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Summer Research Project

Nearness and Continuity

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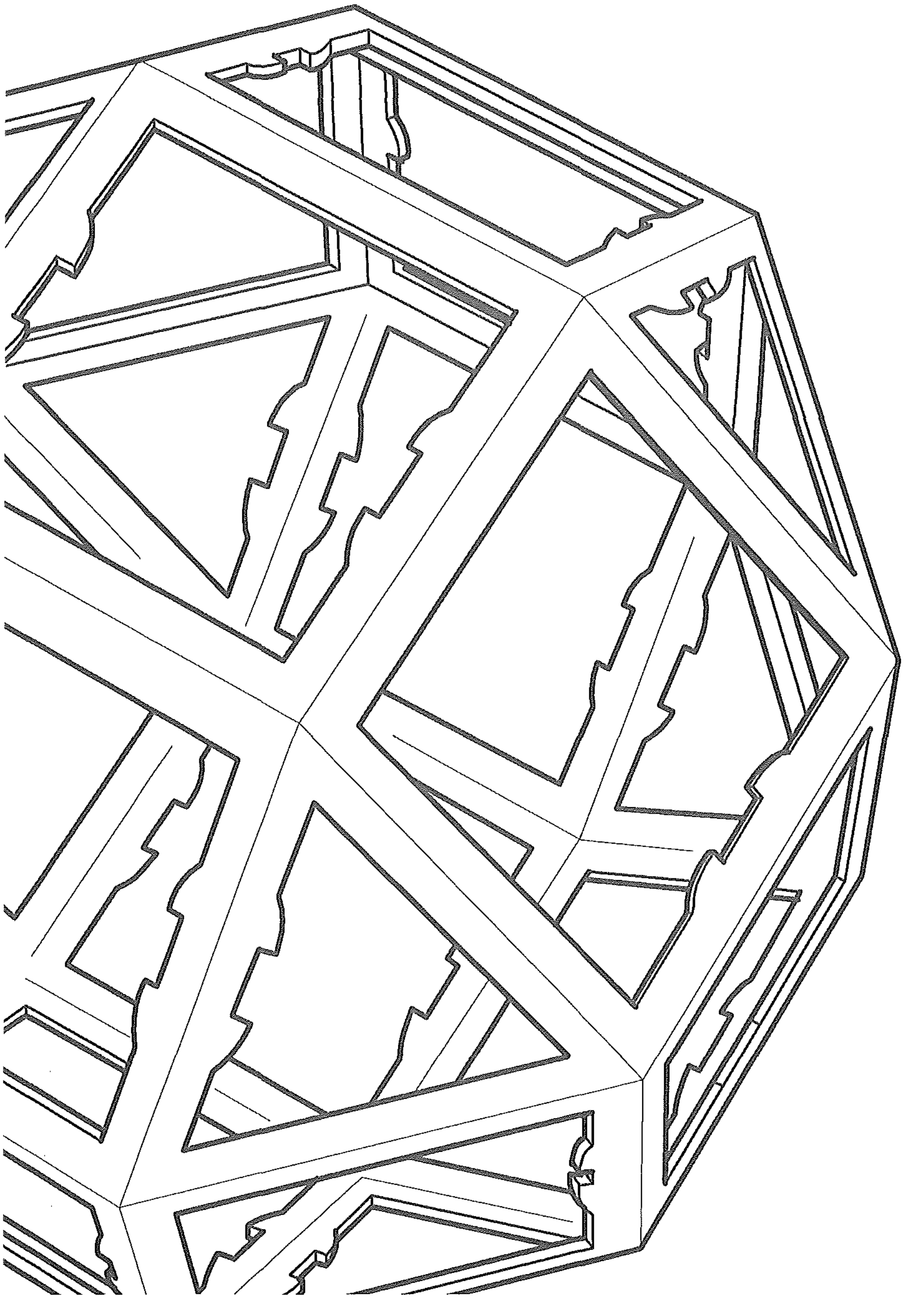

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Summer Project 2010–11: Nearness and Continuity

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Introduction

Nearness relations provide an approach to continuity and limits that makes a clear transition from motivation and intuitive understanding to rigorous analysis. This approach is advocated, in [?], as a means of introducing students to real analysis without excessive abstraction.

