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Spatial point processes and moment dynamics in the life sciences: a parsimonious derivation and some extensions

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Abstract Mathematical models of dynamical systems in the life sciences typically assume that biological systems are spatially well mixed (the mean-field assumption). Even spatially explicit differential equation models typically make a local mean-field assumption. In effect, the assumption is that diffusive movement is strong enough to destroy spatial structure, or that interactions between individuals are sufficiently long-ranged that the effects of spatial structure are weak. However, many important biophysical processes, such as chemical reactions of biomolecules within cells, disease transmission among humans, and dispersal of plants, have characteristic spatial scales that can generate strong spatial structure at the scale of individuals, with important effects on the behaviour of biological systems. This calls for mathematical methods that incorporate spatial structure. Here we focus on one method, spatial-moment dynamics, which is based on the idea that important information about a spatial point process is held in its low-order spatial moments. The method goes beyond dynamics of the first moment, i.e. the mean density or concentration of agents in space, in which no information about spatial structure is retained. By including the dynamics of at least the second moment, the method retains some information about spatial structure. Whereas mean-field models effectively use a closure assumption for the second moment, spatial-moment models use a closure assumption for the third (or a higher-order) moment.

The aim of the paper is to provide a parsimonious and intuitive derivation of spatial-moment dynamic equations that is accessible to non-specialists. The derivation builds naturally from the first moment to the second and we show how it can be extended to higher-order moments. Rather than tying the model to a specific biological example, we formulate a general model of movement, birth and death of multiple types of interacting agents. This model can be applied to problems from a range of disciplines, some of which we discuss. The derivation is performed in a spatially non-homogeneous setting, to facilitate future investigations of biological scenarios, such as invasions, in which the spatial patterns are non-stationary over space.

Keywords agent-based model · integro-differential equation · interacting agents · moment closure · spatio-temporal process · spatial pattern · stochastic process

1 Introduction

Dynamic spatial point processes deal with the behaviour of populations of agents in a space. There are rather few restrictions on the populations, other than that they live in a continuous space, and that the location of an agent can be associated with a point in the space (this does not mean that the agent itself has to be infinitesimal in size). Spatial point processes provide a key to describing the dynamics of spatially structured systems, and have a potentially wide range of applications in biology, from molecules interacting on surfaces in cells, to tissue growth in multicellular organisms, to dynamics of interacting populations in ecology, as well as in other subject areas such as the social sciences.

Typically in biology, dynamic models of populations of agents make use of the first moment as the state variable. This is a spatially averaged density, or intensity, or concentration of agents. Models of the first moment dynamics are referred to as 'mean-field', and classical examples include the logistic model for population growth (Verhulst, 1836), Lotka–Volterra models for ecologically interacting populations (Lotka, 1920; Volterra, 1927) and SIR models for the spread of an epidemic (Kermack and McKendrick, 1927). Spatially explicit models, such as reaction–diffusion equations, allow the first moment to be a function of location in space (Shigesada and Kawasaki, 1997). However, they typically still neglect variations in densities over small spatial scales (i.e. scales commensurate with individual dispersal and interaction) and may be termed 'local mean-field'.

The first moment is silent on matters of spatial structure, as illustrated in Fig. 1. This shows three spatial patterns that all have the same average density of agents, and yet are clearly different. To capture information on spatial structure, the second spatial moment is needed, at least. Unlike the first moment, the second moment is a function of distance, and describes the density $g(r)$ of pairs of agents separated by a distance r, normalised for illustration here by dividing through by the average density squared so that, for large enough r, $g(r) \approx 1$ (Illian et al., 2008). In Fig. 1(a), there is no spatial structure: the agents are all independently located with uniformly distributed Cartesian coordinates (a spatial Poisson process), and $g(r)$ is approximately 1 at all distances (Fig. 1(d)). In Fig. 1(b), agents tend to occur in clusters, with more pairs of agents close to one another than in a Poisson process, and $g(r) > 1$ at short distances (Fig. 1(e)). In contrast, agents in Fig. 1(c) tend to be spaced apart from one another, with fewer pairs of agents close to one another than in a Poisson process, and $g(r) < 1$ at short distances (Fig. 1(f)). The point processes in Fig. 1 are all spatially homogeneous, meaning that the probability of there being an agent in a small area is independent of the location of that region. Of course, point processes can also be non-homogeneous (i.e. have regions of high and low densities) and such point processes may or may not be have spatial covariances.

Spatial structures like those in Fig. 1 become important when the proximity of agents matters, as is often the case in the life sciences. How to describe

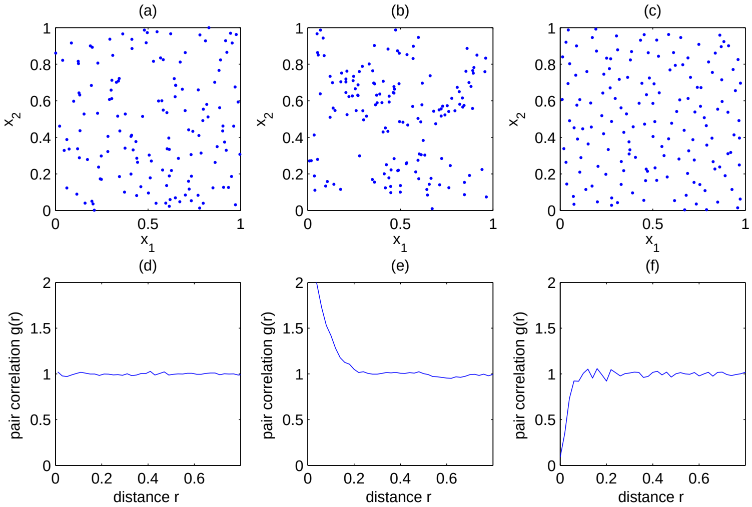

Fig. 1 Examples of spatial patterns (a–c) and the corresponding second spatial moments (d–f). The second moment is normalised by dividing through by the average density squared to give the pair correlation function $g(r)$, which is approximately 1 for large r. All three patterns have the same mean density (first moment) but differ in their second moment: (a) Poisson spatial pattern (all agent's locations are independent); (b) aggregated spatial pattern (agents tend to be arranged in clusters); (c) disaggregated spatial pattern (agents tend to be spaced apart).

the dynamics of neighbourhood interactions is not obvious because the spatial structures both determine and are determined by the interactions. The response to such difficulties has been a general shift away from mathematical formalism towards stochastic, agent-based models with algorithmic rules that can be easily simulated on modern computers (Grimm et al., 2006). However, such models have the drawback of being rather intractable mathematically.

Here, we focus on and review the dynamics of the second spatial moment as a way of going beyond simulations of spatial agent-based models. By working in continuous space, we avoid the need to specify an artificial lattice for the agent locations. The use of lattice-based models is usually for technical convenience rather than biological realism (Bruna and Chapman, 2013) and the choice of lattice can influence model behaviour (Fernando et al., 2010; Plank and Simpson, 2012). The idea behind spatial-moment dynamics is to capture spatial correlations between pairs of agents in the dynamics, moving on from mean-field approaches that ignore spatial correlations altogether. This approach has its roots in statistical physics (Kirkwood, 1935), although the application to biology is more recent (Matsuda et al., 1992; Bolker and Pacala, 1997, 1999; Dieckmann and Law, 2000; Keeling, 2000; Lewis and Pacala, 2000). As in mean-field models, the hierarchy of spatial moments is closed by assumption, but the closure is made at second order, so that the dynamical system is able to hold some basic information on spatial structure as it unfolds over time.

There are other mathematical paths that do not rely on a closure assumption, for instance working directly with the stochastic process, using a perturbation approximation, or working with the full hierarchy of moments, as discussed in Sec. 7 (Blath et al., 2007; Bruna and Chapman, 2012b; Ovaskainen et al., 2014).

In the life sciences, dynamics of the second spatial moment were originally developed for ecological problems where events are influenced by interactions with a small number of neighbours, at rates potentially far from those in a well-mixed, mean-field system (Dieckmann and Law, 2000). Such dynamical systems have the capacity to carry forward the spatial structure that plants and animals respond to, modifying the spatial structure as they do so (Bolker and Pacala, 1997; Law and Dieckmann, 2000). This is important in plant populations and communities because local spatial structure can make a 'plant's-eye' view of its community quite different from a large-scale average (Purves and Law, 2002; Llambi et al., 2004; Law et al., 2009). Second-moment dynamics have therefore been used to analyze the the combined effects of spatial structure and small neighbourhoods on plant communities (Bolker and Pacala, 1999; Bolker et al., 2003; Law et al., 2003; Murrell and Law, 2003). The method has been extended to describe the spatial structure that can emerge in the size distribution of plants (Murrell, 2009; Adams et al., 2013). A similar approach can be applied to the dynamics of animal populations as they become associated with their preferred habitat (Murrell and Law, 2000), and to spatial structures that develop between predators and their prey (Murrell, 2005; Barraquand and Murrell, 2012, 2013). However, the widespread take-up of spatial, agent-based models across the life sciences suggests that spatial-moment dynamics have a potential field of application much broader than ecology.

The purpose of this paper is primarily methodological. Moment-dynamic equations up to second order are already available in the literature, usually as special cases designed to address particular ecological questions, as described above. However, the algebra can appear complicated and the models context-specific and there is a need for a straightforward, general derivation that is not tied to specific ecological applications. With this in mind, we introduce a simple and elegant approach, suggested by Grey (2000, pers. comm.). This approach combines the intuitive appeal of the derivation of Bolker and Pacala (1999) with the rigour of Dieckmann and Law (2000). It is sufficiently transparent to invite extensions to some more complicated problems, some of which we outline. In particular, it allows a conjecture about the equation for the dynamics of the n^{th} spatial moment.

2 Stochastic, agent-based model

Spatial-moment dynamics are approximation schemes for stochastic, spatially explicit, agent-based models. Such models are defined by an initial state and by a set of rules through which properties of agents are updated over time.

A realization of the model gives the state at each point in time, which can potentially be a complicated multi-type spatial pattern. Repeated realizations of the model from the same initial conditions generate an ensemble of spatial patterns at each point in time. The expected values of the spatial moments of these patterns are the state variables of the spatial-moment dynamics.

In the stochastic agent-based model, each agent has a physical location in space. For ease of presentation, we assume that the dynamics take place in a two-dimensional space $\Omega \subseteq \mathbb{R}^2$; other numbers of dimensions are possible. The space should be large relative to the scales over which agents interact and move. The agents can be of different types; they could, for instance, be different types of molecule, cell types, genotypes or species.

