ON THE EXISTENCE OF TOPOLOGICAL OVALS IN FLAT PROJECTIVE PLANES

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Abstract. We show that every flat projective plane contains topological ovals. This is achieved by completing certain closed partial ovals, the so-called quasi-ovals, to topological ovals.

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ABSTRACT. We show that every flat projective plane contains topological ovals. This is achieved by completing certain closed partial ovals, the so-called quasi-ovals, to topological ovals.

1. Introduction

The real Desarguesian projective plane is an (abstract) *projective plane*, that is, a point-line geometry satisfying the following two axioms.

(P1) Two distinct points p and q are contained in a unique line $p \vee q$.
(P2) Two lines L and M intersect in a unique point $L \wedge M$.

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By removing a line from this plane we arrive at the Euclidean plane. Every convex, differentiable simply closed curve in the Euclidean plane is an *oval* in the real projective plane, that is, a set of points such that

- (O1) every line in the projective plane intersects the set in no more than two points, and such that
- (O2) every point p of the set is contained in precisely one line that intersects the set only in p.

Actually, every oval that is homeomorphic to a circle arises in this manner as a convex, differentiable simply closed curve. A line that intersects an oval in no, 1, or 2 points is called *exterior line*, *tangent* or *secant* of the oval, respectively.

The real Desarguesian projective plane is the classical example of a *flat projective plane*, that is, a projective plane which shares the point set with the real Desarguesian projective plane and whose lines are homeomorphic to the circle $\mathbb{S}^1$. For non-isomorphic examples the reader is referred to [3] and [4, Chapter 3]. Using transfinite induction (cf. [2]) it is possible to construct ovals in such a plane. However, these ovals are not very 'nice' from a topological point of view. It is also known (see [1]) that an oval in a flat projective plane is a closed subset of its point set if and only if it is homeomorphic to a circle. Such an oval is called *topological* and automatically shares many of the properties of topological ovals in the real Desarguesian projective plane. We will mention some of these properties in Section 3 of this note. Topological ovals play a crucial role in the theory of flat incidence geometries, in particular in the theory of flat circle planes (cf. [5]). Our goal is to show that topological ovals exist in every flat projective plane.

A projective plane whose point set and line set carry topologies is called *topological* if the operations of joining two distinct points by a line and intersecting two distinct lines in a point are continuous in their respective domains of definition. It is possible to turn any flat projective plane into a topological projective plane in an essentially unique way. Its point set already carries a natural topology and the topology on the line set that does the trick is the so-called Hausdorff topology. The *dual* of a projective plane is the point-line geometry that we arrive at by exchanging the roles of points and lines. It is clear from axioms (P1) and (P2) and from what we just said that the dual of a (topological) projective plane is a (topological) projective plane. In particular, the dual of a flat projective plane is topological and if we regard a point as the set of all lines it is contained in, this dual can be seen to be a flat projective plane, too. For details the reader is referred to [3] and [4].

Here is a brief sketch of our construction of a topological oval in a flat projective plane: We first observe that the set of tangents of such an oval is a *dual oval*, that is, a topological oval in the dual projective plane (cf. [1]). This is in strong contrast to the situation for Mazurkiewicz' ovals (cf. [2]), where all tangents pass through one single point. A pair consisting of a point and a line that passes through the point is called a *flag*. Every point of an oval together with the tangent at this point is a flag. Any set of such flags associated with a topological oval is a quasi-oval (see Section 3). On the other hand, we will show that closed quasi-ovals exist in any flat projective plane, that it is possible to extend every closed quasi-oval by a flag to a larger quasi-oval, that this extension process can be done to eventually give a 'dense' and 'round' quasi-oval, and finally that the topological closure of such a quasi-oval is a topological oval.

For the rest of this note let $\mathcal{P}$ be a flat projective plane with point set P and line set $\mathcal{L}$. Note again that, as topological spaces, both sets are homeomorphic to the point set of the real Desarguesian projective plane. Furthermore, we denote the pencil of all lines through a point p by $\mathcal{L}_p$. If we remove a line from a flat projective plane, the incidence structure we are left with is a *flat affine plane.* Like the Euclidean plane, the point set of a flat affine plane is homeomorphic to $\mathbb{R}^2$ and all its lines are subsets of its point set separating the point set into two open components. Furthermore, as in any affine plane, two distinct points are contained in a unique connecting line and there is a unique parallel line to a given line through some point. Finally, two non-parallel lines in a flat affine plane intersect transversally, that is, they never just 'touch' in their common point. These facts lead to a convexity theory for flat affine planes, that is, a generalization of the convexity theory for the Euclidean plane. In this theory a subset of the point set of a flat affine plane is called convex if any two of its points can be connected by a line segment that is completely contained in the set. For details the reader is referred to [4, Chapter 31].

2. Half-planes, Intervals and Triangles

In this section we compile some further facts about flat projective planes that we will need in the following. All these facts are easy consequences of the convexity theory for flat affine planes.

Let L and M be lines and p, p_1, p_2 and p_3 be points of $\mathcal{P}$. For $L \neq M$ the set $P \setminus (L \cup M)$ consists of two connected components. We call each connected component a *half-plane* defined by L and M. More precisely, if $p \in P \setminus (L \cup M)$, denote the connected component containing p by $H^p_{L,M}$. The set $H^p_{L,M}$ is called the half-plane defined by L, M, and p (in Figure 1.1 the shaded region is this half-plane). For the closure of this half-plane we have $\overline{H^p_{L,M}} = H^p_{L,M} \cup L \cup M$.

