

**ON FRÉCHET DIFFERENTIABILITY OF
SOME NON-LINEAR OPERATORS
OCCURRING IN INVERSE PROBLEMS;
AN IMPLICIT FUNCTION THEOREM
APPROACH**

by

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No. 56

June 19, 1990.

Abstract

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Submitted to: Inverse Problems, The Institute of Physics.

All communications concerning this paper to be addressed to: D.J.N. Wall.

Short Heading: On Fréchet Differentiability.

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1. Introduction

Our interest in this paper is in proving certain regularity results of the non-linear operators occurring naturally in inverse or identification problems. We emphasise that we do so in order to justify the use of computational techniques which are widely used in the solution of such problems. In order for the non-linear operator approach to succeed the minimum requirement must be that the measurements depend continuously upon the function, say ν , to be identified, that is if T is the non-linear operator mapping ν onto the quantity to be measured then it must be continuous. Knowledge of this continuity provides information on the topology of the function spaces upon which T acts. For example if T is continuous from some space only into L^2 it would be no use in utilising point measurements to reconstruct the function to be identified. As the inverse problem is improperly posed, regularisation methods must be utilised for its solution, it then follows that continuity is also important to obtain the required regularisation results; then continuity — in a weak sense — of the inverse of T may be required.

To use a numerical method with rapid convergence properties in the solution of the inverse problem we must require higher regularity properties on T than just continuity. In particular to generate an affine approximation to T it is required that T be Fréchet differentiable. To obtain high order convergence properties of the numerical method this Fréchet derivative must also be Lipschitz continuous.

In this paper we shall show how the *implicit function theorem* can be readily utilised in proving Fréchet differentiability of the appropriate operator in many identification problems. This has been hinted at in previous treatments [6], [20], however they use it only for particular interior measurement problems. Here we extend its usefulness to general situations such as exterior and boundary measurement identification problems. Then we show how the Fréchet differentiability result is readily extended to examine Lipschitz continuity properties of the operator. The versatility

of the implicit function theorem approach for proving Fréchet differentiability and continuity of these non-linear operators does not seem to have been appreciated in the literature where a variety of different techniques have been used to obtain such results.

The first work on proving Fréchet differentiability of a non-linear operator for an inverse problem appears to have been performed in the geophysics literature, Woodhouse [32] considered the expression for the Fréchet derivative of the inverse problem of free oscillations in the Earth. These were derived formally from a variational principle of Rayleigh by Backus and Gilbert [3]. Woodhouse showed that the expression for the Fréchet derivative was not valid when the function to be identified was discontinuous and square integrable, that is existing in L^2 . Parker [25] pointed out if the choice of function space was different, such as L^∞ the Fréchet derivative expression might be valid. These difficulties and a further comment by Anderssen [2] motivated Parker [25] to re-examine the existence of the Fréchet derivative for an inverse problem of the layered Earth which he had derived formally earlier [24]. In all the mentioned examinations the existence of the Fréchet derivative are not carried out in a rigorous manner, also see more recently [5].

In more recent times Fréchet differentiability and continuity results for the non-linear operators from various inverse problems have appeared in the mathematical literature. Apart from the aforementioned work [6] and [20], the results have been proved directly without the use of the implicit function theorem; see for example [10] and [26].

The computational methods for solution of the non-linear operator equations are based on an affine approximation of the operator equation. This affine approximation is required for the two standard methods used — the Newton-Kantorovich method and the optimisation method — this is because with the latter, the standard Gauss-Newton approximation to the Hessian is commonly used [15, chapter 10].

The following are some of the advantages which accrue to the non-linear operator approach to inverse problems.

- (i) It is a method which is applicable to a variety of inverse problems for any type of differential equation and any number of spatial dimensions. As well as determining a spatially varying coefficient in a differential equation, the Newton-Kantorovich method and its variants may be used to solve inverse boundary scattering problems.
- (ii) The approach extends approximate methods of solution, such as the Born approximation, into algorithms providing solutions to the full non-linear problem.
- (iii) Regularisation methods giving the existence and stability of solutions in the presence of measurement noise may be incorporated.
- (iv) Non-linear operator methods may be used to reconstruct functions which are not very smooth. All that is required is a suitable regularity result, allowing one to prove Fréchet differentiability. Other methods for solving inverse problems often require fairly strong smoothness assumptions

on the function being reconstructed [12], [27].

- (v) Importantly for many inverse problems, at present, it seems there is no other way of reconstructing an arbitrary function, that is no direct (non-iterative) method of solution is known.

There is a need to examine theoretical questions in addition to a numerical solution of the inverse problem. There are numerous iterative schemes for the solution of inverse problems in the literature which have been derived in an ad-hoc manner. Many of these on further investigation turn out to be variants of the Newton-Kantorovich method — or perhaps gradient methods. Some authors in the engineering literature derive what they call Newton-Kantorovich methods, but neglect to prove Fréchet differentiability. Such knowledge gives an idea of how the iterative scheme can be expected to behave, and how it may be possible to go about improving its performance. In addition, many approximate methods for the solution of these problems are obtained in an ad-hoc manner. These often may be formalised as a linearisation of the non-linear operator equation about some simple approximation — often a constant function.

We shall not consider the application of the implicit function theorem to boundary scattering problems here — although it has been applied to such problems [9].

Once Fréchet differentiability has been proven, other properties of the non-linear operator can be easily found. We illustrate this in the sequel where applicable. It is important to observe that whatever formulation is utilised in showing Fréchet differentiability of the non-linear operator, the result will hold for any other formulation of the same mapping, however some forms of explicit formulae for the Fréchet differential may be more suitable computationally than others, see for example (3.6) and (3.7).

We shall illustrate the application of our technique to two problems widely examined in the inverse problem literature. Both of which involve the identification of a spatially varying parameter in a partial differential equation. The first problem being examined in §3 and is described through the modified Helmholtz equation and the second problem is examined in §4 and is the time independent diffusion equation.