The state of the system at time t consists of the location $x_n \in \Omega$ and type $i_n \in \{1, \ldots, i_{\max}\}$ of each agent n ($n = 1, \ldots, N(t)$), where $i_{\max}$ is the number of different types. The rules for changing the properties of agents are context-dependent. To be specific, we consider three classes of event: movement, birth and death. This means that an agent's location may change over time through movement, and the agent can give birth and die, changing the total number of agents $N(t)$. Birth events are accompanied by dispersal of the new agent, so that there is never more than one agent at a single location in space. The notion of birth and death can be extended to more general events creating an agent and causing it to disappear, for instance through generating a new molecule in a chemical reaction. Other processes, for example growth of agents or transition of agents from one type to another, are also possible (see Sec. 6).

Movement, birth and death events occur to agent n with rates per unit time $\hat{M}_n$, $\hat{B}_n$ and $\hat{D}_n$ respectively (the $\hat{}$ distinguishes these functions from related ones used in the moment dynamics below). These events are Poisson processes over time, meaning that the probability of the events occurring in a short period of time δt, to leading order in δt, is $\hat{M}_n \delta t$, $\hat{B}_n \delta t$, $\hat{D}_n \delta t$ respectively. When an event happens, the system is updated to a new state and consequently the rates change; the Poisson processes are therefore inhomogeneous over time. The event rates are assumed to comprise an intrinsic component (which may depend on the agent's type i_n and location x_n) and a component that depends on the presence of other agents in the neighbourhood. These two components are often referred to as density-independent and density-dependent respectively.

We denote the intrinsic component of the movement rate of an agent of type i at location x by $m_i(x)$. In addition to this intrinsic component, an agent of type j and location y contributes $w_{ij}^{(m)}(x, y)$ to the movement rate. The overall movement rate of agent n is defined as the sum of the intrinsic component and the contributions of all other agents:

$$\hat{M}_n = m_{i_n}(x_n) + \sum_{l \neq n} w_{i_n i_l}^{(m)}(x_n, x_l). \tag{2.1}$$

When a movement event occurs to an agent of type i at some location u, the agent moves to a new location x drawn from a probability density function

$\mu_i^{(m)}(u,x)$. For simplicity, this movement distribution is assumed to be independent of the types and locations of other agents in the space. In the simplest model, $\mu_i(x,u)$ would be a function of $|x-u|$ (i.e. dependent on the distance from the initial to final location but independent of the direction and of the initial location).

The birth and death rates of agent n are defined similarly to Eq. (2.1):

$$\hat{B}_n = b_{i_n}(x_n) + \sum_{l \neq n} w_{i_n i_l}^{(b)}(x_n, x_l), \tag{2.2}$$

$$\hat{D}_n = d_{i_n}(x_n) + \sum_{l \neq n} w_{i_n i_l}^{(d)}(x_n, x_l). \tag{2.3}$$

When a birth event occurs to an agent of type i at location u, a new agent of the same type as the parent is created. The new agent disperses from the parent and appears at a location x, drawn from a probability density function $\mu_i^{(b)}(u,x)$.

The definitions of the event rates in Eqs. (2.1)–(2.3) are equivalent to those in previous models (e.g. North and Ovaskainen, 2007; Raghib et al., 2011; Barraquand and Murrell, 2012). The weighting functions $w_{ij}(x,y)$ and dispersal functions $\mu_i(x,y)$, that define the agent-based model, describe the core mechanisms generating spatial structure. Usually, these functions will be concentrated at $y=x$ and decay to 0 as $|y-x|$ increases (for example a Gaussian function $w(x,y) = w_0 e^{-k|y-x|^2}$, with $k>0$). This means that agents are strongly influenced by near neighbours and not by more distant neighbours. The breadth of the function sets the spatial scale over which the mechanism operates; for example, smaller values of k in the Gaussian function above would mean that agents influence their neighbours over a greater range. Similarly, the breadths of the functions $\mu_i(x,y)$ set the spatial scales for movement and for dispersal of offspring.

For the general derivation of the moment dynamics below, the weighting and dispersal functions do not have to be specified in detail. The only constraints are that $\hat{M}_n$, $\hat{D}_n$ and $\hat{B}_n$ must never be negative, $\mu_i(x,y) \geq 0$ and $\int \mu_i(x,y)\ dy = 1$. The integrals of the neighbour-weighting functions, $\int w_{i,j}(x,y)\ dy$, are not required to equal 1, but instead can be varied to control the overall strength of the corresponding interaction. (This contrasts Dieckmann and Law (2000), where these functions integrate to unity and the strength of the interaction is controlled by an additional parameter.) The intrinsic rates and neighbour-weighting functions in Eqs. (2.1)–(2.3) have dimensions T^{-1}. The dispersal functions $\mu_i(x,y)$ have dimensions L^{-2}.

3 Spatial moments

Here the spatial moments are defined up to order 3, together with related conditional probabilities. These are needed for the derivation of the spatial moment dynamics that follows. The definitions can be understood in terms of

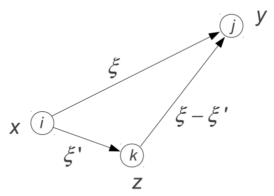

Fig. 2 Geometry of spatial moments up to third order in a two-dimensional space. A type-i agent is located at x, type-j at y, and type-k at z. The probability of finding an agent in a small region δx, of area h, centred on x can be written in terms of the first moment as $Z_{1,i}(x)h$; the probability of finding an agent of type i in δx and an agent of type j in δy is written in terms of the second moment as $Z_{2,ij}(x,y)h^2$, etc. If the system is spatially homogeneous, the physical locations x, y, z can be replaced by displacements $\xi = y - x$ and $\xi' = z - x$.

the geometry of three, small, non-intersecting regions δx, δy and δz containing the points x, y and z respectively (Fig. 2). Each region is assumed to have an area h, with the standard assumption that the probability of there being more than one agent in a region is $O(h^2)$. Note that there is no assumption of homogeneity of the space: the environment may differ from one part of the space to another, as may the density and pattern of the agents themselves.

3.1 Spatial moments at time t

The first three spatial moments are the densities of single agents, pairs and triplets. We assume geometries for the moments as in Fig. 2, indexing the type of agents by i, j, k, but note that this indexing can be ignored if all agents are of the same type. The spatial moments are all functions of time in the dynamics below but, for notational simplicity, we omit time as an argument where there is no ambiguity. We define $N_i(A)$ to be the number of agents of type i in the region $A \subset \mathbb{R}^2$ at time t.

The first spatial moment is defined in terms of the expected number of agents of type i in a small region δx, of area h, centred on x:

$$Z_{1,i}(x) = \lim_{h \to 0} \frac{E\left[N_i(\delta x)\right]}{h} \tag{3.1}$$

In the spatial statistic literature, this is referred to as the intensity, denoted $\Lambda(x)$ (Illian et al., 2008). The second spatial moment, the density of pairs comprising type i at x and type j at y, is defined as:

$$Z_{2,ij}(x,y) = \lim_{h \to 0} \frac{E\left[N_i(\delta x)N_j(\delta y) - \delta_{ij} N_i\left(\delta x \cap \delta y\right)\right]}{h^2}, \tag{3.2}$$

If δx and δy are non-overlapping, the numerator reduces to $E\left[N_i(\delta x)N_j(\delta y)\right]$, which, in the limit $h \to 0$, is equivalent to the probability that there is an agent of type i in δx and an agent of type j in δy. The second term in the numerator is necessary to remove self-pairs that would otherwise create a

Dirac-delta peak in $Z_2(x,y)$ at $x=y$ (Law and Dieckmann, 2000; Illian et al., 2008; Raghib et al., 2011). Here δ_{ij} is the Kronecker-delta symbol. The third moment (density of triplets) is defined similarly as

$$\begin{aligned} Z_{3,ijk}(x,y,z) = \lim_{h\to 0} \frac{1}{h^3} E\Big[& N_i(\delta x)N_j(\delta y)N_k(\delta z) - \delta_{ij}N_i(\delta x \cap \delta y)N_k(\delta z) \\ & -\delta_{ik}N_i(\delta x \cap \delta z)N_j(\delta y) - \delta_{jk}N_j(\delta y \cap \delta x)N_i(\delta x) \\ & + 2\delta_{ijk}N_i(\delta x \cap \delta y \cap \delta z)\Big]. \end{aligned} \tag{3.3}$$

Again, the extra terms in the numerator are needed to remove non-distinct triplets. The definitions above are equivalent to those of Illian et al. (2008), who refer to them as the product densities. In general, the n^{th} spatial moment Z_n has dimensions L^{-2n} and represents the expected number of n-tuplets of agents per unit (area)^n.

3.2 Probabilities of agent presences

As a precursor for the derivation below, it helps to record the probabilities of agents being found in given areas. Since the probability of there being more than one agent in a small region of area h is $O(h^2)$, we have

$$E\left(N_i(x)\right) = P\left(N_i(x)=1\right) + O(h^2)$$

Using (3.1)–(3.3), we can write the probabilities of agents being present in given areas, at any given time, in terms of the spatial moments:

$$\begin{aligned} P\left(N_i(\delta x)=1\right) &= Z_{1,i}(x)h + O(h^2), \\ P\left(N_i(\delta x)=1 \ \&\ N_j(\delta y)=1\right) &= Z_{2,ij}(x,y)h^2 + O(h^3), \\ P\left(N_i(\delta x)=1 \ \&\ N_j(\delta y)=1 \ \&\ N_k(\delta z)=1\right) &= Z_{3,ijk}(x,y,z)h^3 + O(h^4), \end{aligned}$$

provided the regions δx, δy and δz do not overlap.

We can also use the law of conditional probability $P(A|B) = P(A \ \&\ B)/P(B)$ to calculate the probabilities of agents being found in given areas, conditional on the presence of other agents. From the above, the probability that there is agent of type j in δy, given that there is an agent of type i in δx is

$$P\left(N_j(\delta y)=1 \mid N_i(\delta x)=1\right) = \frac{Z_{2,ij}(x,y)\ h}{Z_{1,i}(x)} + O(h^2). \tag{3.4}$$

Similarly, the probability that there is an agent of type k in δz, given that there is an agent of type i in δx and type j in δz is

$$P\left(N_k(\delta z)=1 \mid N_i(\delta x)=1 \ \&\ N_j(\delta y)=1\right) = \frac{Z_{3,ijk}(x,y,z)\ h}{Z_{2,ij}(x,y)} + O(h^2). \tag{3.5}$$

	Movement	Birth	Death	Multi-type	NH
Bolker and Pacala (1997)	No	DI	DI+DD	No	No
Bolker and Pacala (1999)	No	DI	DI+DD	Yes	No
Lewis and Pacala (2000)	No	DI	No	No	Yes
Lewis (2000)	No	DI+DD	No	No	Yes
Dieckmann and Law (2000)	DI	DI+DD	DI+DD	Yes	No
Murrell and Law (2000)	DD	No	No	Yes	No
Bolker (2003)	No	DI	DI+DD	No	No
Murrell and Law (2003)	No	DI	DI+DD	Yes	No
Murrell (2005)	DI	DI+DD	DI+DD	Yes	No

Table 1 Summary of the features included in previous spatial moment-dynamic models. Key: NH = non-homogeneous; DI = density-independent; DD = density-dependent. Our model includes density-independent and density-dependent movement, birth and death of multiple types of agents in a non-homogeneous setting.