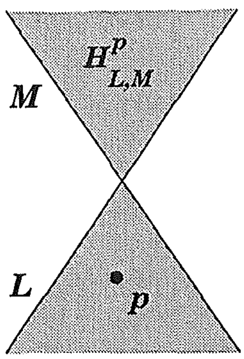

Figure 1.1

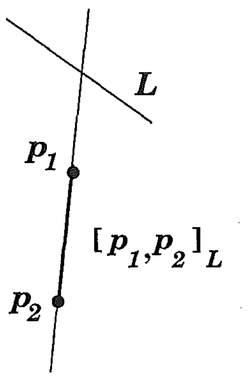

Figure 1.2

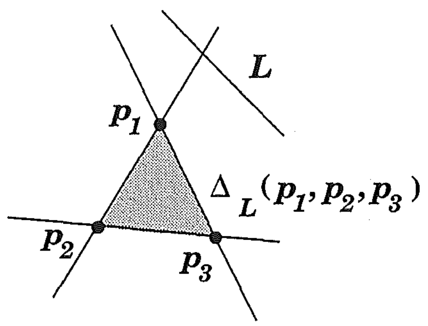

Figure 1.3

Every line of $\mathcal{P}$ is homeomorphic to $\mathbb{S}^1$. Let p_1 and p_2 be two distinct points. We call the two connected components of $(p_1 \vee p_2) \setminus \{p_1, p_2\}$ the *open intervals defined by* p_1 *and* p_2. Let L be a line that does contain neither p_1 nor p_2. We denote the open interval defined by p_1 and p_2 that does not contain the point $(p_1 \vee p_2) \wedge L$ by $]p_1, p_2[_L$ and define

$[p_1,p_2[_L := \{p_1\} \cup]p_1,p_2[_L$, $]p_1,p_2]_L := \{p_2\} \cup]p_1,p_2[_L$ and $[p_1,p_2]_L := \{p_1,p_2\} \cup]p_1,p_2[_L$ (see Figure 1.2). These sets are also connected, because $[p_1,p_2]_L = \overline{]p_1,p_2[_L}$. In analogy to the classical case, they are called *open*, *half-open*, and *closed intervals* defined by p_1 and p_2 relative to L. The set of all open intervals defined by p and points on L is denoted by $\mathcal{I}_{p,L}$ (*half-lines*). Dually we obtain $\mathcal{I}^*_{L,p}$, the set of all *half-pencils* (half-lines in the dual plane) defined by L and lines through p.

For p_1, p_2 and p_3, non-collinear and non-incident with L, the set $\Delta_L(p_1,p_2,p_3) := H^{p_1}_{p_2 \vee p_3, L} \cap H^{p_2}_{p_3 \vee p_1, L} \cap H^{p_3}_{p_1 \vee p_2, L}$ is called the *open triangle* defined by p_1, p_2 and p_3 relative to L, and the points p_1, p_2 and p_3 are called its *vertices* (see Figure 1.3). The intervals defined by the vertices of a triangle relative to the same line are called its *sides*.

The following lemma states some facts we will be using frequently [4, Chapter 31].

Lemma 1. *Let p_1, p_2 and p_3 be three non-collinear points, let L be a line containing none of these points, let $\Delta := \Delta_L(p_1,p_2,p_3)$, let p, q_1, q_2 and q_3 be points, and M, L_1, L_2 and L_3 be lines.*

(1) *Two points are contained in the same half-plane defined by the lines L_1 and L_2 if and only if L_1 and L_2 meet the same open interval defined by the points; see Figure 2.1.*

(2) *Δ, as a set of points, is open and $\partial\Delta = [p_1,p_2]_L \cup [p_2,p_3]_L \cup [p_3,p_1]_L$.*

(3) *If $q_1,q_2,q_3 \in \overline{\Delta}$ are three non-collinear points, then $\Delta_L(q_1,q_2,q_3) \subseteq \Delta$, in particular, $[q_1,q_2]_L \subseteq \overline{\Delta}$ (and $\overline{\Delta}$ and Δ are connected); see Figure 2.3.*

(4) *If $p \in \Delta$, $q \notin \overline{\Delta}$ and I is an open interval defined by p and q, then $|I \cap \partial\Delta| = 1$, in particular, if $M \cap \Delta \neq \emptyset$ then $|M \cap \partial\Delta| = 2$; see Figure 2.2.*

(5) *If $q_1 \in]p_2,p_3[_L$, $q_2 \in]p_1,p_3[_L$ and $q_3 \in]p_1,p_2[_L$, then $\Delta = \Delta_L(p_1,q_2,q_3) \cup \Delta_L(p_2,q_3,q_1) \cup \Delta_L(p_3,q_1,q_2) \cup \Delta_L(q_1,q_2,q_3) \cup]q_1,q_2[_L \cup]q_2,q_3[_L \cup]q_3,q_1[_L$ (this is a disjoint union); see Figure 2.3.*

(6) *If the lines L_1, L_2 and L_3 do not pass through the same point and C is a connected component of $P \setminus (L_1 \cup L_2 \cup L_3)$, then there is a line L such that $C = \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$.*

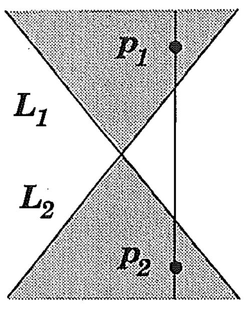

Figure 2.1

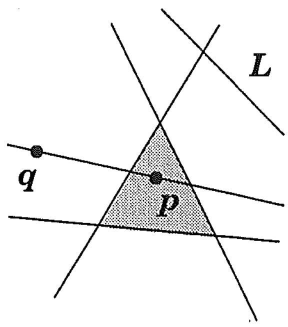

Figure 2.2

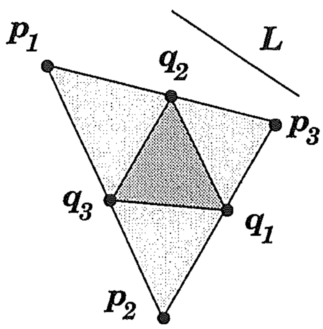

Figure 2.3

Note that two lines are contained in the same connected component of the line set minus the pencils of two points p_1 and p_2 if and only if the lines intersect $p_1 \vee p_2$ in the same open interval defined by the points.

3. Quasi-ovals

In this section we define quasi-ovals and collect some results about them. Let π_1 and π_2 be the projections from the set of flags of $\mathcal{P}$ to P and $\mathcal{L}$, respectively. A set $\mathcal{F}$ of flags in $\mathcal{P}$ is called a *quasi-oval* if there exists a line L of $\mathcal{P}$ such that

(QO) for every flag $(q, M) \in \mathcal{F}$ the set $\pi_1(\mathcal{F} \setminus \{(q, M)\})$ is contained in one of the two connected components of $P \setminus (L \cup M)$ (see Figure 3).