The two Fréchet differentiability results proven in §3 are complementary. The result of §3.1 gives differentiability for a continuous refractive index of arbitrary distance from unity. The result of §3.3, only requires the refractive index to be square integrable however it must be sufficiently close to unity. §3.2 shows how the results of §3.1 can be extended to examine the regularity properties of a boundary measurement inverse problem — namely identification of the refractive index from measurements of the far-field amplitude. These three results illustrate the essential *characteristics* of the implicit function theorem approach. In §4 we examine an inverse problem associated with the time-independent diffusion equation which has been extensively examined in earlier works, see [10] for some discussion of the relevant literature. In §4.1 after examining the Fréchet differentiability, we show that the non-linear operator is a compact mapping and hence is ill-posed, this implies that any direct attempt at linearization will result in an unstable solution. In §4.2 we extend the

results available in the literature for the boundary measurement inverse problem.

2. Preliminaries

For the readers convenience we quote the well known implicit function theorem which is central to our approach.

THEOREM 2.1. (Implicit Function Theorem) *Consider the functional $\xi(v, y)$, $v \in X$, $y \in Y$, where $\xi : X \times Y \mapsto W$ with X, Y, W being Banach spaces. Then suppose that there exists an open subset $X_0 \subset X$, such that for every $v \in X_0$ the equation $\xi(v, y) = 0$ has a unique solution $y = y(v)$ in Y , it then follows*

(a) *The map $v \rightarrow y(v)$, $v \in X_0$, $y \in Y$ defined from $X_0 \rightarrow Y$ is continuous upon satisfaction of the additional assumptions*

(i) *$\xi(v, y)$ is continuous,*

(ii) *$\xi_y(v, y)$ is continuous in v and y ,*

(iii) *$[\xi_y(v, y)]^{-1}$ exists as a bounded mapping $W \rightarrow Y$.*

(b) *Moreover the map is Fréchet differentiable if the conditions in (a) are satisfied and also $\xi_v(v, y)$ is continuous in v and y , with*

$$y'(v) = -[\xi_y(v, y)]^{-1}\xi_v(v, y).$$

Proof: See [33, §12.4]. □

Observe this is the strong form of implicit function theorem in that the Fréchet derivative is given at all $v \in X_0$; the basic form of implicit function theorem only provides information at a point $v_0 \in X_0$. We shall need the following corollary to handle inverse problems in which the measured quantity is a functional of the solution of the direct problem.

COROLLARY 2.1. *If an operator $B(v, y)$ is continuously differentiable and a function $\xi(v, y)$ exists as in Theorem 2.1 then $B(v, y(v))$ is Fréchet differentiable with respect to v . Moreover*

$$B'(v) = B_v(v, y) + B_y(v, y)y'(v),$$

where $y'(v)$ is as given in Theorem 2.1.

Proof: This follows from the chain rule and Theorem 2.1. □

The inverse problem is one of identification of v given an approximate measurement to y , whereas the solution of $\xi(v, y) = 0$ for y , given v , is generally termed the direct problem. Observe

that the assumptions (i) – (iii) of Theorem 2.1 require suitable knowledge of the regularity properties of the direct problem.

Sometimes the nature of the continuity of the non-linear operator is required, for example the Lipschitz constant of Lipschitz continuity. The Lipschitz continuity result of the operator T in our context gives a bound on the change in the solution of the direct problem that results from a change in the coefficient ν . We note such a result for the operator T^{-1} would be very desirable for the inverse problem, but is not one which is obtainable in general, compare with §4.1. The Lipschitz continuity may be easily obtained from the appropriate Fréchet differentiation result via and the mean value theorem for operators which we now quote.

THEOREM 2.2. (Mean Value Theorem) *If X, Y are Banach spaces and D a convex subset of X , then if $B : D \mapsto Y$ is Fréchet differentiable at every point of D , then*

$$\|B(f) - B(g)\|_Y \leq L\|f - g\|_X,$$

where $f, g \in D$ and the Lipschitz constant $L = \sup_{h \in D} \|B'(h)\|_Y$.

Proof: See for example [33, p265, et. seq.], or [18, Lemma 4.47]. □

For use in the sequel we define our standard the operator formulation of the boundary measurement inverse problem as

$$T(v) = B(v, y(v)) = \Sigma, \quad x \in M,$$

here Σ are the measurements of y taken from a set M on the surface S and where the support of v is $\Omega \subset \mathbb{R}^n$ with the boundary of Ω sometimes being S .

3. Modified Helmholtz equation — penetrable wave scattering

The inverse problem under consideration in this section is one of identifying a spatially varying refractive index, n , from the modified Helmholtz equation.

The direct scattering problem described by

$$(\Delta + k^2 n^2(x))u(x) = 0, \quad x \in \mathbb{R}^n, \quad n \in \{2, 3\}, \quad (3.1)$$

with the compact set $\{\text{supp } n = \bar{\Omega} \subset \mathbb{R}^n\}$, and where the total wave function u can be decomposed as

$$u = u^{inc} + u^s. \quad (3.2)$$

The incident wave u^{inc} is produced by sources exterior to $\bar{\Omega}$ so it thereby satisfies a homogeneous Helmholtz equation with constant wave number k within Ω . It therefore follows that u satisfies a non-homogeneous Helmholtz equation with wave number k and the non-homogeneous term $-k^2(n^2 - 1)$ is considered as specifying the source of u . In (3.2) u^s is the scattered wave which can be considered as arising from the presence of the scattering centre Ω with refractive index specified. The scattered wave function must also satisfy the radiation condition

$$\frac{\partial u^s}{\partial r} - ik u^s = o(r^{(1-n)/2}), \quad r \rightarrow \infty,$$

with $r = |x|$ and a suppressed time dependence $\exp(i\omega t)$ of u assumed. The standard representation theorem applied to (3.1) then shows that an integral equation for u may be written as

$$u(x) = u^{inc}(x) + k^2 \int_{\Omega} (n^2 - 1)\gamma(x, x')u(x') dV', \quad x \in D, \quad (3.3)$$

