Deformations with smallest weighted L^p average distortion and Nitsche type phenomena

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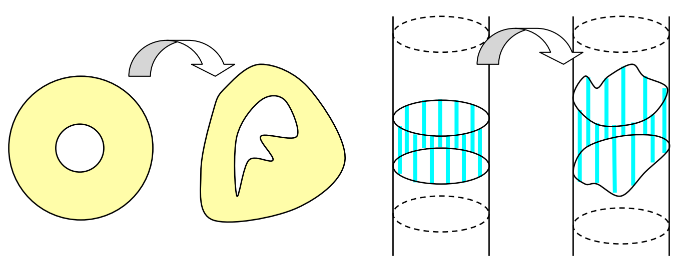
Abstract

The existence and uniqueness properties for extremal mappings with smallest weighted L^p distortion between annuli and the related Grötzsch type problems are discussed. An interesting critical phase type phenomena is observed. When $p < 1$, apart from the identity map, minimizers never exist. When $p = 1$ we observe Nitsche type phenomena; minimisers exist within a range of conformal moduli determined by properties of the weight function and not otherwise. When $p > 1$ minimisers always exist.

Interpreting the weight function as a density or "thickness profile" leads to interesting models for the deformation of highly elastic bodies and tearing type phenomena.

1 Introduction

Consider deforming an annular region in the complex plane with a given conformal metric (viewed as some material property of the region) so as to minimize some weighted L^p-average of the local conformal distortion - a measure of the local anisotropic stretching of the material. This is illustrated below with two different metrics, namely the usual planar metric and the flat metric on $\mathbb{C} \setminus \{0\}$.



Deformations in the plane ($ds = |dz|$) and cylinder ($ds = |dz|/|z|$)

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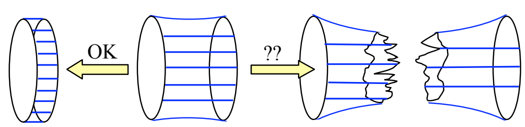
Since the local conformal distortion is unaffected by compostion with a conformal mapping, the classical version of the Riemann mapping theorem for doubly connected domains informs us that we can modify the image to be another round annulus without changing the mean averages of distortion, so long as the single conformal invariant, modulus, is preserved.

In this way our problem is reduced to considering deformations $f : \mathbb{A}_1 \to \mathbb{A}_2$ between round annuli with associated conformal invariants σ_1, σ_2 and given measure, or positive density, on $\mathbb{A}_1$. For these mappings we seek to minimize some functional of distortion - roughly a sort of weighted conformal energy.

The questions we discuss here are generalizations of the problem of minimizing mean distortion initiated in [3] in joint work with Astala, Iwaniec and Onninen. The connection with the 1962 conjecture of Nitsche [12] regarding the existence of harmonic mappings between annuli was identified in [2]. In that paper the surprising phenomenon that minimizers of mean distortion exist only within specific ranges of moduli for the domain and range was observed. (Nitsche's conjecture is equivalent to showing that there are no stationary deformations outside this range, not just minima.) A motivation for this study is to determine whether this phenomenon persists for other conformal energy functionals.



Nitsche phenomenon

Allowable deformations decrease modulus abitrarily (left) or increase modulus within a range(right).

In this work we will identify a critical phase phenomena for the special case of minimising a weighted L^p norm. It turns out that the (inverse of the) case studied by Nitsche is exactly the borderline for existence and nonexistence of minimisers. In related work [11] we showed that, except in the trivial case, minimisers for the related Teichmüller problem for mean distortion never occur.

The results presented here confirm that minimising various averages of distortion functionals exhibit unexpected properties and have the potential to model various nonlinear phenomena in materials science - although we do not discuss that to any great extent here.

Let us now discuss the sorts of deformations we consider and the functional we seek to minimize.

2 Mappings of finite distortion

A homeomorphism $f : \Omega \to \Omega'$ between planar domains of Sobolev class $W^{1,1}_{loc}(\Omega, \Omega')$ has finite distortion if the Jacobian determinant $J(z,f)$ is non-negative and there is a function $\mathbb{K}(z,f)$ finite almost everywhere such that

$$\|Df(z)\|^2 \leqslant \mathbb{K}(z,f)\; J(z,f).$$

The function $\mathbb{K}(z,f)$ is the called the *distortion* of the mapping f. That $\mathbb{K}$ is a measure of the anisotropic local stretching can be seen from the following formula: if we set

$$K = \limsup_{r\to 0} \frac{\max_{|h|=r} |f(z+h) - f(z)|}{\min_{|h|=r} |f(z+h) - f(z)|}$$

the *linear distortion* at $z \in \mathbb{C}$, then

$$\mathbb{K}(z,f) = \frac{1}{2}\Big(K + \frac{1}{K}\Big)$$

The function $\mathbb{K}$ has far better convexity properties than K (see [3]) but as $t \mapsto t + 1/t$ is convex, these functionals will share L^∞ minimizers.

Mappings of finite distortion are generalisations of quasiconformal homeomorphisms and have found considerable recent application in geometric function theory and nonlinear PDEs, [1, 7, 8]. A survey of recent developments -some of which we use here - for the theory of the planar theory of these mappings can be found in P. Koskela's ICM lecture, [10].

2.1 Nitsche type problems

Define annuli

$$\mathbb{A}_1 = \{1 \leqslant |z| \leqslant R\}, \qquad \mathbb{A}_2 = \{1 \leqslant |z| \leqslant S\}$$

with moduli $\sigma_1 = \log(R)$ and $\sigma_2 = \log(S)$. We consider homeomorphisms of finite distortion $f : \mathbb{A}_1 \to \mathbb{A}_2$ mapping the boundary components to each other,

$$f(\{|z| = 1\}) = \{|z| = 1\}, \qquad \text{and} \qquad f(\{|z| = R\}) = \{|z| = S\}.$$

On the annulus $\mathbb{A}_1$ place a positive weight $\eta : \mathbb{A}_1 \to \mathbb{R}_+$ (we view $\eta(z)dz$ as a conformal measure on $\mathbb{A}_1$ or a material property of $\mathbb{A}_1$). In polar coordinates

$$f_z = \frac{1}{2}\, e^{-i\theta} \left(f_\rho - \frac{i}{\rho}\, f_\theta \right), \quad f_{\overline{z}} = \frac{1}{2}\, e^{i\theta} \left(f_\rho + \frac{i}{\rho}\, f_\theta \right) \tag{2.1}$$

and $|f_z|^2 + |f_{\overline{z}}|^2 = \frac{1}{2}(|f_\rho|^2 + \rho^{-2}|f_\theta|^2)$, $J(z,f) = |f_z|^2 - |f_{\overline{z}}|^2 = \frac{1}{\rho}\, \Im(f_\theta \overline{f_\rho})$ which together yield

$$\mathbb{K}(z,f) = \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2 - |f_{\overline{z}}|^2} = \frac{\rho|f_\rho|^2 + \rho^{-1}|f_\theta|^2}{2\Im(f_\theta \overline{f_\rho})}. \tag{2.2}$$

Given a convex function $\varphi : [1,\infty) \to [0,\infty)$ a Nitsche type problem asks us to establish the existence or otherwise of a minimizer (or perhaps stationary point) of the functional

$$f \mapsto \iint_{\mathbb{A}_1} \varphi(\mathbb{K}(z,f))\, \eta(z)\, |dz|^2. \tag{2.3}$$

Thus we seek a deformation of the annulus $\mathbb{A}_1$ to $\mathbb{A}_2$ which minimises some weighted L^φ average of the distortion.