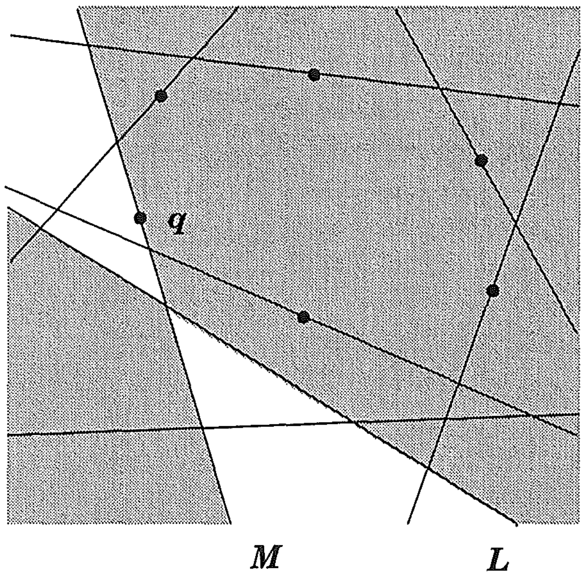

Figure 3

Example 1. We want to construct a quasi-oval containing three given points p_1, p_2 and p_3 that are not collinear. Choose a line L that does not contain any of the points. For $\{i, j, k\} = \{1, 2, 3\}$ we define $L_i := p_i \vee ((p_j \vee p_k) \wedge L)$. By Lemma 1.1 the set $\{(p_1, L_1), (p_2, L_2), (p_3, L_3)\}$ is a quasi-oval.

Note that we can extend a quasi-oval with less than three flags to a quasi-oval with three flags by choosing L to yield the above situation for the already given points and lines.

Example 2. For every topological oval $\mathcal{O}$ the set of pairs (q, M), such that $q \in \mathcal{O}$ and M is the tangent of $\mathcal{O}$ at q, is a quasi-oval. The condition (QO) is easily verified for an exterior line L of $\mathcal{O}$, because $\mathcal{O}$ minus one point is connected (see Figure 4.1). Furthermore, this quasi-oval is a maximal quasi-oval, that is, a quasi-oval that is not properly contained in any other quasi-oval. We will later construct the promised topological ovals by starting off with a quasi-oval from Example 1 and then adding new flags until we arrive at a maximal quasi-oval.

Some care has to be taken here, since not all maximal quasi-ovals yield ovals (see Figures 4.2 and 4.3 for examples of such maximal quasi-ovals). The quasi-oval in Figure 4.2 is an 'oval with a corner', that is, axiom (O1) is satisfied but the uniqueness of the tangent line (O2) does not hold for every point (in Figure 4.2 this only happens in the point where the otherwise smooth curve has a corner). The quasi-oval in Figure 4.3 is 'dual' to the one in Figure 4.2; again (O2) is violated.

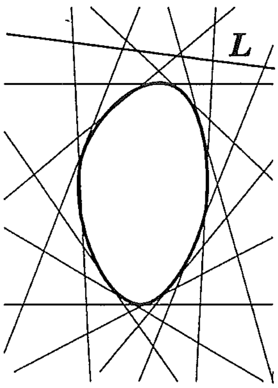

Figure 4.1

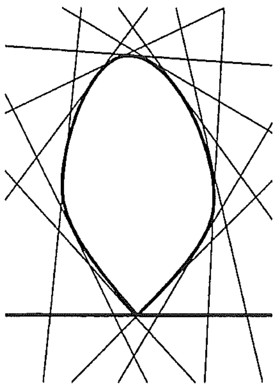

Figure 4.2

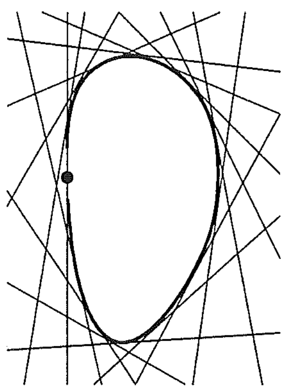

Figure 4.3

In analogy to the terminology of ovals, we call the points in $\pi_1(\mathcal{F})$ the *points of* $\mathcal{F}$, and the lines in $\pi_2(\mathcal{F})$ the *tangents of* $\mathcal{F}$.

Quasi-ovals and topological ovals have many properties in common. Here are some examples.

Proposition 1. *Let $\mathcal{F}$ be a quasi-oval and L a line such that (QO) holds. Then the following hold.*

(1) *Every tangent intersects the point set of $\mathcal{F}$ in exactly one point.*
(2) *Every line in the projective plane intersects the point set of $\mathcal{F}$ in no more than two points.*
(3) *Let (q_1, M_1) and (q_2, M_2) be two distinct flags in $\mathcal{F}$. Then every tangent intersects the line $q_1 \vee q_2$ in a point of $(q_1 \vee q_2) \setminus]q_1, q_2[_L$.*
(4) *For three points p_1, p_2 and p_3 of $\mathcal{F}$ the set $\overline{\Delta_L(p_1, p_2, p_3)} \setminus \{p_1, p_2, p_3\}$ and all tangents of $\mathcal{F}$ are disjoint. In particular, this set is contained in all the half-planes defined in (QO).*

Proof. (1) Suppose that there is a flag $(q, M) \in \mathcal{F}$ such that M intersects $\pi_1(\mathcal{F})$ in another point $q' \neq q$ of $\mathcal{F}$. Then $\pi_1(\mathcal{F} \setminus \{(q, M)\})$ contains the point q' whereas $P \setminus (L \cup M)$ does not. This contradicts (QO). Hence every tangent intersects $\pi_1(\mathcal{F})$ in exactly one point.