4 Spatial-moment dynamics

Spatial moment models describe properties of the ensemble average of stochastic, spatially explicit, agent-based models, of the kind outlined in Sec. 2. This section derives the dynamics of the first and second moments from the stochastic process in Sec. 2, i.e. the expected density of agents at a given point in space, and the expected density of pairs of agents at two given points. The models do not give information on the size or nature of fluctuations around that ensemble average, and they cannot, for instance, be used to estimate the probability that a population will eventually go extinct.

The derivation is similar to those of Bolker and Pacala (1999), Raghib et al. (2011) and others by these groups in that it is based on the expected numbers of agents in small neighbourhoods. This contrasts to the master-equation approach, which describes the spatial point process as a sum of Dirac-delta functions (Dieckmann and Law, 2000; Murrell and Law, 2000). The derivation includes density-dependent movements and a non-homogeneous space, and is related to previous derivations as shown in Table 1. All these approaches can be used in non-homogeneous settings and lead to equivalent systems of equations.

The main differences between the derivation here and others in the literature are the standardisation of the notation for the n^{th} spatial moment as Z_n and the encoding of expected rate functions and transition probabilities separately from the moment dynamic equations. We also adopt a consistent symbol for interaction kernels (w) and for dispersal/movement kernels (μ). This makes the derivation significantly more parsimonious than that in Appendix A of Dieckmann and Law (2000). The transparency and the notational simplifications allow an extension to higher-order moments (see Sec. 6.3).

We avoid specifying a particular closure scheme for the system (see Sec. 5). This is advantageous as it allows the performance of different closures to be readily assessed (Murrell et al., 2004) and contrasts with some other approaches that incorporate a specific closure scheme into the derivation (e.g. Bolker and Pacala, 1997; Bolker, 2003).

Numerical integration of the equations for the first and second moments derived in this section would proceed in the same way as for other approaches (e.g. Bolker and Pacala, 1997; Dieckmann and Law, 2000). This is not a trivial task as the number of terms can be large and many terms require computation of an integral. Nevertheless, this has been accomplished in a variety of scenarios, including multi-type (Murrell, 2005), non-homogeneous (Lewis and Pacala, 2000) and size-structured models (Adams et al., 2013).

4.1 Rate functions for first-moment dynamics

In the agent-based model, an agent located at x has movement, birth and death rates defined by Eq. (2.1)–(2.3). The neighbour-dependent components of these rates were found by summing over all neighbours, weighted by the appropriate kernel function $w(x, y)$, where y is the location of the neighbour. The equivalent expression in the spatial-moment dynamics entails an integration over y of the probability of an agent being located at y conditional on the presence of the agent at x, weighted by $w(x, y)$. The expected movement rate $M_{1,i}(x)$ for an agent of type i located at x is therefore

$$\begin{aligned} M_{1,i}(x) &= m_i(x) + \sum_j \int w_{ij}^{(m)}(x,y) \frac{P(N_j(\delta y) = 1 \mid N_i(\delta x) = 1)}{h} dy \\ &= m_i(x) + \frac{1}{Z_{1,i}(x)} \sum_j \int w_{ij}^{(m)}(x,y)\ Z_{2,ij}(x,y) dy. \end{aligned} \tag{4.1}$$

Eq. (3.4) has been used here to convert the conditional probability into a conditional density of pairs. The expected birth and death rates for an agent of type i located at x have the same structure as Eq. (4.1)

$$B_{1,i}(x) = b_i(x) + \frac{1}{Z_{1,i}(x)} \sum_j \int w_{ij}^{(b)}(x,y) Z_{2,ij}(x,y) dy, \tag{4.2}$$

$$D_{1,i}(x) = d_i(x) + \frac{1}{Z_{1,i}(x)} \sum_j \int w_{ij}^{(d)}(x,y) Z_{2,ij}(x,y) dy. \tag{4.3}$$

The rates are functions of spatial moments, and are therefore functions of time, but we have omitted the time argument t for notational simplicity. We make no assumption that the process is stationary in time. The same applies to higher-order rate terms used in later sections.

4.2 Dynamics of the first moment

The rate of change of $Z_{1,i}(x)$ can be found from the change in the probability that the region δx contains an agent of type i over a short period of time δt. Since movement, birth and death events take place as independent Poisson processes, the probability of more than one event occurring during a short time interval of length δt is $O(\delta t^2)$.

The probability that there is an agent of type i in δx at time $t+\delta t$ can be found by conditioning on two cases: (a) that an agent was present at time t and is still present; (b) that an agent was absent at t and is now present. To write this concisely, we introduce some additional notation. Let $p_1(t)$ [respectively $p_0(t)$] be the probability that there is [respectively is not] an agent in δx at time t. Let $s_{1|1}$ [respectively $s_{1|0}$] be the probability that there is an agent at $t+\delta t$, given that there was [respectively was not] an agent at t. Then we have

$$p_1(t+\delta t) = s_{1|1} p_1(t) + s_{1|0} p_0(t). \tag{4.4}$$

The probabilities of an agent of type i being present, $p_1(t)$, or absent, $p_0(t)$, in δx at time t are related to the first moment via

$$p_1(t) = 1 - p_0(t) = Z_{1,i}(x,t)\ h + O(h^2). \tag{4.5}$$

The probability $s_{1|1}$ that an agent in δx remains in δx is the probability that the agent neither moves nor dies during $[t, t+\delta t]$:

$$s_{1|1} = 1 - \left(M_{1,i}(x) + D_{1,i}(x)\right)\delta t + O(\delta t^2). \tag{4.6}$$

(Here and below we omit the time argument of the functions.) An agent can arrive in δx as a result of either a movement or a birth event (always accompanied by dispersal). The probability $s_{1|0}$ that an agent arrives in δx is the probability that it arrives via a movement event, integrated over all possible starting locations u, plus the probability that it arrives via a birth event, integrated over all possible locations u of the parent:

$$s_{1|0} = h\ \delta t \int \left(\mu_i^{(m)}(u,x) M_{1,i}(u) + \mu_i^{(b)}(u,x) B_{1,i}(u)\right) Z_{1,i}(u) du + O(\delta t^2). \tag{4.7}$$

The rate functions $M_{1,i}(u)$ and $B_{1,i}(u)$ are per capita rates at location u, so the rates per unit area at location u are products of $M_{1,i}(u)$ and $B_{1,i}(u)$ with density $Z_{1,i}(u)$. Inserting Eqs. (4.5)–(4.7) into Eq. (4.4) and letting $h, \delta t \to 0$ gives

$$\begin{aligned} \frac{d}{dt} Z_{1,i}(x) &= -\left(M_{1,i}(x) + D_{1,i}(x)\right) Z_{1,i}(x) \\ &\quad + \int \left(\mu_i^{(m)}(u,x) M_{1,i}(u) + \mu_i^{(b)}(u,x) B_{1,i}(u)\right) Z_{1,i}(u) du. \end{aligned} \tag{4.8}$$

This equation describes the dynamics of the first moment of each type of agent at each location in space. It is a function of the second moment, as well as of the first moment, because the second moment is present in the per capita rates Eqs. (4.1)–(4.3). This means that the dynamics of the first moment are directly influenced by the spatial structure of the system. Eq. (4.8) is equivalent to the expected value of the first jump moment of the first spatial moment in Dieckmann and Law (2000).

4.3 Rate functions for second-moment dynamics

The expected rate of movement $M_{2,ij}(x,y)$ of an agent of type i in δx in a pair with an agent of type j in δy has a structure similar to Eq. (4.1) with intrinsic and neighbour-dependent components. The key difference is that, because the rate is conditional on the presence of the agent in δy, the neighbour-dependent component is a function of the conditional presence of a third agent, of type k in δz:

$$\begin{aligned}M_{2,ij}(x,y) &= m_i(x) + \sum_k \int w_{ik}^{(m)}(x,z) \frac{P(N_k(\delta z)=1 \mid N_i(\delta x)=1 \;\&\; N_j(\delta y)=1)}{h} dz \\ &\quad + w_{ij}^{(m)}(x,y) \\ &= m_i(x) + \frac{1}{Z_{2,ij}(x,y)} \sum_k \int w_{ik}^{(m)}(x,z) Z_{3,ijk}(x,y,z) dz + w_{ij}^{(m)}(x,y).\end{aligned} \tag{4.9}$$

Eq. (3.5) has been used here to convert the conditional probability into a conditional density of triplets. Because the definition of $Z_{3,ijk}(x,y,z)$ in Eq. (3.3) excludes triplets containing a self-pair, the integral term in Eq. (4.9) only measures the contribution of 'third-party' agents, distinct from the pair of agents at in δx and δy. Therefore, the effect $w_{ij}^{(m)}(x,y)$ of the agent in δy on the focal agent in δx must be added to Eq. (4.9) as a separate term (Adams et al., 2013). Using the same reasoning, the expected birth and death rates of an agent of type i in δx, in a pair with an agent of type j in δy are:

$$B_{2,ij}(x,y) = b_i(x) + \frac{1}{Z_{2,ij}(x,y)} \sum_k \int w_{ik}^{(b)}(x,z) Z_{3,ijk}(x,y,z) dz + w_{ij}^{(b)}(x,y), \tag{4.10}$$

$$D_{2,ij}(x,y) = d_i(x) + \frac{1}{Z_{2,ij}(x,y)} \sum_k \int w_{ik}^{(d)}(x,z) Z_{3,ijk}(x,y,z) dz + w_{ij}^{(d)}(x,y). \tag{4.11}$$

4.4 Dynamics of the second moment

The rate of change of $Z_{2,ij}(x,y)$ depends on the change in probability that there is an agent of type i in the region δx and an agent of type j in the region δy, over a short period of time δt. The rate terms for these changes are given by Eqs. (4.9)–(4.11).