The Nitsche phenomena mentioned above is equivalent to the question posed to minimise (2.3) with $\varphi(t) = t$ and $\eta(x) \equiv 1$; minimisers of mean distortion. In our set up it was proven in [2] that if

$$S + \frac{1}{S} \leqslant 2R, \tag{2.4}$$

then there is a unique minimiser (whose inverse is harmonic). It was further shown that outside this range there are no minimisers (and the way a minimising sequence degenerated was explained). Given the symmetry here one expects the minimiser to be a radial mapping; one of the form

$$z = re^{i\theta} \mapsto \rho(r)e^{i\theta}, \qquad \rho(1) = 1, \quad \rho(R) = S \tag{2.5}$$

and indeed the minimiser is

$$z \mapsto \frac{|z| + \sqrt{|z|^2 + \omega}}{1 + \sqrt{1+\omega}}\, \frac{z}{|z|}, \qquad \omega = \frac{2 - SR}{R^2 - 1}$$

2.2 Grötzsch type problems

The classical Grötzsch problem asks one to identify the linear mapping as the homeomorphism of least maximal distortion between two rectangles. Thus we put

$$\mathbb{Q}_1 = [0,\ell] \times [0,1], \qquad \mathbb{Q}_2 = [0,L] \times [0,1]$$

and suppose we have a deformation of finite distortion $f : \mathbb{Q}_1 \to \mathbb{Q}_2$ with

$$\Re e f(0,y) = 0, \quad \Re e f(\ell,y) = L, \quad \Im m f(x,0) = 0, \quad \Im m f(x,1) = 1 \tag{2.6}$$

(so f is orientation preserving and maps edges to edges). This Sobolev map is absolutely continuous on lines and so $\int_0^\ell \Re e(f_x)\, dx = L$ and $\int_0^1 \Im m(f_y)\, dy = 1$ for almost all y and x respectively, and hence

$$\Re e \iint_{\mathbb{Q}_1} f_x(z)\, |dz|^2 = L, \quad \Im m \iint_{\mathbb{Q}_1} f_y(z)\, |dz|^2 = \ell. \tag{2.7}$$

The distortion function is

$$\mathbb{K}(z,f) = \frac{|f_x|^2 + |f_y|^2}{J(z,f)} \geqslant 1. \tag{2.8}$$

A Grötzsch problem seeks a minimiser, satisfying the boundary conditions (2.6), to the functional

$$f \mapsto \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\ \lambda(z)\ |dz|^2 \tag{2.9}$$

for some positive weight function λ.

2.3 Equivalence between Nitsche and Grötzsch problems

The universal cover of an annulus is effected by the exponential map, so $z \mapsto \exp(2\pi z)$ takes $z = x + iy \in [0,L] \times [0,1]$ to $\mathbb{A}_2$ if $\sigma_2 = \log(S) = 2\pi L$. A branch of logarithm must be chosen to define an "inverse" map $[0,\ell] \times [0,1] \to \mathbb{A}_1$. If $f : \mathbb{A}_1 \to \mathbb{A}_2$ is given, then we can define $\tilde{f}(z) = \frac{1}{2\pi}\log(f(\exp 2\pi z))$. A particular point here is that log is conformal (in fact we only really need log to define a univalent conformal mapping from $\mathbb{A}_1 \setminus ([1,S] \times \{0\})$ to $\mathbb{Q}_2$ with edges matching up) so

$$\mathbb{K}(z,\tilde{f}) = \mathbb{K}\left(z, \frac{1}{2\pi}\log\big(f(e^{2\pi z})\big)\right) = \mathbb{K}\big(z, f(e^{2\pi z})\big), \tag{2.10}$$

and hence a change of variables yields

$$\begin{aligned} \iint_{\mathbb{Q}_1} \varphi\big(\mathbb{K}(z,\tilde{f})\big)\lambda(z)\ |dz|^2 &= \iint_{\mathbb{Q}_1} \varphi\big(\mathbb{K}(z, f(e^{2\pi z}))\big)\lambda(z)\ |dz|^2 \\ &= 4\pi^2 \iint_{\mathbb{A}_1} \varphi\big(\mathbb{K}(w,f)\big)\ \lambda(z)e^{-4\pi\Re e(z)}\ |dw|^2. \end{aligned}$$

With the choice

$$\eta(w) = 4\pi^2\ \lambda(z)e^{-4\pi\Re e(z)}, \qquad e^z = w, \tag{2.11}$$

the equivalence between the two problems (with related weight) is seen. Again, the exact branch of log here will be immaterial to our considerations.

Note: In fact the equivalence between Nitsche and Grötzsch problems is only when one assumes periodic boundary behaviour in the Grötzsch problem. We will be fortunate in that the absolute minimisers for the Grötzsch problem in the situations we consider do exhibit this periodicity and so can be lifted.

3 Sublinear distortion functionals

The purpose of this brief section is to establish the claim made in the abstract that minimisers never exist for the L^p-minimisation problem when $p < 1$. We frame the discussion in considerably more generality. We recall from [3, Theorem 5.3] (actually the proof of this result)

Lemma 3.1. *Let $\Psi(t)$ be a positive strictly increasing function of sublinear growth:*

$$\lim_{t\to\infty} \frac{\Psi(t)}{t} = 0$$

Let $\mathbb{B} = \mathbb{D}(z_0, r)$ be a round disk and suppose that $f_0 : \mathbb{B} \to \mathbb{C}$ is a homeomorphism of finite distortion with $\iint_{\mathbb{B}} \Psi(\mathbb{K}(z, f_0)) < \infty$. Then there is a sequence of mappings of finite distortion $f_n : \mathbb{B} \to f_0(\mathbb{B})$ with $f_n(\zeta) = f_0(\zeta)$ near $\partial\mathbb{B}$ and with

- $\mathbb{K}(z, f_n) \to 1$ *uniformly on compact subsets of* $\mathbb{B}$
- $\iint_{\mathbb{B}} \Psi(\mathbb{K}(z, f_n)) \to \iint_{\mathbb{B}} \Psi(1)$ *as* $n \to \infty$.

We now prove the following theorem.

Theorem 3.2. *Let $\Psi(t)$ be a positive strictly increasing function of sublinear growth, let Ω be a domain and let $\lambda(z) \in L^\infty(\Omega)$ be a positive weight. Suppose that $g_0 : \Omega \to \mathbb{C}$ is a homeomorphism of finite distortion with*

$$\iint_\Omega \Psi(\mathbb{K}(z, g_0)) < \infty$$

Then there is a sequence of mappings of finite distortion $g_n : \Omega \to g_0(\Omega)$ with $g_n(\zeta) = g_0(\zeta)$, $\zeta \in \partial\Omega$ with

$$\iint_\Omega \Psi(\mathbb{K}(z, g_n))\, \lambda(z) \to \Psi(1) \iint_\Omega \lambda(z) \qquad \text{as } n \to \infty \tag{3.3}$$

Proof. Let $\epsilon > 0$. Since $\Big(\Psi(\mathbb{K}(z, g_0)) - \Psi(1)\Big)\lambda(z) \in L^1(\Omega)$ we can choose a finite collection of disjoint disks contained in Ω, say $\{B_i\}_{i=1}^N$, so that

$$\Big|\iint_{\Omega \setminus \bigcup B_i} \Big(\Psi(\mathbb{K}(z, g_0)) - \Psi(1)\Big)\lambda(z)\, dz\Big| < \epsilon/2 \tag{3.4}$$

Next, for each i we use Lemma 3.1 in the obvious way to find $h_i : B_i \to \mathbb{C}$ with $h_i = g_0$ in a neighbourhood of ∂B_i and

$$\Big|\iint_{B_i} \Psi(\mathbb{K}(z, h_i))\lambda(z) - \Psi(1) \iint_{B_i} \lambda(z)\Big| < \frac{\epsilon}{2N}$$

Then the map

$$g_\epsilon(z) = \begin{cases} g_0(z) & z \in \Omega \setminus \bigcup B_i \\ h_i(z) & z \in B_i \end{cases}$$

is of finite distortion and

$$\begin{aligned} \Big|\iint_\Omega \Psi(\mathbb{K}(z, g_\epsilon))\, \lambda(z) - \Psi(1) \iint_\Omega \lambda(z)\Big| &= \Big|\iint_{\Omega \setminus \bigcup B_i} \Big(\Psi(\mathbb{K}(z, g_\epsilon)) - \Psi(1)\Big)\, \lambda(z)\, dz \\ &\quad + \sum_{i=1}^N \iint_{B_i} \Big(\Psi(\mathbb{K}(z, h_i)) - \Psi(1)\Big)\, \lambda(z)\, dz\Big| \\ &< \epsilon \end{aligned}$$

The result obviously follows. $\square$

And the next corollary is the easy consequence we seek.