(2) Suppose that there is a line K that intersects $\pi_1(\mathcal{F})$ in at least three points p_1, p_2 and p_3. Let M_i, $i = 1, 2, 3$, be the corresponding tangents; these are distinct from K by (1). Since $K \neq L$, the point $q = K \wedge L$ is well-defined. We label the above points in such a way that q and p_2 are in different connected components of $K \setminus \{p_1, p_3\}$. By Lemma 1.1, the points p_1, p_3 are in different half-planes relative to L and M_2. This contradicts (QO). Hence every line intersects the point set of $\mathcal{F}$ in no more than two points.

(3) The statement is obviously true for M_1 and M_2. Let $M \neq M_1, M_2$ be a tangent of $\mathcal{F}$ and suppose that $M \wedge (q_1 \vee q_2)$ is in $]q_1, q_2[_L$. Then, by Lemma 1.1, the points q_1 and q_2 are in different half-planes determined by L and M. This contradicts (QO).

(4) Let M be a tangent of $\mathcal{F}$. By (3) it is disjoint to $]p_1,p_2[_L \cup]p_2,p_3[_L \cup]p_3,p_1[_L$. Suppose that M meets $\Delta_L(p_1,p_2,p_3)$. Then Lemmata 1.4 and 1.2 show that M meets $[p_1,p_2]_L \cup [p_2,p_3]_L \cup [p_3,p_1]_L$ in precisely two points, which is a contradiction to what we have said above. This shows the first assertion. The set under consideration plus one vertex of the triangle is connected by Lemma 1.3 and therefore contained in every half-plane (which is a connected component) defined by a tangent not containing this vertex. Now this is true for every vertex of the triangle and the second assertion follows. □

Proposition 1.4 implies that an open triangle defined by three points of a quasi-oval contains no points of the quasi-oval. On the other hand, all points of a quasi-oval are contained in the closure of a triangle 'formed' by tangents of the quasi-oval (cf. Lemma 2). In order to employ this observation we will now give another definition of quasi-ovals, which is more convenient to work with. Let $(\mathcal{F}_0, p, L)$ be a triple consisting of a set $\mathcal{F}_0$ of three flags (p_i, L_i), $i = 1,2,3$, of $\mathcal{P}$ together with an *anti-flag* (p, L), that is, $p \notin L$, such that all four lines and points under discussion are distinct and in general position, i.e., no three of the points are collinear and dually for the lines. This triple is called a *complete triangle* if and only if $p_i \in]L_i \wedge L_j, L_i \wedge L_k[_L$ for $\{i,j,k\} = \{1,2,3\}$ and $p \in \Delta_L(p_1,p_2,p_3)$, see Figure 5.

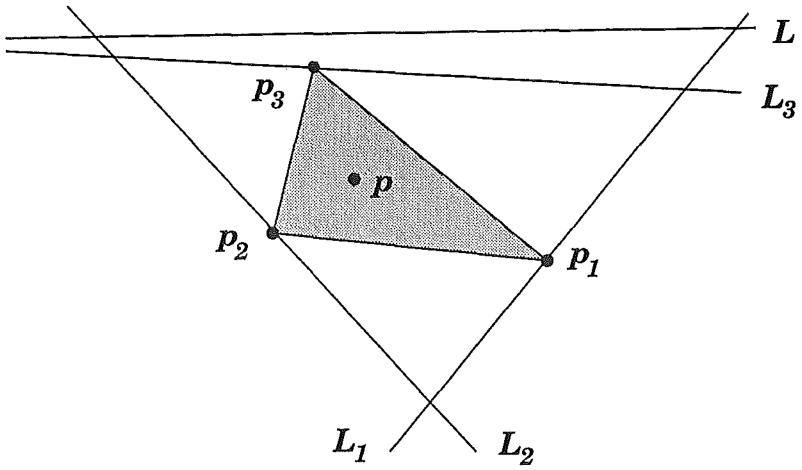

Figure 5

One can think of the line L of a complete triangle intersecting the three lines L_1, L_2 and L_3 as shown in Figure 5. Lemma 1.3 implies that $\Delta_L(p_1,p_2,p_3) \subseteq \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$; furthermore, the 'big triangle formed by the points $L_1 \wedge L_2$, $L_2 \wedge L_3$ and $L_3 \wedge L_1$' does not contain any point of L and the 'small triangle formed by the points p_1, p_2 and p_3' contains the point p.

A set $\mathcal{F}$ of flags in $\mathcal{P}$ is called a *quasi-oval with respect to a complete triangle* $(\mathcal{F}_0, p, L)$ if $\mathcal{F}_0 \subseteq \mathcal{F}$ and (QO) holds.

Note that the set $\mathcal{F}_0$ of flags of a complete triangle $(\mathcal{F}_0, p, L)$ is a quasi-oval with respect to $(\mathcal{F}_0, p, L)$. The following proposition shows that our two definitions of quasi-ovals are equivalent for sets with three or more flags.

Proposition 2. *Let $\mathcal{F}$ be a quasi-oval and $\mathcal{F}_0$ be a subset of $\mathcal{F}$ with three elements. Then there is an anti-flag (p, L) such that $(\mathcal{F}_0, p, L)$ is a complete triangle and $\mathcal{F}$ is a quasi-oval with respect to $(\mathcal{F}_0, p, L)$.*

Proof. Let (p_i, L_i), $i = 1, 2, 3$ be the three flags of $\mathcal{F}_0$. Let L' be a line such that (QO) holds for L' instead of L. For three points q_1, q_2 and q_3 of $\mathcal{F}$ the set $\tilde{\Delta}(q_1, q_2, q_3) := \overline{\Delta_{L'}(q_1, q_2, q_3)} \setminus \{q_1, q_2, q_3\}$ is connected by Lemma 1.3 and disjoint to all tangents of $\mathcal{F}$ by Proposition 1.4. The union of all these sets $\tilde{\Delta}(q_1, q_2, q_3)$ is a connected set C, which is disjoint to every tangent of $\mathcal{F}$. Hence C is contained in a connected component of $P \setminus (L_1 \cup L_2 \cup L_3)$. By Lemma 1.6, there is a line L such that this connected component equals $\Delta := \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$. Now $\tilde{\Delta}(p_1, p_2, p_3) \subseteq C \subseteq \Delta$ and therefore $p_1, p_2, p_3 \in \overline{\Delta}$. Since $p_i \in]L_i \wedge L_j, L_i \wedge L_k[_L$ for $\{i, j, k\} = \{1, 2, 3\}$, the triple $(\mathcal{F}_0, p, L)$ is a complete triangle if we choose $p \in \Delta_L(p_1, p_2, p_3)$.