In the following work, we present the axioms for a nearness space, prove that the natural definition on $\mathbf{R}^n$ does give a nearness relation, and then use nearness to define and study the notion of a continuous function. We also prove the equivalence of the nearness definition of continuity and the standard ϵ-δ definition. Finally, we deal with two pillars of elementary real analysis: the Intermediate Value Theorem and the Extreme Value Theorem.

Nearness Relations

Let X be a nonempty set, and δ a binary relation between points and subsets of X. We say that δ is a **nearness relation** on X if it satisfies the following five axioms.

A1 $x \in A \Rightarrow x \, \delta \, A$.

A2 $x\,\delta\,A \Rightarrow A \neq \varnothing$.

A3 $x\,\delta\,(A \cup B) \Leftrightarrow (x\,\delta\,A \vee x\,\delta\,B)$.

A4 The **Lodato property**: $(x\,\delta\,A \wedge \forall_{y\in A}\,(y\,\delta\,B)) \Rightarrow x\,\delta\,B$.

A5 $x\,\delta\,\{y\} \Leftrightarrow x = y$.

If $x\,\delta\,A$, we say that the point x is **near** the set A. In the contrary case, we write $x\ \not\delta\,A$ and say that x is **apart from**, or **far from**, A. We call the pair $(X,\,\delta\,)$—or, when it is clear what nearness we are dealing with, just the set X—a **nearness space.**

The canonical example of a nearness is the one defined on $\mathbf{R}$ as follows: $x\,\delta\,A$ means that for each $r > 0$, there exists $y \in A$ such that $|x-y| < r$; that is,

$$\forall_{r>0}\exists_{y\in A}\,(|x-y| < r)\,. \tag{1}$$

This condition is equivalent to $\rho(x, A) = 0$, where

$$\rho(x, A) \equiv \inf\{|x-y| : y \in A\}$$

is the distance from the point x to the set A.

Proposition 1 *The relation* δ *defined by* (1) *is a nearness on* $\mathbf{R}$.

Proof. Since $|x-x| = 0 < r$ for each $r > 0$, it is clear that if $x \in A$, then (1) holds; this verifies **A1**. To deal with **A2**, observe that if $x\,\delta\,A$, then, by (1), there exists $y \in A$ with $|x-y| < 1$; whence $A \neq \varnothing$. Next, if $x\,\delta\,A$ and $r > 0$, then there exists $y \in A$ such that $|x-y| < r$; but $A \subset A \cup B$, so $y \in A \cup B$. Thus if $x \in A$—and similarly if $x \in B$—then for each $r > 0$ there exists $y \in A \cup B$ with $|x-y| < r$; whence $x\,\delta\,(A \cup B)$. This proves the implication "$\Leftarrow$" in **A3**. To prove the reverse implication, suppose that $x\ \not\delta\,A$ and $x\ \not\delta\,B$. Then there exists $r_1 > 0$ such that $|x-y| \geqslant r_1$ for all $y \in A$, and there exists $r_2 > 0$ such that $|x-y| \geqslant r_2$ for all $y \in B$. It follows that for all $y \in A \cup B$,

$$|x-y| \geqslant \min\{r_1, r_2\} > 0.$$

Hence $x \not\delta (A \cup B)$. It follows that if $x \, \delta \, (A \cup B)$, then we cannot have both $x \not\delta A$ and $x \not\delta B$, so either $x \, \delta \, A$ or $x \, \delta \, B$. This completes the verification of **A3**.

For **A4**, assume that $x \, \delta \, A$ and that $y \, \delta \, B$ for each $y \in A$. Given $r > 0$, pick $y \in A$ such that $|x - y| < r/2$. Since $y \, \delta \, B$, there exists $z \in B$ with $|y - z| < r/2$. Then

$$\begin{aligned} |x - z| &= |x - y + y - z| \\ &\leqslant |x - y| + |y - z| \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Since $r > 0$ is arbitrary, we now see that $x \, \delta \, B$. This verifies **A4**.

