

RESEARCH REPORT

ISSN 1172-8531

(PSL)
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no. 2000/8

UNIFORM DIAGONALISATION
OF MATRICES OVER
REGULAR RINGS

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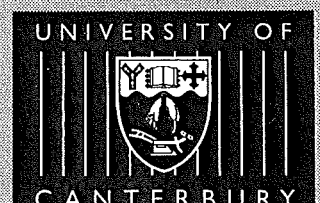
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Uniform Diagonalisation of Matrices over Regular Rings

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Abstract

The fundamental Separativity Problem for von Neumann regular rings is shown to be equivalent to a linear algebra problem: for a field F , is there a “uniform formula”

$$PAQ = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for diagonalising a 2×2 matrix A over $M_n(F)$, independently of n ? Here P and Q are required to be invertible matrices whose entries are fixed regular algebra expressions in the entries of A .

1991 Mathematics Subject Classification. Primary: 16E50,16S50,15A33. Secondary: 16S10,15A09,15A24.

1 Introduction

The concept of separativity has nicely unified approaches to various cancellation problems (relative to direct sum) in a number of different areas e.g. regular rings, exchange rings, C^* -algebras, noetherian rings, torsion-free abelian groups of finite rank ([AGOP1, AGOP2, AGOR, B, OV, Pa, Pe]). We recall that a (von Neumann) regular ring R is *separative* if for all finitely generated projective R -modules

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B,$$

equivalently, multiple-isomorphism ($A^n \cong B^n$ for all $n > 1$) implies isomorphism. The fundamental Separativity Problem [AGOP1] for regular rings asks if such rings are always separative. (Its solution has major ramifications - see [AGOP1].) The theme of the present paper is that the Separativity Problem for regular rings comes right back to linear algebra. For background material on von Neumann regular rings, the reader may consult Goodearl's standard text [VNRR]. We also recommend the excellent surveys by Goodearl [G] and Lam [L] for background on various cancellation problems (including separativity). Our rings have a multiplicative identity.

Since a regular ring is separative if all its indecomposable factor rings are separative, we see that all regular rings are separative if all regular algebras over fields are separative. Let F be a field. By [AGOP2, Theorem 3.4] the Separativity Problem for regular F -algebras is equivalent to diagonalising 2×2 matrices : given a 2×2 matrix A over a regular F -algebra R , do there exist invertible 2×2 matrices P and Q over R such that PAQ is diagonal? In the special case of a matrix algebra $R = M_n(F)$, it is a

basic result in elementary linear algebra that P and Q can be obtained algorithmically so that

$$PAQ = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$$

where E_1, E_2 are $n \times n$ diagonal matrices with 1's and 0's. Here one treats A as a $2n \times 2n$ matrix over F and operates at the level of field entries using the field operations. But what happens if we treat A as a 2×2 matrix and operate at the level of $n \times n$ matrices for the entries, allowing ourselves only the regular algebra operations in R (including quasi-inverses)? It is not at all clear that there is still an algorithm. More specifically, is there a "formula" for the entries of some suitable P and Q in terms of fixed "regular algebra expressions" in the entries a, b, c, d of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$? We will show that for each n , such a formula exists. Moreover, the Separativity Problem is then equivalent to the existence of such a uniform formula which is also INDEPENDENT of n .

Let us be a bit more precise about what we mean by "uniform diagonalisation". We will work with algebras over some fixed commutative ring Λ (but $\Lambda = \mathbf{Z}$ and $\Lambda = F$ are our main interests).

Definition 1. *Let R be a regular Λ -algebra. We say that 2×2 matrices over R can be uniformly diagonalised by regular Λ -algebra operations if for a general 2×2 matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

over R , there exist fixed regular Λ -algebra expressions $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4$ in a, b, c, d such that for all substitutions for a, b, c, d from R , and for all choices of a quasi-inverse operation \prime in R , the matrices

$$P = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}$$

are invertible and

$$PAQ = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

is diagonal. □

Here we are viewing a regular Λ -algebra R from the standpoint of a variety once R is equipped with a particular quasi-inverse operation \prime . So in addition to the usual ring operations $+, -, \cdot, 0, 1$ and the unary scalar operations from Λ , and their laws, the unary operation \prime satisfies $a = aa'a$. "Regular Λ -algebra expressions" are then just words formed from these operations.

Notes

1. For example, the matrix

$$P = \begin{bmatrix} 1 & , & 1+b \\ a' & , & a'b + 1 + a' \end{bmatrix}$$

has regular \mathbf{Z} -algebra expressions in a and b . In fact P is invertible for all choices of a and b and all choices of a quasi-inverse operation $'$, with

$$P^{-1} = \begin{bmatrix} 1 + a' + ba', & -1 - b \\ -a' & , & 1 \end{bmatrix}$$

But we are not allowed to make *different* choices of a' in say the (2, 1) and (2, 2) entries for the same a (!) e.g. for $R = M_2(F)$ and

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

the different choices

$$a' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ in the (2, 1) entry}$$

$$a' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ in the (2, 2) entry}$$

would give a non-invertible

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. In our formulation of uniform diagonalisation, we have not required that the entries of P^{-1} and Q^{-1} should also be fixed regular Λ -algebra expressions in a, b, c, d . The latter (seemingly stronger) requirement could have been adopted because the Separativity Problem is also equivalent to uniform diagonalisation of 2×2 matrices over $M_n(F)$ in this sense. (See proof of Theorem 3 (\Rightarrow) and note that P^{-1} and Q^{-1} will also have fixed regular expressions in the free variables w, x, y, z). However, our weaker version seems more natural, and serves us well for the positive results later.
3. To be completely rigorous, uniform diagonalisation of 2×2 matrices over a regular Λ -algebra $(R, +, \cdot)$ means the following (which is our fall-back position): let T be the free regular Λ -algebra on 4 generators w, x, y, z (the free object in the earlier variety). See [GMM]. Then there exist 2×2 matrices P and Q over T such that under all ways of equipping R with a quasi-inverse operation and for all homomorphisms $\theta : T \rightarrow R$ (in the universal algebra sense), P and Q become invertible in $M_2(R)$ and $P \begin{bmatrix} w & x \\ y & z \end{bmatrix} Q$ becomes diagonal.