It remains to verify (QO) for the line L. Let r and s be two points of $\mathcal{F}$. Because $]r, s[_{L'} \subseteq C \subseteq \Delta$ the open interval $]r, s[_{L'}$ and L are disjoint. Now the statement follows from (QO) for L' and Lemma 1.1. □

For a set $\mathcal{F}$ of flags we define $\mathcal{F}^* = \{(M, q) \mid (q, M) \in \mathcal{F}\}$. This is a set of flags in the dual plane. The following proposition shows that a quasi-oval $\mathcal{F}$ with respect to $(\mathcal{F}_0, L, p)$ is, essentially, a quasi-oval of the dual plane of $\mathcal{P}$.

Proposition 3. *Let $\mathcal{F}$ be a quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$. Then $\mathcal{F}^*$, that is, $\mathcal{F}$ with points and lines exchanged is a quasi-oval of the dual plane of $\mathcal{P}$ with respect to $(\mathcal{F}_0^*, L, p)$.*

Proof. We have to show that $(\mathcal{F}_0^*, L, p)$ is a complete triangle of the dual plane. Let $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 1.3, the line L_i and the side $]p_j, p_k[_L$ are disjoint whereas $p \vee p_i$ meets this side by Lemma 1.4. Hence L_i and $p \vee p_i$ are in different connected components of $\mathcal{L}_{p_i} \setminus \{p_i \vee p_j, p_i \vee p_k\}$. This proves that the first condition of a complete triangle is satisfied.

As mentioned after the definition of a complete triangle we have $p \in \Delta_L(p_1, p_2, p_3) \subseteq \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$ and therefore $]p, L_i \wedge L_j[_L \subseteq \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$ by Lemma 1.3. Thus L and L_k are in the same half-plane of the dual plane defined by p and $L_i \wedge L_j$ (see the remark after Lemma 1). This proves that L is an element of the triangle $\Delta_p(L_1, L_2, L_3)$ of the dual plane.

Let $(q, M) \in \mathcal{F}$. We have to show that every tangent of $\mathcal{F}$ except M intersects the line $p \vee q$ in a point of $(p \vee q) \setminus [p, q]_L$. This is an immediate consequence of Lemma 1.1 and Proposition 1.4, because $p \in \Delta_L(p_1, p_2, p_3)$. □

By the preceding proposition, every statement about quasi-ovals that we prove is also true with points and lines exchanged. For later use we highlight one observation made in the proof of Proposition 3.

Proposition 4. *Let $\mathcal{F}$ be a quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$ and let $(q, M) \in \mathcal{F}$. Then every tangent of $\mathcal{F}$ except M intersects the line $p \vee q$ in a point of $(p \vee q) \setminus [p, q]_L$. In particular, p belongs to all the half-planes defined in (QO).*

Lemma 2. *Let $\mathcal{F}$ be a quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$ where $\mathcal{F}_0 = \{(p_1, L_1), (p_2, L_2), (p_3, L_3)\}$. Let $\Delta_i := \Delta_L(p_j, p_k, L_j \wedge L_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $\pi_1(\mathcal{F}) \subseteq \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{p_1, p_2, p_3\}$. In particular, $\Delta := \Delta_L(p_1, p_2, p_3)$ contains no points of $\mathcal{F}$ and, furthermore, no tangent of $\mathcal{F}$ intersects Δ.*

Proof. It follows from Proposition 4 and $p \in \Delta \subseteq \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$ that

$$\begin{aligned} \pi_1(\mathcal{F}) \setminus \{p_1, p_2, p_3\} &\subseteq H^p_{L,L_1} \cap H^p_{L,L_2} \cap H^p_{L,L_3} \\ &= \Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1). \end{aligned}$$

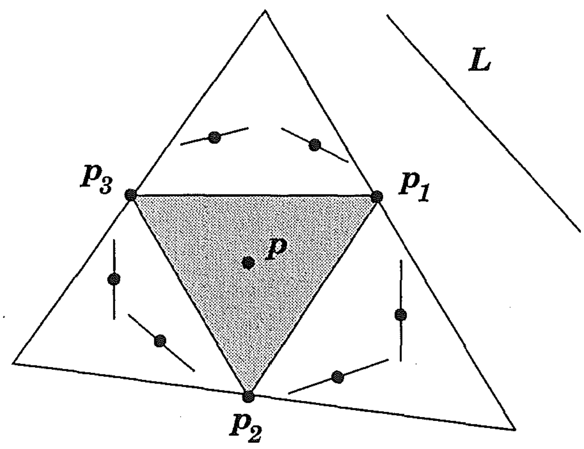

Figure 6

Now, the triangle $\Delta_L(L_1 \wedge L_2, L_2 \wedge L_3, L_3 \wedge L_1)$ is the disjoint union of Δ_1, Δ_2, Δ_3 and the closure of Δ minus its three vertices by Lemma 1.5; cf. Figure 6. The last set of this union is disjoint to all tangents of $\mathcal{F}$ by Proposition 1.4. This proves the second part of the statement and $\pi_1(\mathcal{F}) \subseteq \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{p_1, p_2, p_3\}$. □

Theorem 1. *Let $\mathcal{F}$ be a quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$. Assume that the set of open intervals of $\mathcal{I}_{p,L}$, the set of all open intervals defined by p and points on L, containing points of $\pi_1(\mathcal{F})$ is dense in $\mathcal{I}_{p,L}$, and assume the dual condition. Then $\overline{\mathcal{F}}$, the closure of $\mathcal{F}$ in the flag-space, is a quasi-oval with respect to $(\mathcal{F}_0, p, L)$, and $\pi_1(\overline{\mathcal{F}})$ is a topological oval.*

Proof. We first show that $\overline{\mathcal{F}}$ is a quasi-oval with respect to $(\mathcal{F}_0, p, L)$. Obviously, $\mathcal{F}_0 \subseteq \overline{\mathcal{F}}$. By Lemma 2, no points of $\overline{\mathcal{F}}$ are on L, and dually, no tangents of $\overline{\mathcal{F}}$ pass through p. Let $(q, M) \in \overline{\mathcal{F}}$.