Corollary 3.5. *Let $\Psi(t)$ be a positive strictly increasing function of sublinear growth, let Ω be a domain and let $\lambda(z) \in L^\infty(\Omega)$ be a positive weight. Suppose that $g_0 : \Omega \to \mathbb{C}$ is a homeomorphism of finite distortion with*

$$\iint_\Omega \Psi(\mathbb{K}(z, g_0)) < \infty$$

Then

$$\min_{\mathcal{F}} \iint_\Omega \Psi(\mathbb{K}(z, g))\, \lambda(z) = \Psi(1) \iint_\Omega \lambda(z)\, dz$$

with equality achieved by a mapping of finite distortion if and only if the boundary values of g_0 are shared by a conformal mapping. Here $\mathcal{F}$ consists of homeomorphisms of finite distortion g with $g|\partial\Omega = g_0$.

4 Minimisers of convex distortion functionals

A natural class of homeomorphic mappings between rectangles satisfying the Grötzsch boundary conditions (2.6) are those of the form

$$f_0(z) = u(x) + iy, \tag{4.1}$$

which will correspond to the lifts of the radial stretchings at (2.5). For these mappings we have $(f_0)_x = u_x$ and $(f_0)_y = i$. We will ultimately want to show these mappings are the extremals for our mapping problems, but will will have to deal with degenerate situations as well - in particular where f_0 is not well defined, but has a well defined inverse. Such mappings are topologically monotone and arise naturally as the limits of homeomorphisms, and so for us as limits of minimising sequences. In order to avoid excess technical complications we make the following assumptions:

Let $w = a + ib \in [0, L] \times [0, 1]$ and set

$$g_0(w) = v(a) + ib \tag{4.2}$$

where $v : [0, L] \to [0, \ell]$ is an absolutely continuous, increasing (but not necessarily strictly increasing) surjection. The derivative of v_a of v is a non-negative $L^1([0, \ell])$ function which if it is positive almost everywhere makes v strictly increasing and we may set

$$f_0 = g_0^{-1} : [0, \ell] \times [0, 1] \to [0, L] \times [0, 1] \tag{4.3}$$

We now proceed as follows.

Lemma 4.4. *Set $g_0(w) = v(a) + ib$, where $v : [0, L_0] \to [0, \ell]$ is an absolutely continuous, increasing surjection. Let $f : [0, \ell] \times [0, 1] \to [0, L] \times [0, 1]$ be a homeomorphism of finite distortion satisfying the boundary conditions (2.6). Then*

$$|v_a(a) f_x(g_0(w)) + i f_y(g_0(w))|^2 \geqslant 0 \tag{4.5}$$

Equality holds for f and almost every w if and only if v is strictly increasing $L = L_0$ and $f = g_0^{-1}$.

Proof. We consider

$$h(w) = (f \circ g_0)(w)$$

The mapping $h \in W^{1,1}_{loc}$ and maps $[0, L_0] \times [0, 1] \to [0, L] \times [0, 1]$ respecting the sides. We compute the $\bar{w}$-derivative of h;

$$\begin{aligned} 2h_{\bar{w}}(w) &= f_z(v(a) + ib)v_a(a) + f_{\overline{z}}(v(a) + ib)v_a(a) \\ &\quad + if_z(v(a) + ib) - if_{\overline{z}}(v(a) + ib) \\ &= f_x(g_0(w))v_a(a) + if_y(g_0(w)) = 0 \end{aligned}$$

as an L^1-function. Thus h is analytic by the Looman-Menchoff theorem. The boundary conditions and analyticity imply that h is a homeomorphism of the boundary which must therefore be a homeomorphism of the rectangles. Then $L = L_0$ and h must be the identity since the two rectangles have moduli L_0 and L. The result follows □

For a suitable function v as above, let us write $z = x + iy$ where

$$z = g_0(w), \quad \text{and} \tag{4.6}$$

$$\omega(x) = v_a(a) \tag{4.7}$$

We note that ω is well defined. First, g_0 is a surjection and if $g_0(w_1) = g_0(w_2)$, then w_1 and w_2 lie in a common interval on which v is constant, whereupon $v_a(a_1) = v_a(a_2) = 0$. However if $\omega(x) > 0$, then

$$|\omega(x)f_x(z) + if_y(z)|^2 \geqslant 0 \tag{4.8}$$

with equality almost everywhere if and only if g_0 is a homeomorphism and $f_0 = g_0^{-1}$. Also, when $\omega > 0$, v is strictly increasing,

$$g_0^{-1}(z) = f_0(z) = u(x) + iy \tag{4.9}$$

exits and

$$\omega(x) = \frac{1}{u_x(x)} \tag{4.10}$$

We now suppose that $\omega > 0$ and expand out (4.8).

$$\begin{aligned} 0 &\leqslant |\omega(x)f_x + if_y|^2 = (\omega(x)f_x + if_y)(\omega(x)\overline{f_x} - i\overline{f_y}) \\ &= \omega^2(x)|f_x|^2 + |f_y|^2 - 2\Im m(\omega(x)f_y\overline{f_x}) \end{aligned}$$

which yields

$$\omega^2(x)|f_x|^2 + |f_y|^2 \geqslant 2\omega(x)\,\Im m(f_y\overline{f_x}). \tag{4.11}$$

Notice that if we write $f = U + iV$, then

$$\Im m(f_y\overline{f_x}) = \Im m(U_x(z) - iV_x(z))(U_y(z) + iV_y(z)) = J(z, f),$$

so (4.11) gives us

$$\omega^2(x)|f_x|^2 + |f_y|^2 \geqslant 2\omega(x)J(z, f) \tag{4.12}$$

with equality almost everywhere if and only if $f = f_0$ (with the implication that f_0 is a homeomorphism).

We can rewrite (4.12) in two different ways. Namely

$$\begin{aligned} |f_x|^2 + |f_y|^2 &\geqslant (1-\omega^{-2}(x))|f_y|^2 + 2\omega^{-1}(x)J(z,f), \\ |f_x|^2 + |f_y|^2 &\geqslant (1-\omega^{2}(x))|f_x|^2 + 2\omega(x)J(z,f), \end{aligned}$$

which gives us two estimates on the distortion function (writing $J = J(z,f)$),

$$\begin{aligned} \mathbb{K}(z,f) &\geqslant (1-\omega^{-2}(x))\frac{|f_y|^2}{J} + 2\omega^{-1}(x), \\ \mathbb{K}(z,f) &\geqslant (1-\omega^{2}(x))\frac{|f_x|^2}{J} + 2\omega(x), \end{aligned}$$

Next, when $\omega > 0$ almost everywhere we can define f_0 by (4.9) with (4.10). Then

$$\begin{aligned} \mathbb{K}(z,f_0) &= (1-\omega^{-2}(x))\frac{|(f_0)_y|^2}{J_0} + 2\omega^{-1}(x), \\ \mathbb{K}(z,f_0) &= (1-\omega^{2}(x))\frac{|(f_0)_x|^2}{J_0} + 2\omega(x), \end{aligned}$$

and thus we have our first useful inequalities

Lemma 4.13. *If $\omega(x) > 0$, then*

$$\mathbb{K}(z,f) - \mathbb{K}(z,f_0) \geqslant (1-\omega^{-2}(x))\left[\frac{|f_y|^2}{J} - \frac{|(f_0)_y|^2}{J_0}\right], \tag{4.14}$$

and

$$\mathbb{K}(z,f) - \mathbb{K}(z,f_0) \geqslant (1-\omega^{2}(x))\left[\frac{|f_x|^2}{J} - \frac{|(f_0)_x|^2}{J_0}\right] \tag{4.15}$$

with equality holding almost everywhere in either inequality if and only if $f = f_0$.