4. If 2×2 matrices over a regular Λ -algebra R can be uniformly diagonalised, then the same is true of 2×2 matrices over any regular Λ -subalgebra or any homomorphic image of R .
5. In studying uniform diagonalisation of matrices over regular algebras, there is nothing to stop us from restricting quasi-inverses to being “generalised inverses” (in short, **g-inverses**) in the sense that $a = aa'a$ and also $a' = a'aa'$. (Propositions 7 and 10 are our only results which don't have analogues in this setting.) Recall that given any quasi-inverse a' of a , the new quasi-inverse $a'aa'$ is a generalised inverse of a . See also [GMM, pp 411-412]. (The term “generalised inverse” has a variety of meanings within linear algebra and semigroup theory.)
□

Example 2 (uniformly diagonalising matrices over a field)

One's initial impression is that there isn't a fixed formula for diagonalising $k \times k$ matrices over a field F because the choice of elementary row/column operations is dictated by whether certain terms are nonzero or not. With the aid of a quasi-inverse operation, however, the procedure can be made uniform. Note that in a field, $aa' = 1$ if $a \neq 0$, otherwise $aa' = 0$. Now given elements $a, u, v \in F$ there is a uniform way of specifying an element $w \in F$ such that

$$w = \begin{cases} u & \text{if } a \neq 0 \\ v & \text{if } a = 0. \end{cases}$$

Simply take $w = aa'(u - v) + v$.

For each $x \in F$, consider the statement $S(x) : x = 0$. Any statement which is built up from a finite number of $S(x_1), \dots, S(x_m)$ using the logical connectives \wedge, \vee, \neg is equivalent to a single $S(a)$ where $a \in F$ is a regular ring expression in x_1, \dots, x_m :

$$\begin{aligned} S(x) \wedge S(y) &\equiv S(xx' + yy' - xx'yy') \\ S(x) \vee S(y) &\equiv S(xy) \\ \neg S(x) &\equiv S(1 - xx') \end{aligned}$$

Combined with our earlier observation for specifying w depending on the truth of $S(a)$, this allows for a uniform specification of a sequence of elementary matrices that will diagonalise (by left/right multiplications) an arbitrary $k \times k$ matrix. For instance, if $k = 2$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let

$$E_1 = \begin{bmatrix} 1 & 0 \\ -aa'a'c & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} aa' & 1 - aa' \\ 1 - aa' & aa' \end{bmatrix}.$$

Then

$$E_2 E_1 A = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

so we have a uniform triangularisation

$$\begin{bmatrix} aa' & 1 - aa' \\ 1 - aa' - aa'a'c & aa' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

We could also easily uniformly diagonalise A explicitly. □

2 The Connection of Separativity with Uniform Diagonalisation

We now link the Separativity Problem with uniform diagonalisation.

Theorem 3. *Let F be a field. Then all regular F -algebras are separative if and only if 2×2 matrices over $M_n(F)$ can be uniformly diagonalised independently of n .*

(Note: this means that we require the form of P and Q in

$$P \begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

to be the same for all $a, b, c, d \in M_n(F)$ and for all n .)

Proof (\Rightarrow). This is clear because the free regular F -algebra T on 4 generators w, x, y, z is then separative. Hence by [AGOP2, Theorem 2.5] the matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ can be diagonalised over T , say

$$P \begin{bmatrix} w & x \\ y & z \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for some invertible 2×2 matrices P and Q over T . Note that the entries of P and Q are certain regular F -algebra expressions in w, x, y, z . Although these are not necessarily unique, we can agree to fix some such expressions. Uniform diagonalisation of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over any regular F -algebra R now follows because, for a given quasi-inverse operation on R , there is an algebra homomorphism $\theta : T \rightarrow R$ with $w \mapsto a, x \mapsto b, y \mapsto c, z \mapsto d$ and respecting quasi-inverses. This gives

$$\theta(P) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta(Q) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

where $\theta(P)$ and $\theta(Q)$ are invertible 2×2 matrices over R and whose entries are fixed regular algebra expressions in a, b, c, d , independent of R .

(\Leftarrow) It suffices to establish this for countable fields because for a general field F , fix a countable subfield E of F which contains the (finite number of) fixed scalars in the uniform diagonalisation formula over the $M_n(F)$. Then for a regular F -algebra R and matrix $A \in M_2(R)$, we can simply diagonalise A over the regular E -algebra R . (Note that 2×2 matrices over $M_n(E)$ will inherit the uniform diagonalisation property from $M_n(F)$.)

