Arithmetic, Induction, and the Algebra of Polynomials
Al-Samaw'āl and his "Splendid Book of Algebra"

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Abstract

Al-Samaw'āl (with the complete name Al-Samaw'āl Ibn Yahyā Al-Maghribī), born around 1130 in Baghdad, Iraq, is best known in the History of Mathematics for his seminal work the *Al-Bāhir fī Al-Jabr* (literally *The Splendid Book of Algebra*) which he composed at the prodigious age of nineteen. In this work, following the Euclidean tradition, Al-Samaw'āl put together and advanced many key algebraic rules formulated by his predecessors, notably Al-Khwārizmi, Ibn Turk, Ibn Qurra, Al-Kūhī, Al-Uqlīdīsī, Abū'l-Wafā, Al-Karajī, Ibn Aslam, Al-Sijzī, Ibn Al-Haytham, Qustā Ibn Lūqā, and Al-Harīrī.

Al-Bāhir is a large work and consists of four sections. Section one provides an account of operations on polynomials in one unknown with rational coefficients, section two deals essentially with second-degree equations, indeterminate analysis, and summations, section three concerns irrational quantities, and section four presents the application of algebraic principles to a number of problems. In this thesis, we give close attention to book four from the second section in which Al-Samaw'āl discusses mathematical relations which amount to the binomial theorem and the Pascal triangle and lays out a table of binomial coefficients and demonstrates how to generalise the entries in the table for any desired value. Our main contribution is a complete translation of the Arabic text, the first time this has been done in a European language. We will then provide a detailed mathematical commentary and offer a careful analysis of the status and impact of this work in the History of Mathematics, paying due attention to the historical context in which it was produced.

1 Introduction: Historical Overview

1.1 A Brief Biographical Sketch

Al-Samaw'āl (with the complete name Al-Samaw'āl Ibn Yahyā Al-Maghribī) was the son of Yehuda ben Abun (or Abu'l-Abbās Yahyā Al-Maghribī). Al-Samaw'āl was born around 1130 in Baghdad, Iraq and passed away around 1180 in Maragha, Iran. His father was a Jewish scholar who was educated in religion and Hebrew literature, and who emigrated from Fez (Morocco) to Baghdad. His mother, Anna Isaac Levi was an educated woman who was originally from Basra (Iraq).

With the encouragement of his maternal uncle who was a physician, Al-Samaw'āl took up the study of medicine and exact sciences at the age of thirteen. He studied medicine with Abu'l-Barakāt while he observed the practical work of his uncle. Simultaneously with medical studies, Al-Samaw'āl started to learn mathematics, beginning with Hindu methods of computation, zījes (astronomical tables), arithmetic, and *misāha* (practical techniques for measure determination for use in surveys), and after that algebra and geometry.

Because of his advanced level in mathematics, Al-Samaw'āl was unable to find a teacher to instruct him beyond the level of the first books of Euclid's *Elements* and he was obliged to study independently. Al-Samaw'āl studied, in addition to Euclid, *Algebra* of Abū Kāmil, the book *Al-Badi* of Al-Karajī, and *Arithmetic* of Al-Wasītī (who collaborated in making astronomical observations with Umar Al-Khayyāmī between 1072 and 1092) with the result that by the age of 18 Al-Samaw'āl had a sound knowledge of the advanced works in mathematics.

However, despite his talent in mathematics, Al-Samaw'āl spent the majority of his life as an itinerant physician in and around Maragha, travelling throughout Iraq, Syria, Kūhistan, and Ādharbayjān. Al-Samaw'āl was a successful physician and had emirs among his patients. Al-Samaw'āl created several new medicines, including an almost miraculous theriac (antidote to poison) and wrote an extant medical book *Nuzhat Al-Ashāb* (usually translated as *The Companions' Promenade in the Garden of Love*).

A seminal moment in Al-Samaw'āl's life was his conversion to Islam. This conversion was particularly surprising given the fact that Al-Samaw'āl's father was a rabbi (Jewish high religious authority) and Al-Samaw'āl was raised in accordance with Jewish traditions. However, after many years of questioning, finally after a dream which included visions of the prophet Mohammad, he converted to the Islamic faith.[1] The conversion had a profound effect on his intellectual outlook.

Despite writing many treatises against the Jewish faith,[2] Al-Samaw'āl nurtured a somewhat tolerant and liberal attitude towards different religious traditions. He argued that it was less important what religious tradition a person ascribed to, be it

[1] Under the main title of the article *The Conversion to Islam of Al-Samau'āl Ibn Yahyā Al-Maghribī* we can see written in small characters that he had a vision of the Prophet Mohammad in the night of Friday, *Arafa* of the Arabic year 558. This day of *Arafa* represents in fact the day when Al-Samaw'āl converted from the Jewish religion to Islam. From an European calendar point of view Al-Samaw'āl converted to Islam on 8 November, 1163 (8 November, 558 in the Arabic calendar) prompted by this vision.

[2] After his conversion to the Muslim religion Al-Samaw'āl wrote a polemic against the Jews. The book with the title *Ifhām Tā'ifat Al-Yahūd* is a polemic against all Judaic religions. Later on, Al-Samaw'āl wrote another book *Ghāyat Al-Majhūd fī al-Radd Ala'l-Nasāra wa'l-Yahūd* (*Decisive Refutation of the Christians and the Jews*) where he presents his beliefs against the Christian and Jewish religion. Al-Samaw'āl wrote another treaty against Jewish people *Badhl al-Majhud fī Iqnā al-Yahūd* (*The Effort to Persuade the Jews*) which has been lost unfortunately since World War II. For a long time Al-Samaw'āl did not convert to Islam from respect for his father. Things did change after he lived far away from his father and after he had a dream with the vision of the Prophet Mohammad. Approximately 100 years after Al-Samaw'āl's death, the Iraki Jewish Physician and Philosopher Sād B. Manṣūr B. Kammūna wrote *Critical Inquiry into the Three Faiths* and presented Al-Samaw'āl's work as the most important and typical summary of Muslim polemics against Jews. Al-Samaw'āl's work *The Conversion to Islam of Al-Samau'āl Ibn Yahyā Al-Maghribī* was translated from Arabic into Latin and from Latin in many different European languages. The Latin translation of this document is *Epistola Samuelis Marrocani ad R. Isaacum Contra Errores Judaeorum*. The original version of Al-Samaw'āl's book *The Confutation of the Jews* exists in Tehran, and is dated 685/1286. There is a manuscript present in Cairo and dated 732/1332. It consists the longer (later) version and Al-Samaw'āl's Vita. From this version from Cairo many editions have been reproduced and translated.

Muslim, Christian, or Jewish, but rather the ways in which an individual exemplified social consciousness associated with those religious traditions, including living a good life, contributing to social welfare, the maintenance of public institutions, and the community, and so on. Al-Samaw'āl put much emphasis on the logically equal validity of all faiths and the social value of religion.

We are pleased to see Al-Samaw'āl's wisdom present in his intellectual writings. His way of presenting the text is typical to a well-educated Jewish person as Al-Samaw'āl was born a Jew. He recognises the problems which would normally be ignored by a Muslim educated person and feels responsibility for all of these problems. Like many scholars of his time, Al-Samaw'āl acutely understood the challenges of furthering scholarship and the development of ideas, as he reveals in the following quotes.

Al-Samaw'āl acknowledges the achievements of his predecessors as vital to insight and discovery as follows:

> There is no idea that might enter someone's brain which might not before have entered the brain of someone else. Every intelligent person knows that the fact that someone is able to correct former scholars does not imply that the same man possesses a greater knowledge than themselves in all their branches of knowledge. It merely implies that he has further progressed than themselves in the knowledge of just that particular matter [17, pages 563–564].

He was well aware of the delicate notion of the original text which could easily be altered because of the scribe's ignorance. This observation was to his credit as he himself included translations or paraphrases of his predecessors.

> For how many reasons may errors enter the works of excellent scholars! Some errors may be due to a copyist or scribe who miswrites a word

> or omits something. If many copyists thereafter copy the same text in succession, and each one of them further miswrites or omits words in passages where the original copyist had already made copying mistakes, the result will be very bad and the mistakes patent [17, pages 563–564].

Al-Samaw'āl was also deeply sensitive to issues of translation.

> The situation is even worse in the case of translators who transmitted knowledge from one language to the other. They often understood passages which they attempted to translate differently from what the author had intended them to be understood. They also frequently encountered difficulties in certain passages and treated them as they thought it would be correct. The author, then, is held responsible while, in fact, he is not to blame [17, pages 563–564].

Such sensitivity to these issues and his active interest in a broad number of fields had an interesting impact on his mathematics, as we shall see below.

1.2 Al-Samaw'āl's Mathematical Contributions

Al-Samaw'āl's most important contribution to mathematics was a work called *Al-Bāhir fī Al-Jabr* (literal translation: *The Splendid Book of Algebra*)[3] where he put together the algebraic rules formulated by his predecessors Al-Khwārizmī, Ibn Turk, Ibn Qurra, Al-Karajī, Abū Kāmil Ibn Aslam, Al-Sijzī, Ibn Al-Haytham, Qustā Ibn Lūqā, and Al-Harīrī and developed and furthered their scope. Despite being only 19 when it was composed, this work became Al-Samaw'āl's most famous. In this book Al-Samaw'āl shows that the techniques of arithmetic could be fruitfully applied in

[3]Some critics/scientists consider Al-Samaw'āl's book *Al-Bāhir fī Al-Jabr* to be entitled *Al-Bāhir fī Al-Hisāb* (*The Splendid Book on Calculation*). This is only partly true because Al-Samaw'āl presents in *Al-Bāhir fī Al-Jabr* a section with calculations, but in fact in all the original documents written in Arabic, the title is written as *Al-Bāhir fī Al-Jabr*.

algebra. This has been dubbed the "arithmetisation of algebra" which was initiated by his predecessors. He gave the first description of this development as follows:

> ... with operating on unknowns using all the arithmetical tools, in the same way as the arithmetician operates on the known [1, page 9].

From a general point of view the term of "arithmetisation of algebra" means the transposition and extension of elementary operations of arithmetic, algorithms like Euclidean division, or the extraction of roots of algebraic expressions, in particular polynomials. Applying the arithmetisation of algebra between the tenth and twelfth centuries, mathematicians developed polynomial algebra and reached a clearer understanding of the algebraic structure of real numbers.

From Al-Samaw'āl's Arabic text, we understand that the complex process of arithmetisation of algebra involves giving:

1. The multiplication and division of algebraic powers.
2. The theory of the division of polynomials.
3. The calculus with signs.
4. The binomial coefficients and the binomial formula.

Among the many subjects covered in the work, Al-Samaw'āl presented methods of dividing polynomials and extracting the square root of polynomials. These methods were ingenious and contrasted with European methods of approaching these mathematical problems. He also developed exponent rules, operations with negative numbers, and rationalising fractions with surds in the denominator.

As we have seen, Al-Samaw'āl was indebted to his many predecessors. But the main source of inspiration, and the individual whom he directly credits for many parts of his work is the scholar Al-Karajī (early eleventh century), whose work he quotes and develops. Al-Samaw'āl admired Al-Karajī but also pointed out the deficiencies of his work. Although Al-Karajī invented an algorithm for extracting the square root of polynomials with positive coefficients, Al-Samaw'āl improved this algorithm in the sense that he could extract the square root of polynomials with negative coefficients as well. Al-Karajī did a lot of work in algebra but expressed his numbers in words rather than symbols. This represents a very difficult obstacle in terms of memorising and improving the knowledge in algebra. Al-Samaw'āl overcame this situation using a table where he associated to each power of an unknown x a place in the table. In Al-Samaw'āl's table a polynomial was represented by the sequence of a polynomial's coefficients written in Hindu numerals. Al-Samaw'āl's technique of presenting the coefficients by symbols (that is using the Hindu numerals) represents a decisive step in the development of symbolism and is requisite to the progress of algebra.

The book *Al-Bāhir* consists of four parts. Part one provides an account of operations (for example, multiplication, division, ratio, and the extraction of the root) on polynomials in one unknown with rational coefficients. Part two deals essentially with second-degree equations, indeterminate analysis, binomial coefficients and the binomial formula, and summations. In this part Al-Samaw'āl presents a noteworthy calculation of the coefficients of $(a+b)^n$. Al-Samaw'āl organises these coefficients into a Trapezium which we are inclined to call Al-Samaw'āl's Right Trapezium. These coefficients will be arranged later on in a triangular table known in the western world as Tartaglia's or Pascal's triangle. Al-Samaw'āl solved in Book 2 of *Al-Bāhir* many quadratic equations which had been attacked and solved algebraically previously by mathematicians like Al-Khwārizmī and Al-Karajī. Al-Samaw'āl offered geometric solutions for these types of equations. Part three concerns irrational quantities. Part four presents the application of algebraic principles to a number of problems.