4.1 A key inequality

In [2] the elementary inequality for complex numbers X, X_0 and real J, J_0

$$\frac{|X|^2}{J} - \frac{|X_0|^2}{J_0} \geqslant 2\,\Re e\Big(\frac{\overline{X_0}}{J_0}\,(X - X_0)\Big) - \frac{|X_0|^2}{J_0^2}\big(J - J_0\big), \tag{4.16}$$

with equality holding if and only if $X/X_0 = J/J_0$ is a positive real number, was used to identify minima for the mean distortion. This inequality, obtained by expanding $|X/X_0 - J/J_0|^2 \geqslant 0$, is used to study the function

$$(X,Y,J) \to \frac{|X|^2 + |Y|^2}{J}$$

which is convex on $\mathbb{C}\times\mathbb{C}\times\mathbb{R}_+$; the graph of the function lies above its tangent plane. When X and Y are partial derivatives, the relation to the distortion function is clear. We want to apply the inequality (4.16) and this requires that the coefficient $(1-\omega^{-2}(x))>0$ in the first case or $(1-\omega^2(x))>0$ in the second. Since this depends on u_x for the candidate extremal mapping, we carry along the two inequalities and write $\mathbb{K}_0=\mathbb{K}(z,f_0)$. First note that if $\varphi:\mathbb{R}\to\mathbb{R}$ is convex, then its graph lies above any tangent line:

$$\varphi(\mathbb{K})-\varphi(\mathbb{K}_0)\geqslant\varphi'(\mathbb{K}_0)\big(\mathbb{K}-\mathbb{K}_0\big).$$

Notice that if $\varphi''>0$, equality quality holds here if and only if $\mathbb{K}=\mathbb{K}_0$. This therefore yields the following two inequalities:

$$\begin{aligned}\varphi(\mathbb{K}(z,f))-\varphi(\mathbb{K}(z,f_0)) &\geqslant (1-\omega^{-2}(x))\varphi'(\mathbb{K}_0)\\ &\quad\left[2\,\Re e\Big(\frac{\overline{(f_0)_y}}{J_0}\,\big(f_y-(f_0)_y\big)\Big)-\frac{|(f_0)_y|^2}{J_0^2}\big(J-J_0\big)\right],\\ \varphi(\mathbb{K}(z,f))-\varphi(\mathbb{K}(z,f_0)) &\geqslant (1-\omega^{2}(x))\varphi'(\mathbb{K}_0)\\ &\quad\left[2\,\Re e\Big(\frac{\overline{(f_0)_x}}{J_0}\,\big(f_x-(f_0)_x\big)\Big)-\frac{|(f_0)_x|^2}{J_0^2}\big(J-J_0\big)\right].\end{aligned}$$

Now $(f_0)_y=i$ and $(f_0)_x=1/\omega(x)=J_0$ so these equations read as

$$\begin{aligned}&\varphi(\mathbb{K}(z,f))-\varphi(\mathbb{K}(z,f_0))\\ &\quad\geqslant\ \big(1-\frac{1}{\omega^2(x)}\big)\varphi'(\mathbb{K}_0)\left[\frac{2}{J_0}\,\Im m\big(f_y-1\big)-\frac{J-J_0}{J_0^2}\right]\\ &\quad=\ 2\left(\omega(x)-\frac{1}{\omega(x)}\right)\varphi'(\mathbb{K}_0)\Im m\big(f_y-1\big)\\ &\qquad+\big(\omega^2(x)-1\big)\varphi'(\mathbb{K}_0)(J_0-J)\,,\end{aligned}\tag{4.17}$$

$$\begin{aligned}&\varphi(\mathbb{K}(z,f))-\varphi(\mathbb{K}(z,f_0))\\ &\quad\geqslant\ (1-\omega^2(x))\varphi'(\mathbb{K}_0)\left[2\,\Re e\big(f_x-(f_0)_x\big)-\big(J-J_0\big)\right].\end{aligned}\tag{4.18}$$

Now we want to multiply these two inequalities by a weight function $\lambda(x)$ and integrate. We are naturally led to consider the Euler-Lagrange equation for the variational problem minimising

$$\iint_{\mathbb{Q}}\varphi(\mathbb{K}(z,f))\lambda(x)\;|dz|^2$$

among functions of the form (4.1). This equation reduces to the rather surprising equation in one real variable

$$\frac{d}{dx}\left[\lambda(x)\Big(1-\frac{1}{u_x^2}\Big)\varphi'\Big(u_x+\frac{1}{u_x}\Big)\right]=0\tag{4.19}$$

We would therefore like $\omega(x)$ to be chosen chosen so that

$$\lambda(x)(1-\omega^2(x))\varphi'\Big(\omega(x)+\frac{1}{\omega(x)}\Big) = \alpha \neq 0 \tag{4.20}$$

for a real constant α. It is quite remarkable that this equation implicitly defines ω directly, it does not involve any of its derivatives.

Remark. We postpone the important discussion of boundary values for the solution f_0 (really g_0) that we seek. Set

$$\int_0^\ell \frac{dx}{\omega(x)} = L_0 \tag{4.21}$$

The boundary conditions we want are that $L = L_0$ to identify the minimum. However, if $L_0 < L$, then Lemma 4.4 still applies - and we obtain strict inequality. Also, we note that from (4.20), with an assumption that $\lambda > 0$ and φ' are continuous, that $\omega = 0$ implies that $\lambda(x)\varphi'(\infty) = \alpha$. In particular, we cannot have $\omega(x) = 0$ unless φ' is bounded - a condition we will see again.

We now suppose that we have (4.20) holding almost everywhere and $L_0 < L$. Then (4.20) forces $0 \leq \omega(x) < 1$ for all x or $\omega(x) > 1$ for all x. The case $\omega \equiv 1$, $\alpha = 0$ yielding $g_0 = f_0 = \textit{identity}$. The first case (where we will ultimately have to deal with degeneration as we cannot guarantee the boundary conditions) has $u_x > 1$ and so must correspond to stretching $L > \ell$. In the other case $\ell < L$.

We proceed as follows.

$$\begin{aligned}
\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\,|dz|^2 &\geqslant \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f_0))\lambda(x)\,|dz|^2 - \alpha \iint_{\mathbb{Q}_1} (J_0 - J)\,|dz|^2 \\
&+ 2 \iint_{\mathbb{Q}_1} \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\varphi'(\mathbb{K}_0)\Im m\big(f_y - 1\big)\,|dz|^2, \\
\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\,|dz|^2 &\geqslant \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f_0))\lambda(x)\,|dz|^2 + \alpha \iint_{\mathbb{Q}_1} (J_0 - J)\,|dz|^2 \\
&+ 2\alpha \iint_{\mathbb{Q}_1} \Re e\big(f_x - (f_0)_x\big)\,|dz|^2.
\end{aligned}$$

For an arbitrary Sobolev homeomorphism it is well known that

$$\iint_{\mathbb{Q}_1} J\,|dz|^2 \leqslant |\mathbb{Q}_2| = L = \int_0^\ell u_x(x)\,dx = \iint_{\mathbb{Q}_1} J_0\,|dz|^2$$