Hence assume F is countable and that we have a uniform diagonalisation formula

$$(\#) \quad PAQ = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for all $A \in M_2(M_n(F))$, independently of n . Let $(T, +, \cdot, \iota)$ be the free regular F -algebra on w, x, y, z . It is enough to diagonalise 2×2 matrices over T . Since F is countable, by [GMM, Corollary 2.3] we can view T as an F -subalgebra of $S = \prod_{n=1}^{\infty} M_n(F)$. Let $p_n : S \rightarrow M_n(F)$ be the n^{th} projection map and let $\pi_n : M_2(S) \rightarrow M_2(M_n(F))$ be the projection

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} \mapsto \begin{bmatrix} r_n & s_n \\ t_n & u_n \end{bmatrix}$$

Note the isomorphism $M_2(S) \cong \prod_{n=1}^{\infty} M_2(M_n(F))$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix over T . Form the matrices P and Q (over T) in $(\#)$ by blindly following the recipe. (The entries of P and Q will be certain regular algebra expressions in a, b, c, d .) We aim to show that P and Q are invertible and that PAQ is diagonal by applying $(\#)$ to each $\pi_n(P)\pi_n(A)\pi_n(Q)$. But there is some subtlety involved here, because in order to apply $(\#)$ we must first unambiguously specify a quasi-inverse operation on each $M_n(F)$ to match quasi-inverses induced from T . For instance, if some $a_n = b_n$ but $(a')_n \neq (b')_n$, we can't blindly apply $(\#)$. We use the primeness of T , which was established in [GMM, Theorem 4.4], to circumvent this as follows.

Lemma. *Let Y be a finite subset of T and let $I = \{n \in \mathbb{N} : p_n(y) = p_n(z) \text{ for some distinct } y, z \in Y\}$. Then the projection*

$$q : T \rightarrow \prod_{\mathbb{N} \setminus I} M_n(F)$$

is faithful.

Proof. Enumerate the elements of Y as y_1, \dots, y_m . For $1 \leq j < k \leq m$, let

$$I_{jk} = \{n \in \mathbb{N} : p_n(y_j) = p_n(y_k)\}.$$

Then $I = \bigcup I_{jk}$. When $1 \leq j < k \leq m$ and $I_{jk} \neq \Phi$, clearly the projection of T onto the I_{jk} part has a nonzero kernel. Moreover, the intersection of these (finitely many) nonzero ideals is nonzero because T is prime. This says $\ker(1 - q) \neq 0$. But $\ker(q) \cap \ker(1 - q) = 0$, whence again by primeness of T , $\ker(q) = 0$ and so q is

faithful. □

Proof of Theorem 3 continued. We now apply the Lemma to the (finite) subset Y of T consisting of all elements which occur as subwords of entries in A, P , or Q . (That is, words needed in a successive evaluation of the entries, in terms of the operations of the algebra $(T, +, -, \cdot, 0, 1, \iota)$. For instance, the subwords of $(a'b)' + cd'$ are $a, a', b, a'b, (a'b)', c, d, d', cd', (a'b)' + cd'$.) Since the projection $q : S \rightarrow \prod_{N \setminus I} M_j(F)$ is a faithful F -algebra homomorphism when restricted to T , there is no loss of generality in assuming $I = \Phi$, that is, distinct members of Y have *all* their components distinct: for $y, z \in Y$, $p_n(y) = p_n(z)$ for some $n \Rightarrow y = z$. Now define a quasi-inverse operation ι on each $M_n(F)$ as follows: for $u \in p_n(Y)$, let $u' = (y')_n = p_n(y')$ where $y \in Y$ is the unique element with $u = y_n$ (and y' is the quasi-inverse from T); for $u \in M_n(F) \setminus p_n(Y)$, let u' be any fixed quasi-inverse of u in $M_n(F)$.

We now have a well-defined quasi-inverse operation on each $M_n(F)$ such that the projection maps p_n , when restricted to T , are “locally” regular F -algebra homomorphisms when applied to the entries of P, A , and Q (preserve quasi-inverses). Hence by (#), $\pi_n(P)$ and $\pi_n(Q)$ are invertible and $\pi_n(P)\pi_n(A)\pi_n(Q)$ is diagonal. Therefore PAQ is diagonal and P and Q are invertible in $M_2(S)$. Since $M_2(T)$ is a regular subring of $M_2(S)$, P and Q are also invertible in $M_2(T)$. □

Remark. The form of P and Q in Theorem 3 can in fact be assumed to be products of fixed elementary matrices. This is because over a separative exchange ring, it was shown in [AGOR, Theorem 2.8] that invertible matrices are products of elementary matrices and an invertible diagonal matrix. □

Next we show that there does exist a diagonalisation formula for 2×2 matrices over $M_n(F)$ for a given n .

Theorem 4. *Let R be a regular ring of index of nilpotence at most n . Then 2×2 matrices over R can be uniformly diagonalised, independently of R .*

Proof. By a result of Burgess and Stephenson [BS, Corollary 25] (see also [VNRR, Theorem 7.15]), a regular ring of index at most n is characterised by the property that for each $x \in R$ there exists $y \in R$ with $xyx = x$ and $x^n y = yx^n$. For the remainder of the proof, let ι be any quasi-inverse operation and let \prime be a special quasi-inverse operation satisfying the additional law

$$x^n x' = x' x^n.$$

The class of all $(R, +, \cdot, -, 0, 1, \iota, \prime)$ forms a variety, whose underlying rings are exactly the regular rings of index at most n . A variety always contains a free object - see [J2, Theorem 2.10]. Accordingly, let $(T, +, \cdot, -, \iota, \prime)$ be the free object on 4 generators w, x, y, z for this variety.