Adopting notation similar to that used in Sec. 4.2, let $p_{qr}(t)$ be the probability that there are q agents of type i in δx and r agents of type j in δy at time t. Let $s_{11|qr}$ be the probability that there is 1 agent of type i in δx and 1 of type j in δy at time $t+\delta t$, given that there were q agents of type i in δx and r agents of type j in δy at time t ($q, r \in \{0,1\}$). Using the rules of conditional probability, the probability of finding a pair comprising i in δx and j in δy at time $t + \delta t$ is,

$$p_{11}(t + \delta t) = s_{11|11} p_{11}(t) + s_{11|01} p_{01}(t) + s_{11|10} p_{10}(t) + s_{11|00} p_{00}(t). \quad (4.12)$$

The probability $s_{11|00}$ is $O(\delta t^2)$ because it would involve the occurrence of two Poisson events (arrival of an agent in δx and of another agent in δy) during a time δt. The probability p_{11} of there being an agent present in δx and another agent present in δy is given by the second moment. The probabilities p_{10} and p_{01} of there being an agent present in one region and absent from the other are equal to the probability of there being an agent present in one region, minus the probability that agents are present in both regions:

$$\begin{aligned} p_{11}(t) &= Z_{2,ij}(x, y, t) h^2 + O(h^3), && (4.13)\\ p_{01}(t) &= Z_{1,j}(y, t) h - Z_{2,ij}(x, y, t) h^2 + O(h^3), && (4.14)\\ p_{10}(t) &= Z_{1,i}(x, t) h - Z_{2,ij}(x, y, t) h^2 + O(h^3). && (4.15) \end{aligned}$$

It is sufficient to retain only the order h terms in p_{10} and p_{01} because, as will be seen below, the associated transition probabilities $s_{11|10}$ and $s_{11|01}$ introduce an additional factor of h.

The transition probabilities $s_{11|qr}$ can be constructed in terms of the expected movement, birth and death rates in Eqs. (4.9)–(4.11). The probability $s_{11|11}$ is the probability that neither the agent in δx nor the agent in δy moves or dies. For brevity, only events involving δx are shown below; those for δy are obtained by switching indices i, j and arguments x, y. Therefore we have

$$s_{11|11} = 1 - \left(M_{2,ij}(x, y) + D_{2,ij}(x, y)\right) \delta t - \langle i, j, x, y \to j, i, y, x \rangle + O(\delta t^2), \quad (4.16)$$

where the term $\langle i, j, x, y \to j, i, y, x \rangle$ makes explicit the substitutions needed to incorporate the events in δy that are also taking place (Dieckmann and Law, 2000). Eq. (4.16) is comparable to Eq. (4.6) in the derivation of the first-moment dynamics.

The probability $s_{11|01}$ is the probability that an agent of type i arrives in δx, given that there is an agent of type j in δy. As in the dynamics of the first moment, this can occur via either a movement event or a birth event. For each class of event, the overall expected rate of arrival in δx is found by integrating over all possible locations u of the source of the event, as in Eq. (4.7). The main difference from Eq. (4.7) is that the probability of an agent being located at u is conditional on the presence of an agent at y. This conditional probability is expressed in terms of the second moment $Z_{2,ij}(u, y)$:

$$\begin{aligned} s_{11|01} = h\delta t \Bigg[&\frac{1}{Z_{1,j}(y)} \int \left(\mu_i^{(m)}(u, x) M_{2,ij}(u, y) + \mu_i^{(b)}(u, x) B_{2,ij}(u, y) \right) Z_{2,ij}(u, y) du \\ &+ \delta_{ij} \mu_j^{(b)}(y, x) B_{1,j}(y) \Bigg] + O(\delta t^2). \end{aligned} \quad (4.17)$$

An extra term has been included here to cover the case in which a pair is created by the agent at y giving birth to a new agent at x; the Kronecker delta δ_{ij} stipulates that this can only happen if the two agents are of the same type (i.e. $i = j$). No such term is needed for movement of the agent at y to x as this event would leave δy empty. A similar equation for the probability $s_{11|10}$ is obtained by the making the substitutions $\langle i, j, x, y \to j, i, y, x\rangle$ to (4.17).

Eqs. (4.13)–(4.17) are now substituted into Eq. (4.12), $Z_{2,ij}(x, y)h^2$ is subtracted from both sides, and the resulting equation divided by $h^2\delta t$. Taking the limit $h, \delta t \to 0$, the rate of change of the second moment $Z_{2,ij}(x, y)$ is

$$\begin{aligned}
\frac{d}{dt}Z_{2,ij}(x,y) = &-\left(M_{2,ij}(x,y) + D_{2,ij}(x,y)\right) Z_{2,ij}(x,y) \\
&+\int \left(\mu_i^{(m)}(u,x)M_{2,ij}(u,y) + \mu_i^{(b)}(u,x)B_{2,ij}(u,y)\right) Z_{2,ij}(u,y)du \\
&+\delta_{ij}\mu_j^{(b)}(y,x)B_{1,j}(y)Z_{1,j}(y) \\
&+\langle i, j, x, y \to j, i, y, x\rangle, \qquad (4.18)
\end{aligned}$$

where the term in angle brackets shows the substitutions needed in the previous terms to incorporate events to the second agent in the pair. The second moment has a symmetry $Z_{2,ij}(x, y) = Z_{2,ji}(y, x)$ that can be applied to simplify these additional terms. The similarity of this equation to that describing the first moment dynamics (4.8) is evident, including the feature that the dynamics contain a dependence on the moment of next order, now the density of triplets inside the rate equations (4.9) – (4.11). Eq. (4.18) is equivalent to the expected value of the first jump moment of the second spatial moment in Dieckmann and Law (2000).

4.5 Relation to spatially homogeneous dynamics

Many previous studies of spatial-moment dynamics by ecologists have investigated a spatially homogeneous problem (e.g. Bolker and Pacala, 1997, 1999; Law et al., 2003). Spatial homogeneity does not preclude spatial structure (i.e. departures from a spatial Poisson process): the agents can generate it themselves. Although agent density is spatially uniform on averaging over many independent realizations of the agent-based model, strong spatial correlations, such as the clusters in Fig. 1(b) and spacing in Fig. 1(c), can still be generated by the neighbour-dependent birth, death and movements, or by the correlation between the locations of parent and offspring.

The dynamics in Eqs. (4.8), (4.18) are referenced to locations in physical space x, y, z. We show here that the dynamics of earlier studies are recovered from Eq. (4.18) by making the following assumptions: (i) spatially homogeneous initial conditions; (ii) replacement of the intrinsic event rates $m_i(x)$, $b_i(x)$ and $d_i(x)$ by m_i, b_i, d_i independent of x; and (iii) replacement of the weighting kernels $w_i(x, y)$ and movement distributions $\mu_i(x, y)$ by functions $w_i(\xi), \mu_i(\xi)$ that depend on the displacement $\xi = y - x$ only. In such cases, the first moment $Z_{1,i}$ is independent of space; the second moment $Z_{2,ij}(\xi)$ depends only on the displacement vector ξ of agent j from i (Fig. 2); the third moment depends only on the two displacement vectors ξ and $\xi' = z - x$, and so on.

In the spatially homogeneous setting, the dynamics of the first moment are independent of location x:

$$\frac{dZ_{1,i}}{dt} = (B_{1,i} - D_{1,i})\, Z_{1,i}, \tag{4.19}$$

where

$$B_{1,i} = b_i + \frac{1}{Z_{1,i}} \sum_j \int w_{ij}^{(b)}(\xi) Z_{2,ij}(\xi) d\xi, \tag{4.20}$$

and $D_{1,i}$ is given by (4.20), with b replaced by d. (The property $\int \mu_i(u,x)dx = 1$ has been used in obtaining these dynamics.) The movement terms have cancelled out here because movement does not alter the total number of agents and therefore cannot affect the average agent density.

The dynamics of the second moment now depend only on the displacement vector ξ:

$$\begin{aligned}\frac{d}{dt} Z_{2,ij}(\xi) = &- (M_{2,ij}(\xi) + D_{2,ij}(\xi))\, Z_{2,ij}(\xi) \\ &+ \int \left(\mu_i^{(m)}(\xi') M_{2,ij}(\xi+\xi') + \mu_i^{(b)}(\xi') B_{2,ij}(\xi+\xi') \right) Z_{2,ij}(\xi+\xi') d\xi' \\ &+ \delta_{ij} \mu_j^{(b)}(-\xi) B_{1,j} Z_{1,j} \\ &+ \langle i, j, \xi \to j, i, -\xi \rangle, \end{aligned} \tag{4.21}$$

where

$$M_{2,ij}(\xi) = m_i + \frac{1}{Z_{2,ij}(\xi)} \sum_k \int w_{ik}^{(m)}(\xi') Z_{3,ijk}(\xi,\xi') d\xi' + w_{ij}^{(m)}(\xi) \tag{4.22}$$

and $B_{2,ij}(\xi)$ and $D_{2,ij}(\xi)$ are given by Eq. (4.22) with m replaced by b and d respectively.

5 Moment closure

Eq. (4.8) for the first-moment dynamics contains terms that depend on the second moment (4.1) – (4.3). Eq. (4.18) for the second-moment dynamics contains terms that depend on the third moment (4.9) – (4.11). In general, the dynamics of the n^{th} spatial moment depend on the $(n+1)^{\text{th}}$ moment. In other words, the dynamical system is not closed. The source of this dependence is the integral over the neighbourhood of a focal agent needed to evaluate the aggregate effect of its neighbours.

To obtain a closed system, it is necessary to employ some type of closure scheme to approximate the $(n+1)^{\text{th}}$ moment in terms of the lower-order moments. Closure at first order assumes that there is no spatial structure, i.e.

that there are no spatial correlations in the locations of agents. This is the mean-field assumption, which is widely used in the life sciences and, for the class of models considered in Sec. 2, replaces the second moment $Z_{2,ij}(x,y)$ by the product of first moments $Z_{1,i}(x)Z_{1,j}(y)$. Closure at second order retains some information about spatial correlations and requires an approximation for the third moment in terms of the second and first moments. Although there is no generally accepted way of deriving such a closure scheme (Ovaskainen et al., 2014), several closure approximations have been proposed (Kirkwood, 1935; Bolker and Pacala, 1997; Murrell et al., 2004) or derived using the principle of maximum entropy (Singer, 2004; Raghib et al., 2011). There is still much to learn about suitable closures, and this is a matter of current research beyond the scope of this paper. From a practical point of view, the performance of the closure can be assessed by comparing the results of the spatial-moment dynamics with the ensemble average of realizations of the stochastic, agent-based model. There is a class of closures — the asymmetric, power-2 closures — known to work well over a wide range of spatial structures (Murrell et al., 2004).

The performance of closure schemes for non-homogeneous systems, such as Eq. (4.8) and (4.18), has received relatively little attention. The extension of a particular closure to the non-homogeneous setting seems clear geometrically. For example, where, in a homogeneous system, $Z_{3,ijk}(\xi,\xi')$ is approximated in terms $Z_{2,ij}(\xi)$, $Z_{2,ik}(\xi')$ and $Z_{2,jk}(\xi'-\xi)$, in a non-homogeneous system $Z_{3,ijk}(x,y,z)$ could be approximated in terms of $Z_{2,ij}(x,y)$, $Z_{2,ik}(x,z)$ and $Z_{2,jk}(y,z)$ (see Fig. 2). This hypothesis needs to be tested by comparing the results of agent-based models to solutions of spatial moment dynamic equations.

