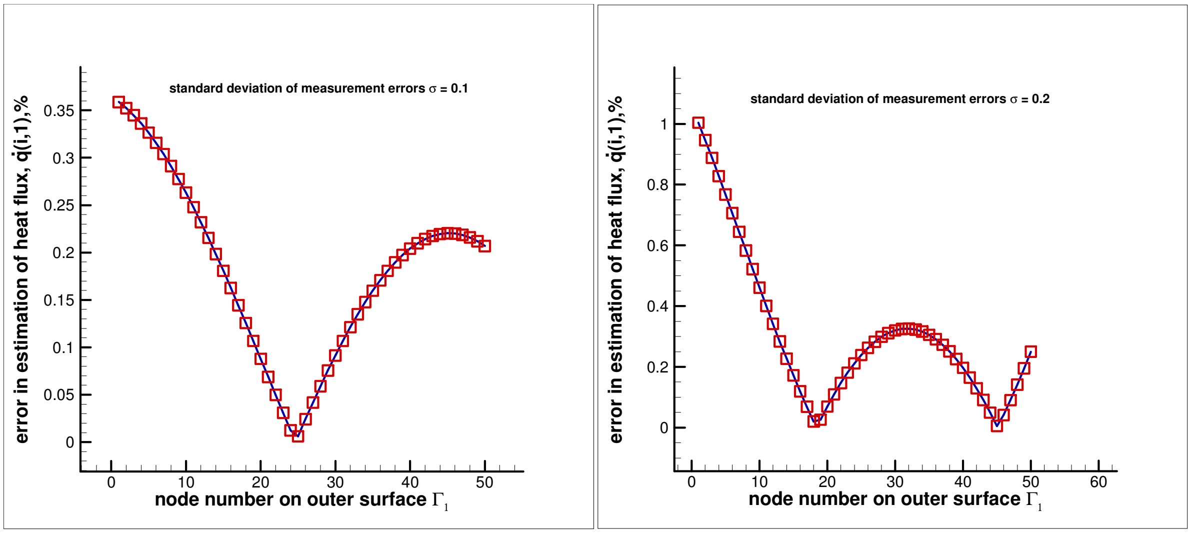
a) b)

Fig. 8 Estimation of a_1, a_2, a_3 ($\dot{q} = a_1 + a_2 x_{i,1} + a_3 y_{i,1}$) and objective function versus iteration number for initial heat flux $\dot{q}_{\text{initial}_4} = 100.0 + 0.0002 x_{i,1} + 0.0001 y_{i,1} (\mathrm{W/\,m^2})$ by considering measurement error ($\sigma = 0.2$) (a), and comparison of initial, optimal, and desired (simulated measured) heat flux distributions (b).

Grid size	**Desired value**	**Initial (guess) value**	**Final value**	**Temperature measurement error**	**Initial value of $\mathcal{J}$**	**Minimum value of $\mathcal{J}$**	**Reduction in objective function & computation time**
40×30	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$a_1 = 200.0$ $a_2 = -1.004$ $a_3 = -5.08$	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$\sigma = 0$	12712.885	9.738×10^{-13}	100% (17m:29s) (50992 iterations)
40×30	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$a_1 = 750.0$ $a_2 = 10.7$ $a_3 = 5.2$	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$\sigma = 0$	36.450	9.602×10^{-13}	100% (16m:27s) (49714 iterations)
40×30	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$a_1 = 500.0$ $a_2 = 0.04$ $a_3 = 0.001$	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$\sigma = 0$	4613.397	9.726×10^{-13}	100% (13m:47s) (50743 iterations)
50×50	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$a_1 = 100.0$ $a_2 = 0.0002$ $a_3 = 0.0001$	$a_1 = 1012.8$ $a_2 = 0.9$ $a_3 = 0.3$	$\sigma = 0.1$	22187.793	0.467	~ 100% (5667 iterations)
50×50	$a_1 = 1000.0$ $a_2 = 1.5$ $a_3 = 0.5$	$a_1 = 100.0$ $a_2 = 0.0002$ $a_3 = 0.0001$	$a_1 = 974.6$ $a_2 = 2.3$ $a_3 = 2.3$	$\sigma = 0.2$	22208.528	1.919	~ 100% (3595 iterations)

Table 2 Results for the estimation of the variable heat flux components a_1, a_2, and a_3.

Here the inverse analysis is examined using three different initial guesses which are selected so that they can reflect the accuracy, efficiency, and robustness of the inverse analysis. As shown in Fig. 4 to Fig. 6, a 100% reduction in the objective function and complete recovering of the values for a_1, a_2, and a_3 (the variable heat flux components) are achieved by starting the optimization process from all three different initial guesses. Despite the large number of iterations for recovering the unknown variables, short time optimization for all three initial guesses reveals the efficiency of the proposed method. The details of the results, including the initial and final values for a_1, a_2, and a_3, the initial and final values of the objective function, the computation time, the number of iterations, and the percentage of the decrease in the objective function, are given in Table 2 (for both cases of no measurement error and measurement error). In case of the measurement error ($\sigma = 0.1$ and $\sigma = 0.2$), there is also an approximately 100% reduction in the objective function. As shown in Table 2 and Fig. 7a and Fig. 8a, the error in recovering the parameter a_1 (constant component of the heat flux) is insignificant whereas the errors in recovering the parameters a_2 and a_3 are significant. Nevertheless, the distribution of retrieved heat flux in the presence of the measurement error is in very good agreement with the desired one (Fig. 7b and Fig. 8b). The errors between the recovered and desired heat flux distributions on the boundary surface Γ_1 at node $i,1$ ($i = 1,\ldots,M$) for the measurement errors of standard deviation of $\sigma = 0.1$ and $\sigma = 0.2$ are demonstrated in Fig. 9a and Fig. 9b, respectively. It can be seen that $\text{error}_{i_{\max}} = 1\%$ for $\sigma = 0.2$. To facilitate the computation of the sensitivity matrix coefficients using the central finite difference relations, the grid nodes $(1, N)$ and (M, N) on corners of the outer surface Γ_2 are excluded from computing the temperature distribution. The results are obtained by a FORTRAN compiler and computations are run on a PC with Intel Core i7 and 16G RAM. A tolerance of 10^{-7} is used in iterative loops to increase the accuracy of results.



a) b)

Fig. 9 Error in estimation of heat flux $\dot{q}_{i,1}$ ($i = 1,\ldots,M$) for $\sigma = 0.1$ (a) and $\sigma = 0.2$ (b).

5. Conclusion

This paper presented an accurate, easy to implement, and very efficient sensitivity analysis scheme in determination of space-dependent heat flux imposed on a part of the boundary of a heat conducting body under specified boundary conditions and variable (temperature-dependent) thermal conductivity. The two-dimensional irregular heat-conducting body was transformed into a regular computational domain to perform all computations related to the direct and inverse heat conduction solution. Then an elliptic grid generation scheme was used to generate a grid over the irregular body. An accurate and very efficient sensitivity analysis scheme was used to calculate sensitivity coefficients needed in a gradient-based optimization method. The explicit expressions for the sensitivity coefficients were derived through the use of chain rule and differentiation with respect to the components of the variable heat flux which allow for the computation of the sensitivity coefficients in one single solve only (at each iteration), regardless of the number of unknown quantities. The conjugate gradient method was used as an optimizer to minimize the objective function expressed in the least squares sense and retrieve the desired quantities. The obtained results revealed that the proposed algorithm is very accurate and efficient.

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