Let S be the subalgebra of $(T, +, \cdot, \prime)$ generated by w, x, y, z . Its elements are words in w, x, y, z involving only the operations $+, -, \cdot, \prime, 0, 1$ (no $\prime\prime$). Since S is a regular subring of T , S also has bounded index of nilpotence and so S is unit-regular by [VNRR, Corollary 7.11]. In particular S is separative by [VNRR, Theorem 4.14]. Hence there exist invertible $P, Q \in M_2(S)$ such that

$$P \begin{bmatrix} w & x \\ y & z \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

and the entries of P and Q are regular ring expressions involving \prime (but not $\prime\prime$). This determines a uniform diagonalisation as follows.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 matrix over a regular ring R of index at most n . Let \prime be any quasi-inverse operation on R . Choose $\prime\prime$ to be a special quasi-inverse operation on R . By freeness of $(T, +, \cdot, \prime, \prime\prime)$ there is a homomorphism $\theta : T \rightarrow R$ which preserves $+, \cdot, \prime, \prime\prime$ and with

$$\theta(w) = a, \theta(x) = b, \theta(y) = c, \theta(z) = d.$$

Now we obtain the uniform diagonalisation

$$\theta(P) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta(Q) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

because the entries of the invertible matrices $\theta(P)$ and $\theta(Q)$ are uniform regular ring expressions in a, b, c, d (relative to $+, \cdot, -, \prime, 0, 1$). \square

Corollary 5. *For any positive integer n and field F , there is a uniform diagonalisation formula for 2×2 matrices over all $M_k(F)$ for $k = 1, 2, \dots, n$. It is independent of the field.* \square

Note where the preceding argument breaks down in trying to establish uniform diagonalisation of 2×2 matrices over unit-regular rings (which by Theorem 3 is equivalent to all regular rings being separative). Here we could introduce unary operations $\prime\prime$ and $\prime\prime\prime$ satisfying

$$xx''x = x \quad \text{and} \quad x'''x'' = 1 = x''x''',$$

take $(T, +, \cdot, \prime, \prime\prime, \prime\prime\prime)$ as the free object on w, x, y, z , and let S be the subalgebra of $(T, +, \cdot, \prime)$ generated by w, x, y, z . The trouble is we can't say that S , just as a subring of a unit-regular ring, is necessarily separative.

For each $n \geq 2$, there is a myriad of possibilities for quasi-inverse operations on $M_n(F)$. Corollary 5 suggests that the real challenge facing a potential uniform diagonalisation formula lies less with the variations in quasi-inverses at a given level n , than with the changes resulting from increasingly bigger n . Our next result supports this contention.

Proposition 6. *Let F be a field. The Separativity Problem has a positive answer for all regular F -algebras if (and only if) the following holds : given one prescribed g -inverse operation \prime on each $M_n(F)$ for $n \in \mathbf{N}$, there is a uniform diagonalisation formula for 2×2 matrices over $M_n(F)$ using these \prime , but independently of n .*

Proof. Assume the property holds. As in the proof of Theorem 3 (\Leftarrow) we can assume F is countable and that the free regular F -algebra T on w, x, y, z relative to a g -inverse operation \prime sits inside $\prod_1^\infty M_n(F)$. (See Note 5 of Introduction.) Notice that T is countable.

Lemma. *Let $Y_1 \subseteq Y_2 \subseteq \dots \subseteq Y_k \subseteq \dots$ be a chain of finite subsets of T whose union is T . Then there is a g -inverse operation \prime on each $M_n(F)$, for $n \in \mathbf{N}$, and a chain $I_1 \subseteq I_2 \subseteq \dots$ of subsets of \mathbf{N} such that for each $k \in \mathbf{N}$:*

- (1) *The projection $q_k : T \rightarrow \prod_{\mathbf{N} \setminus I_k} M_n(F)$ is faithful.*
- (2) *For $n \notin I_k$, the coordinate projections $p_n : T \rightarrow M_n(F)$ are “locally” regular F -algebra homomorphisms on Y_k , that is, $p_n(y') = p_n(y)'$ for all $y \in Y_k$.*

Proof. Let $I_k = \{n \in \mathbf{N} : p_n(y) = p_n(z) \text{ for some distinct } y, z \in Y_k\}$. By the Lemma in the proof of Theorem 3, (1) holds and p_n is 1-1 on Y_k for $n \notin I_k$. Given $n \in \mathbf{N}$, let

$$K_n = \{k \in \mathbf{N} : n \notin I_k\}, \quad X_n = \bigcup_{k \in K_n} Y_k,$$

$$Z_n = p_n(X_n) = \bigcup_{k \in K_n} p_n(Y_k)$$

Observe that p_n is 1-1 on X_n so we can use p_n to induce g -inverses for elements of Z_n : given $a \in Z_n$, let $a' = p_n(y')$ where $y \in X_n$ is the unique element with $p_n(y) = a$. Now extend \prime to all of $M_n(F)$ by assigning g -inverses to the remaining elements in any manner. By construction, (2) is true. \square

Proof of Proposition 6 continued. It is enough to diagonalise a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over T . Let \prime be the g -inverse operation of $M_n(F)$ given in the Lemma, and let

$$(\#) \quad PAQ = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

be the uniform formula that diagonalises 2×2 matrices over $M_n(F)$ relative to these prescribed \prime . The rest of the proof is now just the argument used before in Theorem 3 (\Leftarrow), with one change: replace the original finite subset Y by any Y_k containing it. \square