We will use the first inequality above when $\alpha < 0$ and the second when $\alpha > 0$. Thus, for $\alpha < 0$

$$\begin{aligned}
&\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\,|dz|^2 \\
&\quad\geqslant \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f_0))\lambda(x)\,|dz|^2 + 2\iint_{\mathbb{Q}_1} \lambda\left(\omega - \frac{1}{\omega}\right)\varphi'(\mathbb{K}_0)\Im m\big(f_y - 1\big)\,|dz|^2,
\end{aligned}$$

while for $\alpha > 0$ we have

$$\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\,|dz|^2$$
$$\geqslant \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f_0))\lambda(x)\,|dz|^2 + 2\alpha \iint_{\mathbb{Q}_1} \Re e\big(f_x - (f_0)_x\big)\,|dz|^2$$

Next, from (2.7) we see that

$$\iint_{\mathbb{Q}_1} \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\varphi'(\mathbb{K}_0)\Im m\big(f_y - 1\big)\,|dz|^2$$
$$= \int_0^\ell \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\varphi'(\mathbb{K}_0)\Big[\int_0^1 \Im m\big(f_y - 1\big)dy\Big]\,dx = 0$$

and

$$\iint_{\mathbb{Q}_1} \Re e\big(f_x - (f_0)_x\big)\,|dz|^2 = \int_0^1 \Big[\int_0^\ell \Re e\big(f_x - (f_0)_x\big)dx\Big]\,dy = 0.$$

Thus we have established

Theorem 4.22. *Let $\lambda(x) > 0$ be a positive weight and $\varphi : [1,\infty) \to [0,\infty)$ be convex increasing. Let the function $u : [0,\ell] \to [0,L]$*

$$u(0) = 0, \qquad u(\ell) = L_0 \leq L \tag{4.23}$$

be a solution to the ordinary differential equation

$$\lambda(x)\left(1 - \frac{1}{u_x^2(x)}\right)\varphi'\left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha \tag{4.24}$$

where α is a nonzero constant. Set

$$f_0(z) = u(x) + iy, \qquad f_0 : [0,\ell]\times[0,1] \to [0,L_0]\times[0,1]. \tag{4.25}$$

Let $f : [0,\ell]\times[0,1] \to [0,L]\times[0,1]$ be a surjective homeomorphism of finite distortion with

$$\Re e f(0,y) = 0, \quad \Re e f(\ell,y) = L, \quad \Im m f(x,0) = 0, \quad \Im m f(x,1) = 1.$$

Then

$$\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\,|dz|^2 \geqslant \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f_0))\lambda(x)\,|dz|^2. \tag{4.26}$$

Equality holds if and only if $f = f_0$. In particular, if $L_0 < L$, then this inequality is strict.

Notice $\alpha = 0$ gives the identity mapping - clearly always an absolute minimiser when it is a candidate.

4.2 Degenerate Cases

Theorem 6.3 identifies the extremal homeomorphism of finite distortion when we can find α so that $L_0 = L$. We will see later that this is not always possible and then Theorem 6.3 provides us with the unattainable lower bound $\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z, f_0))\lambda(x)$ - since the inequality is strict. When $L_0 < L$ of course f_0 is not a candidate mapping for the minimisation problem - so it might not be surprising the bound is unattainable. However it might be possible that this *value* is the limit of a minimising sequence of candidates. What we want to do here is to find circumstances in which this happens.

Theorem 4.27. *Suppose that f_0 is defined as in Theorem 6.3 and that*

- *φ' is bounded,*
- *λ is continuous and*
- *for no choice of α is it possible that $L = L_0$*

Then there is a sequence of surjective homeomorphism of finite distortion $f_j : [0, \ell] \times [0, 1] \to [0, L] \times [0, 1]$ such that

$$\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z, f_j))\lambda(x)\ |dz|^2 = \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z, f_0))\lambda(x)\ |dz|^2 \tag{4.28}$$

In particular, under these circumstances there is no extremal homeomorphism of finite distortion for the mapping problem whatsoever.

Remark. We will see in the next few sections the condition ϕ' bounded is necessary for nonexistence of minimisers, but not sufficient. The behaviour of the weight λ near its minimum determines whether we can solve the boundary problem for arbitrary L.

Proof. Our assumption is that φ is convex increasing and thus φ' is positive, continuous and increasing, not necessarily strictly. We may also assume $\lim_{t\to\infty} \varphi'(t) = 1$. The function $t \mapsto (1 - t^{-2})\varphi'(t + 1/t)$ is strictly increasing on $[1, \infty)$ and our solution u^α is obtained by the rule $u^\alpha_x(x) = t_x$ where $(1-t_x^{-2})\varphi'(t_x + 1/t_x) = \alpha/\lambda(x)$. The implicit function theorem implies that for $\alpha < \alpha_0 = \min_x \lambda(x)$ the function $u^\alpha_x \geq 1$ is continuous and bounded, while $u^{\alpha_0}_x$ is continuous in the chordal metric of the extended real line (it is possible that $u^{\alpha_0}_x = +\infty$), and that

$$u^{\alpha_0}_x \to u^{\alpha_0}_x, \qquad \text{uniformly in the chordal metric of } \overline{\mathbb{R}} \tag{4.29}$$

It is easy to see that

$$u^\alpha(\ell) \nearrow u^{\alpha_0}(\ell) = L_0 < L, \qquad \alpha \nearrow \alpha_0$$

Thus for $\alpha \leq \alpha_0$, the family $u^\alpha \in W^{1,1}([0, \ell])$, with a uniform bound. Further $u_0 = u^{\alpha_0}$ is strictly increasing with derivative tending to ∞ as x approaches a minimum, say x_0, of λ (which may be an endpoint of $[0,\ell]$). Thus $f_0(z) = u_0(x) + iy$ is the extremal mapping (of this form!) with largest image.

Let

$$g_0(w) = v_0(a) + ib, \qquad v = u_0^{-1}$$

Then $(v_0)_a(a) = 1/(u_0)_x(x)$ with $u_0(x) = a \in [0, L_0]$. With $u_0(x_0) = a_0$ we have $(v_0)_a(a_0) = 0$. We now define a new function $g : [0,L] \times [0,1] \to [0,\ell] \times [0,1]$ by simply defining $g(w) = v(a) + ib$ to be constant near x_0. That is (with appropriate modification should x_0, the minimum of λ be an endpoint)

$$v(a) = \begin{cases} v_0(a) & a \leq a_0 \\ v_0(a_0) & a_0 \leq a \leq a_0 + L - L_0 \\ v_0(a + L_0 - L) & a_0 + L - L_0 \leq a \leq L \end{cases} \tag{4.30}$$

Then v_a is a continuous non-negative L^1 function valued in $[0,1]$, vanishing on $[a_0, a_0 + L - L_0]$ and with $\|v_a\|_1 = \ell$. Set

$$v_a^j(a) = \frac{1}{1+\frac{1}{j}}\big(v_a(a) + \frac{1}{j}\big)$$

so $\|v_a^j\|_1 = \ell$. Define $v(a) = \int_0^a v_a^j$ to get a homeomorphic mapping of finite distortion $g^j(w) = v^j(a) + ib$. Notice that not only $g^j \to g$ uniformly in $W^{1,1}([0,L] \times [0,1])$, but the derivatives converge uniformly also. Thus

$$\Big(v_a^j(a) + \frac{1}{v_a^j(a)}\Big)\, \lambda(v^j(a))\, v_a^j(a) \to \varphi\Big(v_a(a) + \frac{1}{v_a(a)}\Big)\, \lambda(v(a))\, v_a(a) \quad \text{uniformly}$$

