SOME BEST CHOICE PROBLEMS WITH UNCERTAIN RECALL

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CONTENTS

ABSTRACT

This thesis investigates an extension of the classical best choice problem which permits attempts at recall of any item passed over at an earlier stage. Three variations of the problem of finding a policy which maximises the probability of obtaining the best of N rankable items for given probability distributions over the availabilities of items from stage to stage, are considered. One variation permits only one attempt to obtain any item, even if that attempt is unsuccessful. Another allows any number of unsuccessful attempts at obtaining an item, while the third variation assumes that, at each stage, all the information about the availabilities of items already inspected is known before a decision whether or not to obtain an item is made.

The general solutions of all three problems are given in terms of the optimal control of a Markov chain through at most N steps.

Analyses of the problems are carried out under additional simplifying assumptions about the probability distribution on the order of presentation and availabilities of the items. These analyses include sufficient conditions for the optimal policies to have certain simple forms and also the derivation of some asymptotic properties. In particular, asymptotes for the maximum probability of obtaining the best item are given and the asymptotic optimality of sequences of policies of a simple closed form is proved.

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CHAPTER I

INTRODUCTION

1. THE BEST CHOICE PROBLEM AND SUMMARY OF RELATED WORK

This thesis will be devoted to the study of a generalisation of what we shall refer to as the classical best choice problem. This problem is equivalent to a number of problems that have appeared in the literature under various names; for example, "Googol" in Fox and Marnie (1960), the marriage problem in Lindley (1961), the dowry problem in Gilbert and Mosteller (1966) and the secretary problem in Chow, Robbins and Siegmund (1971). We now state the assumptions of the best choice problem - in terms of choosing a secretary although of course the problem has many other interpretations.

A known fixed number, N, of applicants of differing quality are presented to an employer one at a time in random order, all N! orderings being equally likely. At the time of interview of an applicant, the employer is only able to observe her rank relative to those who have preceded her, and he must then make the decision either to employ the present applicant and make no further interviews, or to interview the next applicant. Recall of applicants already passed over is not permitted. Under these assumptions, the classical best choice problem is to find the policy which maximises the probability that the employer chooses the best applicant.

The solution of the problem is well known. The maximum probability of employing the best, for $N \geqslant 2$, is given by

$$V^N = \frac{s^N}{N} \sum_{n=s^N}^{N-1} 1/n ,$$

where s^N is the positive integer s such that

(1.1) $$\sum_{n=s}^{N-1} 1/n \geq 1 > \sum_{n=s+1}^{N-1} 1/n .$$

A policy which gives this probability is:

do not employ any of the first s^N applicants, but thereafter employ the first applicant who is better than her predecessors.

The bounds for s^N

(1.2) $$N/e - 1 < s^N < N/e + 1 - 1/e ,$$

follow from (1.1) by using the fact that

$$\log (n+1) - \log n < 1/n < \log n - \log (n-1) .$$

It is then straightforward to show that $s^N/N \to e^{-1}$ and $V^N \to e^{-1}$ as $N \to \infty$.

The solution of the classical problem was first published by Moser and Pounder (1960) in Martin Gardner's column, "Mathematical Games", of the journal Scientific American, as the solution to the problem "Googol" which was posed earlier in the same column by Fox and Marnie (1960). The problem has also appeared in this "brain-teaser" role in Mosteller (1965) and in Bissinger and Siegel (1963) (with solution by Bosch (1964)).

It is, however, in its role as an intuitive but non-trivial illustrative example of the use of backward recursion for constructing sequential decisions, that the best choice problem is most famous. It has been treated in this manner by Lindley (1961), where the asymptotic result is first given. Dynkin (1963) uses the problem as an example of his very nice characterisation of the "optimum choice of the instant for stopping a Markov process". Similar coverage of the problem is given in Dynkin and Yushkevitch (1969), De Groot (1970) and Chow, Robbins and Siegmund (1971).