Finally, we deal with **A5**. If $x = y$, then $x \in \{y\}$ and so, by **A1** (which we have already verified), $x \, \delta \, \{y\}$. Conversely, if $x \, \delta \, \{y\}$, then for each $r > 0$, there exists an element of $\{y\}$ whose distance from x is less than r. But the only element of $\{y\}$ is y, so $|x - y| < r$ for all $r > 0$, and therefore $x = y$. ■

More generally, the canonical nearness on n-dimensional Euclidean space $\mathbf{R}^n$ is defined by

$$\mathbf{x} \, \delta \, A \Leftrightarrow \forall_{r>0} \exists_{\mathbf{y} \in A} \left(\|\mathbf{x} - \mathbf{y}\| < r \right) ,$$

where

$$\|\mathbf{x} - \mathbf{y}\| \equiv \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.$$

We omit the proof (very similar to that of Proposition 1) that this relation δ satisfies the axioms for a nearness.

Every nonempty subset S of a nearness space (X, δ) is also a nearness space with nearness defined as the restriction of δ to S (that is, a point of S is near a subset of S if and only if the point and set are near in the original sense on X).

1 Continuity through nearness

Continuity can be defined using nearness, rather than in the standard approach of analysis courses which uses the $\delta - \varepsilon$ definition. The $\delta - \varepsilon$ approach from here on is referred to as the standard definition.

Let X and Y be nonempty subsets of $\mathbf{R}^n$. We say that a function $f : X \to Y$ is **continuous at the point** $x \in X$ if for each $A \subset X$,

$$x \,\delta\, A \Rightarrow f(x) \,\delta\, f(A)$$

or, equivalently,

$$f(x) \,\not\delta\, f(A) \Rightarrow x \,\not\delta\, A.$$

If f is continuous at each point of X, we say that it is **continuous on** X.

Every constant function $f : x \rightsquigarrow c$, where $c \in \mathbf{R}^n$, is continuous on $\mathbf{R}^n$: for if $\mathbf{x} \,\delta\, A$ in $\mathbf{R}^n$, then $f(\mathbf{x}) = c$ and $f(A) = \{c\}$, so $f(\mathbf{x}) \in f(A)$ and therefore, by **A1**, $f(\mathbf{x}) \,\delta\, f(A)$.

We present the first of our two proofs that sums of continuous functions are continuous.

Proposition 2 *Let X be a subset of $\mathbf{R}^n$, let $\mathbf{x} \in X$, and let $f : X \to \mathbf{R}^n, g : X \to \mathbf{R}^n$ be continuous at x. Then the function $f + g$ is continuous at $\mathbf{x}$.*

Proof. Let $A \subset X$, and assume that $(f+g)(\mathbf{x}) \,\not\delta\, (f+g)(A)$. Then there exists $r > 0$ such that

$$|(f+g)(\mathbf{x}) - (f+g)(\mathbf{y})| \geqslant r \qquad (\mathbf{y} \in A)\,.$$

The set A can be written as the union of the two disjoint subsets

$$A_1 \equiv \{\mathbf{y} \in A : |f(\mathbf{x}) - f(\mathbf{y})| < r/2\}\,,$$
$$A_2 \equiv \{\mathbf{y} \in A : |f(\mathbf{x}) - f(\mathbf{y})| \geqslant r/2\}\,.$$

For each $\mathbf{y} \in A_1$ we have

$$\begin{aligned} r &\leqslant |(f+g)(\mathbf{x}) - (f+g)(\mathbf{y})| \\ &\leqslant |f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{x}) - g(\mathbf{y})| \\ &< \frac{r}{2} + |g(\mathbf{x}) - g(\mathbf{y})|\,, \end{aligned}$$

so $|g(\mathbf{x}) - g(\mathbf{y})| > r/2$. Hence $g(\mathbf{x}) \not\delta\, g(A_1)$ and therefore (by the continuity of g at $\mathbf{x}$) $\mathbf{x} \not\delta\, A_1$. On the other hand, it follows from the definition of the set A_2 that $f(\mathbf{x}) \not\delta\, f(A_2)$; the continuity of f at $\mathbf{x}$ now gives $\mathbf{x} \not\delta\, A_1$. Hence $\mathbf{x} \not\delta\, A_1$ and $\mathbf{x} \not\delta\, A_2$. It follows from **A3** that $\mathbf{x} \not\delta\, (A_1 \cup A_2)$; that is, $\mathbf{x} \not\delta\, A$. We now conclude that $f + g$ is continuous at $\mathbf{x}$. ■

An induction argument (omitted) now enables us to prove that if $f_1, \ldots, f_m$ are functions from X to $\mathbf{R}^n$ that are continuous at $x \in X$, then $f_1 + \cdots + f_m$ is continuous at x.