where \bar{D} is a compact region in \mathbb{R}^n , excluding sources of u^{inc} , with $\bar{\Omega} \subset D$ and

$$\gamma = \begin{cases} \frac{\exp(ikR)}{4\pi R} & \text{in } \mathbb{R}^3, \\ \frac{i}{4} H_0^{(1)}(kR) & \text{in } \mathbb{R}^2, \end{cases}$$

$$R = |x - x'|.$$

It is important to appreciate that the integral equation (3.9) has been obtained by utilising the assumption that the refractive index is continuous in \mathbb{R}^n . If this is not the case, transmission conditions must be added to the specification of the problem, and under some conditions surface integral terms occur in the integral equation; see [29]. For later convenience we define the linear integral operators

$$\mathbb{K} = k^2 \int_{\Omega} \gamma(x, x') dV', \quad \mathbb{K}_{\nu} = \mathbb{K}(\nu - 1),$$

and set $\nu = n^2$. Then the solution of the direct problem is also the solution of

$$(I - \mathbb{K}_{\nu})u = u^{inc}. \quad (3.4)$$

3.1: Integral equation solution. To study the inverse problem we first need existence and regularity results for the direct problem. These are provided in Theorem 3.1.

THEOREM 3.1. *If $n \in C^0(\bar{\Omega})$ and $u^{inc} \in C^1(\bar{D})$ then there exists a unique solution $u \in C^1(\bar{D})$ of (3.4). Moreover*

$$\|u\|_{C^1(\bar{D})} \leq C \|u^{inc}\|_{C^1(\bar{D})},$$

for some constant C .

Proof: The existence and boundedness results follows from the Fredholm alternative theorem applied to (3.4). To apply this the compactness of $\mathbb{K}_\nu : C^1(\bar{D}) \mapsto C^1(\bar{D})$ is required. As \mathbb{K}_ν is an integral operator with a weak singularity it follows $\mathbb{K}_\nu : C^0(\bar{D}) \mapsto C^1(\bar{D})$ is bounded [22, p159]. Then as the imbedding $\mathbb{I} : C^1(\bar{D}) \mapsto C^0(\bar{D})$ is compact, and the composition of a bounded operator and a compact operator is compact, namely $\mathbb{K}_\nu \mathbb{I}$, we have the desired compactness. To complete the proof, the Fredholm theorem shows uniqueness implies existence, and the required uniqueness is given by [31] — the work of [21] is used in obtaining it. \square

The functional

$$\xi(\nu, u) = u(\nu; x) - \mathbb{K}_\nu u(\nu; x) - u^{inc}(x) = 0, \quad (3.5)$$

which is obtained from the integral representation for the direct problem equation (3.4) can be utilised with the implicit function theorem to obtain the Fréchet derivative of the mapping $\nu \rightarrow u$. Here in the interior inverse problem u is to be measured throughout D . First appropriate function spaces for the mapping ξ must be defined, so note $\xi : X_0 \times C^1(\bar{D}) \mapsto C^1(\bar{D})$ with $X_0 = \{\nu : \nu \in C^0(\bar{\Omega}), \nu > 0\}$.

THEOREM 3.2. *The map $\nu \rightarrow u$ from X_0 to $C^1(\bar{D})$ is Fréchet differentiable, with Fréchet differential*

$$u'(\nu)s = [I - \mathbb{K}_\nu]^{-1} \mathbb{K}u(\nu)s \quad (3.6)$$

$$= \int_{\Omega} G(\nu; x, x') u(\nu; x') s(x') dV', \quad (3.7)$$

where $G(\nu; x, x')$ is the Green function pertinent for the refractive index ν , see (3.9).

Proof: Observe $u'(\nu)$ is a linear operator, the Fréchet derivative with $s \in X_0$. To prove differentiability we check the conditions of the implicit function theorem. First Theorem 3.1 assures us there is only one solution $u(\nu)$ in $C^1(\bar{D})$ then:

Condition (i). To show ξ is continuous in ν and u let us consider

$$\begin{aligned} \delta\xi &= \xi(\nu + \delta\nu, u + \delta u) - \xi(\nu, u) \\ &= \delta u - \mathbb{K}(\delta\nu u + (\nu - 1)\delta u + \delta\nu\delta u), \end{aligned}$$

then

$$\|\delta\xi\|_{C^1(\bar{D})} \leq \|\delta u\|_{C^1(\bar{D})} + \|\mathbb{K}\| \|\delta\nu u + (\nu - 1)\delta u + \delta\nu\delta u\|_{C^0(\bar{\Omega})},$$

with the standard operator norm

$$\|\mathbb{K}\| = \sup \frac{\|\mathbb{K}u\|_{C^1(\bar{D})}}{\|u\|_{C^0(\bar{\Omega})}} \quad (3.8)$$

being used for \mathbb{K} . It then follows

$$\begin{aligned} \|\delta\xi\|_{C^1(\overline{D})} &\leq \|\delta u\|_{C^1(\overline{D})} + \|\mathbb{K}\|(\|\delta\nu\|_{C^0(\overline{\Omega})}\|u\|_{C^1(\overline{D})} + \|\nu - 1\|_{C^0(\overline{\Omega})}\|\delta u\|_{C^1(\overline{D})}) \\ &\quad + \|\delta\nu\|_{C^0(\overline{\Omega})}\|\delta u\|_{C^1(\overline{D})}, \end{aligned}$$

on using $\|u\|_{C^0(\overline{\Omega})} \leq \|u\|_{C^1(\overline{D})}$.

The limit $\|\delta\nu\|, \|\delta u\| \rightarrow 0$ then gives the result.

Condition (b) and (ii). To show ξ_ν is continuous in ν and u consider the partial Fréchet derivative of (3.5), which follows as

$$\xi_\nu(\nu, u)s = -\mathbb{K}us$$

because (3.5) is linear in ν , also note that $s \in C^0(\overline{\Omega})$.