The originator of the problem would appear to be unknown. In Gilbert and Mosteller (1966), Frederick Mosteller stated that he was told the problem in 1955 by Andrew Gleason who claimed to have heard it from someone else. This person may have been Herbert Robbins, since he set the problem in an examination in the early 1950's (Robbins, 1976). On the other hand it is stated in Fox and Marnie (1960) that they devised the game "Googol" in 1958.

Before looking at some of the variations of the classical best choice problem it is worth noting that a far more difficult sequential decision problem with vague similarities was posed more than 100 years ago by Cayley (1874). In his words, the problem was "A lottery is arranged as follows: There are k tickets representing a, b, c, ... pounds, respectively. A person draws once; looks at his ticket; and if he pleases, draws again (out of the remaining $k-1$ tickets); looks at his ticket, and if he pleases draws again (out of the remaining $k-2$ tickets); and so on, drawing in all not more than n times; and he receives the value of the last ticket. Suppose he regulates his drawings in the manner most advantageous to him according to the theory of probabilities, what is the value of his expectations?" This problem cannot be considered to be a variation or generalisation of the best choice problem since the quantities a, b, c, ... are known in advance and the aim is not the maximisation of the probability of choosing the ticket with the highest value. Nevertheless in Cayley (1875), the solution of the case $k = 4$, $a = 1$, $b = 2$, $c = 3$, $d = 4$, (Expectations are 10/4, 38/12, 85/24, 4 respectively for the cases $n = 1, 2, 3, 4$) and the computational technique for solving the general problem appear to be the first published examples of the use of backward induction for solving a statistical decision problem.

Many possible avenues for variation and generalisation of the classical best choice problem, spring to mind. We now list some of those which have been considered in the literature.

1. More than one applicant can be chosen.

2. A criterion other than the maximisation of the probability of choosing the best may be used.

3. Observations of random variables other than relative ranks may be made.

4. The problem can be turned into a game by allowing an opponent to choose the order of presentation.

5. Costs of observation may be introduced.

6. Total number of applicants, N, has a known prior probability distribution.

7. Rejected applicants can be recalled at a later stage.

8. Offers of employment may be refused by an applicant.

The most thorough and comprehensive article which considers variations of the classical best choice problem, is Gilbert and Mosteller (1966). They commenced by giving a solution of the best choice problem which included narrower bounds for s^N than those given by (1.2). The first variation considered was to allow at most r applicants to be chosen and the aim was to find a policy which maximises the probability that one of the chosen applicants is best (denote this probability by V_r^N). It is shown for r = 2, that the asymptotic form of the optimal policy passes $Ne^{-3/2}$ applicants before making any first choice of a relatively best applicant and passes Ne^{-1} applicants before making any second choice of a relatively best applicant. They also show that $V_2^N \rightarrow 0.591$, and give a general expression for the limit of V_r^N as $N \rightarrow \infty$.

Gilbert and Mosteller also solved the important variation which they referred to as the full information game. Instead of observing relative ranks sequentially, the employer observes the actual qualities which are assumed to be independent values from some known probability distribution. The aim is to maximise the probability of choosing the highest quality applicant when only one choice is permitted. The problem is distribution free in the sense that all distributions of qualities are equivalent to the uniform distribution on (0,1). The optimal policy is determined by a sequence $\{b_i\}$, $i = 0,1,\ldots,$ independent of N, where applicant j is chosen if she is relatively best and her quality is greater than b_{N-j}. Implicit formulas for b_i are derived and first and second order approximations are given. Gilbert and Mosteller also show that the asymptotic probability of choosing the best is about 0.580, and briefly touch on the r - choice full information problem.

Gilbert and Mosteller, while departing a little from the title of their article - "Recognising the Maximum of a Sequence" - also investigated the full information problem from the point of view of maximising the expected quality of the applicant chosen. This variation, however, is not distribution free and cases with uniform, normal, half-normal, exponential and inverse powers distributions on the qualities are treated for 1, 2 and 3 choices. Earlier, Moser (1956) had solved the case when there is one choice and the qualities are uniformly distributed on (0,1).