Next, we prove the continuity of power functions. More generally, we prove

Proposition 3 *Let c be a constant, and k a positive integer. Then the function $f : x \rightsquigarrow cx^k$ is continuous on $\mathbf{R}$.*

Proof. We illustrate with the cases $k = 1$ and $k = 2$. If $c = 0$, then f is the constant function $f(x) = 0$ and is therefore continuous;so we may assume that $c \neq 0$. Let $x\,\delta\,A$. First taking $k = 1$, and given $r > 0$, pick $y \in A$ with $|x - y| < r/|c|$. Then

$$|cx - cy| = |c\,(x - y)| = |c|\,|x - y| < |c|\,\frac{r}{|c|} = r.$$

Since $r > 0$ is arbitrary, it follows that f is continuous at x.

Now take $k = 2$. Given $r > 0$, we need to produce $y \in A$ such that $\left|cx^2 - cy^2\right| < r$. For the moment, let $y \in A$ satisfy $|x - y| < t < 1$. By the triangle inequality,

$$|y| = |(y - x) + x| \leqslant |x - y| + |x| < t + |x|\,. \tag{2}$$

Hence

$$\begin{aligned}\left|cx^2 - cy^2\right| &= |c|\,\left|x^2 - y^2\right| \\ &= |c|\,|x - y|\,|x + y| \\ &\leqslant |c|\,t\,(|x| + |y|) \\ &\leqslant t\,|c|\,(t + 2\,|x|)\,,\end{aligned}$$

the last step following from (2). So if we choose t with

$$0 < t < \min\left\{1, \frac{r}{|c|\,(1+2\,|x|)}\right\},$$

and then $y \in A$ with $|x-y| < t$ (which is possible since $x\,\delta\,A$), we have

$$\left|cx^2 - cy^2\right| < \frac{r}{|c|\,(1+2\,|x|)}\,|c|\,(1+2\,|x|) = r.$$

Since $r > 0$ is arbitrary, we have shown that f is continuous at x.

The continuity of the function $x \rightsquigarrow cx^n$ for each constant c and $n \geqslant 3$ can be proved with estimates similar to, but more complicated than, those used in the case $n = 2$. ■

It follows from Proposition 3 and the remark immediately after the proof of Proposition 2 that a general polynomial function

$$x \rightsquigarrow c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n$$

is continuous on $\mathbf{R}$.

The next lemma leads to an alternative proof of Proposition 2.

Lemma 4 *Suppose that* $f : X \to \mathbf{R}^n$ *and* $g : X \to \mathbf{R}^n$ *are both continuous at* $x \in X \subset \mathbf{R}^n$. *If* $x\,\delta\,A$ *in* X, *then for each* $r > 0$, *there exists* $a \in A$ *such that* $|f(x) - f(a)| < r$ *and* $|g(x) - g(a)| < r$.

Proof. Let $x\,\delta\,A$ in X, and $r > 0$. Assume that no such $a \in A$ exists. Then for each $y \in A$, either $|f(x) - f(y)| \geqslant r$ or $|g(x) - g(y)| \geqslant r$. Writing

$$A_f \equiv \{y \in A : |f(x) - f(y)| \geqslant r\},$$
$$A_g \equiv \{y \in A : |g(x) - g(y)| \geqslant r\},$$

we have $A = A_f \cup A_g$, $f(x) \not\delta\, f(A_f)$, and $g(x) \not\delta\, g(A_g)$. Since both f and g are continuous at x, it follows that $x \not\delta\, A_f$ and $x \not\delta\, A_g$; whence, by **A3**,$x \not\delta\, A$, a contradiction from which we conclude that the desired element a of A exists after all. ■

Here is our **alternative proof of Proposition 2**. Under the hypotheses of that proposition, let $x \, \delta \, A$ in X, and let $r > 0$. By Lemma 4, there exists $a \in A$ such that $|f(x) - f(a)| < r/2$ and $|g(x) - g(a)| < r/2$. Hence

$$\begin{aligned} |(f+g)(x) - (f+g)(a)| &= |f(x) - f(a) + g(x) - g(a)| \\ &\leqslant |f(x) - f(a)| + |g(x) - g(a)| \\ &< \frac{r}{2} + \frac{r}{2} = r \end{aligned}$$

Since $r > 0$ is arbitrary, we have $(f+g)(x) \, \delta \, (f+g)(A)$. Hence $f+g$ is continuous at x.