Suppose that there are points $q_1, q_2 \in \pi_1(\overline{\mathcal{F}})$ that are in opposite half-planes relative to L and M. Because the set of lines intersecting $q_1 \vee q_2$ in a point of $]q_1, q_2[_L$ is open we can find a tangent $M' \in \mathcal{F}$ such that q_1 and q_2 are still in different half-planes relative to L and M'. Then, because half-planes are open, each of the above half-planes also contains a point of $\pi_1(\mathcal{F})$, say p_1 and p_2, which contradicts (QO) for $\mathcal{F}$. This shows that $\pi_1(\overline{\mathcal{F}})$ is entirely contained in the closure of the half-plane $H^p_{L,M}$ by Proposition 4.

Now suppose that there is a point $q_1 \neq q$ such that $q_1 \in M \cap \pi_1(\overline{\mathcal{F}})$. Since the set of open intervals of $\mathcal{I}_{p,L}$ containing points of $\pi_1(\mathcal{F})$ is dense in $\mathcal{I}_{p,L}$ and by the above, $\Delta_L(p,q,q_1)$ contains a point r of $\mathcal{F}$. Let N be the corresponding tangent of $\mathcal{F}$ at r. N cannot intersect $]q,q_1[_L$, as seen above. Hence N must pass through q or through q_1 or N intersects $]p,q[_L$ and $]p,q_1[_L$. In the latter case at least one of the points of the complete triangle is in the half-plane opposite the $H^q_{L,N}$ and we obtain a contradiction.

We can now assume that the tangents to all points of $\mathcal{F}$ in $\Delta_L(p,q,q_1)$ pass through q or through q_1. This then gives us a contradiction as before, since the dual of $\mathcal{F}$ is a quasi-oval. Hence $\overline{\mathcal{F}}$ is a quasi-oval with respect to $(\mathcal{F}_0,p,L)$.

By [1], it only remains to show that $\mathcal{O} := \pi_1(\overline{\mathcal{F}})$ is an oval, because $\mathcal{O}$ is closed. Axiom (O1) is just Proposition 1.2, and Proposition 1.1 guarantees the existence of a tangent at every point of $\mathcal{O}$. The map $\alpha : \mathcal{O} \to \mathcal{I}_{p,L}$, which assigns to a point of $\mathcal{O}$ the half-line of $\mathcal{I}_{p,L}$ containing it, is well-defined and continuous. Since $\mathcal{O}$ is compact and $\alpha(\mathcal{O})$ is dense in $\mathcal{I}_{p,L}$, the continuity of α shows that it is surjective. Hence every line through p, which is, essentially, the union of two half-lines, intersects $\mathcal{O}$ in two points. Therefore α is injective and thus a homeomorphism. Likewise, the map $\alpha^* : \pi_2(\overline{\mathcal{F}}) \to \mathcal{I}^*_{L,p}$, which assigns to a tangent of $\overline{\mathcal{F}}$ the half-pencil of $\mathcal{I}^*_{L,p}$ (the dual of $\mathcal{I}_{p,L}$) containing it, is a homeomorphism.

Now, let $(q,M) \in \overline{\mathcal{F}}$ and let M' be a line through q that meets $\mathcal{O}$ only in q. Because $\mathcal{O} \cong \mathcal{I}_{p,L} \cong \mathbb{S}^1$, we see that $\mathcal{O} \setminus q$ is connected. Hence $(\overline{\mathcal{F}} \setminus \{(q,M)\}) \cup \{(q,M')\}$ is also a quasi-oval. Furthermore $p \notin M'$, because every line through p intersects $\mathcal{O}$ in two points as we saw above. Since α^* is surjective, there is a flag $(r,N) \in \overline{\mathcal{F}}$ such that $\alpha^*(N)$ is the half-pencil containing M'. We assume that $N \neq M'$. Let $s = L \wedge M'$. Then the four lines L, $p \vee s$, M' and N all pass through s and M' and N are in the same component of $\mathcal{L}_s \setminus \{L, p \vee s\}$. Furthermore, because $\mathcal{L}_s \cong \mathbb{S}^1$, either L and M' are in different components of $\mathcal{L}_s \setminus \{p \vee s, N\}$ or L and N are in different components of $\mathcal{L}_s \setminus \{p \vee s, M'\}$. This means that in the former case, p and r are in different half-planes determined by L and M', and that in the latter case p and q are in different half-planes determined by L and N. However, since the quasi-ovals $(\overline{\mathcal{F}} \setminus \{(q,M)\}) \cup \{(q,M')\}$ and $\overline{\mathcal{F}}$ have the same points, Proposition 1.4 gives us a contradiction, because accordingly the points of the respective quasi-ovals and $p \in \Delta_L(p_1,p_2,p_3)$ are in the same half-planes. Therefore we have shown that $N = M'$ and so $M' = M$ equals the tangent of $\overline{\mathcal{F}}$ at q. This proves that there is a unique tangent of $\mathcal{O}$ at q. Hence $\mathcal{O}$ is an oval. □

4. The Construction

The aim of this section is to prove

Theorem 2. *Every closed quasi-oval can be extended to a quasi-oval whose point set is a topological oval.*

As already mentioned in the introduction, Theorem 2 and Example 1 together prove our main result. In fact, there are plenty of topological ovals in every flat projective plane.

Corollary. *For any three non-collinear points of a flat projective plane there is a topological oval passing through these three points.*

The following lemma is the main step in the proof of Theorem 2.

Lemma 3. *Let $\mathcal{F}$ be a closed quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$. Let*

$$\alpha : P \setminus (\{p\} \cup L) \to \mathcal{I}_{p,L}$$

be the map which assigns to each point r the open half-line I containing it. Let (q_1, M_1) and (q_2, M_2) be two elements of $\mathcal{F}$ such that $\alpha(\pi_1(\mathcal{F} \setminus \{(q_1, M_1), (q_2, M_2)\})$ is contained in one connected component of $\mathcal{I}_{p,L} \setminus \{\alpha(q_1), \alpha(q_2)\}$. Then, given a half-line R in the other component, there exists a flag (q, M) such that $q \in R$ and such that $\mathcal{F} \cup \{(q, M)\}$ is a quasi-oval with respect to $(\mathcal{F}_0, p, L)$.