Another interesting variation examined by Gilbert and Mosteller is to introduce game theory to the classical best choice problem by allowing an opponent to choose the order of presentation so as to minimise the probability of choosing the best applicant. (An earlier treatment of a two player game for a variation of the best choice problem appears in Chow, Robbins, Moriguti and Samuels (1964).) Where the opponent has total freedom to choose the order, the value of the game is shown to be $1/N$, which is the probability of choosing the best if the employer selects any applicant at random. Under the restriction that the opponent is only able to select the position of the best applicant with the remaining $N-1$ positions in the order being filled randomly, the minimax probability of choosing the best applicant is shown to be $1/(1+\sum_{n=1}^{N-1} 1/n)$. The value of the game is approximately doubled when the interviewer can make 2 choices.

The problem of maximising the probability that the applicant chosen is one of the r best, is another variation which has received some attention. Gilbert and Mosteller have shown that for $r = 2$, this probability approaches 0.574 approximately as $N \to \infty$, but were dissuaded from considering the problem for general r by the complexity of the algebra. Even the asymptotic solution for general r is not straightforward, and the derivation of some partial results are set as exercises in Dynkin and Yushkevitch (1969) and in De Groot (1970). At about the same time as Gilbert and Mosteller's article appeared, Gusein-Zade (1966) published an article in which this problem was considered as an example. The asymptotic form for the optimal policy and the maximum probability that one of the r best is chosen can be calculated numerically as a special case of the general class of problems considered in Mucci (1973a).

Lindley (1961) in the second of his examples examined the variation of the classical best choice problem in which the aim was to find a policy which minimised the expected rank of the applicant selected. He showed that the optimal policy is given by a sequence of non-negative integers $s_1,\ldots,s_N$ where the employer is to stop and choose the first applicant n whose relative rank is less than or equal to s_n. (s_N is of course N.) The sequence $s_1, \ldots, s_N$ is calculated by means of a relatively simple recursive equation, which also gives V^N, the minimum expected rank of the chosen applicant. Lindley in an endeavour to find approximate values for $s_1, \ldots, s_N$, approximated the recursive equation by a differential equation. It was apparent through this, that V^N had a finite limit but the approximation was too crude for an evaluation to be made.

Chow, Morguti, Robbins and Samuels (1964) showed by direct calculation that, for the problem of minimising the expected rank of the applicant chosen,

$$V^N \nearrow \Pi_{j=1}^{\infty} (1+2/j)^{1/(j+1)} \doteq 3.870 .$$

They also solved the two person game where the opponent can choose the order of presentation in an attempt to increase the expected rank of the applicant chosen. There is an opponents policy, viz. at stage n present either the best or the worst of the remaining applicants with probability 1/2 each, which gives expected rank of $(N+1)/2$ no matter what policy is used by the employer.

Chow et al remarked that they were able to evaluate $\lim V^N$ by approximating the recursion equation by a sequence of differential equations, instead of just one as Lindley had done. However, this technique was heuristic, and it was not obvious how to go about making the argument rigorous. It was this problem of proving that an approximate solution for a generalised class of problems could be derived from a suitable differential equation, that Mucci (1973a, 1973b) solved. These two papers are very concentrated and many of the details are omitted. A complete treatment can be found in Mucci (1972).

Mucci makes the basic assumptions of the classical best choice problem with the generalisation that the employer wishes to find the policy which optimises the expected value of a payoff q which is a monotone function of the absolute rank, r, of the applicant chosen. Let t be any policy based on the relative ranks and let r_t be the absolute rank of the applicant chosen. In Mucci (1973a) it is assumed that $q(1) = 1$ and $q(r) \searrow 0$ and the problem is to determine

$$V^N = \max_t E[q(r_t)] .$$

In the second article (Mucci, 1973b) $q(1) = 1$, $q(r) \nearrow \infty$ and $\Sigma \lambda^r q(r) < \infty$ are the assumptions and the aim is to determine

$$V^N = \min_t E[q(r_t)] .$$

These generalisations include many of the earlier variations of the classical problem. For example, the classical problem is equivalent to taking $q(1) = 1$, $q(r) = 0$, $r \geqslant 2$, and the problem of minimising the expected rank is equivalent to taking $q(r) = r$, $r \geqslant 1$.

