# Slice and match - how to modify 2-dimensional geometries

## NZMS Colloquium 2015

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New Zealand

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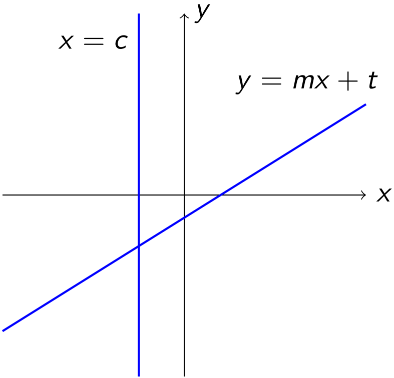
# The Euclidean real affine plane



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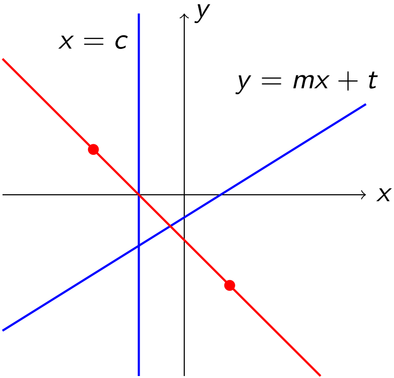

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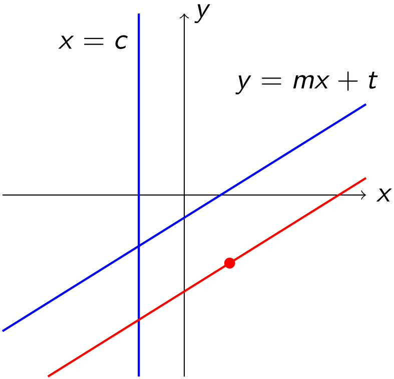

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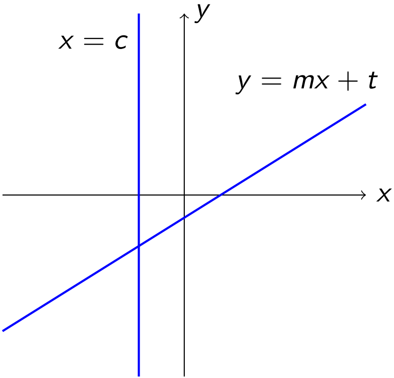
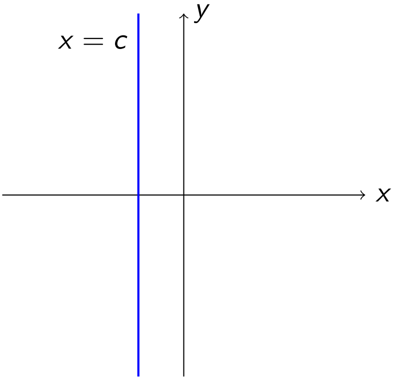



The group of collineations of the Euclidean real affine plane is a Lie group of dimension 6 consisting of the transformations $(x, y) \mapsto (x, y)A + (s, t)$ where $A \in \mathrm{GL}_2(\mathbb{R})$, $s, t \in \mathbb{R}$.

# Generalised Moulton planes



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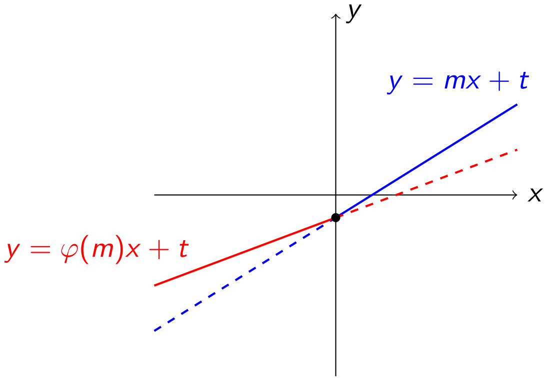