It is important to understand that the dependence of dynamics of the n^{th} spatial moment on the $(n+1)^{\text{th}}$ moment comes from an assumption, that neighbours act additively on the target agent. This is an assumption about the biological system, and may not be applicable in all biological scenarios. For instance, a combination of several reagents and an enzyme, all local in space, might be needed to characterise a reaction rate within a cell. The birth rate of a plant might be a nonlinear function of the number of neighbours (Finkelshtein et al., 2013). The ability of a cell to move within a near-confluent monolayer will depend not only on the number of neighbouring cells, but also on their geometric configuration (Plank and Simpson, 2012; Bruna and Chapman, 2012b). The juxtaposition of several different agents is well known to ecologists, for instance in the need for a natural enemy to be present to achieve a mutualism involving protection of one partner and a home for the other (Bronstein et al., 2003). In such cases, the dynamics of the n^{th} moment may depend on moments of order $n+2$ and higher and the question of how to close the system becomes more difficult.

6 Extensions

We describe here some extensions of the basic model of spatial moment dynamics in Sec. 4.

6.1 Marked agents

The agents may have traits other than agent type; such traits are referred to as marks in the point process literature (Stoyan and Penttinen, 2000; Illian et al., 2008; Law et al., 2009). An agent's mark can change over time depending on the type and location of the agent itself and other agents in the neighbourhood.

Using the model of Adams et al. (2013) for the growth of a stand of plants, we show how the dynamics of agent-marks can be superimposed on the birth, death, movement dynamics of Eqs. (4.8), (4.18). In keeping with earlier sections, the argument is built on the physical location of agents, rather than on their displacements from one another, thereby removing the assumption of spatial homogeneity in Adams et al. (2013). A plant's growth rate depends on its species (type), its local physical environment and properties of neighbouring plants, including their species, sizes and locations.

To proceed, the agent-based model in Sec. 2 is modified so that, at a given time, the n^{th} agent is associated with a mark s_n, representing its size, as well as with a type i_n and location x_n. The mark can change via growth events (we assume the plant cannot shrink), assumed to occur in fixed increments δs as an inhomogeneous Poisson process over time with rate

$$\hat{G}_n = \frac{1}{\delta s}\left(g_{i_n}(x_n, s_n) + \sum_{l \neq n} w_{i_n i_l}^{(g)}(x_n, s_n, x_l, s_l) \right). \tag{6.1}$$

This rate consists of an intrinsic component g and a neighbour-dependent component. The function $w_{ij}^{(g)}(x, s, y, s')$ defines the contribution of an agent of type j and size s' located at y to the growth rate of an agent of type i and size s located at x. The factor of $1/\delta s$ ensures that the average growth rate is not affected by changing δs.

The first spatial moment $Z_{1,i}$ is now a function of location x and size s. If the size increment δs is small, s can be treated as a continuous variable and the growth process results in a convection term in the equations for the spatial-moment dynamics (Adams et al., 2013). The expected growth rate of an agent of type i and size s located at x is:

$$G_{1,i}(x, s) = g_i(x, s) + \frac{1}{Z_{1,i}(x, s)} \sum_j \iint w_{ij}^{(g)}(x, s, y, s') Z_{2,ij}(x, s, y, s') dy ds', \tag{6.2}$$

The rate of change of the first moment, $\partial/\partial t\,(Z_{1,i}(x, s))$, is given by Eq. (4.8), plus the growth term

$$-\frac{\partial}{\partial s}\left(G_{1,i}(x, s) Z_{1,i}(x, s)\right). \tag{6.3}$$

Similarly, the expected growth rate for an agent of type i and size s at x, in a pair with an agent of type j and size s' at y, is

$$G_{2,ij}(x,s,y,s') = g_i(x,s) + w_{ij}^{(g)}(x,s,y,s') + \frac{1}{Z_{2,ij}(x,s,y,s')} \times \sum_k \iint w_{ik}^{(g)}(x,s,z,s'')Z_{3,ijk}(x,s,y,s',z,s'')dzds''. \quad (6.4)$$

The rate of change of the second moment, $\partial/\partial t\,(Z_{2,ij}(x,s,y,s'))$, is given by Eq. (4.18) plus two convection terms, representing growth of the agents at x and y respectively:

$$-\frac{\partial}{\partial s}\left(G_{2,ij}(x,s,y,s')Z_{2,ij}(x,s,y,s')\right) - \frac{\partial}{\partial s'}\left(G_{2,ji}(y,s',x,s)Z_{2,ij}(x,s,y,s')\right). \quad (6.5)$$

The neighbour-dependent components of movement, birth and death rates may also be size-dependent. For example, a large neighbour may have a stronger effect than a smaller neighbour; a large agent may be less susceptible than a small agent to the effects of its neighbours. This type of effect can be included by allowing the interaction kernels $w^{(m,b,d)}$ to depend on the sizes of the agents in the pair, as in the function $w^{(g)}$ defined above. It would also be possible to allow marks to decrease as well as to increase, for example by modelling s_n as a biased random walk. This would result in a diffusive term, in addition to the convection term in Eq. (6.3) (Codling et al., 2008).

6.2 Agents that change type

Most existing models assume that an agent's type, denoted by indices i, j, k, is (a) fixed over the entire lifetime of the agent, and (b) faithfully inherited by its offspring (Bolker and Pacala, 1999; Dieckmann and Law, 2000; Murrell and Law, 2003). This is appropriate for some classifications of agents, such as species, but too restrictive in general.

Some models include special cases of agents switching types, for example infection of a susceptible agent in an epidemic model (Bolker, 1999; Brown and Bolker, 2004). However, other types of switching and mutation are possible (e.g. Champagnat et al., 2006). Agents classified, for instance, by cell type, phenotype or life stage could change type during their lives. Mutation events in cancer cells cause permanent change to the genotype of daughter cells. Mutation events would also have obvious relevance in an evolutionary model. In a stage-structured population, agents of one type (adults) would give birth to agents of another type (juveniles); juveniles would have to become adults before being able to reproduce. In all of these examples, the mutation or switching rates would, in general, be neighbour-dependent.

In this section, we show how assumptions (a) and (b) above can be relaxed to include mutations and switching. Champagnat et al. (2006) and Champagnat and Méléard (2007) derived spatial-moment equations for a model with mutations across a continuous trait space. Here, we consider a discrete set of agent types, indexed $i, j, \ldots$. We define $B_{1,il}(x)$ to be the expected rate at which an agent of type i located at x gives birth to an agent of type l (mutation):

$$B_{1,il}(x) = b_{il}(x) + \frac{1}{Z_{1,i}(x)} \sum_j \int w^{(b)}_{ijl}(x,y) Z_{2,ij}(x,y) dy. \tag{6.6}$$

Similarly, we define $C_{1,il}(x)$ to be the expected rate at which an agent of type i located at x switches to type $l \neq i$.

$$C_{1,il}(x) = c_{il}(x) + \frac{1}{Z_{1,i}(x)} \sum_j \int w^{(c)}_{ijl}(x,y) Z_{2,ij}(x,y) dy. \tag{6.7}$$

Each of these rates contains an intrinsic and a neighbour-dependent component. In the neighbour-dependent component, $w^{(b)}_{ijl}(x,y)$ is the contribution that an agent of type j located at y makes to the rate at which an agent of type i located at x gives birth to an agent of type l. A similar role is played by $w^{(c)}_{ijl}(x,y)$ in the switching rate. We assume that the dispersal kernel $\mu^{(b)}_i(x,y)$ depends only the type of the parent agent (i) and is independent of the type of the daughter agent (l). This model reduces to the fixed-species model on setting $C_{1,il}(x) = 0$, $B_{1,il}(x) = 0$ for $i \neq l$, and $B_{1,ii}(x)$ to be given by Eq. (4.2).

With these new types of event, the equation for the dynamics of first moment becomes

$$\begin{aligned} \frac{d}{dt} Z_{1,i}(x) = &- \left(M_{1,i}(x) + D_{1,i}(x) + \sum_l C_{1,il}(x) \right) Z_{1,i}(x) + \sum_l C_{1,li}(x) Z_{1,l}(x) \\ &+ \int \mu^{(m)}_i(u,x) M_{1,i}(u) Z_{1,i}(u) du + \sum_l \int \mu^{(b)}_l(u,x) B_{1,li}(u) Z_{1,l}(u) du. \end{aligned} \tag{6.8}$$

This is the same as Eq. (4.8) for the fixed-species model except that it contains an additional loss term for agents changing from type i to other types, an additional gain term for agents switching from other types to type i, and the birth term is summed over the possible types of the parent agent.

The expected rate of an agent of type i at x in a pair with an agent of type j at y giving birth to an agent of type l is

$$B_{2,ijl}(x,y) = b_{il}(x) + \frac{1}{Z_{2,ij}(x,y)} \sum_k \int w^{(b)}_{ikl}(x,z) Z_{3,ijk}(x,y,z) dz + w^{(b)}_{ijl}(x,y). \tag{6.9}$$

The corresponding expected switching rate $C_{2,ijl}(x,y)$ is given by a similar equation with b replaced by c. The equation for the dynamics of the second moment is

$$\begin{aligned}\frac{d}{dt}Z_{2,ij}(x,y) = &-\left(M_{2,ij}(x,y) + D_{2,ij}(x,y) + \sum_l C_{2,ijl}(x,y)\right) Z_{2,ij}(x,y)\\ &+\sum_l C_{2,lji}(x,y)Z_{2,lji}(x,y) + \mu_j^{(b)}(y,x)B_{1,ji}(y)Z_{1,j}(y)\\ &+\int \mu_i^{(m)}(u,x)M_{2,ij}(u,y)Z_{2,ij}(u,y)du\\ &+\sum_l \int \mu_l^{(b)}(u,x)B_{2,lji}(u,y)Z_{2,lj}(u,y)du\\ &+\langle i,j,x,y \rightarrow j,i,y,x\rangle. \qquad (6.10)\end{aligned}$$

6.3 Dynamics of higher-order moments

The structure of the moment equations (4.8), (4.18) is transparent enough to allow a continuation up the hierarchy of spatial moments, which would be harder using the approach of Dieckmann and Law (2000) for example. In this section, we sketch a derivation the dynamics for the third spatial moment, i.e. the density of triplets with the geometry shown in Fig. 2. By analogy, we then make a conjecture about the dynamics of the n^{th} spatial moment.