In Mucci (1973a) a differential equation with solution function g, defined on $[0,1]$ and boundary condition $g(1) = 0$ is considered. It is shown that g is unique for each q, and that $\lim_{N\to\infty} V^N = g(0)$. The solution of the equation does not have an explicit form, however, and a recursion is given which will permit the calculation of not only $\lim V^N$ but also the asymptotic form of the policies for any q.

In Mucci (1973b) the unboundedness of q presented additional problems for evaluating $\lim V^N$ from the solution of a particular differential equation. Nevertheless, a recursion which calculates $\lim V^N$ and the asymptotic form of the optimal policy for most cases, is obtained.

Yet another important generalisation of the classical best choice problem with practical applications is to make the assumption that total number of applicants N has a known prior distribution π, over the positive integers. Presman and Sonin (1972) have treated this problem fairly thoroughly. They give sufficient conditions on π in order that the optimal policy should take the form: make no choice before stage s; thereafter choose the first relatively best applicant. This is the same form as the optimal policy for the classical problem. They showed that a policy of this form was optimal in the cases where the distribution π was

1) uniform on $\{1,2,\ldots,M\}$,

2) Poisson (plus one to give a distribution over the positive integers), or

3) geometric, probability p of a success.

The asymptotic properties of one-parameter (usually the mean) families for π for which the optimal policy is of the classical form, is investigated. In particular, it is shown that the limit of the probability of choosing the best applicant is

1) $2e^{-2}$ for the uniform family as $M \to \infty$,

2) e^{-1} for the Poisson family as $\mu \to \infty$, and

3) $\gamma \int_1^\infty e^{-\gamma x} x^{-1}\, dx$ for the geometric family as $1/p \to \infty$, where $\gamma \doteq 0.18$ is the root of the equation

$$0 = \int_1^\infty e^{-\gamma x} (1 - \log x) x^{-1}\, dx .$$

Also in the corresponding cases

1) $s/M \to e^{-2}$,

2) $s/\mu \to e^{-1}$ and

3) $sp \to \gamma$.

A generalised payoff structure in combination with a prior distribution for N over a finite set of integers has been considered by Rasmussen (1975). However, the article contains a serious error (in Lemma 3.2) and this leads to an assumption about the form of the optimal policy which contradicts the findings of Presman and Sonin (1972). The remaining results of Rasmussen are therefore in jeopardy. His assumption is correct if the prior distribution is degenerate, but this case is covered in Mucci (1973a).

The same error is propagated by reference into Rasmussen and Robbins (1975). This is not serious though, since the assumption about the form of the optimal policy is true for the problem considered viz. the problem of choosing the best when the total number of applicants has a uniform prior distribution. Nevertheless this problem had already been solved in Presman and Sonin (1972).

The consideration of sampling costs is usually associated with the observation of random variables other than relative ranks. This generalisation is therefore rather removed from the classical best choice problem. Some examples both with and without recall appear in De Groot (1970).

To complete this section we now consider the generalisation of the best choice problem that is to be studied in this thesis. Apart from the articles Smith (1975), Smith (1977) and Smith and Deely (1975), which include some of the results to be presented later, the only article to consider this generalisation is Yang (1974). He treated the case where an approach may be made to any applicant already interviewed under the assumption that the probability of a successful approach to applicant j at stage n is given by $p(n-j)$, $1 \leq j \leq n \leq N$. The sequence $p(0), p(1), \ldots, p(N-1)$, where $p(0) = 1$, is necessarily non-increasing since Yang assumed that unavailable applicants remained unavailable. The object is to find the policy which maximises the probability of employing the best applicant when only one successful approach is permitted.