Set

$$f^j = (g^j)^{-1} : [0,\ell] \times [0,1] \to [0,L] \times [0,1]$$

The mappings f^j are surjective diffeomorphisms of finite distortion. We calculate, with the change of variables $g^j(w) = z$,

$$\begin{aligned}
\iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f^j))\lambda(z)\,dz &= \iint_{\mathbb{Q}_2} \varphi\Big(\frac{\|Df^j(g^j)\|^2}{J(g^j,f^j)}\Big)\, \lambda(g^j(w))\, J(w,g^j)dw \\
&= \iint_{\mathbb{Q}_2} \varphi\Big(v_a^j(a) + \frac{1}{v_a^j(a)}\Big)\, \lambda(v^j(a))\, v_a^j(a)da \\
&\to \iint_{\mathbb{Q}_2} \varphi\Big(v_a(a) + \frac{1}{v_a(a)}\Big)\, \lambda(v(a))\, v_a(a)da \\
&= \iint_{[0,L_0]\times[0,1]} \varphi\Big((v_0)_a(a) + \frac{1}{(v_0)_a(a)}\Big)\, \lambda((v_0)(a))\, (v_0)_a(a)da \\
&= \iint_{[0,\ell]\times[0,1]} \varphi\Big((u_0)_x(x) + \frac{1}{(u_0)_x(x)}\Big)\, \lambda(x)\,dx \\
&= \iint_{\mathbb{Q}_1} \varphi\Big(\mathbb{K}(z,f_0)\Big)\, \lambda(z)\,dz
\end{aligned}$$

5 The Nitsche phenomenon

Before moving on to discuss the theory in more generality we provide a couple of interesting applications based around the classical Nitsche problem.

Theorem 6.3 strongly motivates us to study the ordinary differential equation (4.24) for solutions will identify minima of out Nitsche and Grötzsch type problems. Note also that the transformation from the Nitsche type problem to the Grötzsch problem yields a significantly simpler equation to study—in fact it's not really an ODE at all.

5.1 Weighted mean distortion

Let us first observe how the Nitsche phenomenon arises, here we have (ignoring multiplicative constants) $\lambda(x) = e^{4\pi x}$ as $\eta(w) = 1$. We are minimising

$$\iint \mathbb{K}(z, f)\, \lambda(x)\, dz$$

so $\varphi' \equiv 1$ and we have

$$1 - \frac{1}{u_x^2(x)} = \alpha e^{-4\pi x}, \qquad u_x(x) = \frac{1}{\sqrt{1 - \alpha e^{-4\pi x}}}$$

$$u(x) = \int \frac{e^{2\pi x}\, dx}{\sqrt{e^{4\pi x} - \alpha}} = \frac{1}{2\pi} \int \frac{dt}{\sqrt{t^2 - \alpha}}, \qquad t = e^{2\pi x}.$$

So

$$u(x) \quad = \quad \frac{1}{2\pi} \log\Big(\frac{e^{2\pi x} + \sqrt{e^{4\pi x} - \alpha}}{1 + \sqrt{1 - \alpha}}\Big), \qquad \alpha \neq 0$$

noting $u(0) = 0$. Recall $u : [0, \ell] \to [0, L]$ and we must solve $u(\ell) = L$, that is

$$L = \frac{1}{2\pi} \log\Big(\frac{e^{2\pi \ell} + \sqrt{e^{4\pi \ell} - \alpha}}{1 + \sqrt{1 - \alpha}}\Big) \tag{5.1}$$

by choice of our free parameter α. Notice that α is not bounded from below, and as $\alpha \to -\infty$ we can make the right hand side of (5.1) as small as we like. Thus there is always a minimiser if $L \leqslant \ell$. If $\alpha > 0$ we see that (4.24) requires $\alpha \leq 1$ so that

$$L \leq \frac{1}{2\pi} \log\Big(\frac{e^{2\pi \ell} + \sqrt{e^{4\pi \ell} - 1}}{1 + \sqrt{1 - \alpha}}\Big)$$

and when unwound, these are precisely the Nitsche bounds. Theorem 4.27 asserts that beyond these bounds there is no minimiser.

For more general weights $\lambda(x)$,

$$1 - \frac{1}{u_x^2(x)} = \frac{\alpha}{\lambda(x)}, \qquad u_x(x) = \sqrt{\frac{\lambda(x)}{\lambda(x) - \alpha}}$$

and we must typically study the behaviour of an integral like

$$u(x) = \int_0^\ell \sqrt{\frac{\lambda(x)}{\lambda(x)-\alpha}}\, dx.$$

Again, as $\alpha \to -\infty$ and if λ is not too bad, we can make this integral as small as we wish. Notice that $\alpha/\lambda(x) < 1$, so if we put $\lambda_0 = \min_{x\in[0,\ell]} \lambda(x)$, then this integral is dominated by the one with the choice $\alpha = \lambda_0$ and the issue is to decide whether

$$\int_0^\ell \sqrt{\frac{\lambda(x)}{\lambda(x)-\lambda_0}}\, dx < \infty.$$

If this integral is finite, then we will observe Nitsche type phenomena; non-existence of minima outside a range of moduli.

Supposing that $\lambda_0 > 0$, the principal issue concerns the integral

$$\int_0^\ell \frac{dx}{\sqrt{\lambda(x)-\lambda_0}} < \infty, \tag{5.2}$$

and without going into excessively fine details, convergence will require that

$$\lambda(t) \approx \lambda_0 + t^{2s}, \qquad s < 1$$

near the minimum.

In particular, if λ is a smooth positive weight and $\lambda'(x) = 0$ at it's minimum (which may well occur at the endpoints in which case we choose the appropriate left or right derivative), then we can always solve the deformation problem.

5.2 An application

We saw above that for $\lambda(x) = e^{4\pi x}$ we observed the classical Nitsche phenomenon for annuli. Also, if $\lambda(x)$ is constant, $u_x(x)$ is constant and therefore $u(x)$ will be a linear mapping that can be stretched to any length, as determined by the constant α.

Let us discuss other weights in the Grötzsch setting. Here the weight function $\lambda(x)$ can also be viewed as some physical property, eg. density, of the material; if seen as a thickness, an object with a "cut" gives a little more insight to Nitsche-type phenomenon.

Consider the weight function on $[0, d]$

$$\lambda(x) = \begin{cases} 1 - \frac{x}{d} & \text{if} \quad 0 \leqslant x < \frac{d}{2} \\ \frac{x}{d} & \text{if} \quad \frac{d}{2} \leqslant x \leqslant d \end{cases}$$

Consider the three dimensional solid of an elastic material defined by

$$Q = \{(x, y, z) : 0 \le x \le d, 0 \le y \le 1, 0 \le z \le \lambda(x, y)\}$$

(so λ defines the thickness of the object over its rectangular base - so Q is really just the region under the graph of λ). We deform Q by stretching along the x

axis so as to minimise the weighted mean distortion $\iint_Q \mathbb{K}(z,f)\lambda(z)$. There is no vertical compression, so the image is $Q' = \{(x',y',z') : (x',y') = f(x,y), z' = z\}$ and f is a deformation of the base. We make the further assumption that λ depends on x alone. We shall soon see that a minimising sequence (and minimisers should they exist) will have the form $f(x,y) = (f(x),y)$, thus reducing the problem to two-dimensional considerations.