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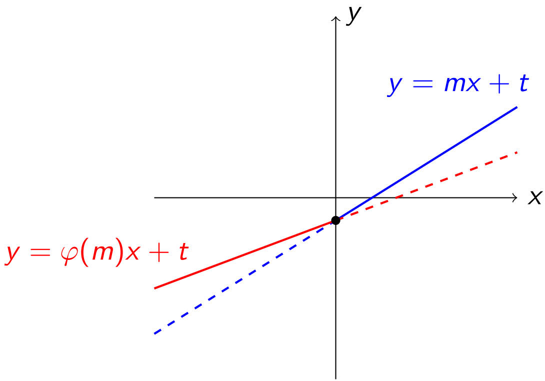
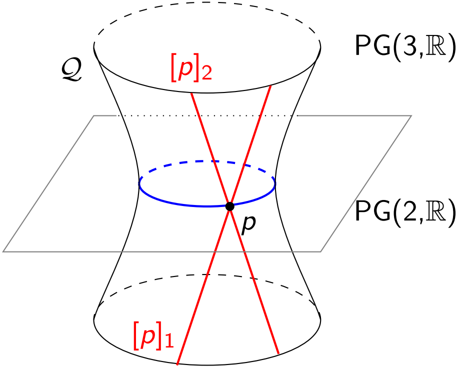



The group of collineations of a generalised Moulton plane is a Lie group of dimension at least 2 containing the transformations $(x, y) \mapsto (x, y) \cdot r + (0, t)$ where $r, t \in \mathbb{R}$, $r > 0$. (Pierce 1961, S. 1985)

# The classical real Minkowski plane

The *miquelian* or *classical real Minkowski plane* is obtained as the geometry of non-trivial plane sections of a ruled quadric $\mathcal{Q}$ in 3-dimensional real projective space.



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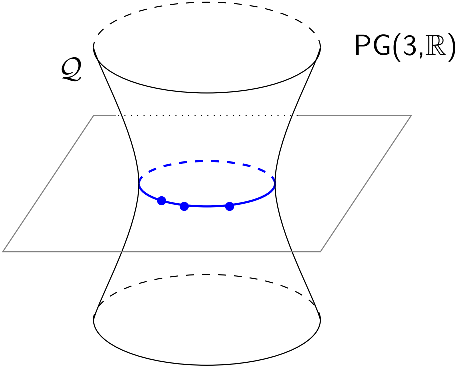

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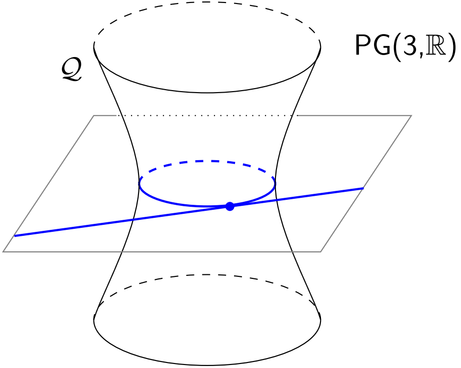
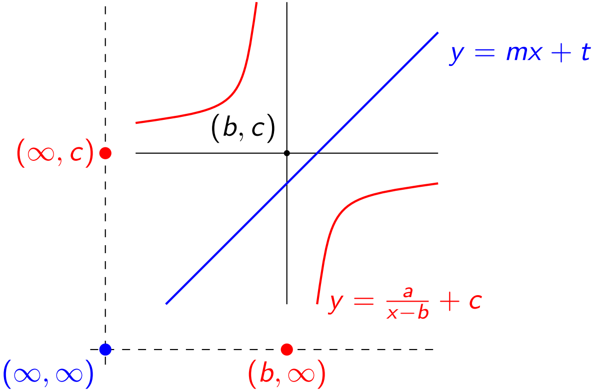


# The hyperbola model of the classical real Minkowski plane



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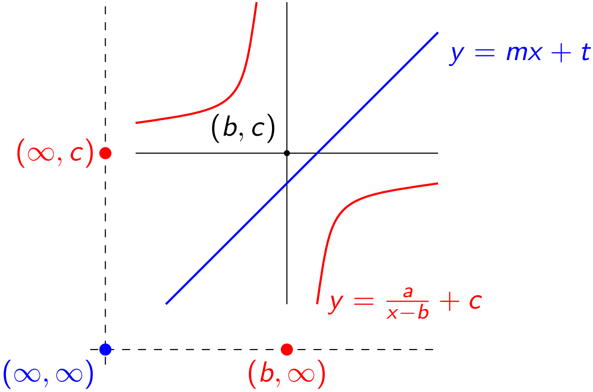


The group of automorphisms of the classical real Minkowski plane is a Lie group of dimension 6 containing the transformations $(x, y) \mapsto (\alpha(x), \beta(y))$ where $\alpha, \beta \in \mathrm{PGL}_2(\mathbb{R})$, the group of fractional linear maps on $\mathbb{R} \cup \{\infty\}$.