Yang shows that no approach should be made before stage s^N, where s^N is given by (1.1). It is also shown that the policy which interviews all N applicants and then approaches the best is optimal if and only if $p(k+1)/p(k) > (N-2)/(N-1)$, for $k = 0,1,\ldots,N-2$. The optimal policies with the corresponding probabilities of employing the best applicant, are obtained for the special cases where $p(k) = p^k$, $(0 < p < 1)$ and $p(k) = p$, $(0 \leqslant p < 1$, $k \geqslant 1)$ (the classical best choice problem is equivalent to taking $p = 0$ in this case).

Sections 3.3 to 3.5 of this thesis contain results which include those of Yang as special cases. These results were obtained independently.

2. STATEMENT OF THE PROBLEMS

We now list the general assumptions of the problems to be studied in this thesis. The first assumption applies to each of the problems, but the last three assumptions are alternatives which distinguish the problems.

Assumption 1.1.

1) A known number N, of applicants of differing quality are presented one at a time for interview by an employer.

2) At stage n, that is at the completion of the interview of applicant n, the employer observes the rank of applicant n relative to her predecessors.

3) The employer may approach at stage n, any of the applicants already interviewed or he may choose to interview the next applicant

(provided $n < N$) .

4) There is a known distribution on the availabilities of each applicant at and subsequent to her interview and on the order of presentation of the applicants according to their quality.

Assumption 1.2. A. Interviews must cease at the time of a successful approach but after an unsuccessful approach the employer may approach another of the applicants already interviewed or continue with further interviews.

Assumption 1.2. B. At stage n and before any decision whether or not to make an approach, the employer may ascertain the availability or not of any or all of the applicants interviewed so far.

Assumption 1.2. C. Only one approach is permitted and interviews cease whether or not it was successful.

The problem of finding a policy which maximises the probability of employing the best applicant under Assumption 1.1 together with Assumption 1.2.A, shall be referred to as Problem A. Problem B and Problem C differ from Problem A only in that Assumption 1.2.A is replaced by Assumption 1.2.B and Assumption 1.2.C respectively.

The essential differences between the three problems are: in Problem B the employer can determine whether or not his approach will be successful before he has to commit himself, whereas in Problem A the only way to determine the availability of an applicant is to make an approach, which if accepted is binding; however in Problem C the "expense" of making an approach is such that no further approaches can be contemplated.

It should be remarked that at present no assumptions are made about the form of the distribution on the order of presentation and availabilities of applicants. It is possible for applicants who are unavailable at some stage to become available at a later stage and vice versa. Additional assumptions about the form of the distribution will be made in Chapters III and IV. Also part 4) of Assumption 1.1 carries the implication that the order of presentation and the avail= ability of any applicants are not influenced by any decisions made by the employer.

We now describe some examples, lettered correspondingly, where the assumptions of the various problems might apply in an idealised way.

<u>Example 1.1.A.</u> (An extension of the marriage problem in Lindley (1961).) A batchelor wishes to maximise the probability of marrying the most compatible of the N spinsters who live in his district. He courts them one at a time and before courting the next spinster he must decide whether or not to propose to any of the spinsters whom he has courted so far. The only way he can determine whether any particular spinster will marry him is to propose, and then if the proposal is accepted his courting days are over. The probability that a proposal is accepted is likely to depend not only upon when it is made but also on the compatibility of the couple and perhaps even upon the order in which courtships were made. The distribution of the order of court= ships may also depend upon compatibilities since he may have some pre= conceived idea of these and choose the order accordingly.

<u>Example 1.1.B.</u> A person who wishes to buy a house, contacts a real estate agent who has a list of N suitable properties which are currently on the market. The houses are inspected one by one over a period of time and before the buyer makes a decision either to buy one of the houses already seen or to inspect another house, he is able to

ascertain which if any of the houses already inspected are no longer on the market. This may give him some idea of the current state of the market and hence likely future trends and thus help in the making of the decisions. Again, the buyer seeks the policy which maximises the probability of buying the best of the N houses.