Our definition of uniform diagonalisation requires that the procedure should work for all choices of a quasi-inverse operation. If we were to relax this requirement, then we can indeed uniformly diagonalise 2×2 matrices over any $M_n(F)$, independently of n and F , but this doesn't seem to resolve the Separativity Problem. In fact we have:

Proposition 7. *There is a uniform formula for diagonalising 2×2 matrices over a unit-regular ring if we restrict the choice of quasi-inverses to unit-quasi-inverses. The formula is independent of the ring.*

Proof. This is along similar lines to the proof of Theorem 4. Let $(T, +, \cdot, //, ///)$ be the free unit-regular ring on w, x, y, z where the unary operations $//$ and $///$ satisfy the laws

$$t//t = t \quad \text{and} \quad t///t// = 1 = t//t///.$$

Note that since $t//$ is a unit, we must have $(t//)/// = (t//)^{-1} = t///$ so any subset of T closed under $//$ is also closed under $///$. Hence the subalgebra of $(T, +, \cdot, //)$ generated by w, x, y, z is all of T . Now diagonalise $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ over T (possible because T is separative), say

$$P \begin{bmatrix} w & x \\ y & z \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}.$$

For a general unit-regular ring $(R, +, \cdot, //, ///)$ and matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, map $w \mapsto a, x \mapsto b$ etc to get a uniform diagonalisation using $//$ but not $///$. \square

It is not difficult to arrive at a specific formula in Proposition 7, by an argument similar to [MM, Lemma 11]. We illustrate part of this.

Example 8 (Uniform triangularisation of 2×2 matrices over unit-regular rings)

Let R be a unit-regular ring and $'$ a quasi-inverse operation on R . Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R).$$

Let

$$e = a'a, \quad g = (c(1-e))'c(1-e), \quad g_1 = (1-e)g.$$

By a standard argument [VNR pp 1-2] e and g_1 are orthogonal idempotents with $Ra + Rc = R(e + g_1)$. Let $t = (1-e)(c(1-e))'$ and $s = c(1-e)$. Consider the sequence of row operations on A corresponding to left multiplications by

$$E_1 = \begin{bmatrix} 1 & 0 \\ -ca' & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} a' & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}.$$

We have

$$\begin{aligned}
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\rightarrow \begin{bmatrix} a & * \\ c(1-e) & * \end{bmatrix} \rightarrow \begin{bmatrix} e & * \\ c(1-e) & * \end{bmatrix} \rightarrow \begin{bmatrix} e+g_1 & * \\ c(1-e) & * \end{bmatrix} \\
&\rightarrow \begin{bmatrix} e+g_1 & * \\ 0 & * \end{bmatrix}
\end{aligned}$$

Hence for $E = E_4E_3E_2E_1$, we have EA triangular. Moreover the entries of E can be expressed as uniform regular ring expressions in a and c . Clearly E_1, E_3, E_4 are invertible, while E_2 is invertible if a' is a unit-quasi-inverse of a . Hence, if we restrict quasi-inverses to unit-quasi-inverses, then we get a uniform triangularisation of A which is independent of the ring R .

Furthermore, one can achieve uniform diagonalisation by next clearing out the (1,2) entry using three elementary column operations. No unit-quasi-inverses are needed this time, because the (1,1) entry is idempotent, but one choice of a g-inverse is made with a view to the final step, which is cleaning up the first column again by one row operation. (Of the four independent choices of quasi-inverse in the formula, all but ONE can be completely general.) \square

One might suspect that the reason a uniform formula works as in Proposition 7 (using unit-quasi-inverses) is that in addition to the usual regular ring operations on the entries of a matrix, one now has access to appropriate units in a uniform way. This ought to make diagonalisation easier. However, with a generalised inverse operation \prime on $M_n(F)$, each a' has the same rank as a (in fact, this is an equivalent formulation), and therefore uniform diagonalisation should be more difficult. The following result is therefore a little surprising, in view of Proposition 6, if one is expecting a negative answer to the Separativity Problem.

Proposition 9. *For each positive integer n and field F , there is a g-inverse operation \prime on $M_n(F)$ in terms of which there is not only a uniform diagonalisation formula for all 2×2 matrices over $M_n(F)$, but the formula is also independent of n and F .*

Proof. Fix n and F . We shall specify the desired \prime by well-defining it on the elements of any given coset $a + \mathbf{Z} \cdot 1$ of $n \times n$ matrices modulo integral scalar matrices. So fix $a \in M_n(F)$. Choose a similarity transformation $\phi : M_n(F) \rightarrow M_n(F)$ such that $\phi(a)$ is a block diagonal matrix

$$(J_1, \dots, J_m, U)$$

where each J_i is an elementary Jordan matrix

$$\begin{bmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & & \ddots & \\ & & & 1 & \lambda_i \end{bmatrix}$$

and, if the characteristic polynomial of a doesn't split into linear factors, U is an invertible matrix with no eigenvalues in F . (For example, make use of the appropriate rational and Jordan canonical forms.) Notice that for any integer k , $\phi(a+k)$ is also of this form with the same block structure.