From our discussion at Section 5.1, we note that for each x, $\alpha \leqslant \lambda(x)$ and hence $\alpha \leqslant \frac{1}{2}$. Recall that (4.24) yields

$$u = \int \sqrt{\frac{\lambda(x)}{\lambda(x)-\alpha}}\, dx. \tag{5.3}$$

Taking $\lambda_1 = 1 - \frac{x}{d}$ and $\lambda_2 = \frac{x}{d}$, and simplifying,

$$u_1 = \int \sqrt{\frac{d-x}{d-x-d\alpha}} \tag{5.4}$$

$$u_2 = \int \sqrt{\frac{x}{x-d\alpha}}, \tag{5.5}$$

observing that $u_1(d/2) = u_2(d/2)$. The change of variables $s = d - x$ in u_1 shows that the analysis of u_1 is similar to that of u_2. Next, changing variables by $t = \sqrt{s - d\alpha}$, and integrating,

$$u_1 = -\sqrt{d-x}\sqrt{d-x-d\alpha} - d\alpha \log(\sqrt{d-x-d\alpha} + \sqrt{d-x}) + C$$

for some constant C found using the boundary condition $u(0) = 0$. Rearranging gives

$$u_1 = \left[\sqrt{1-\alpha} - \sqrt{\left(1-\frac{x}{d}\right)^2 - \alpha\left(1-\frac{x}{d}\right)} - \alpha \log\left(\frac{\sqrt{1-\frac{x}{d}} + \sqrt{1-\frac{x}{d}-\alpha}}{1+\sqrt{1-\alpha}}\right)\right] d. \tag{5.6}$$

Similarly, as $u(d) = D$

$$u_2 = D - \left[\sqrt{1-\alpha} - \sqrt{\left(\frac{x}{d}\right)^2 - \alpha\left(\frac{x}{d}\right)} - \alpha \log\left(\frac{\sqrt{\frac{x}{d}} + \sqrt{\frac{x}{d}-\alpha}}{1+\sqrt{1-\alpha}}\right)\right] d. \tag{5.7}$$

Now we require that $u_1(d/2) - u_2(d/2) = 0$ (i.e. they meet in the middle). Thus

$$D = \left[2\sqrt{1-\alpha} - \sqrt{1-2\alpha} - 2\alpha \log\left(\frac{1+\sqrt{1-2\alpha}}{\sqrt{2}\,(1+\sqrt{1-\alpha})}\right)\right] d,$$

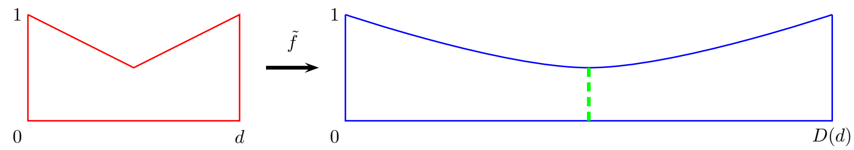
and hence D can be made as small (though positive) as desired, by letting α tend to $-\infty$. However, there is an upper limit on D;

$$\alpha \leqslant \min_{x\in[0,d]} \lambda(x) = \frac{1}{2},$$

and thus

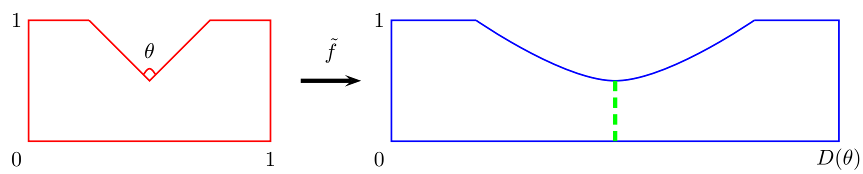
$$D_{\max} = \left[\sqrt{2} - \log\left(\sqrt{2}-1\right)\right] d.$$

This is a Nitsche-like bound on the maximal stretch. It is the value α that determines how far the final stretch can be; the maximum value of α determines a limit (if any) on D.



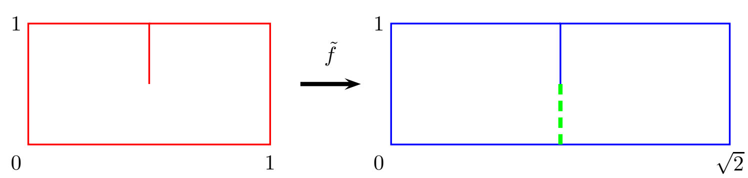
Stretching of a cut of base length d ($\alpha = \frac{1}{2}$).

As d tends to 0, so does D.



Stretching of a block with an open cut ($\alpha = \frac{1}{2}$).

Here we illustrate what happens as θ approaches 0. In the sequence suggested by the above picture (with $\theta \to 0$), D tends to $\sqrt{2}$.



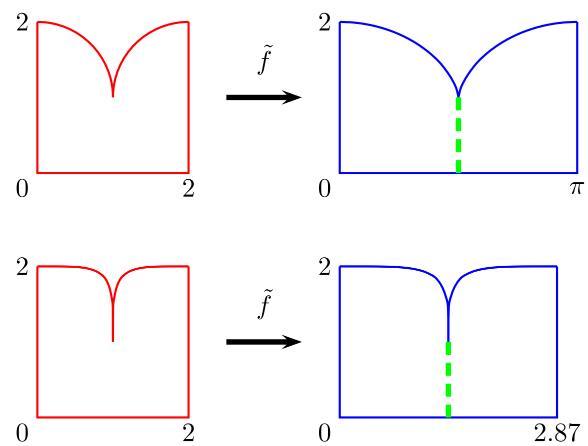
Stretching of a block with a straight-line cut ($\alpha = \frac{1}{2}$).

Once again we observe that α determines how far the final stretch is; it is in fact $\sqrt{\frac{1}{1-\alpha}}$. That is, the deeper the cut, the smaller the maximal stretch - it would seem that this calculation provides a possible test of this model for stretching elastic bodies with cuts. We will address this elsewhere.

A different sequence of weight functions with the same limiting case as above is a sequence of cusps;

$$\lambda_1(x) = 1 + (1 - x^n)^{1/n} \qquad \lambda_2(x) = 1 + \left(1 - (2-x)^n\right)^{1/n} \tag{5.8}$$

The minimum of the weight function λ determines the critical value $\alpha = 1$.



Stretching of a cusp ($\alpha = 1$, $n = 2$ (top), $n = 5$ (bottom)).

Here the maximum stretch is not very far (although greater than $\sqrt{2}$). Carrying out the calculation for the sequence of weight functions by letting n vary shows that this sequence also decreases to a limit of $\sqrt{2}$.

Again we stress that beyond the computed bounds, Theorem 4.27 asserts that there is no minimiser whatsoever.

6 φ' unbounded

In this section we show that should the convex function φ have unbounded derivative, then there is always a minimiser, with mild assumptions on the weight function λ. In particular we do not see the Nitsche phenomenon for the L^p–norms of mean distortion.

Let us first observe that when φ is smooth and convex increasing, the function

$$F(t) = \left(1 - \frac{1}{t^2}\right) \varphi'\left(t + \frac{1}{t}\right)$$

is increasing for $t > 0$, indeed

$$F'(t) = \frac{2}{t^3}\varphi'\left(t + \frac{1}{t}\right) + \left(1 - \frac{1}{t^2}\right)^2 \varphi''\left(t + \frac{1}{t}\right) > 0$$

Next, if φ' is unbounded, it is monotone and then

$$\lim_{t \searrow 0} F(t) = -\infty, \qquad \lim_{t \to +\infty} F(t) = +\infty$$

The intermediate value theorem implies that for each $x \in (0, \ell)$ and $\alpha \in \mathbb{R}$ we can find $t_x > 0$ so that

$$F(t_x) = \frac{\alpha}{\lambda(x)}$$

We then define a function v_α by the rule

$$v_\alpha(x) = t_x > 0$$

Then v is a positive function which certainly satisfies

$$\lambda(x)\left(1 - \frac{1}{v^2(x)}\right)\varphi'\left(v(x) + \frac{1}{v(x)}\right) = \alpha \tag{6.1}$$

The regularity of the function v_α depends on that of λ. The function u that we are looking for define the mapping f is an antiderivative of v. For f to be a mapping of finite distortion, we'll need that u is absolutely continuous. These conditions are all easily seem to be true if λ (and hence v_α) is piecewise continuous.

We then define

$$u_x(x) = v_\alpha(x) \tag{6.2}$$

If λ is bounded and bounded away from 0, then it is easy to see that v_α is uniformly large when α is chosen large, while v_α is uniformly small if α is chosen large and negative. Further

$$u(x) = \int_0^x v_\alpha(s)\; ds$$

depends continuously on α (as v_α depends piecewise continuously). Thus $u(\ell)$ can be made to assume any positive value - in particular we can solve $u(\ell) = L$, and so we don't see the Nitsche phenomena. Here is a theorem summarising this discussion. The reader will see that we have not striven for maximum generality.