Example 1.1.C. The instruments of a defensive missile site have detected the launch of an enemy MIRV (Multiple Independent Re-entry Vehicle) which is known to have N warheads. The computer is only able to calculate the trajectory of the warheads one at a time. The controller of the site, who wishes to maximise the probability of successfully intercepting the warhead with the most valuable target, is faced at each stage with decision whether to fire his one remaining ABM (Anti Ballistic Missile) at one of the warheads already tracked or whether to calculate the trajectory of another warhead. The probability of a successful interception will obviously depend upon the delay before the ABM is fired, and since there is only one missile, only one interception attempt can be made.

CHAPTER II

SOLUTIONS OF THE GENERAL PROBLEMS

While Problems B and C can easily be formulated as optimal stopping problems, Problem A will have to be treated slightly differently. Here the alternative decision to continuation is not to stop but instead to make an approach. It is not known beforehand though, whether or not an approach will be successful. One way of overcoming this difficulty might be to consider a randomised stopping policy. Instead, we shall treat all three problems as special cases of the following general problem.

1. A MARKOV CHAIN DECISION PROBLEM

The problem now to be examined is an extension of the problem of optimal stopping for a finite Markov chain, to allow for a decision which may or may not result in the stopping of the chain.

Consider a Markov chain with a finite state space S, for which termination will occur at the Nth stage if it has not occurred earlier. Let us also assume for ease of calculation, that S is partitioned into the non-empty sets $S_1, S_2, \ldots, S_N$, where S_n is the set of possible states of the chain at the nth stage. This is not a severe restriction on the model since the states of any finite chain can be redefined to include the stage number. On S there is an initial probability mass function f (where $f(\sigma) = 0$, $\sigma \notin S_1$) and a payoff function U where $U(\sigma)$ is the payoff received if the chain terminates in state σ. In any state $\sigma \in S \setminus S_N$ there is a choice of two decisions δ_0, δ_1 which have the following results. If δ_0 is chosen when in state σ then the chain proceeds with transition probability function $f_0(\sigma,\cdot)$. If, however, δ_1 is chosen then the chain terminates in state σ with probability $f_1(\sigma)$ or otherwise it proceeds with transition probability function $f_1(\sigma,\cdot)$. Thus for $\sigma \in S_n$, $n \in \{1,2,\ldots,N-1\}$,

$$(2.1) \qquad f_1(\sigma) + \Sigma_{\sigma' \in S_{n+1}} f_1(\sigma,\sigma') = 1 .$$

The chain terminates for every $\sigma \in S_N$ and thus $f_1(\sigma) = 1$, $\sigma \in S_N$. For the sake of completeness we shall assume that δ_1 is chosen for every $\sigma \in S_N$.

Hence any policy can be represented by a set Δ, $S_N \subseteq \Delta \subseteq S$, where decision δ_1 is chosen when the chain is in state σ if and only if $\sigma \in \Delta$.

If we denote by $\sigma(\Delta)$, the random state in which the chain terminates under policy Δ, then what we seek is a policy Δ^N, called an optimal policy, such that

$$E[U(\sigma(\Delta^N))] = \max_{S_N \subseteq \Delta \subseteq S} E[U(\sigma(\Delta))] ,$$

where the expectation is with respect to the distribution of the terminating state, determined by f, f_0, f_1. Because of the finiteness of S, U is bounded and hence $E[U(\sigma(\Delta))]$ always exists. Since there is only a finite number of distinct policies, an optimal policy exists and thus the problem is well posed.

The solution is a straightforward application of the inductive technique of Dynamic Programming; see for example Bellman (1957).