Al-Bāhir is an extension of the algebra of polynomials presented previously by Al-Karajī. The difference between Al-Samaw'āl's work and Al-Karajī's work is that Al-Samaw'āl included negative powers and coefficients. Al-Karajī, in contrast, treated only polynomials with positive powers and coefficients. Al-Samaw'āl could tackle even challenging problems. At one point he discusses the solution of 210 simultaneous equations in ten unknowns.

Al-Karajī invented the algorithm of square root extraction but he could not succeed in applying it for polynomials with subtractive coefficients. It is argued that a great obstacle in this context is the fact that Al-Karajī's algebra lacked symbols. Al-Samaw'āl initiated a symbolic style of reasoning in algebra in the sense that he used a visualisation in which he associated to each power of "x" a place in a table in which a polynomial was represented by the sequence of its coefficients, written in Hindu numerals.

In part 2 of *Al-Bāhir*, Al-Samaw'āl gave geometrical solutions to the six types of equations (which we represent using modern notation):

3 simple types: $ax = b$, $x^2 = bx$, and $x^2 = a$,

3 difficult (complex) types: $a + x^2 = bx$, $a = bx + x^2$, and $x^2 = a + bx$.

Al-Karajī gave algebraic solutions to these equations.

Later, we are going to see how Al-Samaw'āl presented a remarkable calculation of the coefficients of $(a+b)^n$. Al-Karajī already discovered these coefficients after 1007 but unfortunately his original computations have been lost by being destroyed.[4] For this reason Al-Samaw'āl's results are of particular interest for the history of mathematics.

In his most famous book *Al-Bāhir fī Al-Jabr* (*The Splendid Book on Algebra*) Al-Samaw'āl develops the study of polynomials. He starts defining powers like x, x^2, x^3, $\ldots, x^{-1}$, x^{-2}, $x^{-3}, \ldots$ and after defining polynomials Al-Samaw'āl describes operations with polynomials for addition, subtraction, multiplication, and division. He also gives methods for the extraction of the roots of polynomials.

Al-Samaw'āl created a table which assists with the multiplication of different exponents (see Table 1).

Note: For clarity, we reproduce the table using modern notation. Al-Samaw'āl's notation is different from the classical European one. Instead of writing x^{-8} for example, he wrote "part māl cube cube" and instead of writing x^8 Al-Samaw'āl wrote "māl cube cube" (see section 2.1).

As we observe, in the first row of the table he wrote the absolute values of the exponents of x. In the second row, he wrote x to different powers starting from x^9 and decreasing the powers until x^{-9}. In the third row, he substituted x by 2 starting from $2^9 = 512$ and finishing with $2^{-9} = \frac{1}{2^9} = \frac{1}{512}$. In the fourth row, he substituted x by 3 and started from $3^9 = 19683$ and finished with $3^{-9} = \frac{1}{3^9} = \frac{1}{19683}$.

Al-Samaw'āl used the table above to explain the law of exponents $x^m \cdot x^n = x^{m+n}$ as follows. He describes the law with reference to the position of the factors in the table.

The distance of the order of the product of the two factors from the

[4] الدكتور علي عبدَالله الدفَاع، المدخل إِلَى تَاريخ الريَاضيَات عند العرب و المسلمين (Ali Abdulallah Aldfā, *The History of Mathematics of Arabs and Muslims (1989).*)

9	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	8	9
x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1	x^{-1}	x^{-2}	x^{-3}	x^{-4}	x^{-5}	x^{-6}	x^{-7}	x^{-8}	x^{-9}
512	256	128	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
19683	6561	2187	729	243	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{243}$	$\frac{1}{729}$	$\frac{1}{2187}$	$\frac{1}{6561}$	$\frac{1}{19683}$

Table 1: Table of exponents (In the original table, Al-Samaw'āl does not use modern symbolic notations for $x, x^2, \ldots, x^{12}$, but we use them for clarity.)

> order of one of the two factors is equal to the distance of the order of the other factor from the unit. If the factors are in different directions, then we count the distance from the order of the first factor towards the unit; but, if they are in the same direction, we count away from the unit [2, page 141].

For example, in order to multiply x^3 by x^4, we count four orders to the left of column 3 and get the result as x^7. To multiply x^3 by x^{-2}, we count two orders to the right from column 3 and get the answer x^1. Using these rules, Al-Samaw'āl could easily multiply polynomials in x and $\frac{1}{x}$ as well as divide such polynomials by monomials. He was also able to divide polynomials by polynomials using a similar chart.

1.3 Al-Samaw'āl's Approach Towards the Development of the Binomial Expansion Formula

The most impressive contribution of Al-Samaw'āl is the development of the coefficients of the binomial expansion of $(a+b)^n$. In his exposition, he presents 5 propositions which begin at the case $(a+b)^1$ and works to $(a+b)^4$ and describes a mathematical induction-like process to establish higher powers. Along with the propositions, Al-Samaw'āl presents a table with the coefficients of the binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} C_k^n a^{n-k} b^k,$$

where n is a positive integer and the values C_k^n are the binomial coefficients. In this table Al-Samaw'āl presents the coefficients from $(a+b)^1$ up to $(a+b)^{12}$. Al-Samaw'āl's table is similar to the one famously known as Pascal's triangle which was presented by Blaise Pascal (1623-1662), almost 500 years after Al-Samaw'āl.

Al-Samaw'āl wrote the expressions for the binomial expansions in words. For example, in the case $n = 4$, Al-Samaw'āl writes:

> Any number divided into two parts, its square-square is equal to the square-square of each part, four times the product of each by the cube of the other, and six times the product of the squares of each part [16, page 65].

Here the term square-square refers to x^4. This phrase is equivalent to

$$(a + b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2.$$

He uses the well-known Arabic terminology "thing", "māl", "cube" and so on up to "cube cube cube cube". In contemporary mathematics we would write the previous terms using symbols consequently as x, x^2, x^3, and x^{12}.

To give some insight into his approach, let us see how Al-Samaw'āl obtains the formula for $(a + b)^4$. Assuming that $c = a + b$, and since $c^4 = c \cdot c^3$ and c^3 is already given by

$$c^3 = (a + b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b,$$

it follows that

$$(a + b)^4 = (a + b)(a + b)^3 = (a + b)(a^3 + b^3 + 3ab^2 + 3a^2b).$$

By using repeatedly the result $(r + s)t = rt + st$, a result which he quotes from Euclid's *Elements*, book II, Al-Samaw'āl finds that this last quantity equals

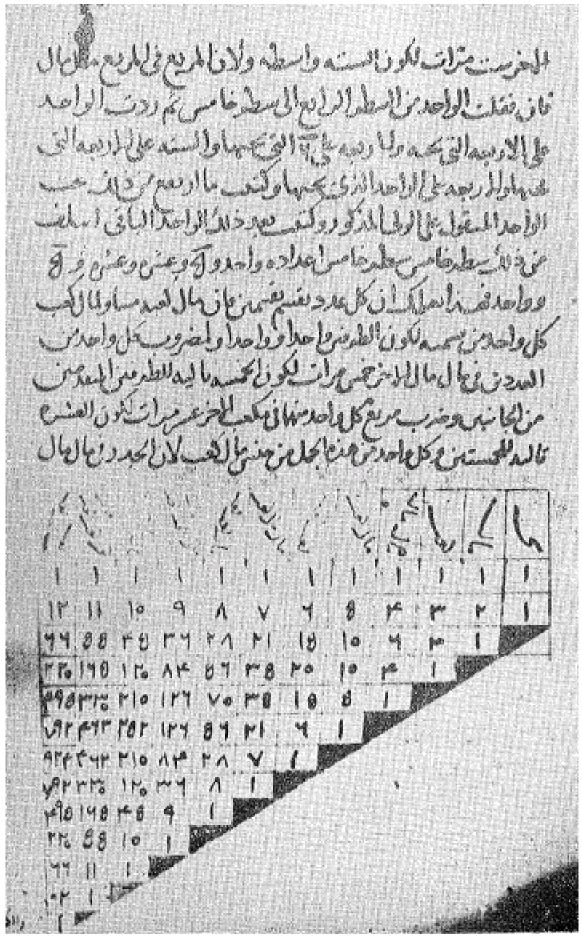


Figure 1: The version of the table as produced in the manuscript.

$$
\begin{aligned}
&(a+b)a^3 + (a+b)b^3 + (a+b)3ab^2 + (a+b)3a^2b \\
&= a^4 + a^3b + ab^3 + b^4 + 3a^2b^2 + 3ab^3 + 3a^3b + 3a^2b^2 \\
&= a^4 + b^4 + 4ab^3 + 4a^3b + 6a^2b^2.
\end{aligned}
$$

The procedure described by Al-Samaw'āl for constructing this table is the familiar one, that any entry comes from adding the entry to the left of it to the entry just above that one. He then notes that one can use the table to read off the expansion of any power up to the twelfth of a number divided into two parts. Al-Samaw'āl did not have recourse to the techniques of induction to generalise this result. However, he presents a number of generalising examples which show how to expand binomials to any desired power.

Unfortunately for the future development of Mathematics, some of the most important work of Islamic Mathematicians, including the work of Al-Bīrūnī, Al-Samaw'āl, Al-Khayāmī, Sharaf Al-Dīn Al-Tūsī, and most of that of Ibn Al-Haytham were not made widely available outside the Arabic tradition. As a consequence, rather than building on these Islamic contributions, European mathematicians were compelled to rediscover much of the same material centuries later.

2 Text and Translation

2.1 Introduction to Translation

Text

Al-Samaw'āl's *Al-Bāhir* exists in two manuscripts, one in the Aya Sofya Library in Istanbul, Turkey and the other manuscript in the Esat Afandi Library in Cairo. The manuscript from Turkey has the number 3155 and the manuscript from Cairo has the number 2118. A critical edition has been made by Roshdi Rashed based on these two manuscripts, with selected passages translated into French and an accompanying mathematical commentary. Roshdi Rashed translated most of Al-Samaw'āl's results adding his personal flair by not keeping the order or by not presenting Al-Samaw'āl's results in the same manner. Roshdi Rashed mentioned Proposition 3 and Proposition 5 but he did not translate them. Where the translation is done, the translation is not always a literal one. This critical edition has been used as the basis of the translation presented here, with several minor emendations. In the text emendations have been indicated by use of square brackets $[\cdots]$ when something critical is missing in the Arabic version. In the English translation, glosses or supplementary material necessary for sense have been indicated by $(\cdots)$ when we explain in English an expression or a word written either in Arabic or English. Additions which have been deemed crucial for intelligibility have been indicated in footnotes. In translating these passages, we have attempted to be as literal as possible to convey the fullest impression of the original text for non-Arabic speakers. However, on occasion there was need to make minor modifications to the text for the mathematical integrity or consistency of the work. These have been clearly identified and noted.

Below, we present the Arabic text taken directly from the critical edition, a literal translation, and a mathematical commentary. For most propositions we have "paraphrased" the mathematical content. The first paraphrase attempts to stick as close as possible to the original using modern notation. The second (where appropriate) is a modern symbolic explanation of the same reasoning. For flow and clarity, we have done this proposition by proposition. The passage from *Al-Bāhir* we have concentrated on is the material covering the rules for expansion leading up to and including the construction of the table of binomial coefficients. This is the first time the extended section has been translated as a whole into English. ([1], pages 104-112)

Numerals and Numbers

The text uses three distinct ways to represent numbers: number symbols or glyphs, number words, and *abjad* notation, a numerical system in which the 28 letters of the Arabic alphabet are assigned a numerical value. In the Arabic alphabet there are 28 letters, 22 are common to other alphabets like Phoenecian, Hebrew, Aramaic, and Greek and 6 letters are specific to the Arabic alphabet. The order of the assignment of letter to numerical values is not based on the classical ordering of the Arabic alphabet, but rather follows the Greek (Ionian) tradition. The first letters of these 2 alphabets are the same, " ا ", (which is pronounced as "aliph") and "ب" (which is pronounced as "bā"). From a classical perspective, the letters following " ا ", "aliph" and "ب", "bā" are "ج", (which is pronounced as "gīm"), and "د", (which is pronounced as "dāl"). This notation was widespread especially in mathematics. Most notably, it is not a place value system of numeration, but works in the following way: Numbers 1 to 9 in the first row are the first nine letters of the Arabic equivalents of the Greek alphabet. The second row contains the next 9 letters with numbers from 10 to 90, the third row contains 9 letters with numbers from 100 to 900, and the fourth row contains the 28^{th} letter "غ" which is associated with the number 1000. The notation is then as in Table 2:

ا	ب	ج	د	ه	و	ز	ح	ط
A	B	G	D	H	W	Z	Ḥ	Ṭ
1	2	3	4	5	6	7	8	9
ي	ك	ل	م	ن	س	ع	ف	ص
Y	K	L	M	N	S	Ạ	F̣	Ṣ
10	20	30	40	50	60	70	80	90
ق	ر	ش	ت	ث	خ	ذ	ض	ظ
Q	R	Sh	T	Th	Kh	Dh	Ḍ	Ẓ
100	200	300	400	500	600	700	800	900
				غ				
				Gh				
				1000				

Table 2: Arabic alphabetic (*abjad*) notation

The other form of notation, which is based on the Hindu-Arabic base-10 place-value notation, has 10 glyphs as follows:

0	٠
1	١
2	٢
3	٣
4	٤
5	٥
6	٦
7	٧
8	٨
9	٩

The following Arabic-English table shows the letters most commonly used in our translations.