The expected rate of movement $M_{3,ijk}(x,y,z)$ of an agent of type i in δx in a triplet with type j in δy and type k in δz is given by:

$$\begin{aligned}M_{3,ijk}(x,y,z) = m_i(x) &+ \frac{1}{Z_{3,ijk}(x,y,z)} \sum_l \int w_{il}^{(m)}(x,u)Z_{4,ijkl}(x,y,z,u)du\\ &+w_{ij}^{(m)}(x,y) + w_{ik}^{(m)}(x,z), \qquad (6.11)\end{aligned}$$

where Z_4 is the fourth spatial moment, and can be defined as an extension of the sequence of moments in Sec. 3. The effect of neighbour agents in δy and δz on the focal agent in δx are added in as separate terms in (6.11). The corresponding expected birth rate $B_{3,ijk}(x,y,z)$ and expected death rate $D_{3,ijk}(x,y,z)$ are given by replacing m in (6.11) with b and d respectively.

Extending the notation in Sec. 4.4, the probability of finding a triplet comprising i in δx, j in δy and k in δz at time $t+\delta t$ is,

$$p_{111}(t+\delta t) = s_{111|111}p_{111}(t) + s_{111|011}p_{011}(t) + s_{111|101}p_{101}(t) + s_{111|110}p_{110}(t). \qquad (6.12)$$

We have omitted terms that would involve more than one Poisson event during the time period δt as the probability of such events is of order $O(\delta t^2)$. The term $s_{111|qrs}$ is the probability of a single agent in each of δx, δy, δz at time $t+\delta t$, given q in δx, r in δy and s in δz at time t ($q,r,s \in \{0,1\}$), the probability of each of these configurations at time t being $p_{qrs}(t)$. These configuration probabilities at t are known from the moments:

$$\begin{aligned} p_{111}(t) &= Z_{3,ijk}(x,y,z,t)h^3 + O(h^4), && (6.13)\\ p_{011}(t) &= Z_{2,jk}(y,z,t)h^2 + O(h^3), && (6.14)\\ p_{101}(t) &= Z_{2,ik}(x,z,t)h^2 + O(h^3) && (6.15)\\ p_{110}(t) &= Z_{2,ij}(x,y,t)h^2 + O(h^3). && (6.16) \end{aligned}$$

where only terms of order h^2 are retained in p_{011}, p_{101}, p_{110} because an extra factor of h is introduced by the associated transition probabilities.

The probabilities for the states at $t + \delta t$ conditional on the states at t depend on the rate terms M_3, B_3, D_3 (see Eq. (6.11)). The term $s_{111|111}$ is the probability that there is no death or movement of an agent from δx, δy or δz from t to $t + \delta t$:

$$\begin{aligned} s_{111|111} = 1 &- \left(M_{3,ijk}(x,y,z) + D_{3,ijk}(x,y,z)\right)\delta t\\ &-\langle i,j,k,x,y,z \to j,i,k,y,x,z\rangle\\ &-\langle i,j,k,x,y,z \to k,i,j,z,x,y\rangle + O(\delta t^2). \qquad (6.17) \end{aligned}$$

The first part of the right-hand side deals with events in δx and the angle brackets show the changes in indices and arguments needed for events in δy and δz. We have made use of a symmetry in the expected movement rate: $M_{3,ijk}(x,y,z) = M_{3,ikj}(x,z,y)$. The conditional probability for entry into δx is

$$\begin{aligned} s_{111|011} = h\delta t \Bigg[&\frac{1}{Z_{2,jk}(y,z)} \int \left(\mu_i^{(m)}(u,x)M_{3,ijk}(u,y,z) + \mu_i^{(b)}(u,x)B_{3,ijk}(u,y,z)\right)\\ &\times Z_{3,ijk}(u,y,z)du + \delta_{ij}\mu_j^{(b)}(y,x)B_{2,jk}(y,z) + \delta_{ik}\mu_k^{(b)}(z,x)B_{2,kj}(z,y)\Bigg]\\ &+O(\delta t^2), \qquad (6.18) \end{aligned}$$

Corresponding equations for $s_{111|101}$ and $s_{111|110}$) (i.e. entry into δy and δz respectively) are obtained by making the same interchanges of indices and arguments as in Eq. (6.17). The final step substitutes Eqs. (6.13) –(6.18) into (6.12), subtracts $Z_{3,ijk}(x,y,z)$ from both sides, divides through by $h^3\delta t$, and takes the limit as $h^3\delta t \to 0$, giving

$$\begin{aligned} \frac{d}{dt}\; Z_{3,ijk}(x,y,z) = &-\left(M_{3,ijk}(x,y,z) + D_{3,ijk}(x,y,z)\right) Z_{3,ijk}(x,y,z)\\ &+ \int \left(\mu_i^{(m)}(u,x)M_{3,ijk}(u,y,z) + \mu_i^{(b)}(u,x)B_{3,ijk}(u,y,z)\right) Z_{3,ijk}(u,y,z)du\\ &+ \left(\delta_{ij}\mu_j^{(b)}(y,x)B_{2,jk}(y,z) + \delta_{ik}\mu_k^{(b)}(z,x)B_{2,kj}(z,y)\right) Z_{2,jk}(y,z)\\ &+\langle i,j,k,x,y,z \to j,i,k,y,x,z\rangle + \langle i,j,k,x,y,z \to k,i,j,z,x,y\rangle. \qquad (6.19) \end{aligned}$$

Thus the dynamics of the third moment are a straightforward extension of those of the second moment in Eq. (4.18). They contain a dependence on the moment of next order, which is the density of quadruplets, in the rate equations M_3, B_3, D_3 (see Eq. 6.11), together with two extra terms at x caused by births from parents at y and z, and with all events repeated at y and z, as indicated by the substitutions in angle brackets.

The sequence of terms in equations for the dynamics of the first, second and third moments, (4.8), (4.18), (6.19), is clear. For a configuration of points (singleton, pair or triplet), there are terms for: (i) loss of the agent at x due to movement or death; (ii) arrival of an agent at x due to movement or reproduction of an agent not in the configuration. For configurations other than singletons, there are two further terms: (iii) arrival of an agent at x due to reproduction by one of the other agents in the configuration; (iv) symmetric sets of terms for the same classes of events occurring at each other node of the configuration.

This common structure allows a conjecture about the dynamics of the n^{th} moment. An n-tuplet of agents is described by the vector $\mathbf{x} = (x_1, \ldots, x_n)$ of agent locations and the vector $\mathbf{i} = (i_1, \ldots, i_n)$ of agent types. The expected rate of movement $M_{n,\mathbf{i}}(\mathbf{x})$ of the agent of type i_1 in δx_1 in this n-tuplet is:

$$\begin{aligned} M_{n,\mathbf{i}}(\mathbf{x}) = m_{i_1}(x_1) &+ \frac{1}{Z_{n,\mathbf{i}}(\mathbf{x})} \sum_l \int w^{(m)}_{i_1 l}(x_1, u) Z_{n+1,\mathbf{i}l}(\mathbf{x}, u) du \\ &+ \sum_{k=2}^{n} w^{(m)}_{i_1 i_k}(x_1, x_k), \end{aligned} \tag{6.20}$$

with similar expressions for the expected birth and death rates obtained by replacing m by b and d respectively. This leads to the following equation for the rate of change of the n^{th} moment:

$$\begin{aligned} \frac{d}{dt}\; Z_{n,\mathbf{i}}(\mathbf{x}) &= -\left(M_{n,\mathbf{i}}(\mathbf{x}) + D_{n,\mathbf{i}}(\mathbf{x})\right) Z_{n,\mathbf{i}}(\mathbf{x}) \\ &+ \int \left(\mu^{(m)}_{i_1}(u, x_1) M_{n,\mathbf{i}}(u, \mathbf{x}_{2\ldots n}) + \mu^{(b)}_{i}(u, x_1) B_{n,\mathbf{i}}(u, \mathbf{x}_{2\ldots n})\right) Z_{n,\mathbf{i}}(u, \mathbf{x}_{2\ldots n}) du \\ &+ \sum_{k=2}^{n} \delta_{i_1 i_k} \mu^{(b)}_{i_k}(x_k, x_1) B_{n-1,\mathbf{i}_{k2\ldots n}}(\mathbf{x}_{k2\ldots n}) Z_{n-1,\mathbf{i}_{2\ldots n}}(\mathbf{x}_{2\ldots n}) \\ &+ \sum_{k=2}^{n} \langle \mathbf{i}, \mathbf{x} \to \mathbf{i}_{k1\ldots n}, \mathbf{x}_{k1\ldots n} \rangle, \end{aligned} \tag{6.21}$$

where we have used the shorthand $\mathbf{x}_{2\ldots n} = (x_2, x_3, \ldots, x_n)$,
$\mathbf{x}_{k1\ldots n} = (x_k, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ and
$\mathbf{x}_{k2\ldots n} = (x_k, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. This equation contains the same four types of term as in the second- and third-order dynamics, appropriately modified for order n:

(i) The movement and death rates of an agent at x_1 in an n-tuplet multiplied by the density of such n-tuplets (i.e. the n^{th} moment).

(ii) The arrival of an agent at x_1 due to movement or reproduction of an agent not in the n-tuplet. This is given by the movement and birth rates of an agent at u in an n-tuplet with other agents at $x_2, \ldots x_n$, multiplied by the density of this n-tuplet, multiplied by the probability of movement/dispersal from u to x_1, integrated over all possible locations u.

(iii) The arrival of an agent at x_1 due to reproduction by one of the other agents in the n-tuplet. This is given by the density of an $(n-1)$-tuplet of agents at $x_2, \ldots, x_n$, multiplied by the birth rate of the agent at x_k in this $(n-1)$-tuplet, multiplied by the probability of dispersal from x_k to x_1, summed over all agents in the $(n-1)$-tuplet $k = 2, \ldots, n$.

(iv) Symmetric terms for the loss/arrival of an agent at x_k obtained by interchanging x_1 and x_k (and i_1 and i_k) and summing over $k = 2, \ldots, n$.

7 Discussion

Simulations of stochastic, agent-based models are now widely used in the life sciences and social sciences (Niazi and Hussain, 2011), and are perceived as a key route to understanding complex processes where agents interact with neighbours (Grimm et al., 2006). Although such simulations can give hints about the causes of emerging patterns, clear-cut answers usually entail going to the underlying mathematics.