Lemma 4 enables us to prove, relatively smoothly, the continuity of differences, products, and (where the denominator is nonzero) quotients of continuous functions. We refer to [3] for details.

The definition of continuity that we adopted earlier is not the standard one. We can, however, prove that it is equivalent to the standard one.

Proposition 5 *Let X be a subset of $\mathbf{R}^m$, x a point of X, and f a mapping of X into $\mathbf{R}^n$. Then the following conditions are equivalent.*

(i) *f is continuous at x.*

(ii) *For each $\varepsilon > 0$, there exists $\alpha > 0$ such that if $y \in X$ and $\|\mathbf{x} - \mathbf{y}\| < \alpha$, then $\|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon$.*

Proof. Assuming (i), let $\varepsilon > 0$ and define

$$A \equiv \{\mathbf{y} \in X : \|f(\mathbf{x}) - f(\mathbf{y})\| \geqslant \varepsilon\}.$$

Then $f(\mathbf{x}) \not\delta \, f(A)$, so $\mathbf{x} \not\delta \, A$ and therefore there exists $\alpha > 0$ such that $\|\mathbf{x} - \mathbf{y}\| \geqslant \alpha$ for all $\mathbf{y} \in A$. It follows that if $\mathbf{y} \in X$ and $\|\mathbf{x} - \mathbf{y}\| < \alpha$, then $\mathbf{y} \notin A$ and therefore $\|\mathbf{x} - \mathbf{y}\| < \varepsilon$. Thus (i) $\Rightarrow$ (ii).

Conversely, assuming (ii), let $\mathbf{x} \, \delta \, A$ in X. Given $\varepsilon > 0$, pick $\alpha > 0$ as in (ii). Since $\mathbf{x} \, \delta \, A$, there exists $\mathbf{y} \in A$ with $\|\mathbf{x} - \mathbf{y}\| < \alpha$; whence $\|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $f(\mathbf{x}) \, \delta \, f(A)$. Hence (ii) $\Rightarrow$ (i). ∎

2 Two fundamental continuity theorems

We end the report by proving two major theorems of elementary analysis, beginning with

Theorem 6 The Intermediate Value Theorem: *Let $f : [0,1] \to \mathbf{R}$ be continuous, and let $f(0)f(1) < 0$. Then there exists ξin the open interval $(0,1)$ such that $f(\xi) = 0$.*

Proof. We may assume without loss of generality that $f(0) < 0 < f(1)$. Let

$$A \equiv \{x \in [0,1] : f(x) \geqslant 0\},$$
$$B \equiv \{x \in [0,1] : f(x) < 0\}.$$

Clearly, $1 \in A$ and 0 is a lower bound for A; so, by the least-upper-bound principle, A has an infimum m. We prove that $0 < m < 1$ and $f(m) = 0$. By definition of *infimum*, for each $\varepsilon > 0$ there exists $x \in A$ with $m \leqslant x < m + \varepsilon$; so $m\,\delta\,A$ and therefore, by the continuity of f at m, $f(m)\,\delta\,f(A)$. If $f(m) < 0$, then since $f(A) \subset [0,\infty)$, $f(m)\ \not\delta\,f(A)$—a contradiction. Hence $f(m) \geqslant 0$. In particular, $f(m) > f(0)$, so $m \neq 0$ and therefore $m > 0$. Suppose that $f(m) > 0$. Then $f(m)\ \not\delta\,f(B)$, so, by the continuity of f, $m\ \not\delta\,B$ and therefore there exists $r > 0$ such that $|m - x| \geqslant r$ for all $x \in B$. Thus if $|m - x| < r$, then $x \notin B$ and so $f(x) \geqslant 0$. Taking

$$x = \max\left\{\frac{m}{2}, m - \frac{r}{2}\right\},$$

we see that $m - r < x < m$, so $|m - x| < r$ and therefore $f(x) \geqslant 0$. Hence $x \in A$, which is absurd since $x < m = \inf A$. We conclude that $f(m) \not> 0$, so $f(m) \leqslant 0$. Since we have also shown that $f(m) \geqslant 0$, it follows that $f(m) = 0$. Moreover, $f(m) < f(1)$ and so $m < 1$. ■