Then

$$\begin{aligned} \|\delta\xi_\nu\|_{C^1(\overline{D})} &= \|\xi_\nu(\nu + \delta\nu, u + \delta u)s - \xi_\nu(\nu, u)s\|_{C^1(\overline{D})} = \|\mathbb{K}\delta us\|_{C^1(\overline{D})} \\ &\leq \|\mathbb{K}\| \|\delta us\|_{C^0(\overline{\Omega})} \\ &\leq \|\mathbb{K}\| \|\delta u\|_{C^1(\overline{D})} \|s\|_{C^0(\overline{\Omega})}. \end{aligned}$$

It therefore follows ξ_ν is continuous in ν and u .

The partial Fréchet derivative of ξ with respect to u is

$$\xi_u(\nu, u)s = (I - \mathbb{K}_\nu)s$$

as (3.5) is linear in u , and with $s \in C^1(\overline{D})$. ξ_u can be shown to be continuous in ν and u in a similar manner to ξ_ν .

Condition (iii). It follows $[\xi_u(\nu, u)]^{-1}$ is bounded from Theorem 3.1.

The explicit expression for the Fréchet derivative is given by the implicit function theorem, and (3.6) can be arranged by use of the integral equation satisfied by the Green function $G(\nu; x, x')$ — the fundamental solution of (3.1) — namely

$$G(\nu; x, x') = \gamma(x, x') + k^2 \int_{\Omega} [\nu(x'') - 1] \gamma(x, x'') G(\nu; x'', x') dV'', \quad x, x' \in D, \quad (3.9)$$

to yield equation (3.7). □

3.2: The far-field inverse problem. We now show how the results from §3.1 can be used for the boundary measurement inverse problem where now u is to be measured on a ball in the far-field. Consider an incident plane wave $u^{inc} = \exp(i\mathbf{k}^i \cdot \mathbf{x})$ where the direction of the incident wave vector is given by \mathbf{k}^i and $k = |\mathbf{k}|$. We shall denote $kx/|x|$ by \mathbf{k}^s , then the asymptotic far-field behaviour is given by

$$u^s(\mathbf{x}) = \frac{e^{ikr}}{r} g(\nu; \mathbf{k}^i, \mathbf{k}^s)$$

with the complex scattering amplitude $g(\nu; \underline{k}^i, \underline{k}^s)$ described by

$$g(\nu; \underline{k}^i, \underline{k}^s) = \frac{k^2}{4\pi} \int_{\Omega} e^{-i\underline{k}^s \cdot \underline{x}'} [\nu(x') - 1] u(x') dV'.$$

The non-linear mapping to be considered is $T : \nu \rightarrow \tilde{g}$

$$g(\nu; \underline{k}^i, \underline{k}^s) - \tilde{g}(\underline{k}^i, \underline{k}^s) = 0,$$

where \tilde{g} are the measured values of $g(\nu^*)$, ν^* being the true refractive index to be determined.

THEOREM 3.3. *The map $\nu \rightarrow g(\nu; \underline{k}^i, \underline{k}^s)$ from $X_0 \rightarrow \mathbb{R}$ is Fréchet differentiable with the Fréchet differential*

$$g'(\nu)s = \frac{k^2}{4\pi} \int_{\Omega} u(\nu; \underline{k}^i) u(\nu; -\underline{k}^s) s dV. \quad (3.10)$$

Proof: If

$$B(\nu, u) = g(\nu; \underline{k}^i, \underline{k}^s),$$

with $\Sigma = \tilde{g}$ then from Corollary 2.1

$$g'(\nu)s = \frac{k^2}{4\pi} \int_{\Omega} e^{-i\underline{k}^s \cdot \underline{x}'} u(\nu; \underline{k}^i) s(x') dV' + \frac{k^2}{4\pi} \int_{\Omega} e^{-i\underline{k}^s \cdot \underline{x}'} [\nu(x') - 1] u'(\nu; \underline{k}^i) s(x') dV'. \quad (3.11)$$

The second integral can be written as

$$\frac{k^2}{4\pi} \int_{\Omega} [k^2 \int_{\Omega} G(\nu; x', x'') [\nu(x') - 1] e^{-i\underline{k}^s \cdot \underline{x}''} dV'] u(\nu; \underline{k}^i, x'') s(x'') dV''$$

on use of (3.7) and Fubini's theorem.

But

$$u(\nu; \underline{k}^s, x'') - e^{-i\underline{k}^s \cdot \underline{x}'} = k^2 \int_{\Omega} G(\nu; x', x'') [\nu(x') - 1] e^{-i\underline{k}^s \cdot \underline{x}'} dV'$$

so that using the symmetry of the Green function $G(\nu; x', x'') = G(\nu; x'', x')$ it follows that the second integral can now be written as

$$\frac{k^2}{4\pi} \int_{\Omega} [u(\nu; -\underline{k}^s, x'') - e^{-i\underline{k}^s \cdot \underline{x}''}] u(\nu; \underline{k}^i, x'') s(x'') dV''$$

so on use of this expression in (3.11) we find (3.10).

This result implies the non-linear operator T is both differentiable and continuous and in the next section we will illustrate how it is possible to imply Lipschitz continuity of such an operator.

3.3: Born Series Solution. One method of solving the direct scattering problem is via successive approximations to the integral equation (3.4), or equivalently via the Neumann series. The resulting series for this problem is sometimes known as the Born series and its truncation

after one term as the Born approximation. As we will show this approach converges when the wavenumber is small enough, and when the refractive index does not vary too far from unity (in the L^2 norm). Extensive numerical solutions of the direct problem has been made by use of the Born approximation in both the electrical engineering and optical literature. In two dimensions [16] and three dimensions [13] use has been made of this approach to solve the inverse scattering problem.

We shall formalise the Born approximation used in the inverse problem solution as the Fréchet derivative (that is the formal linearisation) of the refractive index to field map about a unity refractive index. This view differs from that of most authors who consider the Born approximation as an approximate solution of the direct problem, and then use this to obtain an approximate solution of the inverse problem.