The use of spatial-moment dynamics is one of several ways of charting the ground between spatial, agent-based models and mathematical analysis. Reaction–diffusion equations have been used for many years in various branches of the life sciences (Murray, 1989; Shigesada and Kawasaki, 1997), allowing the first moment to be a function of space. However, the assumption of local mean-field dynamics in the reaction terms of these partial differential equations means that they do not deal with small-scale spatial structure. This is sometimes referred to as the hydrodynamic limit, corresponding to an assumption that dispersal occurs on a much faster timescale than population dynamics (Cantrell and Cosner, 2004). A classical example of a local mean-field model is the Fisher–Kolmogorov equation (Fisher, 1937; Kolmogorov et al., 1937) for a population undergoing motility and logistic growth; this model ignores correlations between agent locations that can affect the dynamics (Simpson and Baker, 2011). Other examples may be found in reaction–diffusion models in ecology (Okubo et al., 1989), cell biology (Murray, 1989) and epidemiology (Noble, 1974). Some models incorporate a spatially distributed (i.e. non-local) reaction process but still ignore pairwise correlations (Medlock and Kot, 2003). Spatial-moment dynamics in non-homogeneous settings allow large- and small-scale spatial structure to be combined (Lewis and Pacala, 2000) and deserve more attention.

To an ecologist, the use of spatial-moment dynamics has the advantage that the second spatial moment, often expressed as a pair correlation function (Illian et al., 2008), is a core measure of spatial structure in plant communities

(Law et al., 2009). However, from a mathematical perspective, the method of low-order spatial moments does not have priority over other methods of incorporating spatial structure into dynamics. The method has the drawback that a closure of the hierarchy of moment equations is needed, just as a closure has been used ubiquitously to avoid dealing with spatial structure altogether (the mean-field assumption). Closures at second order can give acceptable approximations to the ensemble average of stochastic processes over a wide range of spatial structures (Murrell et al., 2004), but our understanding of them is still limited and they are a matter of ongoing mathematical research (Singer, 2004; Raghib et al., 2011).

Previous mathematical work on spatial point processes has focused primarily on homogeneous spaces, in which the expected density (and higher-order moments) are independent of physical location and the spatial structure comes from spatial covariances between agents (e.g. Bolker and Pacala, 1997; Dieckmann and Law, 2000; Adams et al., 2013). Non-homogeneous processes are important in several areas of biology, for example ecological invasions (Shigesada and Kawasaki, 1997), *in vitro* cell invasion assays (Simpson et al., 2013), embryogenesis (Young et al., 2004) and wound healing (Khain et al., 2007). All these processes involve colonisation of a region by a population of agents that is initially spatially confined. Lewis and Pacala (2000) and Lewis (2000) modelled ecological invasions, although their results are restricted to a birth/dispersal process with short-range interactions and without movement or density-dependent death. Murrell and Law (2000) modelled beetle movement in a heterogeneous environment, assumed to be fixed. Outside these special cases, relatively little is known about spatial moment dynamics for non-homogeneous systems.

Processes of interest in the life sciences typically operate in continuous space. However there are circumstances in which the discretisation of space can be helpful, and spatial-moment dynamics have their counterpart in discrete-space, lattice models (Matsuda et al., 1992). For instance, computations are more straightforward on lattices. Also, lattice models can have crowding effects built in through exclusion of more than one individual from a lattice cell (Liggett, 1999; Simpson et al., 2007). Exclusion models often use the mean-field assumption in deriving a continuum limit (Deroulers et al., 2009; Simpson et al., 2009, 2010), on the basis that, for instance, unbiased, random movements overwhelm spatial effects of births and deaths. Otherwise, the continuum limit needs to keep track of second-order spatial correlations, at least. The pair approximation has been used for this purpose, for instance in lattice-based models of Lotka–Volterra and logistic dynamics (Matsuda et al., 1992; Ellner, 2001). Related second-order closures also have been applied to network models of epidemics where ‘space’ becomes a non-trivial lattice topology (Keeling et al., 1997; Keeling, 1999; Van Baalen, 2000; Kiss et al., 2005), and extensions to higher-order correlation structures have been made (Petermann and De Los Rios, 2004). Kirkwood’s superposition approximation has been used as a second-order closure for cell proliferation (Baker and Simpson, 2010), and has been extended to biased movement in a non-homogeneous space (Simpson and Baker, 2011) and to cell adhesion (Johnston et al., 2012).

A more rigorous way forward is a formal analysis of the stochastic process on which the agent-based model simulations are based. For instance, Blath et al. (2007) investigated a stochastic, lattice model of two competing species by means of stochastic differential equations, to see if coexistence could be achieved purely by spatial properties of competition, as had been previously been observed in a numerical study of a lattice-free moment model Murrell and Law (2003). Among other things, this work illustrates the importance of parsimony in constructing agent-based models. Blath et al. (2007) made a conjecture on coexistence (Conjecture 2.5), but the full stochastic competition model of Murrell and Law (2003) was too complicated to admit a rigorous proof.

Another way forward is to approximate the stochastic process via a perturbation method, using a small parameter $\epsilon \ll 1$ to characterise the system. For example, Bruna and Chapman (2012b) examined the dynamics of finite-sized, non-overlapping particles undergoing Brownian motion. Taking the occupied-volume fraction as a small parameter ϵ, they used matched asymptotic expansions in ϵ to derive a nonlinear diffusion equation, and found that the diffusion coefficient for collective movement of the population was an increasing function of ϵ. Bruna and Chapman (2012a) extended the model to deal with multiple species, each with its own diffusivity, and Bruna and Chapman (2013) considered the case where the particles are moving in a severely confined domain (e.g. a narrow channel whose width is comparable to the diameter of the particles). This approach has the advantage that it can capture exactly the steric interactions of finite-sized particles undergoing Brownian motion, without the need for a closure assumption. However, it can only handle short-range interactions (collisions), is limited to low-density situations in which the occupied volume fraction is small ($\epsilon \ll 1$), and deals only with movement of agents (not proliferation or death).

At a given time, the set of spatial moments of all orders gives an exact description of the ensemble average of the stochastic process (Finkelshtein et al., 2009). Thus, in principle, the time evolution of the ensemble average is known exactly from the dynamics of the set of all spatial moments. A perturbation expansion around the spatial mean-field model (Ovaskainen and Cornell, 2006; North and Ovaskainen, 2007; Cornell and Ovaskainen, 2008) can be put on a rigorous mathematical basis using techniques from Markov evolutions (Kondratiev and Kuna, 2002; Finkelshtein et al., 2009, 2012), allowing a closed system of equations for moments of all orders to be derived (Ovaskainen et al., 2014). The perturbation method rescales the kernels defining the spatial range of pairwise interactions by a parameter ϵ. As $\epsilon \to 0$, the kernels become increasingly flat and long-ranged, corresponding to the spatial mean-field case. This allows an $O(\epsilon^d)$ (d is the number of spatial dimensions) correction to the mean-field solution to be obtained, without the need for a closure assumption. Working with two-point configurations may lead to loss of accuracy when ϵ is relatively large, i.e. when there are strong, short-range interactions among agents. Ovaskainen et al. (2014) conjectured that accuracy may be improved by moving to three-point configurations and including an $O(\epsilon^{2d})$ term. Results are currently available for a population of unmarked agents of a single type, but could be extended to include marks and multiple types.

In summary, spatial-moment dynamics and related techniques are helpful in giving insight into seemingly difficult problems in which behaviour of agents is determined by processes that are local in space. Such systems are not well characterised by a mean-field assumption based on spatially averaged densities. Such problems crop up repeatedly in the life sciences, because processes often take place locally in spaces where agents are not well mixed, and we anticipate that the techniques have many applications outside the subject area of ecology for which they were originally developed. The tools are flexible, and can be extended to deal with problems in which the environment is non-homogeneous, and to problems of invasion and retreat of agents in which spatial structures are not stationary over space. The notion of space itself needs no more than a measure of distance between neighbours and the application to other spaces such as those in networks could also be considered.

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References

T. Adams, E.P. Holland, R. Law, M.J. Plank, M. Raghib, On the growth of locally interacting plants: differential equations for the dynamics of spatial moments. Ecology **94**, 2732–2743 (2013)

R.E. Baker, M.J. Simpson, Correcting mean-field approximations for birth-death-movement processes. Physical Review E **82**(4), 041905 (2010)

F. Barraquand, D.J. Murrell, Intense or spatially heterogeneous predation can select against prey dispersal. PLoS ONE **7**, 28924 (2012)

F. Barraquand, D.J. Murrell, Scaling up predator–prey dynamics using spatial moment equations. Methods in Ecology and Evolution **4**, 276–289 (2013)

F. Barraquand, D.J. Murrell, Evolutionarily stable consumer home range size in relation to resource demography and consumer spatial organization. Theoretical Ecology **5**(4), 567–589 (2012)

J. Blath, A. Etheridge, M. Meredith, Coexistence in locally regulated competing populations and survival of branching annihilating random walk. Annals of Applied Probability, 1474–1507 (2007)

B. Bolker, S.W. Pacala, Using moment equations to understand stochastically driven spatial pattern formation in ecological systems. Theoretical Population Biology **52**(3), 179–197 (1997)

B.M. Bolker, Analytic models for the patchy spread of plant disease. Bulletin of Mathematical Biology **61**(5), 849–874 (1999)

B.M. Bolker, Combining endogenous and exogenous spatial variability in analytical population models. Theoretical Population Biology **64**(3), 255–270 (2003)

B.M. Bolker, S.W. Pacala, Spatial moment equations for plant competition: understanding spatial strategies and the advantages of short dispersal. American Naturalist **153**(6), 575–602 (1999)

B.M. Bolker, S.W. Pacala, C. Neuhauser, Spatial dynamics in model plant communities: what do we really know? American Naturalist **162**, 135–148 (2003)

J.L. Bronstein, W.G. Wilson, W.F. Morris, Ecological dynamics of mutualist/antagonist communities. American Naturalist **162**, 24–39 (2003)

D.H. Brown, B.M. Bolker, The effects of disease dispersal and host clustering on the epidemic threshold in plants. Bulletin of Mathematical Biology **66**(2), 341–371 (2004)

M. Bruna, S.J. Chapman, Diffusion of multiple species with excluded-volume effects. Journal of Chemical Physics **137**(20), 204116 (2012a)

M. Bruna, S.J. Chapman, Excluded-volume effects in the diffusion of hard spheres. Physical Review E **85**(1), 011103 (2012b)

M. Bruna, S.J. Chapman, Diffusion of finite-size particles in confined geometries. Bulletin of Mathematical Biology, 1–36 (2013)

R.S. Cantrell, C. Cosner, Deriving reaction–diffusion models in ecology from interacting particle systems. Journal of Mathematical Biology **48**(2), 187–217 (2004)

N. Champagnat, S. Méléard, Invasion and adaptive evolution for individual-based spatially structured populations. Journal of Mathematical Biology **55**(2), 147–188 (2007)