ا	ب	ج	د	ه	ز	ح	ط	ك	ل	م	ن
a	b	c	d	e	f	g	h	i	j	k	l

Terminology

At the outset of his work, Al-Samaw'āl explains his technical terminology which forms his language of algebra. These terms are not original to him, but are part of the established "algebraic tradition". He states, noting the interchangeability of terms:

> ...when we multiply every number by itself then the result from the multiplication will be called *māl* or square or radicand and that the number which is multiplied by itself will be called side or thing or root. We call the number which is a combination of the multiplication of 3 numbers cube if the 3 numbers are equal. If the 3 numbers are not equal we will call the number solid [1, pages 17-18, author's own translation].

The building blocks of his system are:

x is defined as side, thing, or root (i.e., modern "x"). The Arabic versions for x are respectively ضلع, شيء, or جذر which are respectively pronounced as *ḍalʿ*, *shay*, or *jadhr*.

x^2 is defined as quantity, square, or radicand. The Arabic version for x^2 are respectively مَال, مربع, or مجذور which are respectively pronounced as *māl*, *murabaʿ*, or *majdhūr*.

x^3 is defined as cube. The Arabic version for x^3 is مكعب which can be pronounced as *muka'ab*.

x^4 is defined as square square. The Arabic version for x^4 is مَال مَال which can be pronounced as *māl māl* or "square square".

$x_1 \cdot x_2 \cdot x_3$ is defined as a solid. The Arabic version for $x_1 \cdot x_2 \cdot x_3$ is مجسم which can be pronounced as *mujasam*.

As we can observe the above definitions given by Al-Samaw'āl are completely interchangeable from an English linguistic view-point. For example we can interchange "side" with "thing" or "root" in order to get the same idea of x. Al-Samaw'āl himself uses the words interchangeably. He does not have a fixed technical vocabulary.

Furthermore, Al-Samaw'āl's vocabulary reveals he is still tied to the geometrical context that he has inherited from his Greek predecessors. For example, he has two terms that he uses to express the concept of "product". The first is "*ḍrb*" which is from the verb to multiply, and the second is "*musataḥ*" which literally translates as surface. These are used interchangeably. Furthermore, sometimes he talks about a "side", which is "*ḍal'*" and sometimes he talks about a "number" to express the same thing also. However, proposition two, for example, shows us that he is simply thinking of numbers.

Al-Samaw'āl's use of voice is interesting too in this respect. In line with the Euclidean tradition, he often invokes the first person singular, especially to state a proposition, for example "I say that...". Often he invokes the third person plural imperative ("Let us multiply..."), which too echoes the Euclidean mode of expression. He seems to be using these words because they are traditional.

Most propositions are accompanied by a simple diagram. In some cases it is not immediately obvious how the diagram supports the mathematical action in the text as Al-Samaw'āl makes no mention of these visual aids in the text, except for the long passage on how to construct the triangle of binomial coefficients. It is clear they have a purpose, usually pedagogical or to clarify the procedure described in the text.

2.2 Title and opening statement

Text

أَلبَاب الرَابع من المـَقَاله الثَانيَة

في برَاهِين هندسية يستعَان بهَا علَى استخرَاج المجهولَات العدَدِية

وهو فنَان

الفن الأَول من البَاب الرَابع من المقَاله الثَانية في الَاصول العدَدِية

Translation

Chapter 4 from Section 2

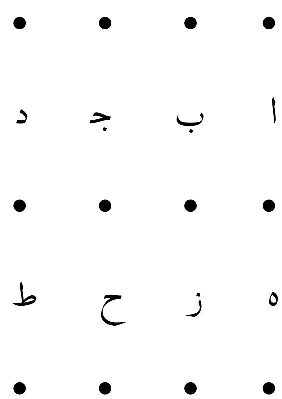
About the Geometrical Demonstrations Used to Extract the Numerical Unknowns

There are two methods. The first method from Chapter 4 from Section 2 consists of the numerical foundation (algebraic proof).

2.3 Proposition One

Text

كل أَربعة اعدَاد فَان ضرب مسطح الأَول و الثَاني في مسطح الثَالث و الرَابع مسَاو لضرب مسطح الأَول و الثَالث في مسطح الثَاني و الرَابع فنفرض اربعة اعدَاد ا ب جـ د . و لنضرب ا في ب و ليخرج ه ، و لنضرب ا في جـ و ليخرج ز ، و لنضرب جـ في د و ليخرج ط ، [و لنضرب ب في د و ليخرج ح]، فَأَقول إِن ضرب ه في ط مسَاو لضرب ز في ح وبرهَانه: ان عدَد ا ضرب في عدَدي ب جـ فخرج من الضرب عدَدَا ه وَ ز . فنسبة ه الَى ز كنسبة ب الى جـ . وَايضَاً فَان عدَد د ضرب في عدَدي ب جـ فخرج من الضرب عدَدَا ح ط فنسبة ح الَى ط كنسبة ب الَى جـ وقد كَانت نسبة ه الَى ز كنسبة ب الَى جـ فنسبة ه الَى ز كنسبة ح الَى ط ، فمسطح ه في ط مسَاو لمسطح ز في ح وذلك مَا اردنَا ان نبين .



Translation

[For] every four numbers, the product of the surface [5] of the first and the second by the surface of the third and the fourth is equal to the product of the surface of the

[5] Al-Samaw'āl has two ways to refer to the product of two numbers. He can use the Arabic word "مسطح" which literally means surface, or he can use the word "ضرب" which means "product". In order to capture the difference, we preserve his choice even though the mathematical meaning is the same.

first and the third by the surface of the second and the fourth.

Let us consider four numbers $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$. Let us multiply $\bar{a}$ by $\bar{b}$ to get $\bar{e}$ and let us multiply $\bar{a}$ by $\bar{c}$ to get $\bar{f}$. Let us multiply $\bar{c}$ by $\bar{d}$ to get $\bar{g}$. [Let us multiply $\bar{b}$ by $\bar{d}$ to get $\bar{h}$.] Then I say that the product of $\bar{e}$ and $\bar{g}$ is equal to the product of $\bar{f}$ and $\bar{h}$.

Its demonstration: [When] the number $\bar{a}$ is multiplied [respectively] by the two numbers $\bar{b}$ and $\bar{c}$, then there results from the multiplication [respectively] 2 numbers $\bar{e}$ and $\bar{f}$. Then the ratio of $\bar{e}$ to $\bar{f}$ is the same as the ratio of $\bar{b}$ to $\bar{c}$. Moreover, multiplying $\bar{d}$ by $\bar{b}$ and $\bar{d}$ by $\bar{c}$, we obtain [respectively] $\bar{h}$ and $\bar{g}$. Then the ratio of $\bar{h}$ to $\bar{g}$ is the same as the ratio of $\bar{b}$ to $\bar{c}$. We know that the ratio of $\bar{e}$ to $\bar{f}$ is the same as the ratio of $\bar{b}$ to $\bar{c}$. Then the ratio of $\bar{e}$ to $\bar{f}$ is the same as the ratio of $\bar{h}$ to $\bar{g}$. Then, the surface of $\bar{e}$ and $\bar{g}$ is equal to the surface of $\bar{f}$ and $\bar{h}$. This is what we wished to explain.

Mathematical Commentary

In contemporary symbolism Al-Samaw'āl is going to prove that

$$(ab)(cd) = (ac)(bd).$$

First paraphrase: using modern symbols, but staying close to his style of reasoning.

Setting out

Let $a,\ b,\ c,\ d$ be four numbers.

Let $ab = e$ and $ac = f$, and let $cd = g$ and $bd = h$.

Then I say that $eg = fh$.

Demonstration

Multiplying a by b and c gives e and f [respectively].

Hence the ratio $e : f$ is the same as the ratio $b : c$.

Moreover, multiplying d by b and c gives h and g [respectively].

Hence the ratio $h : g$ is the same as the ratio $b : c$.

Thus the ratio $e : f$ is the same as the ratio $h : g$.

Hence $eg = fh$ which is what we wished to explain.

Second paraphrase: using symbolic algebra.

Claim: $(ab)(cd) = (ac)(bd)$.

Demonstration

Since

$$\frac{ab}{ac} = \frac{b}{c} = \frac{bd}{cd}$$

we have

$$(ab)(cd) = (ac)(bd).$$

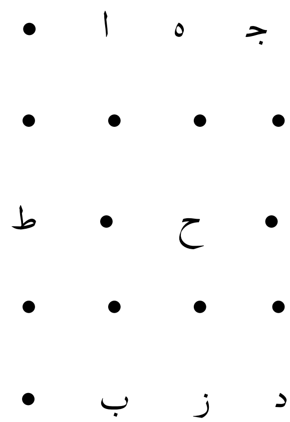
$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ d & c & b & a \\ \bullet & \bullet & \bullet & \bullet \\ g & h & f & e \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

The above geometric figure created by Al-Samaw'āl can be interpreted in various ways. One way is to say that geometrically speaking we associate to each letter of the alphabet a point. We start with the first 8 letters of the English alphabet a, b, c, d, e, f, g, and h and we can keep on continuing. As the reader would already know, the letters into the Arabic alphabet are written from right to left, different from the English alphabet. From another perspective we can see that the table represents an image of our above proposition in the sense that the ratio of $\bar{h}$ to $\bar{g}$ is also the same as the ratio of $\bar{b}$ to $\bar{c}$ and the ratio of $\bar{e}$ to $\bar{f}$ is the same as the ratio of $\bar{h}$ to $\bar{g}$ and those ratios are symmetrically presented into the table.

2.4 Proposition Two

Text

مسطح ضلعي كل مكعبين مسَاو [لمكعب] مسطحهمَا ، فليكن العدَدَان المكعبَان عدَدي ا ب
وليكن ضلعَاهمَا جـ د وليكن مربعَاهمَا ه ز ولنضرب جـ في د وليخرج عدَد ح ولنضرب ا في
بوليخرج ط ، فَأقول ان عدَد ط مسَاو لمكعب عدَد ح . برهَانه: انه قد تبين من المقَالَات
العدَدية ، ان عدَد ه المربع اذَا ضرب في عدَد ز المربع خرج من الضرب مربع عدَد ح
المسطح فَاذَا ضرب الحَاصل في ذلك في مسطح جـ في د ، أعني في عدَد ح ، حصل من
ذلك مكعب عدَد ح ، وهو من ضرب مسطح ه في ز في مسطح جـ في د . لكن الحَاصل
من ضرب مسطح ه في ز في مسطح جـ في د مسَاو لَلحَاصل من ضرب مسطح جـ في ه
في مسطح [ز في] د كمَابينَا في الشكل الذي قبل هذَا . فَالحَاصل من ضرب مسطح ه في
جـفي مسطح د في ز مسَاو لمكعب عدَد ح . لكن الحَاصل من ضرب جـ في ه [هو عدَد ا
] و مسطح د في ز هو عدَد ب . فمسطح ا في ب أَعني عدَد ط ، مسَاو لمكعب عدَد ح
وذلك مَا اردنَا ان نبين .



Translation

The surface of two sides each cubed is equal to the cube of their surface. Let the 2 cubic numbers be the 2 numbers $\bar{a}$ and $\bar{b}$ and let their sides be $\bar{c}$ and $\bar{d}$ and let their square be $\bar{e}$ and $\bar{f}$ and let us multiply $\bar{c}$ by $\bar{d}$ and we get the number $\bar{g}$ and let us multiply $\bar{a}$ by $\bar{b}$ and we get $\bar{h}$. Then I say that the number $\bar{h}$ is equal to the cube of the number $\bar{g}$.

Its demonstration: Indeed, it was indicated in the arithmetical sections that if the square number $\bar{e}$ is multiplied by the square number $\bar{f}$ there results from this multiplication the square of the number $\bar{g}$, the surface. Then, if the previous result is multiplied by the surface of $\bar{c}$ and $\bar{d}$, I mean by the number $\bar{g}$, there results from this the cube of the number $\bar{g}$, and it is from the product of the surface of $\bar{e}$ and $\bar{f}$ and the surface of $\bar{c}$ and $\bar{d}$. But the result from the product of the surface of $\bar{e}$ and $\bar{f}$ and the surface of $\bar{c}$ and $\bar{d}$ is equal to the result from the product of the surface $\bar{c}$ and $\bar{e}$ and the surface of $\bar{f}$ and $\bar{d}$, as we explained in the previous result.