For an elementary Jordan matrix J (of any size), let

$$J^+ = \begin{cases} \text{transpose of } J \text{ if } J \text{ is nilpotent;} \\ J^{-1} \text{ if } J \text{ is invertible.} \end{cases}$$

Observe that J^+ is a g-inverse of J (analogous to its Moore-Penrose inverse in the real case). We extend this definition to block diagonal $n \times n$ matrices A , with the fixed block structure of $\phi(a)$ above, in the case where the blocks are either elementary nilpotent or invertible; namely we take A^+ as the block diagonal matrix obtained by taking the transposes of the nilpotents and inverses of the invertibles. Again a g-inverse results, but more interestingly we have a uniform formula

$$A^+ + (1 - A^+A)(1 + A)^+(1 - AA^+)$$

which gives a unit-quasi-inverse for any such A . One only has to check this for an elementary nilpotent matrix and an invertible matrix. (Curiously, this formula fails for the Moore-Penrose inverse of a general real square matrix, such as $A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$! See also Proposition 10.)

We can now define ι on the coset $a + \mathbf{Z} \cdot 1$: if $b = a + k$, with k integer, let

$$b' = \phi^{-1}(\phi(b)^+).$$

Clearly b' is a g-inverse of b . By construction, ϕ becomes "locally" a homomorphism of regular rings on the coset $a + \mathbf{Z} \cdot 1$, with respect to the quasi-inverse operations ι and $^+$. In particular

$$\phi(a') = \phi(a)^+ \text{ and } \phi((1+a)') = (1 + \phi(a))^+.$$

Therefore, from the earlier uniform expression in terms of A and $^+$, we get the uniform formula

$$u(a) = a' + (1 - a'a)(1 + a)')(1 - aa')$$

which produces a unit-quasi-inverse for each $a \in M_n(F)$, as a regular ring expression involving only $+, \cdot, -, \iota, 0, 1$. The existence of a uniform diagonalisation formula in terms of ι , which is independent of n and F , now follows directly from Proposition 7. (By the arguments of Example 8, one could actually write down an explicit formula.) \square

Remark. In the case of the real field, it would be interesting to know whether one could take the Moore-Penrose inverse as the ι in Proposition 9. \square

Here is an interesting curiosity, which comes as a by-product of some of our earlier arguments.

Proposition 10. *Let F be a field and n a positive integer. There is a uniform formula which gives a unit-quasi-inverse U for each $n \times n$ matrix A over F as a regular ring expression in terms of A and any quasi-inverse operation on $M_n(F)$. There does NOT exist such a formula which is independent of n .*

Proof. We proceed as in the proof of Theorem 4. Let $(T, +, \cdot, \iota, \prime)$ be the free object there, but taken on a single generator x . Let S be the subalgebra of $(T, +, \cdot, \iota)$ generated by x . Then S is unit-regular so x has a unit-quasi-inverse u in S . Moreover u is some regular ring expression in x involving $+, \cdot, -, \iota, 0, 1$. Now let $A \in M_n(F)$ and let ι be a quasi-inverse operation on $M_n(F)$. Choose a special quasi-inverse operation \prime on $M_n(F)$. By freeness of $(T, +, \cdot, \iota, \prime)$ there is a homomorphism $\theta : T \rightarrow M_n(F)$ which preserves ι and takes $x \mapsto A$. Let $U = \theta(u)$. This is a unit-quasi-inverse of A and is a fixed regular ring expression in A and the given quasi-inverse operation ι .

Now suppose there is such an expression which is independent of n . In Example 8, given a quasi-inverse operation ι on $R = M_n(F)$, replace every quasi-inverse that is in the triangularisation formula by the expression for a unit-quasi-inverse in terms of ι . Then E_2 is invertible and we get a uniform diagonalisation

$$E \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

of all 2×1 matrices over $M_n(F)$ (in terms of ι) which is independent of n . By the argument in Theorem 3 (\Leftarrow), all 2×1 matrices over the free regular algebra T can now be diagonalised. But this says T is Hermite (in fact strongly separative; see [AGOP1]), which is a contradiction because T has infinite stable rank (see [MM, Proposition 8]). Hence no such formula exists. \square

Word of caution. Separativity of a regular ring R can be characterised by the property that 2×2 matrices over corner rings eRe are equivalent to diagonal matrices [AGOP2, Theorem 3.4]. This allows the class of separative regular Λ -algebras, over a commutative ring Λ , to be viewed as a variety in the following way. First we restrict a quasi-inverse operation ι on R to have the additional property that aa' and $a'a$ always commute. (Every regular ring supports such a quasi-inverse). The idempotents of R are then precisely the aa' . To capture diagonalisation

$$P \begin{bmatrix} ere & ese \\ ete & eue \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

over the corner rings eRe for $e = aa'$, we introduce sixteen 5-ary operations on R to pick out for each 5-tuple (r, s, t, u, a) the various entries of P, Q, P^{-1}, Q^{-1} . One then formulates invertibility of P and Q and the diagonalisation using the obvious 18 identities. (One needs another 32 identities of the form $entry \times aa' = entry = aa' \times entry$ to ensure the entries of P, Q, P^{-1}, Q^{-1} are in eRe !) Now choose a free

object S in this variety on 4 generators w, x, y, z . Separativity of S guarantees a diagonalisation

$$P \begin{bmatrix} w & x \\ y & z \end{bmatrix} Q = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for some invertible 2×2 matrices P and Q over S . But we can't infer from the argument in Theorem 3 (\Rightarrow) that the mapping $w \mapsto a, x \mapsto b, \dots$ etc leads to a uniform diagonalisation of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over any separative regular Λ -algebra R , independently of R (in particular for all $M_n(F)$). There is a type of uniform diagonalisation that comes from this, but it potentially requires use of all the previously described operations (including those sixteen 5-ary ones)! \square

3 Rings admitting a uniform diagonalisation via ring operations: the case with no quasi-inverses

The discussion in [AGOP1, section 6] strongly suggests that the fundamental Separativity Problem for regular rings has a negative answer. Our Theorem 3 suggests a possible approach to showing this: prove that there is no uniform formula for diagonalising 2×2 matrices over $M_n(F)$ which is independent of n . One could attempt to show that there can't be a bound on the number of quasi-inverses that occur in the entries of P and Q (in a formula $PAQ = \text{diagonal}$). The first step would be to show that we can't get by with zero quasi-inverses, i.e. where the entries in P and Q are just fixed F -algebra expressions in a, b, c, d . This will follow from a more general result of this section, namely, that the algebras over which 2×2 matrices can be uniformly diagonalised in this fashion are precisely the m -algebras.