Theorem 6.3. *Let $\lambda(x)$ be a piecewise continuous positive weight bounded and bounded away from 0. Let $\varphi : [1,\infty) \to [0,\infty)$ be smooth and convex increasing with $\varphi'(s)$ unbounded as $s \to \infty$. Then the minimisation problem*

$$\min_{f\in\mathcal{F}} \iint_{\mathbb{Q}_1} \varphi(\mathbb{K}(z,f))\lambda(x)\; |dz|^2 \tag{6.4}$$

has a unique solution of the form $f(z) = u(x) + iy$. Here $\mathcal{F}$ is the family of all mappings of finite distortion satisfying the boundary conditions described in 2.2

We then have the following corollary about the weighted L^p-norms of distortion functions.

Corollary 6.5. *Let $\lambda(x)$ be a piecewise continuous positive weight bounded and bounded away from 0. Then the minimisation problem*

$$\min_{f\in\mathcal{F}} \iint_{\mathbb{Q}_1} \mathbb{K}^p(z,f)\lambda(x)\; |dz|^2 \tag{6.6}$$

has a unique solution of the form $f(z) = u(x) + iy$. Here $\mathcal{F}$ is the family of all mappings of finite distortion satisfying the boundary conditions described above.

7 Critical phase case: φ' bounded

Examining the above argument we see that in this case we can always find a solution to the minimisation problem of the given form if $L < \ell$ by varying α among negative values, $\alpha = 0$ produces the identity mapping.

However, in this case there are further subtleties. The reader will quickly get to a condition on the integrability of $\psi(\lambda_0/\lambda(x))$ where ψ is the inverse of the bounded increasing function $t \mapsto \varphi'(t+t^{-1})(1-t^{-2})$ with $\lambda_0 = \min_{[0,\ell]} \lambda$. Let us give two illustrative examples in the standard (Nitsche) case with $\ell = 1$, $\lambda(x) = e^{-4\pi x}$. We may assume that $\varphi'(t) \nearrow 1$ and the limiting case $\alpha = e^{4\pi}$:

Case: $\varphi(t) = t - \log(t)$, $\varphi'(t) = 1 - \frac{1}{t}$, $a = a(x) = e^{4\pi(x-1)} \leqslant 1$.

We choose u_x to be the largest real root of the polynomial:

$$\begin{aligned} \Big(1 - \frac{1}{t+t^{-1}}\Big)\big(1 - \frac{1}{t^2}\big) &= a \\ p(t) = -1 + t - at^2 - t^3 + (1-a)t^4 &= 0. \end{aligned}$$

Since

$$p(\frac{1}{1-a}) = -1 + \frac{1}{1-a} - \frac{a}{(1-a)^2} - \frac{1}{(1-a)^3} + \frac{1}{(1-a)^3} = -\frac{a^2}{(1-a)^2} < 0$$

the largest real root

$$u_x(x) > \frac{1}{1-a(x)}$$

and

$$\int_0^x u_y(y) > \int_0^x \frac{1}{1-e^{4\pi(y-1)}} \approx \frac{1}{4\pi}\log\Big(\frac{1}{1-x}\Big)$$

and this diverges as $x \to 1$. Therefore with appropriate choice of α we can always solve $u(0) = 0$ and $u(1) = L$. Hence there is no Nitsche phenomena.

Case: $\varphi(t) = t + \frac{1}{(p-1)t^{p-1}}, p > 0, p \neq 1$.

We have $\varphi'(t) = 1 - \frac{1}{t^p}$, $0 < a = a(x) = e^{-4\pi x} < 1$ for $0 < x < 1$, and hence u_x is the largest real root of the polynomial

$$P(t) = \left(1 - \frac{1}{(t+t^{-1})^p}\right)\left(1 - \frac{1}{t^2}\right) - a = 0. \tag{7.1}$$

Note that when $t > 0$, $P(t)$ is a continuous monotonically increasing function of t. Also note that $P(1) = -a < 0$, and $\lim_{t\to\infty} P(t) = 1 - a > 0$, so that P has exactly one real positive root $u_x > 1$.

First let us deal with $0 < p < 1$. Observe that

$$\big(1-(1-a)^2\big)\left(\big(1+(1-a)^2\big)-(1-a)\right) - a\left(1+(1-a)^2\right) = -a^2(1-a)^2 < 0.$$

This may be rewritten as

$$\left(1-\frac{1}{\left(\frac{1}{1-a}\right)^2}\right)\left(1-\frac{1}{\frac{1}{1-a}+\frac{1-a}{1}}\right)-a \;<\; 0$$

Now using the fact that $0<p<1$, we see that

$$P\left(\frac{1}{1-a}\right)=\left(1-\frac{1}{\left(\frac{1}{1-a}\right)^2}\right)\left(1-\frac{1}{\left(\frac{1}{1-a}+\frac{1-a}{1}\right)^p}\right)-a<0$$

and hence the largest real root

$$u_x>\frac{1}{1-a}.$$

The integral of the right hand side diverges (see the reasoning for the case $\varphi'=1-t^{-1}$). Thus with appropriate choice for α we can always solve $u(0)=0, u(1)=L$ and therefore we see no Nitsche phenomena for $p<1$.

Next, take $p\geqslant 2$. Recall (7.1). Note that $\left(t+\frac{1}{t}\right)^p>\left(t+\frac{1}{t}\right)^2>t^2$. Choose $Q(t)$ as

$$P(t)=\left(1-\frac{1}{(t+t^{-1})^p}\right)\left(1-\frac{1}{t^2}\right)-a>\left(1-\frac{1}{t^2}\right)^2-a=Q(t).$$

The largest real root of $P(t)$ is therefore dominated by the largest real root of $Q(t)$. Solving $Q(t)=0$ gives

$$\int_0^1 u_x\,dx<\int_0^1\frac{1}{\sqrt{1-e^{-2\pi x}}}\,dx=\log\left(e^{\pi}+\sqrt{e^{2\pi}-1}\right),$$

a finite number. Therefore, when $p\geqslant 2$, $u_x(x)$ is dominated by an integrable function and we must see the Nitsche phenomenon. It is no coincidence that the value of the integral here is strongly reminiscent of that for the "standard" Nitsche case (5.1); the integrands for that case and the estimate here are very similar.

It remains to cover the case where $1<p<2$. Note that for $p>1$,

$$1-\frac{1}{(t+t^{-1})^p}>1-\frac{1}{t^p},$$

and for $p<2$,

$$1-\frac{1}{t^2}>1-\frac{1}{t^p}.$$

Therefore the polynomial

$$P(t)=\left(1-\frac{1}{(t+t^{-1})^p}\right)\left(1-\frac{1}{t^2}\right)-a>\left(1-\frac{1}{t^p}\right)^2-a=Q(t),$$

and the largest real root of $P(t)$ is again dominated by the largest real root of $Q(t)$. Solving $Q(t) = 0$ yields

$$u_x < \frac{1}{\left(1 - \sqrt{a(x)}\right)^{1/p}}.$$

Near $x = 0$, $\sqrt{a(x)} = e^{-2\pi x} \approx 1 - 2\pi x$ and so

$$\int_0^1 \frac{1}{\left(1 - \sqrt{a(x)}\right)^{1/p}}\, dx \approx \left(\frac{1}{2\pi}\right)^{1/p} \int_0^1 \frac{1}{x^{1/p}}\, dx,$$

which converges if and only if $p > 1$. Therefore in this case, too, u_x is dominated by an integrable function and Theorem 4.27 asserts that we must see a critical Nitsche-type phenomenon.

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