Therefore the result from the product of the surface of $\bar{e}$ and $\bar{c}$ and the surface of $\bar{d}$ and $\bar{f}$ is equal to the cube of the number $\bar{g}$. But, the result from the product $\bar{c}$ and $\bar{e}$ [is the number $\bar{a}$] and the surface of $\bar{d}$ and $\bar{f}$ is the number $\bar{b}$. Therefore the surface of $\bar{a}$ and $\bar{b}$, I mean the number $\bar{h}$, is equal to the cube of number $\bar{g}$ and this is what we wished to explain.

Mathematical Commentary

Another mathematical fact which is proved by Al-Samaw'āl is:

The cube of a surface of two numbers is equal to the surface of the cube of the first one and the cube of the second one.

In modern mathematics Al-Samaw'āl proves that $(cd)^3 = c^3d^3$.

First paraphrase: using modern symbols, but staying close to his style of reasoning.

Setting out

Let a and b be cubes: $a = c^3$ and $b = d^3$.

Let $c^2 = e$ and $d^2 = f$.

Let $cd = g$ and $ab = h$.

Then I say that $h = g^3$.

Demonstration

We already know that

$$ef = g^2.$$

Multiplying by $cd = g$ gives

$$(ef)(cd) = g^3.$$

By Proposition 1 we have

$$(ef)(cd) = (ce)(fd)$$

and so

$$(ce)(fd) = g^3.$$

But $ce = a$ and $fd = b$. Thus

$$ab = h = g^3$$

which is what we wished to explain.

Second paraphrase: using symbolic algebra.

Claim: $c^3d^3 = (cd)^3$.

Demonstration

We already know that

$$c^2d^2 = (cd)^2.$$

Multiplying by cd gives

$$(c^2d^2)(cd) = (cd)^3.$$

By Proposition 1 we have

$$(c^2d^2)(cd) = (c^2c)(d^2d)$$

and so

$$(c^2c)(d^2d) = (cd)^3$$

But $c^2c = c^3$ and $d^2d = d^3$. Thus

$$c^3d^3 = (cd)^3$$

which is what we wanted.

The proof using symbolic algebra makes it easier to see how we might use induction to prove that $c^nd^n = (cd)^n$.

$$
\begin{array}{cccc}
\bullet & a & e & c \\
\bullet & \bullet & \bullet & \bullet \\
h & \bullet & g & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & b & f & d
\end{array}
$$

The above table symmetrically presents all identities from the proof of Proposition 2 like for example

$$ef = g^2,$$

$$cd = g,$$

$$ce = a,$$

$$fd = b,$$

$$ab = h.$$

From the first two above equalities, we obtain that

$$efcd = g^3.$$

Therefore

$$h = cefd = g^3.$$

2.5 Proposition Three

Text

كل عدَد يقسم بقسمين فَان مكعبه مسَاو لَلمكعبين الكَائِنين من قسميه وَ ضرب كل وَاحد من قسميه في مربع الآخر ثلَاث مرَات .

مثَاله: ان عدَد اب قُسم علَى نقطة جَ فَأقول إن مكعب اب مسَاو لمكعب اجَ وَ مكعب ج بوَ ضرب اجَ في مربع ج ب ثلَاث مرَات وَ ضرب ج ب في مربع اجَ ثلَاث مرَات .

برهَانه: فلَأن مربع اب مسَاو لمربع اجَ وَ مربع ج ب وَ ضرب اجَ في ج ب مرتين فَاذَا ضُرب اب في مربعه خرج مكعبه. فمكعب اب مسَاو لضرب اب في مربع اجَ وَ مربع ج بوَ ضرب اجَ في ج ب مرتين. وَ كل مَا يضُرب في اب فهو مثل مضروبه في اجَ ج ب كمَا بين اقليدوس في الشكل الَاول من المقَالة ٢ ، فضرب مربع اجَ في اجَ وَ في ج ب وَ ضرب مربع ج ب في ج ب وَ في اجَ وَ ضرب ضعف السطح الذي يحيط به اجَ ج ب في اجوَ ضربه ايضَاً في ج ب مثل مكعب اب ، لكن ضرب مربع [اب] في [اب] هو مكعب اجَ وَ ضرب مربع اجَ في ج ب وَ ضرب مربع ج ب في اجَ وَ ضرب ضعف السطح الذي يحيط به اجَ ج ب في كل وَاحد من اجَ ج ب [وَ مكعب ج ب] . لكن كل مسطح يضُرب في احد ضلعيه فَان الحَاصل من الضرب مسَاو لمَا يرتفع من ضرب مربع ذلك الضلع في الضلع الآخر، لَأن كل ثلَاثة اعدَاد فَان مضروب الَاول في الثَاني ثم في الثَالث مسَاو لمضروب الَاول في الثَالث ثم في الثَاني، فَالمكعب الكَائِن من اب مسَاو لمكعب اجَ وَ مكعب ج ب وَ ضرب اجَ في مربع ج ب ثلَاث مرَات وَ ضرب ج ب في مربع اجَ ثلَاث مرَات وَذلك مَا اردنَا ان نبين.

Translation

[When] any number is divided into two parts, then its cube is equal to the sum of the cubes of its two parts and the product of each of its parts and the square of the other part taken three times.

Its example: If a number $\overline{ab}$ is divided at the point $\bar{c}$, then I say that the cube of $\overline{ab}$ is equal to the cube of $\overline{ac}$ and the cube of $\overline{cb}$ and the product of $\overline{ac}$ and the square of $\overline{cb}$ three times and the product of $\overline{cb}$ and the square of $\overline{ac}$ 3 times.[6]

Its demonstration: Indeed, the square of $\overline{ab}$ is equal to the square of $\overline{ac}$ and the square of $\overline{cb}$ and the product of $\overline{ac}$ and $\overline{cb}$ taken twice.

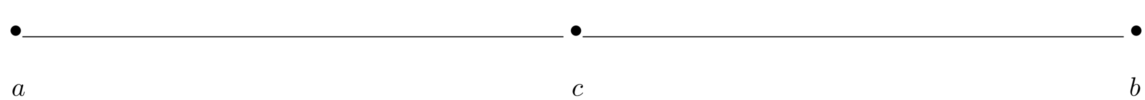
If we multiply $\overline{ab}$ by its square, we obtain its cube. The cube of $\overline{ab}$ is equal to the product of $\overline{ab}$ and the square of $\overline{ac}$, the square of $\overline{cb}$ and the product of $\overline{ac}$ and $\overline{cb}$ taken twice.

The product of $\overline{ab}$ and any number is equivalent to the product of $\overline{ac}$ and $\overline{cb}$ by this number, as Euclid explained in the first proposition of Book 2.

The product of the square of $\overline{ac}$ by $\overline{ac}$ and by $\overline{cb}$ and the product of the square of $\overline{cb}$ by $\overline{cb}$ and by $\overline{ac}$ and the product of double the surface encompassed by $\overline{ac}$ and $\overline{cb}$ by $\overline{ac}$ and the product of this also by $\overline{cb}$ is equivalent to the cube of $\overline{ab}$.

(So, expressing differently), the product of the square of [7][$\overline{ab}$] and [$\overline{ab}$] is the cube of $\overline{ac}$ and the product of the square of $\overline{ac}$ and $\overline{cb}$ and the product of the square of $\overline{cb}$ and $\overline{ac}$ and the product of double the surface that is encompassed by $\overline{ac}$ and $\overline{cb}$ by

[6]Observation: The notations used by Al-Samaw'āl are of strict geometric significance. $\overline{ab}$ signifies the segment $[ab]$, $\overline{ac}$ signifies the segment $[ac]$, and $\overline{cb}$ signifies the segment $[cb]$. From a geometric point of view, we can represent this below:



[7]We suggest that the text is corrupted here and for sense we replace $\overline{ac}$ and $\overline{ac}$ by $\overline{ab}$ and $\overline{ab}$.

each one of $\overline{ac}$ and [8] $\overline{cb}$ [and the cube of $\overline{cb}$].[9]

But, for every surface, if we multiply the surface by one of its sides then the result from this product is equal to the product of the square of that side and the other side, since for every three numbers, the product of the first and the second and then by the third is equal to the product of the first and the third then by the second.

Thus, the existing cube of $\overline{ab}$ is equal to the cube of $\overline{ac}$ and the cube of $\overline{cb}$ and the product of $\overline{ac}$ and the square of $\overline{cb}$ taken three times and the product of $\overline{cb}$ and the square of $\overline{ac}$ taken three times and this is what we wished to explain.

Mathematical Commentary

Symbolically speaking, let us see how Al-Samaw'āl proves that

$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2.$$

Paraphrase: using symbolic algebra.[10]

Claim: $(a+b)^3 = a^3 + b^3 + 3(ab^2 + a^2b)$.

Demonstration

We already know that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

[8] As we observe Al-Samaw'āl realised the properties of commutativity of addition and multiplication as he changed the expression from the above paragraph into the expression from this paragraph.

[9] For transcription reasons $\overline{cb}^3$ has not been written into Al-Samaw'āl's original document.

[10] We have replaced Al-Samaw'āl's line segments by algebraic symbols. To stay close to his notation we have used the following correspondences:
The line AB represents the number $a+b$.
The line AC represents the number a.
The line CB represents the number b.

Multiplying by $a+b$ we get

$$(a+b)^3 = (a+b)(a^2+b^2+2ab).$$

For any number x we have $(a+b)x = ax+bx$, as Euclid explains in *Elements II, 1.*

So we get

$$a^2a + a^2b + b^2a + b^2b + (2ab)a + (2ab)b = (a+b)^3.$$

But $(xy)x = x^2y$ since for any three numbers $p,\ q, r$ we have $(pq)r = (pr)q$.

Thus

$$(a+b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b,$$

which is what we wished to explain.

2.6 Proposition Four

Text

كل عدَد يقسم بقسمين فَان مربع مربع العدَد المقسوم مسَاو لمربع مربع كل وَاحد من القسمين وَ ضرب كل وَاحد من القسمين في مكعب الآخر أَربع مرَات وَ ضرب مربع وَاحدهمَا في مربع الآخر ست مرَات .

مثَاله: ان عدَد اب قسم بقسمين وهمَا اج ج ب فَان مربع مربع اب مسَاو لمربع مربع اج وَ مربع مربع ج ب وَ ضرب اج في مكعب ج ب أَربع مرَات وَ ضرب ج ب في مكعب اجأَربع مرَات وَ ضرب مربع اج وَ مربع ج ب ست مرَات .

برهَانه: ان مَال مَال اب هو من ضرب اب في مكعبه، وقد بينَا في الشكل الذي قبل هذَا ان مكعب اب مسَاو لمكعب اج وَ مكعب ج ب وَ ضرب اج في مربع ج ب ٣ مرَات وَ ضرب ج ب في مربع اج ثلَاث مرَات، وَ مضروب اب في كل عدَد مسَاو لمضروب ذلك العدَد في اج وَ في ج ب ، فمضروب مكعب اج في اج وهو مَال مَال اج وَ في ج ب وَ

مضروب مكعب جـ ب في جـ ب وهو مَال مَال جـ ب وَ في اجـ وَ ضرب مسطح مربع جـ بفي اجـ ثلَاث مرَات في اجـ وَ في جـ ب مثل مَال مَال اب . لكن ثلَاثة امثَال ضرب سطح مربع اجـ في جـ ب في اجـ ثلَاثة امثَال ضرب مكعب اجـ في جـ ب وَايضَاً فَان ثلَاثة امثَال مسطح ضرب مربع اجـ في جـ ب في جـ ب ثلَاثة امثَال ضرب مربع اجـ في مربع جـ بوَايضَاً فَان ثلَاثة امثَال ضرب مسطح مربع جـ ب في اجـ في اجـ مسَاو لثلَاثة امثَال ضرب مربع اجـ في مربع جـ ب وَ ثلَاثة امثَال ضرب مسطح مربع جـ ب في اجـ في جـ ب مسَاو لثلَاثة أَمثَال ضرب مكعب جـ ب في اجـ . فمَال مَال اب مسَاو لمَال مَال اجـ وَ مَال مَال جـ بوَ ضرب اجـ في مكعب جـ ب اربع مرَات وَ ضرب جـ ب في مكعب اجـ اربع مرَات وَ ضرب مربع اجـ في مربع جـ ب ست مرَات وذلك مَا أَردنَا أَن نبين .