The maximum expected payoff,

$$V^N = \max_{S_N \subseteq \Delta \subseteq S} E[U(\sigma(\Delta))] ,$$

and an optimal policy Δ^N are generated by means of the recursion:

$$(2.2) \qquad V(\sigma) = f_1(\sigma)U(\sigma) , \quad \sigma \in S_N .$$

For $n \in \{N-1, N-2, \ldots, 1\}$, $\sigma \in S_n$,

$$(2.3) \qquad V_0(\sigma) = \Sigma_{\sigma' \in S_{n+1}} f_0(\sigma,\sigma')\, V(\sigma'),$$

$$(2.4) \qquad V_1(\sigma) = f_1(\sigma)U(\sigma) + \Sigma_{\sigma' \in S_{n+1}} f_1(\sigma,\sigma')V(\sigma'),$$

$$(2.5) \qquad V(\sigma) = \max\{V_0(\sigma), V_1(\sigma)\}.$$

Finally

$$(2.6) \qquad V^N = \Sigma_{\sigma \in S_1} f(\sigma)\, V(\sigma).$$

It follows that any policy Δ^N satisfying

$$(2.7) \qquad S_N \subseteq \Delta^N,$$

$$(2.8) \qquad \sigma \in \Delta^N \Rightarrow V_1(\sigma) \geqslant V_0(\sigma),$$

and

$$(2.9) \qquad \sigma \notin \Delta^N \Rightarrow V_1(\sigma) \leqslant V_0(\sigma),$$

is optimal.

It can be seen that $U(\sigma)$ only appears in equations (2.2) and (2.3) and then only through the product $f_1(\sigma)U(\sigma)$. This product can be replaced by the function $\overline{U}(\sigma)$, $\sigma \in S$, which we shall call the <u>expected immediate payoff function for δ_1</u>. Only $\overline{U}(\sigma)$ need be specified and then, if required, $f_1(\sigma)$ and $U(\sigma)$ could be derived using (2.1).

The Markov chain considered above is stationary and only stationary policies have been considered since any state is hit at most once. While in some situations this approach may appear to be artificial it lends itself more easily to the solution of the problem by backward induction than does the case of a non-stationary Markov chain with a smaller state space and non-stationary policies.

In the application of the Markov chain formulation to Problems A, B and C the number of transitions, i.e. the number of observations, will naturally appear in the description of the states since the problems are essentially those of sequential sampling.

The Markov chain decision problem we have treated includes the pure stopping problem if we make δ_1 the terminating decision by setting the termination probability function $f_1(\sigma) = 1$, $\sigma \in S$. General stationary finite Markov decision problems and stopping problems have been treated in many books and articles, for example, Derman (1970) and Dynkin and Yushkevitch (1969). However, the particular problem that we have looked at, being of a rather specialised nature, does not appear to have been considered elsewhere.

We shall introduce some notation for use in the remaining work. Denote the set of positive integers by I and for each $n \in I$, denote the set $\{1,2,\ldots,n\}$ by I_n . The convention that $I_n = \phi$ for $n \notin I$, will be adopted.

2. SOLUTION OF PROBLEMS A, B AND C.

All three problems will be formulated as special cases of the Markov chain decision problem treated in section 2.1. Once we have specified the appropriate decisions, state space, the initial and transition probability functions, and the expected immediate payoff function the solutions of the problems will be given by (2.2) to (2.9) inclusive. The state space, the initial and transition probability functions f, f_0, f_1 and the expected immediate payoff function $\overline{U}$ will all be determined for each of Problems A, B and C by the probability space of part 4) of Assumption 1.1.

This is the probability space $(\Omega, \mathfrak{F}, P)$, where $\mathfrak{F}$ is the class of all subsets of Ω, P is a known arbitrary probability distribution on $\mathfrak{F}$

and Ω is defined as follows:

Ω is the set of points

$$\omega = (\omega(0),\omega(1),\ldots,\omega(N)) ,$$

where (i) $\omega(0) = (\omega_1(0),\omega_2(0),\ldots,\omega_N(0))$ is a permutation of the set I_N, representing the order of presentation of the N applicants according to quality, e.g. $\omega_n(0) = k$ means that applicant number n is the kth best (of absolute rank k), and

(ii) for $j \in I_N$,

$$\omega(j) = (\omega_j(j),\omega_{j+1}(j),\ldots,\omega_N(j)) ,$$

is a vector of 1's and 0's representing the availabilities of applicant j at stages $j,j+1,\ldots,N$. For example, $\omega_n(j) = 1$ means that an approach made at stage n to applicant j will be successful, $(1 \leqslant j \leqslant n \leqslant N)$.