Again we must define appropriate function spaces — with $\nu \in L^2(\Omega)$ and $u \in C^0(\bar{\Omega})$ — the boundedness of the operators $\mathbb{K} : L^2(\Omega) \mapsto C^0(\bar{\Omega})$ and $\mathbb{K}_\nu : C^0(\bar{\Omega}) \mapsto C^0(\bar{\Omega})$ follows from the weak singularity of their kernels. We shall need operator norms other than (3.8), on the integral operators involved, so we shall define the standard uniform operator norm of \mathbb{K}_ν by $\|\mathbb{K}_\nu\|_\infty$ and the operator norm for \mathbb{K} as

$$\|\mathbb{K}\| = \sup_{\|u\|_{2,\Omega}=1} \|\mathbb{K}u\|_{C^0(\bar{\Omega})}.$$

It immediately follows from these norms that

$$\|\mathbb{K}_\nu\|_\infty \leq \|\mathbb{K}\| \|\nu - 1\|_{2,\Omega}. \quad (3.12)$$

We shall consider the operator $\tilde{\mathbb{K}}$ is defined as \mathbb{K} , but now mapping $L^2(\Omega) \rightarrow C^0(\bar{D})$, with the standard operator norm.

THEOREM 3.4. *If $\|\mathbb{K}_\nu\|_\infty < 1$ then there exists a unique solution $u \in C^0(\bar{\Omega})$ to the integral equation (3.4) with*

$$u = (I + \sum_{n=1}^{\infty} \mathbb{K}_\nu^n) u^{inc} \quad (3.13)$$

and

$$\|u\|_{C^0(\bar{\Omega})} \leq \|u^{inc}\|_{C^0(\bar{\Omega})} / (1 - \|\mathbb{K}_\nu\|_\infty). \quad (3.14)$$

Proof: This follows directly from application of Banach's lemma to (3.4) (see [18]). \square

The Born approximation to the solution is given by the first iterate in (3.13), namely

$$u = (I + \mathbb{K}_\nu) u^{inc}, \quad (3.15)$$

with error

$$\|u - (I + \mathbb{K}_\nu) u^{inc}\|_{C^0(\bar{\Omega})} \leq \|\mathbb{K}_\nu\|_\infty^2 \|u^{inc}\|_{C^0(\bar{\Omega})} / (1 - \|\mathbb{K}_\nu\|_\infty).$$

Now we shall need the following lemma, which is an extension of a result of Colton [7, p 40] to give sufficiency conditions that $\|\mathbb{K}_\nu\|_\infty < 1$ in Theorem 3.3. We shall restrict the remainder of this section to the three dimensional case, that is in \mathbb{R}^3 . Assume first that Ω is a ball, that is $\Omega = \{x \in \mathbb{R}^n : |x| \leq a\}$, there is no difficulty if Ω is not a ball, then the smallest ball containing Ω may be considered.

LEMMA 3.1. *Let $\mu = \|\nu - 1\|_{2,\Omega}$, then $\|\mathbb{K}_\nu\|_\infty < 1$ whenever $k^2 < 1/(\mu a)$.*

Proof: Now

$$|(\mathbb{K}_\nu u)(x)| = k^2 \left| \int_\Omega \gamma(x, x') [\nu(x') - 1] u(x') dV' \right|,$$

and

$$\begin{aligned} |(\mathbb{K}_\nu u)(x)| &\leq \frac{k^2}{4\pi} \|u\|_{C^0(\bar{\Omega})} \int_\Omega \frac{|\nu(x') - 1|}{|x - x'|} dV' \\ &\leq \frac{k^2 \mu}{4\pi} \|u\|_{C^0(\bar{\Omega})} \int_\Omega \frac{dV'}{|x - x'|^2} \end{aligned}$$

from using the Schwartz inequality. Now from [22, pp159] the integral on the right-hand side is less than or equal to $4\pi a$ hence it follows $|(\mathbb{K}_\nu u)(x)| \leq k^2 \mu a \|u\|_{C^0(\bar{\Omega})}$ or $\|\mathbb{K}_\nu\|_\infty \leq k^2 \mu a$ and the result follows. The analogous result for a one-dimensional scattering problem is contained in [4]. \square

We now require a result for the solution (3.13) continued into \bar{D} via (3.4).

LEMMA 3.2. (Colton) *If $\|\mathbb{K}_\nu\|_\infty < 1$ then there exists a unique solution $u \in C^0(\bar{D})$ where all sources of u^{inc} are external to the compact domain \bar{D} .*

Proof: The equation (3.4) gives a unique continuation of u onto \bar{D} (see [7, p 39]). \square

We shall consider ν as belonging to the open set X_2 defined by

$$X_2 = \{\nu \in L^2(\Omega), \|\nu - 1\|_{2,\Omega} < 1/\|\mathbb{K}\|\},$$

and if Ω is a ball of radius a it follows from Lemma 3.1 $\|\mathbb{K}\| = ak^2$. We shall then need the following regularity result.

LEMMA 3.3. *If $\nu \in X_2$ then there exists a unique solution $u \in C^0(\bar{D})$ to (3.4), moreover*

$$\|u\|_{C^0(\bar{D})} \leq C \|u^{inc}\|_{C^0(\bar{D})}, \quad (3.16)$$

for some constant C .

Proof: As $\nu \in X_2$ it follows $\|\mathbb{K}_\nu\|_\infty < 1$, hence Lemma 3.1 shows there exists a unique solution $u \in C^0(\bar{D})$. Now $u(x) = u^{inc}(x) + \mathbb{K}(\nu - 1)u(x)$, $x \in \bar{D}$ so on taking norms of this equation and using (3.14) we can show

$$\|u\|_{C^0(\bar{D})} \leq \left[1 + \frac{\|\mathbb{K}\| \|\nu - 1\|_{2,\Omega}}{1 - \|\mathbb{K}_\nu\|_\infty} \right] \|u^{inc}\|_{C^0(\bar{D})}, \quad (3.17)$$

so that (3.16) follows.