Translation

[When] any number is divided into two parts, then the square square of the divided number is equal to the square square of each one of the two parts and the product of each one of the two parts and the cube of the other part taken four times and the product of the square of one of them and the square of the other taken six times.

Its example: When a number $\overline{ab}$ is divided into two parts, and they are $\overline{ac}$ and $\overline{cb}$, then the square square of $\overline{ab}$ is equal to the square square of $\overline{ac}$ and the square square of $\overline{cb}$ and the product of $\overline{ac}$ and the cube of $\overline{cb}$ taken four times, and the product of $\overline{cb}$ and the cube of $\overline{ac}$ taken four times, and the product of the square of $\overline{ac}$ and the square of $\overline{cb}$ taken six times.

Its demonstration: Indeed, the square square of $\overline{ab}$ is the product of $\overline{ab}$ and its cube and we explained in the previous proposition that the cube of $\overline{ab}$ is equal to the cube of $\overline{ac}$ and the cube of $\overline{cb}$ and the product of $\overline{ac}$ and the square of $\overline{cb}$ taken three times and the product of $\overline{cb}$ and the square of $\overline{ac}$ taken three times, and (we know that) the product of the number $\overline{ab}$ and any number is equal to the product of that number and $\overline{ac}$ and $\overline{cb}$.

Therefore, the product of the cube of $\overline{ac}$ by $\overline{ac}$, which is the square square of $\overline{ac}$, and by $\overline{cb}$ and the product of the cube of $\overline{cb}$ by $\overline{cb}$, which is the square square of $\overline{cb}$, and $\overline{ac}$ and the product of the surface of the square of $\overline{cb}$ and $\overline{ac}$ taken three times by $\overline{ac}$ and by $\overline{cb}$ [and the product of the surface of the square of $\overline{ac}$ by $\overline{cb}$ taken 3 times by $\overline{ac}$ and by $\overline{cb}$] is the square square of $\overline{ab}$.

But, three times the product of the surface of the square of $\overline{ac}$ and $\overline{cb}$ by $\overline{ac}$ (is equal to) three times the product of the cube of $\overline{ac}$ and $\overline{cb}$.

Similarly, three times the surface of the product square of $\overline{ac}$ and $\overline{cb}$ by $\overline{cb}$ (is equal to) three times the product of the square of $\overline{ac}$ and the square of $\overline{cb}$. Similarly, three times the product of the surface of the square of $\overline{cb}$ and $\overline{ac}$ by $\overline{ac}$ is equal to three times the product of the square of $\overline{ac}$ and the square of $\overline{cb}$. And three times the product of the surface of the square of $\overline{cb}$ and $\overline{ac}$ by $\overline{cb}$ is equal to three times the product of the cube of $\overline{cb}$ and $\overline{ac}$.

Therefore, the square square of $\overline{ab}$ is equal to the square square of $\overline{ac}$ and the square square of $\overline{cb}$, and the product of $\overline{ac}$ and the cube of $\overline{cb}$ taken four times, and the product of $\overline{cb}$ and the cube of $\overline{ac}$ taken four times, and the product of the square of $\overline{ac}$ and the square of $\overline{cb}$ taken six times. And this is what we wished to explain.

Mathematical Commentary

Let us go further in more difficult proofs and see how Al-Samaw'āl is going to prove that:

$$(a+b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2.$$

Paraphrase: using symbolic algebra.[11]

Claim:

$$(a+b)^4 = a^4 + b^4 + 4ab^3 + 4a^3b + 6a^2b^2.$$

Demonstration

We know that

$$(a+b)^4 = (a+b)(a+b)^3$$

and we saw in Proposition 3 that

$$(a+b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b.$$

Therefore

$$(a+b)^4 = (a+b)(a^3 + b^3 + 3a^2b + 3ab^2).$$

We know that, for any number x, we have $(a+b)x = ax + bx$.

So we get

$$a^3a + a^3b + b^3a + b^3b + (3a^2b)a + (3a^2b)b + (3ab^2)a + (3ab^2)b = (a+b)^4.$$

Here $a^3a = a^4$ and $b^3b = b^4$.

But $(3a^2b)a = 3a^3b$.

[11] As in Proposition 3, we have replaced Al-Samaw'āl's line segments by algebraic symbols.

Similarly $(3a^2b)b = 3a^2b^2$ and $(3ab^2)a = 3a^2b^2$ and $(3ab^2)b = 3ab^3$.

Thus

$$(a+b)^4 = a^4 + b^4 + 4ab^3 + 4a^3b + 6a^2b^2$$

which is what we wished to explain.

2.7 Proposition Five

Text

مَال مَال مسطح كل عدَدين مسَاو لمسطح مَالي مَاليهمَا . فليكن العدَدَان عدَدي ا ب وَ مسطحهمَا عدَد جـ فَأقول ان مَال مَال جـ مسَاو لضرب مَال مَال ا في مَال مَال ب برهَانه: انَا نضرب ا في نفسه وَ ليخرج د وَ نضرب ا في د وَ ليخرج ه وَ نضرب ه في ا وَ ليخرج ز فـ ز هو مَال مَال ا . وَ لنضرب ب في نفسه وَ ليخرج ح وَ لنضرب ح في ب وَ ليخرج ط وَ نضرب ط في ب وَ ليخرج ك فهو مَال مَال ب ، وَ نضرب جـ في نفسه وَ ليخرج ل وَ نضرب ل في جـ وَ يخرج م وَ نضرب م في جـ وَ ليخرج ن فهو مَال مَال جـ . فَأقول انه مسَاو لمسطح ز في ك . فلَأن ضلعي جـ عدَدَا ا ب وَ ضلعي ل همَا عدَدَا د ح المربعَان يكون نسبة مسطح جـ الَى مسطح ل مُؤلفة من نسبة ا الَى د وَ من ب الَى ح . لكن نسبة االَى د كنسبة د الَى ه لَأن عدَدي ا د ضربَا في ا فخرج د ه وَ نسبة ب الَى ح كنسبة ح الَى ط لَأن عدَدي ب ح ضربَا في عدَد ب فخرج ح ط [وَ نسبة جـ الَى ل كنسبة ل الَى م لَأن عدَدي جـ ل ضربَا في عدَد جـ فخرج ل م] فنسبة جـ الَى ل مُؤلفة من نسبة د الَى ه وَ من نسبة ح الَى ط . لكن ل هو مسطح د في ح فـ م هو مسطح ه في ط كمَا يظهر من عكس الشكل ه من المقَالة ٨ من كتَاب الَاسطقسَات . لكن نسبة د الَى ه كنسبة ه الَى ز وَ نسبة ح الَى ط كنسبة ط الَى ك وَ نسبة ل الَى م كنسبة م الَى ن فنسبة م الَى ن مُؤلفة من نسبة ه الَى ز وَ من نسبة ط الَى ك ، لكن ه ضرب في ط فخرج م فـ ز يضرب في ك فيخرج ن وذلك مَا اردنَا ان نبين .

Translation

The *māl māl* of the surface of every two numbers is equal to the surface of the *māl māl* of each of them.

Let the numbers be the two numbers $\bar{a}$ and $\bar{b}$ and their surface be the number $\bar{c}$. Then, I say that the *māl māl* of $\bar{c}$ is equal to the product of the *māl māl* of $\bar{a}$ and the *māl māl* of $\bar{b}$.

Its demonstration: We multiply $\bar{a}$ by itself and we get $\bar{d}$ and we multiply $\bar{a}$ by $\bar{d}$ to get $\bar{e}$ and we multiply $\bar{e}$ by $\bar{a}$ to get $\bar{f}$, then $\bar{f}$ is *māl māl* of $\bar{a}$.

Let us multiply $\bar{b}$ by itself and we get $\bar{g}$ and let us multiply $\bar{g}$ by $\bar{b}$ to get $\bar{h}$ and let us multiply $\bar{h}$ by $\bar{b}$ to get $\bar{i}$ then $\bar{i}$ is *māl māl* of $\bar{b}$.

We multiply $\bar{c}$ by itself and we get $\bar{j}$ and we multiply $\bar{j}$ by $\bar{c}$ to get $\bar{k}$ and multiply $\bar{k}$ by $\bar{c}$ to get $\bar{l}$ then $\bar{l}$ is *māl māl* of $\bar{c}$. Then, I say that $\bar{l}$ is equal to the surface of $\bar{f}$ and $\bar{i}$.

Therefore, because the two sides of $\bar{c}$ are the two numbers $\bar{a}$ and $\bar{b}$ and the two sides of $\bar{j}$ are the two numbers $\bar{d}$ and $\bar{g}$ which are squared, we obtain that the ratio of the surface $\bar{c}$ to the surface $\bar{j}$ is compounded of the ratio of $\bar{a}$ to $\bar{d}$ and the ratio of $\bar{b}$ to $\bar{g}$.

Thus, the ratio of $\bar{a}$ to $\bar{d}$ is equivalent to the ratio of $\bar{d}$ to $\bar{e}$ because the two numbers $\bar{a}$ and $\bar{d}$ are multiplied by the (same) number $\bar{a}$ and the result is $\bar{d}$ and $\bar{e}$.

And the ratio of $\bar{b}$ to $\bar{g}$ is equivalent to the ratio of $\bar{g}$ to $\bar{h}$ because the two numbers $\bar{b}$ and $\bar{g}$ are multiplied by the (same) number $\bar{b}$ and the result is $\bar{g}$ and $\bar{h}$. [And the ratio of $\bar{c}$ to $\bar{j}$ is equivalent to the ratio of $\bar{j}$ to $\bar{k}$ because when the two numbers $\bar{c}$ and $\bar{j}$ are multiplied by the same number $\bar{c}$ and the result is $\bar{j}$ and $\bar{k}$.] Therefore, the ratio of [12] [$\bar{j}$ to $\bar{k}$] is compounded of the ratio of $\bar{d}$ to $\bar{e}$ and the ratio of $\bar{g}$ to $\bar{h}$.

But $\bar{j}$ is the surface of $\bar{d}$ and $\bar{g}$ so $\bar{k}$ is the surface of $\bar{e}$ and $\bar{h}$. This was evident from the converse of Proposition 5 from Book VIII of *The Elements*.[13] But, the ratio of $\bar{d}$ to $\bar{e}$ is equivalent to the ratio of $\bar{e}$ to $\bar{f}$ and the ratio of $\bar{g}$ to $\bar{h}$ is equivalent to the ratio of $\bar{h}$ to $\bar{i}$. And the ratio of $\bar{j}$ to $\bar{k}$ is equivalent to the ratio of $\bar{k}$ to $\bar{l}$.

The ratio of $\bar{k}$ to $\bar{l}$ is compounded of the ratio of $\bar{e}$ to $\bar{f}$ and the ratio of $\bar{h}$ to $\bar{i}$. As when we multiply $\bar{e}$ by $\bar{h}$ we get $\bar{k}$ so when we multiply $\bar{f}$ by $\bar{i}$ we get $\bar{l}$. And this is what we wished to explain.

Mathematical Commentary

First paraphrase: using modern symbols, but staying close to his style of reasoning.

Setting out

Let the two numbers be a and b, and let their product be c.

Then I say that $c^4 = a^4 b^4$.

[12] For mathematical reasons we substituted the ratio of $\bar{c}$ to $\bar{j}$ by the ratio of $\bar{j}$ to $\bar{k}$.

[13] Proposition 5 from Book VIII of *The Elements*: "Plane numbers have to one another the ratio compounded of the ratios of their sides."

Demonstration

Construction

Let $a^2 = d$ and $ad = e$ and $ae = f$, so that $f = a^4$.

[Similarly] let $b^2 = g$ and $bg = h$ and $bh = i$, so that $i = b^4$.

And let $c^2 = j$ and $cj = k$ and $ck = l$, so that $l = c^4$.

Then I say that $l = fi$.

Explanation

[Now] since $c = ab$ and $j = dg$, the ratio $c : j$ is the ratio compounded of the ratios $a : d$ and $b : g$[14].

But the ratio $a : d$ is equal to the ratio $d : e$ since a and d multiplied by a give d and e [respectively][15].

[Similarly] the ratio $b : g$ is equal to the ratio $g : h$ since b and g multiplied by b give g and h.

[Similarly, the ratio $c : j$ is equal to the ratio $j : k$ since c and j multiplied by c give j and k.]

Therefore, the ratio [$j : k$] is the ratio compounded of the ratios $d : e$ and $g : h$.

But $j = dg$ so $k = eh$.

This follows from a converse to *Elements VIII, 5*.[16]

But the ratio $d : e$ equals the ratio $e : f$,

and the ratio $g : h$ equals the ratio $h : i$,

and the ratio $j : k$ equals the ratio $k : l$.