# Geometric and topological properties

The residual incidence structure $\mathcal{M}_p$ at a point $p$ of a Minkowski plane $\mathcal{M}$ is an affine plane, the *derived affine plane* at $p$. Its points are the points of $\mathcal{M}$ not on a generator through $p$ and lines come from the generators $\neq [p]_1, [p]_2$ and circles through $p$. A circle not passing through $p$ induces a hyperbolic curve $\mathcal{M}_p$.

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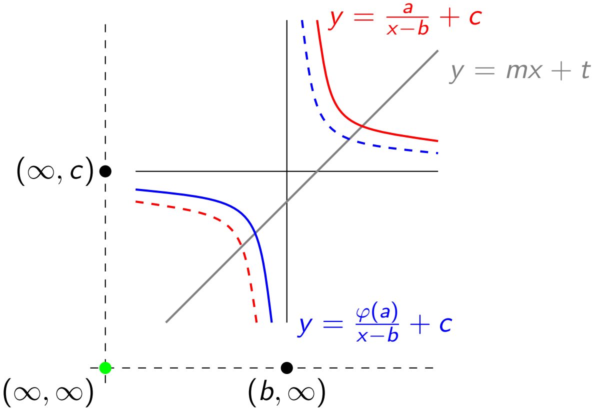
Circles in the classical real Minkowski plane are graphs of fractional linear maps on $\mathbb{S}^1 \simeq \mathbb{R} \cup \{\infty\}$. The circle space of the classical real Minkowski plane is homeomorphic to $\mathrm{PGL}_2(\mathbb{R})$.

The circle space $\mathcal{C}$ of a 2-dimensional Minkowski plane $(\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C})$ has two connected components $\mathcal{C}^+$ and $\mathcal{C}^-$, the former consisting of graphs of orientation-preserving homeomorphisms of $\mathbb{S}^1$ and the latter consisting of graphs of orientation-reversing homeomorphisms of $\mathbb{S}^1$.

# Modified classical real Minkowski planes w.r.t. a point



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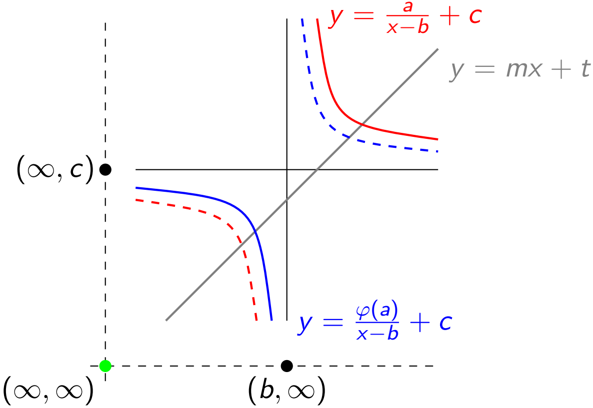
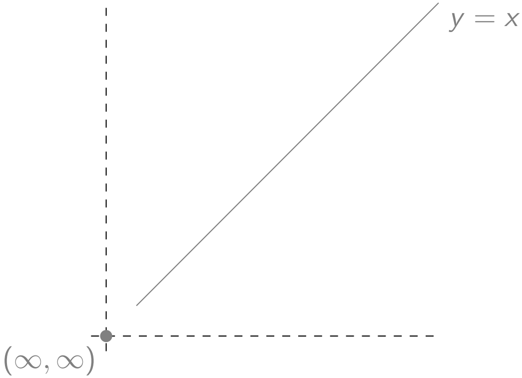



The group of automorphisms of a modified classical real Minkowski plane w.r.t. the point $(\infty, \infty)$ is a Lie group of dimension at least 3 containing the transformations $(x, y) \mapsto (rx + s, \frac{y}{r} + t)$ where $r, s, t \in \mathbb{R}$, $r > 0$. (S. 1985)