Definition 11. *We say that 2×2 matrices over a Λ -algebra R can be uniformly diagonalised by Λ -algebra operations if for a general 2×2 matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

over R , there exist fixed expressions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$, which are (non-commuting) polynomials in a, b, c , and d , with coefficients from Λ such that for all substitutions for a, b, c, d from R , the matrices

$$P = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}$$

are invertible in $M_2(R)$ and the matrix product PAQ is diagonal. For ease of reference in the next few results, we call such Λ -algebras "good". \square

The following is immediate:

Remark 12. *The rings for which a given diagonalisation (i.e. Λ -formula) works form a variety. In particular a homomorphic image of a good Λ -algebra is good, as is any direct product of copies of a good Λ -algebra. \square*

We recall that if D is an infinite commutative domain, then the coefficients of a polynomial function $g : D \rightarrow D$, $x \rightarrow g(x)$, are uniquely determined. We use this fact to get the following:

Lemma 13. *Let R be a good Λ -algebra. Then R cannot contain, as a Λ -subalgebra, an infinite commutative domain.*

Proof. We will use the same notation as in Definition 11. Substitution of zero for three of the four entries in A turns the α_i and β_i into polynomials over Λ in one indeterminate, and the fact that PAQ is diagonal yields two equations. For example using an arbitrary a in R , and $b = c = d = 0$, gives that $\alpha_1(a, 0, 0, 0) a \beta_2(a, 0, 0, 0) = 0$ and $\alpha_3(a, 0, 0, 0) a \beta_1(a, 0, 0, 0) = 0$. These are equations in one unknown over an infinite domain, so the first shows that the constant term is zero in either α_1 or in β_2 . The second equation shows that either α_3 or β_1 has constant term zero. The three other ways of setting three of the variables equal to zero, each give two equations. The six other conclusions are that the constant term must be zero in either: α_1 or β_4 , α_3 or β_3 , α_2 or β_2 , α_4 or β_1 , α_2 or β_4 , and α_4 or β_3 . Clearly we have zero constant term in both α_1 and α_2 , or in β_2 and β_4 . However the matrices P and Q must be invertible under the substitutions $a = b = c = d = 0$, which means that the matrix of their constant terms cannot have a row of zeroes, or a column of zeroes. Thus we have a contradiction. \square

Lemma 14. *Let R be a good Λ -algebra. Then R cannot contain a non-zero nilpotent element.*

Proof. Again we use the same notation as in Definition 11, and assume, if possible, that the ring contains a non-zero element t , with $t^2 = 0$. We shall make four substitutions into the matrix A by letting one variable be t and the other three be zero. Each substitution will yield two equations. It is clear that such substitutions will transform any α_i or β_i into the form $\lambda_1 t + \lambda_2$, where λ_1 , and λ_2 belong to the (central) image of Λ in R . For example, suppose that $\alpha_i(t, 0, 0, 0) = c_i t + a_i$, and $\beta_i(t, 0, 0, 0) = f_i t + b_i$. Then from the diagonalisation equation for

$$\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}$$

one gets $a_1 t b_2 = a_3 t b_3 = 0$. The other three substitutions yield $a_1 t b_4 = a_3 t b_3 = 0$, $a_2 t b_2 = a_4 t b_1 = 0$, and $a_2 t b_4 = a_4 t b_3 = 0$.

However the matrices

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \text{ and } \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

are invertible since they correspond to the substitution of zero for all variables. This implies that a_1 and a_2 cannot be annihilated by any non-zero element of R . But tb_2 and tb_4 annihilate both a_1 and a_2 . Thus $tb_2 = tb_4 = 0$. This, in turn implies that $t = 0$ because t now annihilates the second column of the second matrix. \square

Definition 15. [P, page 47]. Recall that a Λ -algebra R is called an m -ring (for some $m > 1$) if the elements of R satisfy the identity $x^m = x$. \square

These rings are commutative regular, and are closed under taking direct products (for fixed m).

Lemma 16. *An m -ring is good (as a \mathbf{Z} -algebra).*

Proof. One has a uniform way of choosing a quasi-inverse of a that is a unit, namely $u = a^{m-2} + 1 - a^{m-1}$, with inverse $a + 1 - a^{m-1}$. However we know that this means that the matrices in $M_2(R)$ can be uniformly diagonalised (Proposition 7). \square

In order to get our next proposition, we need a little commutative algebra.

Lemma 17. *Let R be a commutative regular Λ -algebra and let P be a minimal prime ideal of Λ . Then R has a prime ideal Q whose inverse image under the action of Λ on R is P .*

Proof. Assume without loss of generality that Λ is a subring of R . Consider the inclusions:

$$\Lambda \subseteq \Lambda_1 \subseteq \Lambda_2 \subseteq R,$$

where Λ_1 is the integral closure of Λ in R , and Λ_2 is the classical ring of quotients of Λ_1 . Note the following:

- (i) Λ_1 contains all the idempotents of R ,
- (ii) a non-zero divisor of Λ_1 remains such in Λ_2 ,
- (iii) Λ_2 is itself regular.