---

[14]by *Elements VIII, 5*

[15]by *Elements VII, 17*

[16]In this context, *Elements VIII, 5* says that if $j = dg$ and $k = eh$ then the ratio $j : k$ is the same as the ratio $d : e$ compounded with the ratio $g : h$. The converse which is needed here should say that if the ratio $j : k$ is the same as the ratio $d : e$ compounded with the ratio $g : h$ and if $j = dg$, then $k = eh$.

Hence the ratio $k : l$ is the ratio compounded of the ratios $e : f$ and $h : i$.

But [we know that] $k = eh$ so we get $l = fi$

which is what we wished to explain.

Second paraphrase: using symbolic algebra.

We know that

$$\frac{ab}{(ab)^2} = \frac{a}{a^2}\frac{b}{b^2}$$

But

$$\frac{a}{a^2} = \frac{a^2}{a^3}$$

and

$$\frac{b}{b^2} = \frac{b^2}{b^3}$$

and similarly[17]

$$\frac{ab}{(ab)^2} = \frac{(ab)^2}{(ab)^3}.$$

Hence

$$\frac{(ab)^2}{(ab)^3} = \frac{a^2}{a^3}\frac{b^2}{b^3}.$$

But we know that $(ab)^2 = a^2b^2$ so we must have [18] $(ab)^3 = a^3b^3$.

Now

$$\frac{a^2}{a^3} = \frac{a^3}{a^4}$$

and

$$\frac{b^2}{b^3} = \frac{b^3}{b^4}$$

and similarly

$$\frac{(ab)^2}{(ab)^3} = \frac{(ab)^3}{(ab)^4}.$$

---

[17] We have incorporated some changes to the manuscript at this point.

[18] We observe that in the above paragraph Al'Samaw'āl proved that $(ab)^3 = a^3b^3$ using a different method from the one already given in Proposition 2.

Hence

$$\frac{(ab)^3}{(ab)^4} = \frac{a^3}{a^4}\frac{b^3}{b^4}.$$

But we know that $(ab)^3 = a^3b^3$ so we must have[19] $(ab)^4 = a^4b^4$.

Al-Samaw'āl creates a mathematical table in order to summarise the properties specified in the proved theorem.

The table looks like:

ا	د	ه	ز
ج	ل	م	ن
ب	ح	ط	ك

The Arabic letters are written in classical Arabic per column starting first row, third row, and second row.

When he explains the equalities (properties) present on these three rows, Al-Samaw'āl starts with the first row, third row, and second row. If we translate the table's Arabic letters into English letters, the table looks like:

$\underline{a}$	$\underline{d}$	$\underline{e}$	$\underline{f}$
$\underline{c}$	$\underline{j}$	$\underline{k}$	$\underline{l}$
$\underline{b}$	$\underline{g}$	$\underline{h}$	$\underline{i}$

As we observe the alphabetical order is not preserved into the English alphabet as it is into the Arabic classical alphabet.

[19] Once again it can be seen easily how we might use induction to prove that $(ab)^n = a^nb^n$.

The table can be interpreted as follows:

First row:

d is obtained multiplying a by itself ($d = a^2$).

e is obtained multiplying a by d ($e = ad$).

f is obtained multiplying a by e ($f = ae$).

f is the *māl māl* (square square) of a ($f = a^4$).

Third row:

g is obtained multiplying b by itself ($g = b^2$).

h is obtained multiplying b by g ($h = bg$).

i is obtained multiplying b by h ($i = bh$).

i is the *māl māl* (double square) of b ($i = b^4$).

Second row:

j is obtained multiplying c by itself ($j = c^2$).

k is obtained multiplying c by j ($k = cj$).

l is obtained multiplying c by k ($l = ck$).

l is the *māl māl* (double square) of c ($l = c^4$).

Explanation:

We used the same order "First row", "Third row", and "Second Row" as Al-Samaw'āl used in his table. As we observe, respecting the alphabetical order, the first row deals with a, the third with b, and the second row with c.

Making the above substitutions, the previous table is going to look like:

$$\begin{array}{cccc} \underline{a} & \underline{a}^2 & \underline{a}^3 & \underline{a}^4 \\ \underline{c} & \underline{c}^2 & \underline{c}^3 & \underline{c}^4 \\ \underline{b} & \underline{b}^2 & \underline{b}^3 & \underline{b}^4 \end{array}$$

As we observe Al-Samaw'āl obtained some other properties like:

c is the surface of a and b ($c = ab$).

j is the surface of d and g ($j = dg$).

k is the surface of e and h ($k = eh$). (1)

l is the surface of f and i ($l = fi$). (2)

Al-Samaw'āl explains that the ratio of c to j is equal to the ratio of a to d times the ratio of b to g.

$$\frac{c}{j} = \frac{a}{d} \times \frac{b}{g}. \qquad (3)$$

Al-Samaw'āl explains again how he created his table as

$$\frac{a}{d} = \frac{d}{e} \qquad (4)$$

and

$$\frac{b}{g} = \frac{g}{h}. \qquad (5)$$

From (3), (4), and (5) we obtain that the ratio of c to j is equal to the ratio of d to e multiplied by the ratio of g to h, which can be written symbolically as

$$\frac{c}{j} = \frac{d}{e} \times \frac{g}{h}.$$

To structure the information on the table, we can write that

$$\frac{a}{d} = \frac{d}{e} = \frac{e}{f}.$$

$$\frac{b}{g} = \frac{g}{h} = \frac{h}{i}.$$

$$\frac{c}{j} = \frac{j}{k} = \frac{k}{l}.$$

Al-Samaw'āl proved as well that

$$\frac{k}{l} = \frac{e}{f} \times \frac{h}{i}.$$

Observation: The above equality can be easily proved applying (1) and (2).

From the previous results (Section 2.7, First paraphrase, Mathematical commentary), we obtain that:

$$\frac{j}{k} = \frac{d}{e} \times \frac{g}{h}.$$

Therefore, although Al-Samaw'āl did not prove, we can prove that

$$\frac{j}{k} = \frac{c^2}{c^3} = \frac{1}{c}. \qquad (6)$$

$$\frac{d}{e} = \frac{a^2}{a^3} = \frac{1}{a}. \qquad (7)$$

$$\frac{g}{h} = \frac{b^2}{b^3} = \frac{1}{b}. \qquad (8)$$

From the equalities (7) and (8), we obtain that

$$\frac{d}{e} \times \frac{g}{h} = \frac{1}{a} \times \frac{1}{b} = \frac{1}{ab} = \frac{1}{c}. \qquad (9)$$

From (6) and (9), we obtain that

$$\frac{j}{k} = \frac{d}{e} \times \frac{g}{h},$$

and the proof is done.

2.8 Cases n = 5 and Higher

Text

و بمثل هذَا البيَان يبرهن علَى ان مَال كعب مسطح كل عدَدين مسَاو لمسطح مَال كعب احدهمَا في مَال كعب الآخر وعلَى هذَا فصَاعدَاً .
ومن فهم مَا قلنَاه فقد يمكنه ان يبرهن علَى ان كل عدَد يقسم بقسمين فَان مَال كعب مسَاو لمَال كعب كل وَاحد من قسمين ، وَ ضرب كل وَاحد منهمَا [في] مَال مَال الآخر خمس مرَات وَ مربع كل وَاحد منهمَا في مكعب الآخر عشرة مرَات وَ مَا يتلوَا ذلك مضَاعفَاً .

Translation

By the same method it can be demonstrated that the *māl* cube of the surface of any two numbers is equal to the surface of the *māl* cube of one of them by the *māl* cube of the other, and so on in increasing order.

For a person who understands what we have done then that person can demonstrate that for every number divided into two parts the *māl* cube is equal to the *māl* cube of each of the two parts and the product of each one by the *māl* *māl* of the other one taken 5 times and the product of the square of each of them by the cube of the other taken 10 times, and so on for the next ascending cases.

Mathematical Commentary

Observation: As can be observed from the above 2 passages, we can see that mathematical induction is present again in Al-Samaw'āl's work. This leads us to recognise Al-Samaw'āl's contribution towards the development of mathematical induction. Although Al-Samaw'āl did not use symbols like for example the concept of "n" or "k", he used in a certain sense the language and modes of expression available to him in order to make us think that he knew about mathematical induction.

From a symbolical point of view, although Al-Samaw'āl did not prove it, he observed that the same methods could be used to show that:

$$(1) \quad (ab)^5 = a^5 b^5,$$

and for $n>5$,

$$(ab)^n = a^n b^n.$$

$$(2) \quad (a+b)^5 = a^5 + b^5 + 5a^4 b + 5ab^4 + 10a^2 b^3 + 10a^3 b^2,$$

and for $n>5$,

$$(a+b)^n = a^n + b^n + na^{n-1}b + nab^{n-1} + \cdots .$$

In conclusion, Al-Samaw'āl tells us that we are going to observe the coefficients of the development of each binomial. In fact, as we are going to see, Al-Samaw'āl will build his table starting with $(a+b)^1$ and going up to $(a+b)^{12}$.

2.9 The Methodology of How to Construct the Table

Text

و لنذكر الآن اصلاً يعرف به عدَد المرَات التى [تلزم] لضرب هذه المرَاتب بعضهَا عند بعض في كل عدَد يقسم بقسمين .

قَال الكرجي: اذَا اردت ذلك وضعت علَى التخت وَاحدَاً و وَاحدَاً تحته ، ثم نقلت الوَاحد الَى سطر آخر و ضَمَمت الوَاحد الَى الوَاحد الذي تحته يكون اثنين وضعته تحته ، ثم وضعت الوَاحد الآخر تحته فيصير وَاحدَاً وَاثنين و وَاحدَاً فهذَا يدلك أَن كل عدَد مُؤلف من عدَدين اذَا ضربت كل وَاحد منهمَا في نفسه مرة وَاحدة لكون الطرفين وَاحدَاً و وَاحدَاً و ضربت احد العدَدين في الآخر مرتين لكون الوَاسطة ٢ بلغ مربع ذلك العدَد . ثم نقلنَا الوَاحد من السطر الثَاني الَى سطر آخر ، وضَمَمنَا الوَاحد الَى الَاثنين يصير ثلَاثة ، كتبنَاه تحت الوَاحد ، وضَمَمنَا الَاثنين الَى الوَاحد الذي تحتهَا فتصير ثلَاثة كتبنَاهَا تحت الثلَاثة فيخرج من ذلك سطر ثَالث تكون آحَاده وَاحدَاً وثلَاثة وثلَاثة و وَاحدَاً . فهذَا يعلمك ان مكعب كل عدَد مُؤلف من عدَدين هو ان يكعب كل وَاحد منهمَا و يضرب كل وَاحد منهمَا في مربع الآخر ثلَاث مرَات . وَ نقلنَا الوَاحد الذي في السطر الثَالث الَى سطر آخر ، ثم ضَمَمنَا الوَاحد الَى الثلَاثة التي تحته تكون اربعة كتبتهَا تحت الوَاحد ، ثم ضَمَمت الثلَاثة الَى الثلَاثة التي تحتهَا يكون ٦ كتبتهَا تحت الَاربعة ثم ضَمَمت الثلَاثة الثَانية الَى الوَاحد يكون اربعة كتبتهَا تحت الستة ثم نقلت الوَاحد الَى تحت الَاربعة فيَأتلف من ذلك سطر آخر يكون اعدَاده وَاحد وَاربعة و ٦ [و اربعة] و وَاحدَاً . فهذَا يعلمك تركيب مَال مَال من عدَد مؤُلف من عدَدين وهو ان تجعل كل وَاحد منهمَا مَال مَال لكون الوَاحد في الطرفين ثم ضربت كل عدَد في مكعب الآخر اربع مرَات لكون الَاربعة تَالية لَلطرفين اللَذين همَا وَاحد [و] وَاحد ، لَأن الجذر في المكعب يكون مَال مَال ، ثم ضربت مربع احدهمَا في مربع الآخر ست مرَات لكون الستة وَاسطة ولَأن المربع في المربع مَال مَال . فَان نقلت الوَاحد من السطر الرَابع الَى سطر خَامس ثم زدت الوَاحد علَى الَاربعة التي تحته و الَاربعة علَى ٦ التي تحتهَا و الستة علَى الَاربعة التي تحتهَا و الَاربعة علَى الوَاحد الذي تحتهَا و كتبت مَا ارتفع من ذلك تحت الوَاحد المنقول علَى الولي المذكور و كتبت بعد ذلك الوَاحد البَاقي إِئتلف من ذلك سطر خَامس اعدَاده وَاحد و ٥ و عشرة و عشرة و ٥ و وَاحد

. فهذَا يعلمك ان كل عدَد يقسم بقسمين فَان مَال كعب مسَاو لمَال كعب كل وَاحد من قسميه لكون الطرفين وَاحداً و وَاحداً وَ لمضروب كل وَاحد من العدَدين في مَال مَال الآخر خمس مرَات لكون الخمسة تَالية لَلطرفين المتقدمين من الجَانبين وَ ضرب مربع كل وَاحد منهمَا في مكعب الآخر عشر مرَات لكون العشرة تَالية لَلخمستين و كل وَاحد من هذه الجمل من جنس مَال كعب لأَن الجذر في مَال مَال وَ المكعب في المَال يرتفع من كل وَاحد منهمَا مَال كعب و بهذَا العمل يعرف عدَد المرَات في التمويل وَ التكعيب الَى أَي نهَاية شئِنَا وهذه صورة ذلك :

كعب كعب كعب كعب	مال كعب كعب كعب	مال مال كعب كعب	كعب (1) كعب كعب	مال كعب كعب	مال مال كعب	كعب كعب	مال كعب	مال مال	كعب	مال	شيء
١	١	١	١	١	١	١	١	١	١	١	١
١٢	١١	١٠	٩	٨	٧	٦	٥	٤	٣	٢	١
٦٦	٥٥	٤٥	٣٦	٢٨	٢١	١٥	١٠	٦	٣	١	
٢٢٠	١٦٥	١٢٠	٨٤	٥٦	٣٥	٢٠	١٠	٤	١		
٤٩٥	٣٣٠	٢١٠	١٢٦	٧٠	٣٥	١٥	٥	١			
٧٩٢	٤٦٢	٢٥٢	١٢٦	٥٦	٢١	٦	١				
٩٢٤	٤٦٢	٢١٠	٨٤	٢٨	٧	١					
٧٩٢	٣٣٠	١٢٠	٣٦	٨	١						
٤٩٥	١٦٥	٤٥	٩	١							
٢٢٠	٥٥	١٠	١								
٦٦	١١	١									
١٢	١										
١											

Table 3: Table of the coefficients of the binomial expression $(a+b)^n$ with n from 1 to 12 in modern Arabic writing.