# Modified classical real Minkowski planes w.r.t. a circle



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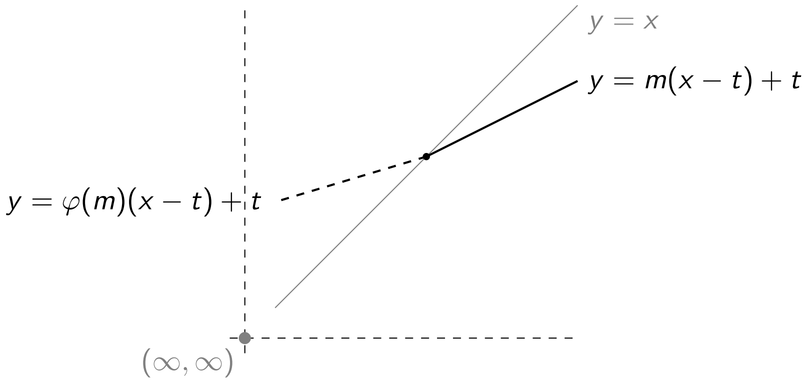

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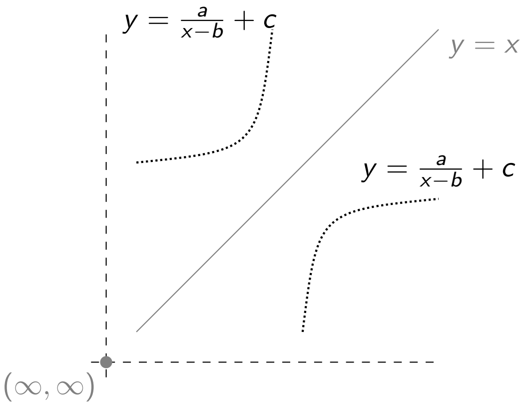

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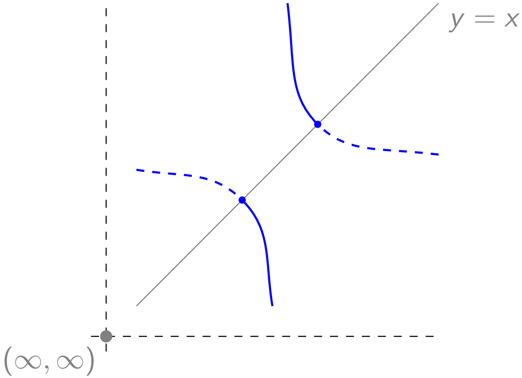

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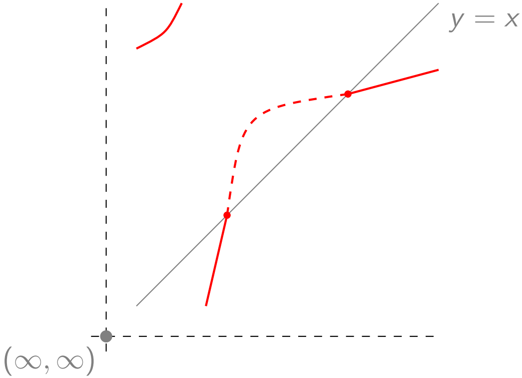

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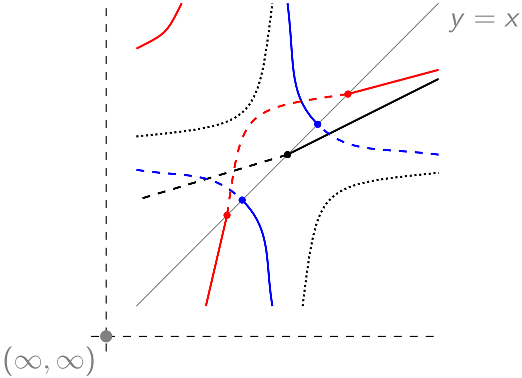


The group of automorphisms of a modified classical real Minkowski plane w.r.t. the circle $y = x$ is a Lie group of dimension at least 3 containing the transformations $(x, y) \mapsto (\alpha(x), \alpha(y))$ where $\alpha \in \mathrm{PSL}_2(\mathbb{R})$. (S. 2015)

# Other types of modified classical real Minkowski planes

## Theorem (Swapping halves)

*If* $(\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}_i)$, $i = 1, 2$, *are two 2-dimensional Minkowski planes, then the geometry* $(\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}_1^+ \cup \mathcal{C}_2^-)$ *is a 2-dimensional Minkowski plane.*

The above process of 'swapping halves' can be applied to any two of the modified classical real Minkowski planes in order to produce a 2-dimensional Minkowski plane whose circles are pieces of up to two circles of the classical real Minkowski plane.

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Piecewise projective 2-dimensional Minkowski planes are planes whose circles are made up of finitely many pieces of circles of the classical real Minkowski plane. There are models of piecewise projective 2-dimensional Minkowski planes whose automorphism groups are trivial.