THEOREM 3.5. *The map $\nu \rightarrow u(\nu)$ from $X_2 \rightarrow C^0(\overline{D})$ is Fréchet differentiable with the Fréchet differential*

$$u'(\nu)s = [I - \tilde{\mathbb{K}}(\nu - 1)]^{-1} \tilde{\mathbb{K}}u(\nu)s. \quad (3.18)$$

Proof: We examine the conditions of the implicit function theorem where $\xi(\nu, u)$ is as given in (3.5) but u is to be understood to be defined through (3.13) and then $\xi : X_2 \times C^0(\overline{D}) \mapsto C^0(\overline{D})$.

Observe that Lemma 3.2 assures us that there is only one solution $u(\nu)$ in $C^0(\overline{D})$ then:

Condition (i). To show ξ is continuous in ν and u consider

$$\begin{aligned} \delta\xi &= \xi(\nu + \delta\nu, u + \delta u) - \xi(\nu, u) \\ &= \delta u - \tilde{\mathbb{K}}(\delta\nu u + (\nu - 1)\delta u + \delta\nu\delta u), \end{aligned}$$

then

$$\|\delta\xi\|_{C^0(\overline{D})} \leq \|\delta u\|_{C^0(\overline{D})} + \|\tilde{\mathbb{K}}\| \|\delta\nu u + (\nu - 1)\delta u + \delta\nu\delta u\|_{2,\Omega}.$$

It then follows

$$\|\delta\xi\|_{C^0(\overline{D})} \leq \|\delta u\|_{C^0(\overline{D})} + \|\tilde{\mathbb{K}}\| (\|\delta\nu\|_{2,\Omega} \|u\|_{C^0(\overline{D})} + \|\nu - 1\|_{2,\Omega} \|\delta u\|_{C^0(\overline{D})} + \|\delta\nu\|_{2,\Omega} \|\delta u\|_{C^0(\overline{D})}),$$

on using $\|u\|_{C^0(\overline{\Omega})} \leq \|u\|_{C^0(\overline{D})}$.

The limit $\|\delta\nu\|, \|\delta u\| \rightarrow 0$ then gives the result.

Condition (b) and (ii). To show ξ_ν is continuous in ν and u consider the partial Fréchet derivative of (3.5) which is

$$\xi_\nu(\nu, u)s = -\tilde{\mathbb{K}}us, \quad (3.19)$$

because (3.5) is linear in ν , also note that $s \in L^2(\Omega)$.

Then

$$\begin{aligned} \|\delta\xi_\nu\|_{C^0(\overline{D})} &= \|\xi_\nu(\nu + \delta\nu, u + \delta u)s - \xi_\nu(\nu, u)s\|_{C^0(\overline{D})} = \|\tilde{\mathbb{K}}\delta us\|_{C^0(\overline{D})} \\ &\leq \|\tilde{\mathbb{K}}\| \|\delta us\|_{2,\Omega} \\ &\leq \|\tilde{\mathbb{K}}\| \|\delta u\|_{C^0(\overline{D})} \|s\|_{2,\Omega}. \end{aligned}$$

It therefore follows ξ_ν is continuous in ν and u as $\tilde{\mathbb{K}}$ is bounded.

The partial Fréchet derivative of ξ with respect to u is

$$\xi_u(\nu, u)s = (I - \mathbb{K}_\nu)s,$$

as (3.5) is linear in u , with $s \in C^0(\overline{D})$. ξ_u may be shown to be continuous in ν and u in a similar manner to ξ_ν .

Condition (iii). It follows $[\xi_u(\nu, u)]^{-1}$ is bounded from Theorem 3.3.

The explicit expression for the Fréchet derivative is given by the implicit function theorem, and is to be interpreted as a series like the solution for the direct problem, namely (3.13). \square

The Fréchet derivative at a refractive index of unity is much simpler than (3.18).

COROLLARY 3.1. *When $\nu = 1$ the Fréchet derivative $u'(1) : L^2(\Omega) \mapsto C^0(\overline{D})$ with the differential given by*

$$\begin{aligned} u'(\nu)s &= \mathbb{K}u^{inc}s \\ &= k^2 \int_{\Omega} \gamma u^{inc}s dV'. \end{aligned}$$

This corollary shows that the Born approximation (3.15) gives the formal linearization of $u(\nu)$ about a refractive index of unity. Most other authors consider the Born approximation to be an approximate solution of the direct problem; obtained as the leading term of the Neumann series. This then provides a linear integral equation which can be used to solve for an approximate solution to the inverse problem. We have shown here that this particular linearization is in fact the Fréchet derivative — that is a uniform linear approximation — to the operator equation about a unity refractive index. This Fréchet differential can be utilised in a modified Newton-Kantorovich scheme to solve the inverse scattering problem, this is done in [30], however there is no guarantee of convergence of such a scheme.

As Fréchet differentiability implies continuity we have from Theorem 3.4 continuity of the appropriate operator for $\nu \in X_2$. To obtain Lipschitz continuity a more restrictive sub-space than X_2 must be used. We shall illustrate the application of the implicit function theorem result in proving Lipschitz continuity of this operator via the mean value theorem Theorem 2.2.

COROLLARY 3.2. *The map $\nu \rightarrow u(\nu)$ for $\nu \in X_1$, where $X_1 = \{\nu \in L^2(\Omega), \|\nu - 1\|_{2,\Omega} \leq M < 1/\|\mathbb{K}\|\}$, is Lipschitz continuous with Lipschitz constant L .*

Proof: The Lipschitz continuity follows directly from the Theorem 2.2 as X_1 is a convex set. Theorem 2.2 shows

$$L = \sup_{\nu \in X_1} \|u'(\nu)\|_{C^0(\overline{D})} = \sup_{\nu \in X_1} \|[I - \mathbb{K}(\nu - 1)]^{-1}\mathbb{K}u\|_{C^0(\overline{D})},$$

so from Lemma 3.3 and (3.17) this implies

$$L = L_0^2 \|\mathbb{K}\| \|u^{inc}\|_{C^0(\overline{D})}$$

where

$$L_0 = 1 + \frac{\|\mathbb{K}\| \|\nu - 1\|_{2,\Omega}}{1 - \|\mathbb{K}\nu\|_{\infty}}.$$

We have proved Lipschitz continuity within the Born series regularity theory but with the requirement that ν belong to a bounded subset of $L^2(\Omega)$. This result is complementary to the result of [31], who required ν to belong to a compact subset of $C^0(\overline{\Omega})$. Our result requires that ν be sufficiently close to unity, [31] has no such L^2 limit on ν .