Translation

Let us now mention a principle for knowing the number of times that are necessary to multiply these degrees by each other for any number divided into two parts.

Al-Karajī says: in order to achieve that, you place "one" on a table and "one" below it, then move the first one into another column and add the first one to the one below it, then you obtain "two" and you put the two under the [translated] one, and you place the second one below the two, then you have one, two, and one. This indicates that for every number consisting of two numbers and, if you multiply each of them by itself once, since the two ends are one and one, and if you multiply one of them by the other twice, since the middle term is two, we obtain the square of that number.

Then we transfer the one in the second column into another column, add the first one [from the second column] to the two [under it], we obtain three and we write it under the one [in the third column]. Therefore, we add the two [from the second column] to the one below it, we obtain three, we write it below the first three, [and then write one under this three]. Thus, we obtain the third column which contains: one, three, three, and one. This teaches us that the cube of any number consisting of two numbers is the cube of each of them and the product of each of them by the square of the other taken three times.

Then, we transfer the one from the third column to another column, then we add the "one" (from the third column) to the three below it, we obtain four to be written under the one, then we add three to the three below it, we obtain six to be written under the four, add the second three to the one below it, we obtain four to be written under the six, then we transfer the one under the four. Then, the result from this is another column which contains the numbers: one, four, six, four, and one. This teaches us that the construction of *māl māl* from a number consisting of two numbers is the *māl māl* of each of them, since the "one" is in the two ends, then you multiply each number by the cube of the other taken four times, since the "four" follows ones at the two ends, since the root multiplied by the cube is *māl māl*, then, you multiply the square of one of them by the square of the other taken six times, since the six is the middle and since the square multiplied by the square is *māl māl*.

Then, if we transfer the one from the fourth column into the fifth column and add the one (in the fourth column) to the four below it, and four to six below it, six to the four below it and the four to the one below it, then we write down the results of that continuously respectively under the transferred one and end by writing the remaining one, we obtain from that the fifth column, its numbers are: 1, 5, 10, 10, 5, and 1. This teaches us that for any number divided into two parts, its *māl* cube is equal to the *māl* cube of each part, since the two ends are one and one, and the product of each of them by the *māl māl* of the other taken 5 times, since fives are in the immediate vicinity of the two end ones and the product of the square of each one by the cube of the other taken 10 times, since the numbers 10 are in the immediate vicinity of the two fives. Each of these terms belong to the type *māl* cube as the product of the root by *māl māl* and the product of the cube by *māl* both give *māl* cube.

Therefore, we can continue the algorithm to determine the number of *māls* and cubes of any power that we wish to obtain and below we are going to write the associated diagram.

Manuscript copies depict the table as follows:

x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12
	1	3	6	10	15	21	28	36	45	55	66
		1	4	10	20	35	56	84	120	165	220
			1	5	15	35	70	126	210	330	495
				1	6	21	56	126	252	462	792
					1	7	28	84	210	462	924
						1	8	36	120	330	792
							1	9	45	165	495
								1	10	55	220
									1	11	66
										1	12
											1

Table 4: Table of the coefficients of the binomial expression $(a+b)^n$ with n from 1 to 12

3 Discussion: Induction and the Binomial Theorem

3.1 General Ideas about Al-Samaw'āl's Contribution in a Middle Eastern and Worldwide Mathematical Context

We focused mostly on Al-Samaw'āl's contribution in the development of Mathematics in Islam and worldwide mostly because of the high demand for information about Islamic Mathematics in the European world. Al-Samaw'āl's work has not been translated extensively in English, especially when it is about Pascal's triangle and this is the reason why we explored this side.

Regarding Al-Samaw'āl's participation in the development of the binomial theorem, we are inclined to say that he did the pioneering research in this area. He proved the steps for $n = 3$ up to $n = 4$ and afterwards he specified the binomial expansion for $n = 5$ and continued for the case $n > 5$.

It is excellent for posterity to be able to read in English what an Islamic mathematician like Al-Samaw'āl could create in terms of new and different mathematical concepts. Analysing Al-Samaw'āl's Right Trapezium, we observe the coefficients of the binomial expansion from $(a+b)^1$ up to $(a+b)^{12}$. Al-Samaw'āl does not just give the table, he explains how the coefficients have been created. His method of presenting the coefficients of the binomial $(a+b)^n$ is different and unique and occurred before Pascal created Pascal's Triangle in a similar manner. This method consists in Al-Samaw'āl's Right Trapezium.

From an induction point of view, Al-Samaw'āl has constructed the initial steps of verifying for $n = 3$ up to $n = 4$. Al-Samaw'āl suggests that the proofs can be done for $n = 5$ and for $n > 5$. He specifies that the mathemtical process can continue for ascending order cases above 5.

Al-Samaw'āl gives in his table the coefficients up to $(a+b)^{12}$, which represents a complex verification of the induction steps. Al-Samaw'āl specifies in writing that the process can continue for greater powers. Analysing Al-Samaw'āl's work, we can say that for his historic times, Al-Samaw'āl's work is very innovative.

In his book *Eléments D'histoire des Mathématiques* published in 1960 Bourbaki considers that the principle of mathematical induction had been clearly conceived and employed for the first time by the Italian F. Maurolico in the 16^{th} century. In spite of this, we would be tempted to think that because of the historic context and because of the difficult communication, the information and in our case mathematical science could not travel among continents. It was definitely easier for the information to circulate inside the same continent.

We would claim that Al-Samaw'āl is the father of mathematical induction. Mathematical induction is present in Al-Samaw'āl's work on his right trapezium and around it, in the sense that Al-Samaw'āl does the verification part up to $(a+b)^{12}$ and specifies in writing that the process can keep going.

Mathematical induction is also present in Al-Samaw'āl's proof of

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i(i-1),$$

which is written in Al-Samaw'āl's original way as:

> The sum of the squares of the numbers that follow one another in natural order from one is equal to the sum of these numbers and the product of each of them by its predecessor [1, page 127].

The proof is done for $n = 4$.

3.2 Selected Applications from Al-Samaw'āl's Arithmetical Treatise *Al-Qiwāmī fī Al-Hisāb Al-Hindī*

In his treatise *Al-Qiwāmī fī Al-Hisāb Al-Hindī* Al-Samaw'āl used the table of binomial coefficients to solve problems posed later in his work. His techniques were notably accurate. He used this technique to extract the side of a square or a cube, introducing the problem as follows. To use Al-Samaw'āl's own words:

> When you extract the side of a square or a cube or any other *marātib*[20] and you know the integer part, I mean the closest side of square, cube, or another *martāba*[21], which is the closest to the required side and if there is a remainder left, it indicates that the side is a surd (irrational).
>
> (Next, we are going to obtain the decimal part of the number), taking the numbers from the rule of that side and multiplying each integer part to the associated *martāba* (power) by the numbers given by rule (namely the table of binomial coefficients).
>
> (Finally), you add all new results and increase the sum by 1. The new quantity is the denominator of the remainder. [13, pages 110-114; author's own translation]

[20]The meaning of the Arabic word *marātib* is the plural for power.

[21]The meaning of the Arabic word *martāba* is the singular for power.

So, Al-Samaw'āl states a general rule for approaching the non-integer part (decimal part) of the rational root of an integer by fractions.

Al-Samaw'āl illustrates this with some examples. For instance in Book Al-Qiwāmī fī Al-Hisāb Al-Hindī, Chapter 15, Section 5, *Extracting the Fractional Part of an Irrational Side which Is Closely Approaching the Integer*, he describes:

> 1. Extracting the square root of 60, we find the closest (perfect) square less than 60 which is 49 and we subtract 49 from 60 in order to get 11. Next, we multiply 2 from the rule of [22] *māl* by 7, (which is $\sqrt{49}$) and get 14. Adding 1 to 14, we obtain 15. Then, we determine 11 parts of 15 which is in fact the ratio between 11 and 15. In conclusion $\sqrt{60}$ is 7 and 11 parts of 15 [13, pages 110-114; author's own translation].

For the case $n = 2$, Al-Samaw'al's binomial coefficients in his table are 1, 2, and 1 which he uses to extract the second root.

In other words he seeks x such that

$$x^2 = 60$$

and his calculations proceed as follows:

$$\sqrt{49} = 7;$$

$$60 - 49 = 11;$$

$$7 \times 2 = 14;$$

$$14 + 1 = 15;$$

[22]Please refer to the column associated with x^2 from Al-Samaw'āl's table of the coefficients of the binomial expression.

$$\frac{11}{15} \simeq 0.733...3.$$

Thus $\sqrt{60} \simeq 7.733...3$.

Modern techniques show that $\sqrt{60} = 7.745966$, so that Al-Samaw'āl's technique is accurate to 2 significant figures.

Another example, which reveals the precision of Al-Samaw'āl's techniques is as follows:

> 2. (An interesting example) is extracting the side of a cube of (volume) 10 (that is $\sqrt[3]{10}$), we find the closest (perfect) cube less than 10 which is 8 and we subtract 8 from 10 in order to get the integer part 2. We take 3 and 3 from the line of cube[23] and multiply the first 3 by the integer part 2 and the second 3 by the square of the integer part 2^2. We add it all together and increase it by 1 in order to get 19. This 19 is the denominator of the remainder. Then, we determine 2 parts of 19 which is in fact the ratio between 2 and 19. In conclusion $\sqrt[3]{10}$ is 2 and 2 parts of 19 [13, pages 110-114; author's own translation].

For the case $n = 3$, Al-Samaw'al's binomial coefficients in his table are 1, 3, 3, and 1 which he uses to extract the third root.

In other words he seeks x such that

$$x^3 = 10$$

and his calculations proceed as follows:

$$\sqrt[3]{8} = 2;$$

[23]Please refer to the column allocated to x^3 from Al'Samaw'āl's table of the coefficients of the binomial expression.

$$10 - 8 = 2;$$

$$3 \times 2 = 6;$$

$$3 \times 2^2 = 12;$$

$$6 + 12 = 18;$$

$$18 + 1 = 19;$$

$$\frac{2}{19} \simeq 0.105263157.$$

Thus $\sqrt[3]{10} \simeq 2.105263157$.

Modern techniques show that $\sqrt[3]{10} = 2.15443469$, so that Al-Samaw'āl's technique is accurate to 2 significant figures.

> 3. Another example is extracting the *māl māl* root (the root of order 4) of 40. We will find the closest (perfect) base of a *martāba* (power) of order 4 less than 40 which is 2 (as $2^4 = 16$). The remainder is 24 (as $40 - 16 = 24$). We find the numbers of the rule of *māl māl* as 4, 6, and 4.[24] Then, we multiply the first number 4 by the integer part 2, we multiply the second number 6 by the square of the integer part which is 4, and the third number 4 by the cube of the integer part which is 8. Adding these values all together and increasing by 1, we obtain 65, which represents the denominator of the remainder. In conclusion, the side will be 2 and 24 parts of 65 [13, pages 110-114; author's own translation].

For the case $n = 4$, Al-Samaw'al's binomial coefficients in his table are 1, 4, 6, 4, and 1 which he uses to extract the fourth root.