Our final result is

Theorem 7 The Extreme Value Theorem: *Let X be a nonempty closed, bounded interval in $\mathbf{R}$ and let $f : X \to \mathbf{R}$ be continuous. Then*

(i) f *is* **bounded:** *that is, there exist* m, M *such that* $m \leqslant f(\mathbf{x}) \leqslant M$ *for all* $\mathbf{x} \in X$.

(ii) f **attains its bounds:** *that is, there exist* $\xi, \eta \in X$ *such that* $f(\xi) = \sup f$ *and* $f(\eta) = \inf f$.

Proof. Clearly, we can assume that $a < b$. Define

$$A \equiv \{x \in [a,b] : f \text{ is bounded on the interval } [a,x]\}.$$

Then $a \in A$ and b is an upper bound for A, so, by the least-upper-bound principle, $s \equiv \sup A$ exists; clearly, $\sigma \in [a,b]$. We first prove that $\sigma = b$. Suppose that $\sigma < b$. Then for each positive integer n, there exists x_n such that

$$\sigma < x_n < \min\left\{b, \sigma + \frac{1}{n}\right\}$$

and f is unbounded on the interval $[a, x_n]$; we can therefore pick $t_n \in [a, x_n]$ such that $|f(t_n)| > |f(\sigma)| + n$. Let

$$T \equiv \{t_n : n \geqslant 1\}.$$

Since $|f(\sigma) - f(x_n)| > n \geqslant 1$ for each n, we have $f(\sigma) \not\delta f(T)$. It follows from the continuity of f that $\sigma \not\delta T$, which is absurd since $|\sigma - t_n| < 1/n$ for each n. Hence $\sigma \nless b$ and so $\sigma = b$.

To complete the proof of (i), it will suffice to show that $b \in A$. Suppose the contrary. Then f is unbounded on $[a,b]$, so for each positive integer n we can choose $y_n \in [a,b]$ such that $|f(y_n)| > |f(b)| + n$. Writing

$$Y \equiv \{y_n : n \geqslant 1\},$$

we see that $f(b) \not\delta f(Y)$ and hence, by continuity, that $b \not\delta Y$. Thus there exists r such that $0 < r < b - a$ and $|b - y_n| \geqslant r$, and therefore a$\leqslant y_n \leqslant b - r$, for each n. But this is absurd, since f is bounded on the interval $[a, b-r]$ and yet $|f(y_n)| \to \infty$ as $n \to \infty$. We conclude that $b \in A$, as desired.

Now let

$$M \equiv \sup\{f(x) : x \in [a,b]\},$$

which exists, by the least-upper-bound principle. Suppose that there is no $\xi \in [a,b]$ with $f(\xi) = M$. Then $f(x) < M$ for each $x \in [a,b]$, so the function $x \rightsquigarrow M - f(x)$ is continuous and everywhere positive. Hence

$$g(x) \equiv \frac{1}{M - f(x)}$$

defines a continuous, positive-valued function on $[a,b]$. In view of (i), there exists $s > 0$ such that, for each $x \in [a,b]$, $g(x) \leqslant s$ and therefore $M - \frac{1}{s} \geqslant f(x)$. Thus $M - \frac{1}{s}$ is an upper bound of the range of f, which contradicts our definition of M as the least such upper bound. It follows that there must exist $\xi \in [a,b]$ with $f(\xi) = M$. A similar argument shows that there exists $\eta \in [a,b]$ with $f(\eta) = \inf f$. This completes the proof of (ii). ∎

Further Directions

Nearness between points and sets can be used to prove many theorems of analysis that are traditionally proved with the ε-δ definition of continuity. The theory extends naturally to an axiomatic one of proximity/nearness between subsets of an ambient set; see [4]. There is also an axiomatic constructive development, based on a primitive notion of apartness rather than nearness [1].

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