Consideration of the far-field inverse problem within the Born series solution can be carried out in a similar manner to §3.2.

4. Time independent Diffusion equation

The direct problem to solve is

$$\begin{aligned}\nabla \cdot (\nu \nabla u) &= \rho, & x \in \Omega, \\ u &= g & x \in \partial\Omega,\end{aligned}\tag{4.1}$$

for u , given the diffusivity ν , ρ , and g . We shall only consider the Dirichlet boundary value problem here, the Neumann boundary value problem can be handled in a similar manner. The bounded domain Ω is assumed to be convex and to have a smooth boundary curve $\partial\Omega$ of finite length with

$$\partial\Omega = \overline{\Omega} \setminus \Omega, \quad \partial\Omega \in C^{2,\alpha},$$

here the Hölder spaces $C^{m,\alpha}$ are equipped with the standard norm which is denoted by $\|\cdot\|_{m,\alpha}$, $m \in \mathbb{Z}_+ \cup \{0\}$, $\alpha \in \mathbb{R}_+$.

We shall consider classical solutions of (4.1) and to achieve the required regularity results the additional assumptions that $\rho \in C^{0,\alpha}(\overline{\Omega})$, $g \in C^{2,\alpha}(\partial\Omega)$ and ν belonging to the open set

$$X_0 = \{\nu : \nu \in C^{1,\alpha}(\overline{\Omega}), \nu > 0\},\tag{4.2}$$

will allow existence and regularity results for the direct problem to be provided by the following theorem.

THEOREM 4.1. *For $\nu \in X_0$ there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (4.1), moreover when $g \equiv 0$*

$$\|u\|_{2,\alpha} \leq C(\nu, \Omega) \|\rho\|_{0,\alpha}.$$

Proof: Follows from [14, p331 et. seq.] □

We first examine the inverse problem with measurements in the interior and then show that the boundary measurements problem can be analysed subsequently with the use of these results.

4.1: Interior Measurement. We examine the Fréchet differentiability of the operator with classical solutions of the direct problem. To apply the implicit function theorem u must belong

to a linear sub-space and as u defined through (4.1) it follows u belongs to an affine sub-space. This may be rectified by the following considerations. Extend $g \in C^{2,\alpha}(\partial\Omega)$ to $\tilde{g} \in C^{2,\alpha}(\overline{\Omega})$ see for example [17, p 92], so then $\tilde{u} = u - \tilde{g}$ satisfies

$$\begin{aligned}\nabla \cdot (\nu \nabla \tilde{u}) &= \rho - \nabla \cdot (\nu \nabla \tilde{g}) \equiv \tilde{\rho}, \quad x \in \Omega, \\ \tilde{u} &= 0, \quad x \in \partial\Omega.\end{aligned}$$

Observing that $u'(\nu)s = \tilde{u}'(\nu)s$ we need only consider the homogeneous boundary condition problem further. We will drop the tildes in the sequel and require the solution to the direct problem to belong to Y_0 with $Y_0 = \{u \in C^{2,\alpha}(\overline{\Omega}), u = 0 \text{ on } \partial\Omega\}$.

Then the functional

$$\xi(\nu, u) = \nabla \cdot (\nu \nabla u) - \rho = 0, \quad (4.3)$$

which is obtained directly from (4.1) can be utilised with the implicit function theorem to obtain the Fréchet derivative of the mapping $\nu \rightarrow u$. First note $\xi : X_0 \times Y_0 \mapsto C^{0,\alpha}(\overline{\Omega})$.

THEOREM 4.2. *The map $\nu \rightarrow u(\nu)$ from $X_0 \rightarrow C^{2,\alpha}(\overline{\Omega})$ is Fréchet differentiable with the Fréchet differential*

$$u'(\nu)s = \int_{\Omega} G(\nu; x, x') \nabla' \cdot [s(x') \nabla' u(\nu; x')] dV'. \quad (4.4)$$

Proof: We examine the conditions of the implicit function theorem. Observe that Theorem 4.1 assures us that there is only one solution $u(\nu)$ in Y_0 .

Condition (i). To show ξ is continuous in ν and u consider

$$\begin{aligned}\|\delta\xi\|_{0,\alpha} &= \|\xi(\nu + \delta\nu, u + \delta u) - \xi(\nu, u)\|_{0,\alpha} \\ &= \|\nabla \cdot (\nu \nabla \delta u) + \nabla \cdot (\delta\nu \nabla u) + \nabla \cdot (\delta\nu \nabla \delta u)\|_{0,\alpha} \\ &\leq \gamma(\|\nu\|_{1,\alpha} \|\delta u\|_{2,\alpha} + \|\delta\nu\|_{1,\alpha} \|u\|_{2,\alpha} + \|\delta\nu\|_{1,\alpha} \|\delta u\|_{2,\alpha}),\end{aligned}$$

where use has been made of $\|\nabla \cdot (\nu \nabla u)\|_{0,\alpha} \leq \gamma \|\nu\|_{1,\alpha} \|u\|_{2,\alpha}$ which may be derived by expanding $\nabla \cdot (\nu \nabla u)$ and using the properties of the Hölder norm. The limit $\|\delta\nu\|, \|\delta u\| \rightarrow 0$ then gives the result.