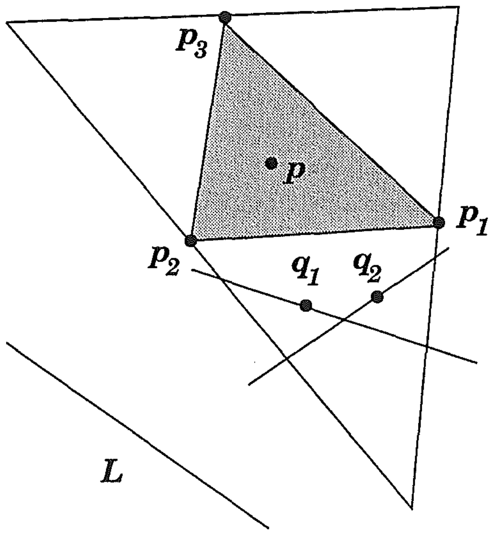

Figure 7

Proof. The strategy is simple: We choose a point $q \in R \cap \Delta_L(q_1, q_2, M_1 \wedge M_2)$ and a line M through q that intersects $q_1 \vee q_2$ in a point of $(q_1 \vee q_2) \setminus [q_1, q_2]_L$. However, we have to make sure that the above triangle is well-defined, that our choices of q and M exist and that a quasi-oval results. Before we do this in a number of steps, let us fix some notation. Let (p_1, L_1), (p_2, L_2), (p_3, L_3) be the flags in the set $\mathcal{F}_0$ and let $\Delta_0 := \Delta_L(p_1, p_2, p_3)$. Furthermore, let $\Delta_i := \Delta_L(p_j, p_k, L_j \wedge L_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $q_1, q_2 \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{p_1, p_2, p_3\}$ by Lemma 2.

We first show that q_1, q_2 are in the same extended triangle $\Delta_i \cup \{p_j, p_k\}$. Suppose otherwise, say $q_1 \in \Delta_1$ and $q_2 \in \Delta_3 \cup \{p_1\}$. Then $\alpha(q_1) \in \alpha(]p_2, p_3[_L)$ and $\alpha(q_2) \in \alpha(]p_1, p_2[_L)$ (compare step (2) below). Hence $\alpha(q_1)$ and $\alpha(q_2)$ are in different connected components of $\mathcal{I}_{p,L} \setminus \{\alpha(p_2), \alpha(p_3)\}$, because $\alpha(]p_2, p_3[_L)$ and $\alpha([p_1, p_2[_L)$ are disjoint by Lemma 1.4. This is a contradiction to the assumption of the Lemma.

In the following steps we assume that, without loss of generality, $q_1, q_2 \in \Delta_3 \cup \{p_1, p_2\}$; compare Figure 7.

(1) $M_1 \wedge M_2 \in \overline{\Delta_3}$ and $\Delta := \Delta_L(q_1, q_2, M_1 \wedge M_2)$ is well-defined: If q_1 or q_2 equals one of p_1 or p_2, then $M_1 \wedge M_2 \in \overline{\Delta_3}$, by Lemma 1.2 and Lemma 1.4. We now

consider the remaining case $q_1, q_2 \in \Delta_3$. By Proposition 1.3, the lines M_1 and M_2 are disjoint from $[p_1, p_2]_L$. Therefore, by Lemma 1.4, both lines meet $]p_1, L_1 \wedge L_2[_L$. Furthermore, by the dual of Proposition 1.2, they meet the set in different points. We may assume that M_1 meets $]p_2, L_2 \wedge M_2[_L$; see Figure 7. Because of axiom (QO), M_1 does not meet $[p_2, q_2]_L$. Therefore it meets the interval $]q_2, L_2 \wedge M_2[_L \subset M_2$, which, by Lemma 1.3, is also contained in $\overline{\Delta_3}$. Thus $M_1 \wedge M_2 \in \overline{\Delta_3}$.

(2) $\alpha(\Delta) = \alpha(]q_1, q_2[_L)$: Let $t \in \Delta$. Then $]p, t[_L$ contains a point of $]q_1, q_2[_L$. Hence $\alpha(\Delta) \subset \alpha(]q_1, q_2[_L)$. The sets Δ and $]q_1, q_2[_L$ are both connected. Since α is open and continuous and Δ is open, the images of the two sets are open intervals of $\mathcal{I}_{p,L} \cong \mathbb{S}^1$ (one contained in the other). All elements of $]q_1, q_2[_L$ are boundary points of Δ. Hence the two intervals $\alpha(\Delta)$ and $\alpha(]q_1, q_2[_L)$ have to coincide.

(3) $\Delta \subseteq H^p_{M_1,M_2}$: This readily follows from Proposition 4 and Lemma 1.1, since $\Delta \subseteq H^{q_1}_{L,M_2} \cap H^{q_2}_{L,M_1} \subseteq H^p_{L,M_2} \cap H^p_{L,M_1} \subseteq H^p_{M_1,M_2}$.

(4) If a line N meets Δ, but not $[q_1, q_2]_L$, then $N \cap H^p_{M_1,M_2} \subseteq \Delta$: Suppose that there are points $r \in N \cap \Delta$ and $s \in (N \cap H^p_{M_1,M_2}) \setminus \Delta$. Then, by (3) and Lemma 1.1, the interval $[r, s]_{M_1}$ is contained in $H^p_{M_1,M_2}$, and therefore, by Lemma 1.3, meets $]q_1, q_2[_L$. This contradicts the assumption of (4).