Now we must produce Q . By [ZS, Theorem 3, page 257], there is a prime ideal P_1 of Λ_1 that contracts to P in Λ . Since P is minimal in Λ , P_1 is minimal in Λ_1 by [ZS, Section 1, page 259] and therefore it consists of zero-divisors. Since P_1 consists of zero-divisors, it “survives” when one forms Λ_2 , and therefore it is the contraction

to Λ_1 of a prime P_2 of Λ_2 [ZS, Corollary 1, page 224]. Now one has the prime P_2 inside the regular ring Λ_2 contracting to P . It is easy to see that R has a prime Q , lying over P_2 —it is, in fact, just P_2R . \square

Corollary 18. *Let Λ be Noetherian, and let it be a subring of R a regular good Λ -algebra. Then Λ is a finite product of finite fields.*

Proof. By Lemma 14, Λ is a semiprime Noetherian ring, so its prime radical is zero. By [ZS, Theorem 5, p 209], the radical is a finite intersection of prime ideals. If necessary replace the non-minimal primes with minimal primes, to get (0) as a finite intersection of minimal primes P_i . By Lemma 17 choose for each i , a prime Q_i in R lying over P_i . Then the ring R/Q_i is a good algebra over the Λ -domain, Λ/P_i . Therefore Λ/P_i is finite by Lemma 13, and hence a field. Thus Λ is a finite subdirect product of the fields Λ/P_i , so it is a finite product of fields. \square

Proposition 19. *If R is a good Λ -algebra, then it is commutative regular in fact, it is an m -ring.*

Proof. We can assume that Λ is finitely generated as a ring (hence Noetherian), since the diagonalisation formula uses only finitely many Λ -coefficients in the formula.

Consider first the case where R is prime. Its centre is a domain and a Λ -subalgebra, and thus a finite field F by Lemma 13. An element a in R must be algebraic over F by Lemma 13. So $F[a]$ is finite-dimensional without nilpotents (Lemma 14), in particular regular. Therefore R is a prime regular ring without nilpotents, hence a division ring [VNRR 3.2]. Now R is an algebraic division algebra over F , so R is a field by [J1, page 183], and therefore finite by Lemma 13.

In the general case, since R has no nilpotents, it is a subdirect product of prime Λ -algebras R_i . Since R_i is a homomorphic image of R , R_i is also a good Λ -algebra, hence a finite field by the special case. At this point, there is no loss of generality in assuming R is also regular because we could replace R by $\prod R_i$, which is also a good Λ -algebra by Remark 12, and then show the latter ring is an m -ring. Now Corollary 18 applies and shows that the image of Λ in R is a finite product of finite fields. Thus we can further assume that Λ is a finite field, say $GF(p^k)$, and that each field homomorphic image of R has the form $GF(p^n) \supseteq GF(p^k)$. If the n are bounded then R is an m -ring. If the n are not bounded, choose u_n in each $GF(p^n)$, a primitive element of order $p^n - 1$. Now observe that the Λ -algebra $\prod GF(p^n)$ is good by Remark 12 and it contains $\prod (u_n)$ which is not algebraic over $GF(p^k)$ in contradiction to Lemma 13. \square

In summary we have the following:

Theorem 20. *Let R be a Λ -algebra. Then the following are equivalent:*

- (i) 2×2 matrices over R can be uniformly diagonalised by Λ -algebra operations alone.
- (ii) R is an m -ring.
- (iii) R is unit regular and every element a has a unit quasi-inverse that is given by a (fixed) polynomial in a over Λ .
- (iv) R is regular and each element a has a quasi-inverse that is given by a (fixed) polynomial in a over Λ .

Proof. One uses Proposition 19 and the following implications. (ii) \Rightarrow (iii): Let $a' = a^{m-2}$ and $u = a' + 1 - aa'$. (iii) \Rightarrow (i): The unit quasi-inverse can be used as in Proposition 7 to give a uniform rule for diagonalisation. (iv) \Rightarrow (iii): Let $p(a)$ be the polynomial quasi-inverse for a . The ring $\Lambda[a]$ is commutative and in it one easily computes that $u = p(a) + 1 - ap(a)$ is a polynomial quasi-inverse with inverse $a + 1 - ap(a)$. \square

Corollary 21. *A Λ -algebra R is an m -ring if and only if there exists a polynomial $p(x)$ in $\Lambda[x]$ such that $a = ap(a)a$ for all $a \in R$.* \square

Remark 22. *Fix p . The ring generated by $\bigoplus_{k=1}^{\infty} GF(p^k)$ and the identity shows that the class of m -rings is strictly smaller than the class of rings which contain no infinite commutative domains and no non-zero nilpotents.* \square

Remark 23. *By Theorem 20, the class of rings over which a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be uniformly diagonalised “polynomially in a, b, c, d ” is indeed very restricted. If however one allows expressions that are “polynomial in a, b, c, d and their first order primes a', b', c', d' ”, one can uniformly diagonalise 2×2 matrices over, for example, all strongly regular rings (regular without nonzero nilpotents). A reworking of Example 8 confirms this.* \square

ACKNOWLEDGEMENT

The second author gratefully acknowledges support from the University of Canterbury and the NSERC (Canada).

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