Condition (b) and (ii). To show ξ_ν is continuous in ν and u consider the partial Fréchet derivative of (4.3) which is

$$\xi_\nu(\nu, u)s = \nabla \cdot (s \nabla u)$$

because (4.1) is linear in ν , also note that $s \in C^{1,\alpha}(\overline{\Omega})$. Then

$$\begin{aligned}\|\delta\xi_\nu\|_{0,\alpha} &= \|\xi_\nu(\nu + \delta\nu, u + \delta u)s - \xi_\nu(\nu, u)s\|_{0,\alpha} = \|\nabla \cdot (s \nabla \delta u)\|_{0,\alpha} \\ &\leq \gamma \|\delta u\|_{2,\alpha} \|s\|_{1,\alpha}.\end{aligned}$$

It therefore follows ξ_ν is continuous in ν and u .

The partial Fréchet derivative of ξ with respect to u is

$$\xi_u(\nu, u)s = \nabla \cdot (\nu \nabla s)$$

as (4.1) is linear in u , with $s \in C^{2,\alpha}(\overline{\Omega})$. ξ_u may be shown to be continuous in ν and u in a similar manner to ξ_ν .

Condition (iii). It follows $[\xi_u(\nu, u)]^{-1}$ is bounded from Theorem 4.1.

The explicit expression for the Fréchet derivative is given by the implicit function theorem by noting

$$[\xi_u^{-1}(\nu, u)]\rho = \int_{\Omega} G(\nu; x, x')\rho(x') dV,$$

where $G(\nu; x, x')$ is the Green function pertinent for the diffusivity ν and satisfying homogeneous Dirichlet data on $\partial\Omega$. □

Chavent [6] proves an analogous result for the same problem by also using the implicit function theorem but for weak solutions $\{\nu \in L^\infty(\Omega), \nu > 0\}$, $u \in H^1(\Omega)$. Chavent's result and Theorem 4.2 are complementary, although his result has a weaker assumption on the function ν to be reconstructed, however our Theorem 4.2 has a stronger function space for the range. For example, Theorem 4.2 allows for point measurements whereas Chavent's does not in \mathbb{R}^n , $n \geq 2$.

Theorem 4.1 can now be used to show Lipschitz continuity of the operator in a similar manner as in Corollary 3.2 [9], however instead, we will illustrate that the operator $T(\nu)$, as defined through (4.3), is compact when $T : X_0 \mapsto Z$ where Z may be $L^2(\Omega)$ or $C^0(\overline{\Omega})$. As the inverse of a compact operator is unbounded this implies the well known property of inverse problems, they are improperly posed. Observe this result is for the non-linear operator, often in the literature the ill-posedness is shown only for the linearisation of this operator and this does not always imply that the underlying non-linear operator is in fact compact. Our result implies *whatever formulation* is utilised for the derivation of a Fréchet differential the linearisation will be ill-posed.

We first must show $T : \nu \rightarrow u(\nu)$ is bounded, and to do this we require a tighter function space than X_0 , that is we require ν to be bounded below so we shall now define $X_1 = \{\nu \in C^{1,\alpha}(\Omega), \nu \geq c > 0\}$.

LEMMA 4.1. *The map $\nu \rightarrow u(\nu)$ is bounded.*

Proof: To show that bounded sets of X_1 are mapped to bounded sets of $C^{2,\alpha}$, that is if $\nu \in X_1$ and $\|\nu\|_{1,\alpha} \leq M_0$ then $\|u\|_{2,\alpha} \leq N_0$ for some N_0 . Now note from [14, p335]

$$\|u(\nu)\|_{2,\alpha} \leq C(\|\rho\|_{0,\alpha} + \|g\|_{2,\alpha}),$$

with $C = C(\alpha, c, M_0, \text{diameter}(\Omega))$. The result follows. □

THEOREM 4.3. *The operators $T(\nu) : X_1 \mapsto Z$ and $T'(\nu) : C^{1,\alpha}(\overline{\Omega}) \mapsto Z$ with $\nu \in X_0$ are compact.*

Proof: The operators $T(\nu) : X_1 \mapsto C^{2,\alpha}(\overline{\Omega})$ and $T'(\nu) : C^{1,\alpha}(\overline{\Omega}) \mapsto C^{2,\alpha}(\overline{\Omega})$ are bounded from Lemma 4.1 and Theorem 4.2, respectively. As the imbedding $C^{2,\alpha}(\overline{\Omega})$ to Z is compact, the result follows as the composition of a compact operator and a bounded operator is compact. \square

In particular this result shows the inverse problem will be unstable for point measurements of u (from setting $Z \equiv C^0(\overline{\Omega})$) or distributed measurements (from setting $Z \equiv L^2(\Omega)$).

4.2: Boundary Measurements The Fréchet differentiability result of Theorem 4.2 can be easily extended to the boundary measurement inverse problem, where in the case considered here $\nu \frac{\partial u}{\partial n}$, the normal flux (current) is measured on the boundary — with Dirichlet boundary data being specified. So if

$$B(\nu, u) = \nu \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega$$

with

$$u(\nu) = g, \quad \text{on } \partial\Omega,$$

being specified, we can show.

THEOREM 4.4. *The map $T : X_0 \mapsto C^{1,\alpha}(\partial\Omega)$ is Fréchet differentiable with*

$$B'(\nu)s = \nu \frac{\partial u'(\nu)s}{\partial n} \Big|_{\partial\Omega} + s \frac{\partial u}{\partial n} \Big|_{\partial\Omega}, \quad (4.5)$$

where $u'(\nu)s$ is given by (4.4).

Proof: From Theorem 4.2 $u(\nu)$ is Fréchet differentiable with derivative (4.4) then the differentiability of $B(\nu)$ with differential (4.5) follows from Corollary 2.1. \square

This result allows for point measurements of $\nu \frac{\partial u}{\partial n} \Big|_{\partial\Omega}$, or as distributional measurements in $L^2(\partial\Omega)$, that is point measurements.

The Newton-Kantorovich method for the determination of ν resulting from equation (4.5) and Theorem 4.4 will again require the calculation of Green functions, a more computational efficient method is described in [10]. This reference also provides a derivation of the Fréchet derivative, although by methods other than the implicit function theorem.

Acknowledgement

T.J. Connolly acknowledges the receipt of a N.Z. U.G.C. Postgraduate Scholarship.

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