N. Champagnat, R. Ferrière, S. Méléard, Unifying evolutionary dynamics: from individual stochastic processes to macroscopic models. Theoretical Population Biology **69**(3), 297–321 (2006)

E.A. Codling, M.J. Plank, S. Benhamou, Random walk models in biology. Journal of the Royal Society Interface **5**(25), 813–834 (2008)

S.J. Cornell, O. Ovaskainen, Exact asymptotic analysis for metapopulation dynamics on correlated dynamic landscapes. Theoretical Population Biology **74**, 209–225 (2008)

C. Deroulers, M. Aubert, M. Badoual, B. Grammaticos, Modeling tumor cell migration: From microscopic to macroscopic models. Physical Review E **79**(3), 031917 (2009)

U. Dieckmann, R. Law, Relaxation projections and the method of moments, in *The Geometry of ecological interactions: simplifying spatial complexity*, ed. by U. Dieckmann, R. Law, J.A.J. Metz (Cambridge University Press, Cambridge, UK, 2000), pp. 412–455

S.P. Ellner, Pair approximation for lattice models with multiple interaction scales. Journal of Theoretical Biology **210**, 435–447 (2001)

A.E. Fernando, K.A. Landman, M.J. Simpson, Nonlinear diffusion and exclusion processes with contact interactions. Physical Review E **81**(1), 011903 (2010)

D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Individual based model with competition in spatial ecology. SIAM Journal on Mathematical Analysis **41**(1), 297–317 (2009)

D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Semigroup approach to birth-and-death stochastic dynamics in continuum. Journal of Functional Analysis **262**(3), 1274–1308 (2012)

D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Establishment and fecundity in spatial ecological models: statistical approach and kinetic equations. Infinite Dimensional Analysis, Quantum Probability and Related Topics **16**(02) (2013)

D.L. Finkelshtein, Y.G. Kondratiev, M.J. Oliveira, Markov evolutions and hierarchical equations in the continuum. I: one-component systems. Journal of Evolution Equations **9**(2), 197–233 (2009)

R.A. Fisher, The wave of advance of advantageous genes. Annals of Eugenics **7**(4), 355–369 (1937)

D. Grey, 2000, personal communication

V. Grimm, U. Berger, F. Bastiansen, S. Eliassen, V. Ginot, J. Giske, J. Goss-Custard, T. Grand, S.K. Heinz, G. Huse, A. Huth, J.U. Jepsen, C. Jørgensen, W.M. Mooij, B. Müller, G. Pe'er, C. Piou, S.F. Railsback, A.M. Robbins, M.M. Robbins, E. Rossmanith, N. Rüger, E. Strand, S. Souissi, R.A. Stillman, R. Vabø, U. Visser, D.L. DeAngelis, A standard protocol for describing individual-based and agent-based models. Ecological Modelling **198**(1), 115–126 (2006)

J. Illian, A. Penttinen, H. Stoyan, D. Stoyan, *Statistical analysis and modelling of spatial point patterns* (John Wiley & Sons, Chichester, 2008)

S.T. Johnston, M.J. Simpson, R.E. Baker, Mean-field descriptions of collective migration with strong adhesion. Physical Review E **85**(5), 051922 (2012)

M.J. Keeling, D.A. Rand, A.J. Morris, Correlation models for childhood epidemics. Proceedings of the Royal Society of London B **264**(1385), 1149–1156 (1997)

M.J. Keeling, Multiplicative moments and measures of persistence in ecology. Journal of Theoretical Biology **205**(2), 269–281 (2000)

M.J. Keeling, The effects of local spatial structure on epidemiological invasions. Proceedings of the Royal Society of London B **266**(1421), 859–867 (1999)

W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London A **115**(772), 700–721 (1927)

E. Khain, L.M. Sander, C.M. Schneider-Mizell, The role of cell-cell adhesion in wound healing. Journal of Statistical Physics **128**(1-2), 209–218 (2007)

J.G. Kirkwood, Statistical mechanics of fluid mixtures. Journal of Chemical Physics **3**, 300–313 (1935)

I.Z. Kiss, D.M. Green, R.R. Kao, Disease contact tracing in random and clustered networks. Proceedings of the Royal Society of London B **272**(1570), 1407–1414 (2005)

A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. Moscow University Bulletin of Mathematics **1**, 1–25 (1937)

Y.G. Kondratiev, T. Kuna, Harmonic analysis on configuration space I: General theory. Infinite Dimensional Analysis, Quantum Probability and Related Topics **5**(02), 201–233 (2002)

R. Law, U. Dieckmann, A dynamical system for neighborhoods in plant communities. Ecology **81**, 2137–2148 (2000)

R. Law, D.J. Murrell, U. Dieckmann, Population growth in space and time: spatial logistic equations. Ecology **84**(1), 252–262 (2003)

R. Law, J. Illian, D.F.R.P. Burslem, G. Gratzer, C.V.S. Gunatilleke, I.A.U.N. Gunatilleke, Ecological information from spatial patterns of plants: insights from point process theory. Journal of Ecology **97**(4), 616–628 (2009)

M.A. Lewis, Spread rate for a nonlinear stochastic invasion. Journal of Mathematical Biology **41**(5), 430–454 (2000)

M.A. Lewis, S. Pacala, Modeling and analysis of stochastic invasion processes. Journal of Mathematical Biology **41**(5), 387–429 (2000)

T.M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes* (Springer, Berlin, 1999)

L.D. Llambi, R. Law, A. Hodge, Temporal changes in local spatial structure of late-successional species: establishment of an Andean caulescent rosette plant. Journal of Ecology **92**, 122–131 (2004)

A.J. Lotka, Undamped oscillations derives from the law of mass action. Journal of the American Chemical Society **42**, 1595–1599 (1920)

H. Matsuda, N. Ogita, A. Sasaki, K. Sato, Statistical mechanics of population – the lattice Lotka-Volterra model. Progress of Theoretical Physics **88**, 1035–1049 (1992)

J. Medlock, M. Kot, Spreading disease: integro-differential equations old and new. Mathematical Biosciences **184**(2), 201–222 (2003)

J.D. Murray, *Mathematical biology* (Springer, New York, 1989)

D.J. Murrell, Local spatial structure and predator-prey dynamics: counterintuitive effects of prey enrichment. The American Naturalist **166**, 354–367 (2005)

D.J. Murrell, On the emergent spatial structure of size-structured populations: when does self-thinning lead to a reduction in clustering? Journal of Ecology **97**, 256–266 (2009)

D.J. Murrell, R. Law, Beetles in fragmented woodlands: a formal framework for dynamics of movement in ecological landscapes. Journal of Animal Ecology **69**(3), 471–483 (2000)

D.J. Murrell, R. Law, Heteromyopia and the spatial coexistence of similar competitors. Ecology Letters **6**(1), 48–59 (2003)

D.J. Murrell, U. Dieckmann, R. Law, On moment closures for population dynamics in continuous space. Journal of Theoretical Biology **229**(3), 421–432 (2004)

M. Niazi, A. Hussain, Agent-based computing from multi-agent systems to agent-based models: a visual survey. Scientometrics **89**, 479–499 (2011)

J.V. Noble, Geographic and temporal development of plagues. Nature **250**, 726–729 (1974)

A. North, O. Ovaskainen, Interactions between dispersal, competition, and landscape heterogeneity. Oikos **116**, 1106–1119 (2007)

A. Okubo, P.K. Maini, M.H. Williamson, J.D. Murray, On the spatial spread of the grey squirrel in britain. Proceedings of the Royal Society of London B **238**(1291), 113–125 (1989)

O. Ovaskainen, S.J. Cornell, Asymptotically exact analysis of stochastic metapopulation dynamics with explicit spatial structure. Theoretical Population Biology **69**(1), 13–33 (2006)

O. Ovaskainen, D. Finkelshtein, O. Kutoviy, S. Cornell, B. Bolker, Y. Kondratiev, A general mathematical framework for the analysis of spatiotemporal point processes. Theoretical Ecology **7**, 101–113 (2014)

T. Petermann, P. De Los Rios, Cluster approximations for epidemic processes: a systematic description of correlations beyond the pair level. Theoretical Ecology **229**, 1–11 (2004)

M.J. Plank, M.J. Simpson, Models of collective cell behaviour with crowding effects: comparing lattice-based and lattice-free approaches. Journal of the Royal Society Interface **9**(76), 2983–2996 (2012)

D.W. Purves, R. Law, Fine-scale spatial structure in a grassland community: quantifying the plants-eye view. Journal of Ecology **90**, 121–129 (2002)

M. Raghib, N.A. Hill, U. Dieckmann, A multiscale maximum entropy moment closure for locally regulated space–time point process models of population dynamics. Journal of Mathematical Biology **62**(5), 605–653 (2011)

N. Shigesada, K. Kawasaki, *Biological invasions: theory and practice* (Oxford University Press, Oxford, 1997)

M.J. Simpson, R.E. Baker, Corrected mean-field models for spatially dependent advection-diffusion-reaction phenomena. Physical Review E **83**(5), 051922 (2011)

M.J. Simpson, K.A. Landman, B.D. Hughes, Multi-species simple exclusion processes. Physica A **388**(4), 399–406 (2009)

M.J. Simpson, K.A. Landman, B.D. Hughes, Cell invasion with proliferation mechanisms motivated by time-lapse data. Physica A **389**(18), 3779–3790 (2010)

M.J. Simpson, A. Merrifield, K.A. Landman, B.D. Hughes, Simulating invasion with cellular automata: Connecting cell-scale and population-scale properties. Physical Review E **76**(2), 021918 (2007)

M.J. Simpson, K.K. Treloar, B.J. Binder, P. Haridas, K.J. Manton, D.I. Leavesley, D.L.S. McElwain, R.E. Baker, Quantifying the roles of cell motility and cell proliferation in a circular barrier assay. Journal of the Royal Society Interface **10**(82) (2013)

A. Singer, Maximum entropy formulation of the Kirkwood superposition approximation. Journal of Chemical Physics **121**, 3657–3666 (2004)

D. Stoyan, A. Penttinen, Recent applications of point process methods in forestry statistics. Statistical Science **15**(1), 61–78 (2000)

M. Van Baalen, Pair approximations for different spatial geometries, in *The Geometry of ecological interactions: simplifying spatial complexity*, ed. by U. Dieckmann, R. Law, J.A.J. Metz (Cambridge University Press, Cambridge, UK, 2000), pp. 359–387

P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement. Correspondence Mathematique et Physique **10**, 113–121 (1836)

V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi* 1927

H.M. Young, A.J. Bergner, R.B. Anderson, H. Enomoto, J. Milbrandt, D.F. Newgreen, P.M. Whitington, Dynamics of neural crest-derived cell migration in the embryonic mouse gut. Developmental Biology **270**(2), 455–473 (2004)