[24]Please refer to the column of x^4 from Al-Samaw'āl's table of coefficients for the binomial expression.

In other words he seeks x such that

$$x^4 = 40$$

and his calculations proceed as follows:

$$\sqrt[4]{16} = 2;$$

$$40 - 16 = 24;$$

$$4 \times 2 = 8;$$

$$6 \times 2^2 = 6 \times 4 = 24;$$

$$4 \times 2^3 = 4 \times 8 = 32;$$

$$8 + 24 + 32 = 64;$$

$$64 + 1 = 65;$$

$$\frac{24}{65} \simeq 0.369230769.$$

Thus $\sqrt[4]{40} \simeq 2.369230769$.

Modern techniques show that $\sqrt[4]{40} = 2.514866859$, so that Al-Samaw'āl's technique is accurate to one significant figure.

> 4. (Let us) extract the *māl* cube root (the root of order 5) of 250. We will determine the closest (perfect) *māl* cube less than 250 which is 3 (as $3^5 = 243$). The remainder is 7 (as $250 - 243 = 7$). We find the numbers of the rule of *māl* cube as 5, 10, 10, and 5.[25] Then, we multiply the first number 5 by the integer part 3, the second number 10 by the square of 3, the third number 10 by the cube of 3, and the fourth number 5 by the

[25]This affirmation is based on the column of x^5 from Al-Samaw'āl's table regarding the coefficients of the binomial expression.

> *māl māl* of 3 (the fourth power of 3). Adding these values all together and increasing by 1, we obtain 781, which represents the denominator of the remainder. In conclusion, the side will be 3 and 7 parts of 781, and the same rule followed for the other *marātib* (powers) [13, pages 110-114; author's own translation].

For the case $n = 5$, Al-Samaw'al's binomial coefficients in his table are 1, 5, 10, 10, 5, and 1 which he uses to extract the fifth root.

In other words he seeks x such that

$$x^5 = 250$$

and his calculations proceed as follows:

$$\sqrt[5]{243} = 3;$$

$$250 - 243 = 7;$$

$$5 \times 3 = 15;$$

$$10 \times 3^2 = 90;$$

$$10 \times 3^3 = 270;$$

$$5 \times 3^4 = 405;$$

$$15 + 90 + 270 + 405 = 780;$$

$$780 + 1 = 781;$$

$$\frac{7}{781} \simeq 0.00896287.$$

Thus $\sqrt[5]{250} \simeq 3.00896287$.

Modern techniques show that $\sqrt[5]{250} = 3.017088168$, so that Al-Samaw'āl's technique is accurate to 2 significant figures. According to Al-Samaw'āl, the same algorithm applies for powers greater than 5.

3.3 Al-Samaw'āl's Way of Inspiring Other Generations of Mathematicians

There are 3 separate themes where different scientists talk about Al-Samaw'āl's work.

1) Authors in the Islamic tradition who knew Al-Samaw'āl's work and used it.

For example Al-Samaw'āl's book *Al-Bāhir fī Al-Jabr* represents an inspiration for Al-Kāshī's book *Miftah-Al-Hisab* (*Key of Arithmetic*), published in 1427 [7, page 93].

2) Authors in the Western tradition who did not know Al-Samaw'āl but who duplicated some of his ideas.

There are plenty of European mathematicians who discuss the same mathematical and generally speaking scientific results as Al-Samaw'āl. The French mathematician Blaise Pascal proved by complete mathematical induction (the basis (base case) and the inductive step) that for $k \leq n$:

$$C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

which makes us think that the research done by plenty of mathematicians before Pascal has been improved in time until Pascal did the complete induction.

Pascal organised the binomial coefficients into a triangle called Pascal's (Tartaglia's) Triangle. Pascal's Triangle is a fixed picture in which the coefficients of the binomial expression $(a+b)^n$ are organised in a triangle. What is interesting to mention is that Al-Karajī discovered the same triangle (which we are inclined to call Al-Karajī's Triangle) and did the initial research in this area more than 500 years before Pascal discovered it (this happened after 1007). The Islamic scientist Al-Samaw'āl created a right trapezium (which we can call Al-Samaw'āl's Right Trapezium) where we can observe the coefficients of all binomials starting from $(a+b)^1$ to $(a+b)^{12}$. His complex table, which is organised as a right trapezium, has the coefficients (numbers) organised extremely precisely from a mathematical point of view. In reality the History of Mathematics tells us that European mathematicians had little access to the research done by Islamic mathematicians for many years because many Islamic mathematical scripts have been lost. This is the reason why we would think that Al-Karajī's Triangle was 'reinvented' by Pascal. Later on, the scientist Isaac Newton (1642-1727) found the general formula for expanding out the binomial $(a+b)^n$. This formula is given by

$$\begin{aligned}(a+b)^n &= a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \cdots + C_n^k a^{n-k}b^k + \cdots + C_n^{n-1}ab^{n-1} + b^n \\ &= \sum_{k=0}^{n} C_n^k a^{n-k}b^k, 0 \le k \le n, k, n \in \mathbb{N},\end{aligned}$$

and is known as the Binomial Theorem.

The French mathematician Levi ben Gerson (1288-1344) worked on combinatorial theorems and their proofs. Two of the most important theorems are the ones that deal with associativity and commutativity of multiplication. In the proofs of these theorems ben Gerson introduces, more explicitly than the previous mathematicians, the most important parts of the method of mathematical induction, what he defined as the process of "rising step-by-step without end". In general, when ben Gerson uses such a proof, he first proves the inductive step that makes the transition from k to $k+1$, then notes that the process begins at some small value of k, then finally states the complete result.

3) Modern writers who have commented on Al-Samaw'āl's work.

The most prominent research regarding Al-Samaw'āl is done by the Arabic mathematicians Salah Ahmad and Roshdi Rashed in their book *Al-Bāhir en Algèbre d'As-Samaw'al*, where Al-Samaw'āl is presented from a bibliographical and scientific point of view [1]. This is not the only book where Rashed presents Al-Samaw'āl as he also presented Al'Samaw'āl's research in recent publications [15 and 16].

Al-Samaw'āl's work is outlined into Victor J. Katz's book *A History of Mathematics*. In this book we can see Al-Samaw'āl's contribution to serious mathematical work like the law of exponents, the division of polynomials, and Pascal's triangle [8].

Al-Samaw'āl's results are present in another book written by Victor J. Katz, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*. In this book we observe again Al-Samaw'āl's contribution to techniques involving the binomial coefficients and Pascal's triangle [9].

Another source of inspiration is the *Dictionary of Scientific Biography* where Al-Samaw'āl has his own space [7].

Al-Samaw'āl's research is present into J. L. Berggren's book *Episodes in the Mathematics of Medieval Islam*, a book published under the prestigious editor Springer-Verlag. In *Episodes in the Mathematics of Medieval Islam*, we can see how Al-Samaw'āl worked on the law of exponents and on the division of polynomials. All of Al-Samaw'āl's work is interesting in its own originality [2].

For example, Karine Chemla discusses in the source *Chinese and Arabic Mathematical Writings* from the journal *Arabic Sciences and Philosophy*, volume 4 (1994) about Al-Samaw'āl's research [3]. As we already specified, Professor Franz Rosenthal presented Al-Samaw'āl's work into the article *Al-Asturlābī and As-Samaw'al on Scientific Progress* [17]. William C. Waterhouse published the article *Note on a Method of Extracting Roots in As-Samaw'Al* which is another great representation of Al-Samaw'āl's methods of doing Mathematics [19].

In conclusion, from the piece of research presented above, we can say that Al-Samaw'āl can be regarded as the "father" of mathematical induction and his work on binomial expansions was advanced through a type of thearetical reasoning, demonstration, and examples that were developed and presented in a unique way.

References

[1] Salah Ahmad and Roshdi Rashed, *Al-Bāhir en Algèbre d'As-Samaw'al*, Imp. de l'Université de Damas, 1972.

[2] J. L. Berggren, *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York, Inc., 2003.

[3] Karine Chemla, *Similarities Between Chinese and Arabic Mathematical Writings: (I)Root Extraction*, Arabic Sciences and Philosophy, **4**, 207-266, 1994.

[4] J. J. O'Connor and E. F. Robertson, *Ibn Yahya Al-Maghribi Al-Samawal*, JOC/EFR, 1999.

[5] Euclid, *Euclid's Elements*, **3**, ed. by Th. Heath, Dover, New York, 1966.

[6] John Fauvel and Jeremy Gray, *The History of Mathematics: A Reader*, Macmillan Education LTD and The Open University, 1987.

[7] Charles Coulston Gillispie, *Dictionary of Scientific Biography*, American Council of Learned Societies, New York: Scribner, **12**, 91-95, 1970-1980.

[8] Victor J. Katz, *A History of Mathematics*, Pearson Education, Inc., 2004.

[9] Victor J. Katz, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, A Sourcebook*, Princeton University Press, New Jersey, 2007.

[10] David A. King, *The History of Science Society*, Isis, **67**, 307-308, No. 2, 1976.

[11] Al-Samaw'āl Al-Maghribī, *Al-Bāhir fī Al-Jabr*, Istanbul, Suleymaniye Library, MS Aya Sofia **2118**, 1149.

[12] Al-Samaw'āl Al-Maghribī, *Al-Maqāla Al-Thālitha fī'ilm Al-Misāha Al-Hindiyya*, Florence, Laurentian Library, MS **238**.

[13] Al-Samaw'āl Al-Maghribī, *Al-Qiwāmī fī Al-Hisāb Al-Hindī*.

[14] M. Perlmann, *Al-Samau'āl, Ifḥām al-Yahūd*, Proceedings of the American Academy for Jewish Research, New York, **32**, 1964.

[15] Roshdi Rashed, *L'Extraction de la racine n-ième et l'invention des fractions décimales (XI^e-XII^e siècles)*, Arch. Hist. Exact Sci., **18**, 191-244, 1978.

[16] Roshdi Rashed, *The Development of Arabic Mathematics: Between Arithmetic and Algebra*, Kluwer Academic Publishers, (Robert S. Cohen, Boston University, ed.), Transl. A. F. W. Armstrong, **156**, 1994.

[17] Franz Rosenthal, *Al-Asturlābī and As-Samaw'al on Scientific Progress*, Osiris, The University of Chicago Press, The History of Science Society, **9**, 555-564, 1950.

[18] Cat. F. Sayyid, *Ifhām ṭā'ifat al-yahūd, ("Confutation of the Jews"), [We looked into both the Arabic and the English version of this article.]*, Cairo MS, American Academy for Jewish Research, **32**, 1964.

[19] William C. Waterhouse, *Note on a Method of Extracting Roots in As-Samaw'Al*, Communicated by C. Truesdell, Department of Mathematics, The Pennsylvania State University, University Park, 1978.

[20] دَار المشرق، المنجد في اَللغة و الأَعلَام، المكتبة الشرقية، بيروت، ١٩٨٨.

[21] الدكتور علي عبدَالّله الدفَاع، المدخل إِلَى تَاريخ الريَاضيَات عند العرب و المسلمين، مُؤسَسة الرسَالة، بيروت، ١٩٨٩.

[22] تَاريخ الريَاضيَات العربية بين الجبر و الحسَاب لَلدكتور رشدي رَاشد، ترجمة الدكتور حسين زين الدين، مركز درَاسَات الوحدة العربية، بيروت، ١٩٨٩.

ARABIC-ENGLISH GLOSSARY

We specify about those words which have mathematical or technical meaning in the treatise.

two	اثنين
four	اربعة
to encompass	احَاط
to obtain	ارتفع
to extract	استخرج
remainder	بَاقي
Proof, demonstration	برهَان
under, below	تحت
construction	تركيب
three	ثلَاثة

root	جذر
result, sum, product	جمع
kind, type	جنس
to get	حصل
result	حَاصل
to get, to obtain, to produce	خرج
five	خمسة
to add	زَاد
six	ستة
thing (x), variable (x)	شيء
to increase, to ascend	صعد
integer	صحيح جمعه صحَاح

surd, irrational	صمَاء
multiplication, product	ضرب
to double	ضعّف
side	ضلع
number	عدَد جمعه اعدَاد
ten	عشرة
to converse	عَكَس
to suppose	فَرَض
rule	قَانون
to divide	قسّم
power	قوة جمعه قوَى
fraction	كسر جمعه كسور

each, every	كل
since, but	لكون
quantity (x^2)**, square/radicand** (x^2)	مَال
example	مثَال
equal	مثل
times (adverb)	مرَات
square $(\square)$	مربع
measurement	مسَاحة
surface	مسطح
is equal to/equals	مسَاو
cube (x^3)	مكعب
is consisting of/consists of	مُؤلف

ratio	نسبة
point	نقطة
to transfer	نَقَل
one	واحد