(5) $N \cap \Delta = \emptyset$ for all $N \in \pi_2(\mathcal{F})$: We can assume that $N \neq M_1, M_2$. Since, by Proposition 1.3, $N \cap [q_1, q_2]_L = \emptyset$, the assumption $N \cap \Delta \neq \emptyset$ implies $N \cap H^p_{M_1,M_2} \subseteq \Delta$ by (4) and $N \neq M_1, M_2$. Then the point q of $\pi_1(\mathcal{F})$ on N belongs to $H^p_{L,M_2} \cap H^p_{L,M_1} \subseteq H^p_{M_1,M_2}$ by (QO). Hence q must be contained in $\Delta \subseteq \Delta_3$ and therefore $\alpha(q) \in \alpha(]q_1, q_2[_L) \subseteq \alpha(]p_1, p_2[_L)$ by (2). But $\alpha(p_3) \notin \alpha([p_1, p_2]_L)$. Therefore, $\alpha(q)$ and $\alpha(p_3)$ are in different connected components of $\mathcal{I}_{p,L} \setminus \{\alpha(q_1), \alpha(q_2)\}$, which is a contradiction to the assumption of the Lemma.

(6) $C := \bigcup\{[r, s]_L | r, s \in \pi_1(\mathcal{F}), r \neq s\} \subseteq (H^p_{M_1,M_2} \cup \{q_1, q_2\}) \setminus \Delta$: By Propositions 1.3 and 4, $]r, s[_L \subseteq H^p_{L,M_2} \cap H^p_{L,M_1} \subseteq H^p_{M_1,M_2}$ for all $r, s \in \pi_1(\mathcal{F}), r \neq s$. Hence $C \subseteq H^p_{M_1,M_2} \cup \{q_1, q_2\}$. We now assume that there are points $r, s \in \pi_1(\mathcal{F}), r \neq s$, such that $[r, s]_L$ contains a point t of Δ. Then $t \neq r, s$ and $]t, r[_L,]t, s[_L \subseteq H^p_{M_1,M_2}$. Obviously, we have $\{r, s\} \neq \{q_1, q_2\}$. If $\{r, s\} \cap \{q_1, q_2\} = \emptyset$, then, by Lemma 1.3, the intervals $]t, r[_L$ and $]t, s[_L$ contain a point of $]q_1, q_2[_L$, which is not possible, because $r \vee s$ and $q_1 \vee q_2$ meet only in one point. In the remaining cases, we have one intersection point by assumption and would obtain a second one by an application of Lemma 1.3.

Now we choose a point $q \in R \cap \Delta$ (this is possible by (2)) and a line M through q that meets $q_1 \vee q_2$ in a point outside of $[q_1, q_2]_L$. By (4), this choice implies $M \cap H^p_{M_1,M_2} \subseteq \Delta$ and therefore M and C are disjoint, by (6). Furthermore, $C' := C \cup [q, q_1]_L \cup [q, q_2]_L \subseteq H^p_{M_1,M_2} \cup \{q_1, q_2\}$, by the choice of q and by (6). It follows from (5) that $N \cap C' = \{r\}$ for $(r, N) \in \mathcal{F} \cup \{(q, M)\}$. Now (QO) follows, since $C' \setminus \{r\}$ is connected. □

Proof of Theorem 2. Let $\mathcal{F}_1$ be a closed quasi-oval. If $\mathcal{F}_1$ has less than three elements we can add flags to $\mathcal{F}_1$ (as mentioned after Example 1) such that the set becomes a quasi-oval with respect to a complete triangle $(\mathcal{F}_0, p, L)$ by Proposition 2. Let $\alpha^* : \mathcal{L} \setminus (\{L\} \cup \mathcal{L}_p) \to \mathcal{I}^*_{L,p}$ be the map dual to α defined in Lemma 3. We have $\mathcal{I}_{p,L} \cong \mathcal{I}^*_{L,p} \cong \mathbb{S}^1$. Let D and

D^* be countable dense subsets of $\mathcal{I}_{p,L}$ and $\mathcal{I}^*_{L,p}$, respectively, and let $\gamma : \mathbb{N} \to D$ and $\gamma^* : \mathbb{N} \to D^*$ be bijections. Let $n \in \mathbb{N}$. We will construct a closed quasi-oval $\mathcal{F}_{n+1}$ by induction, such that $\gamma(n) \in \alpha(\pi_1(\mathcal{F}_{n+1}))$ and $\gamma^*(n) \in \alpha^*(\pi_2(\mathcal{F}_{n+1}))$ and $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$.

Assume that such an $\mathcal{F}_n$ has already been constructed. If $\gamma(n) \in \alpha(\pi_1(\mathcal{F}_n))$, we define $\mathcal{F}_{n+\frac{1}{2}} := \mathcal{F}_n$. Now suppose that $\gamma(n) \notin \alpha(\pi_1(\mathcal{F}_n))$. Since α and π_1 are continuous, and $\mathcal{F}_n$ is closed and hence compact, $\alpha(\pi_1(\mathcal{F}_n))$ is closed. We can therefore find half-lines $I_1, I_2 \in \alpha(\pi_1(\mathcal{F}_n))$, such that the connected component of $\mathcal{I}_{p,L} \setminus \{I_1, I_2\}$ containing $\gamma(n)$ contains no other element of $\alpha(\pi_1(\mathcal{F}_n))$. Now Lemma 3 (with (q_1, M_1) and (q_2, M_2) being the two flags of $\mathcal{F}_n$ with $q_1 \in I_1$ and $q_2 \in I_2$) yields an extension $\mathcal{F}_{n+\frac{1}{2}}$ of $\mathcal{F}_n$ such that $\gamma(n) \in \alpha(\pi_1(\mathcal{F}_{n+\frac{1}{2}}))$. Dually we can find an extension $\mathcal{F}_{n+1}$ of $\mathcal{F}_{n+\frac{1}{2}}$ as desired.

Since $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$ is a chain of quasi-ovals with respect to the same complete triangle $(\mathcal{F}_0, p, L)$, so is $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Furthermore, this quasi-oval satisfies the assumptions of Theorem 1. Hence $\overline{\mathcal{F}}$, the closure of $\mathcal{F}$ in the flag-space, is a quasi-oval and $\pi_1(\overline{\mathcal{F}})$ is a topological oval containing the points of the quasi-oval $\mathcal{F}_1$ we started off with.

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