In each of the problems the relative rank of applicant n is observed at stage n. We shall denote this random variable by $X_n(0)$ where

$$X_n(0) = \sum_{k=1}^{n} \chi_{\{\omega\in\Omega\,:\,\omega_k(0)\leqslant\omega_n(0)\}} ,$$

and for $E \in \mathcal{F}$, χ_E denotes the indicator function of E. Since the best applicant is sought it will be useful to denote by Y_n, the number of the applicant who is relatively best at stage n. Thus

$$Y_n = \max\,\{k \in I_n : X_k(0) = 1\} .$$

Using an extension of the terminology of Gilbert and Mosteller (1966) we shall refer to applicant Y_n as the <u>candidate at stage n</u>.

It is stated in Assumption 1.2.B that the availability of the first n applicants at stage n may be observed. It will be assumed that the employer makes these observations since he loses nothing by doing so. For $1 \leq j \leq n \leq N$, $X_n(j)$ is the random variable denoting the availability of applicant j at stage n, and is defined by

$$X_n(j) = \chi_{\{\omega \in \Omega : \omega_n(j) = 1\}} .$$

Under the assumptions for Problem B the random variables $(X_n(0), X_n(1), \ldots, X_n(n))$ are observed at stage n before any decision is made. For the sake of brevity we shall denote this vector of random variables by X_n .

In all three problems the availability of the candidate at stage n, whether observed directly or not, will be of interest. We shall denote this random variable by Z_n , and thus $Z_n = X_n(Y_n)$.

We now need to define the possible decisions available at each stage. It is the aim in each problem to employ the best applicant and it seems reasonable that if an approach is to be made at any stage, then it should only be made to the current candidate. This is because at stage n either the best applicant has not yet appeared or if she has then she will be the candidate. That is, for every $\omega \in \Omega$,

$$\omega_j(0) = 1 \Rightarrow Y_n(\omega) = j , \quad n \geq j .$$

In Problem C only one approach is permitted and hence it should only be made to the current candidate. It only remains to decide at what stage this approach should be made and thus Problem C reduces to a stopping problem.

While any number of unsuccessful approaches are permitted, in Problem B it will be known beforehand whether or not an approach will be successful. Clearly, since nothing is to be gained from an unsuccessful approach, it follows that if an approach is to be made at stage n then it should be made to the candidate and then only if she is available. Thus Problem B also reduces to a pure stopping problem.

The justification of the fact that there are only two "sensible" decisions at each stage is not so straightforward for Problem A. A cursory examination of the problem suggests that it may even be advantageous to approach an applicant other than the candidate at some early stage. This might be true for a particularly pathological distribution P where the availabilities and positioning of the best applicant may be highly dependent upon the availability of some other applicant at an early stage. However, it can be shown that any policy which includes the possibility of an approach to a non-candidate at a particular stage, is <u>inadmissible</u> in the sense that for each distribution P on $\mathfrak{J}$, there exists another policy for which the probability of employing the best applicant is at least as great.

We shall have to consider the possibility of approaching more than one applicant at any stage in which case it must also be decided in what order a multiple approach should be carried out. The specifying of the order is necessary only to avoid the possibility of making more than one successful approach. There is no point in making more than one approach to a single applicant at any one stage since the later approaches at that stage will also be unsuccessful. Clearly there are $\sum_{k=0}^{n} n!/(n-k)!$ different decisions at stage n.

Let D be a policy and suppose that by stage n the relative ranks $X_1(0) = x_1, \ldots, X_n(0) = x_n$, have been observed. Also suppose that under policy D a set of applicants $J_k, J_k \subseteq I_k$, has been unsuccessfully approached at stage k for each $k \in I_{n-1}$. We shall refer to this situation as state σ. In state σ, suppose further that D requires the applicants in the ordered set $\mathcal{J}_n$ to be approached and that $\mathcal{J}_n$ includes applicant m, where $m \neq \max\{k \in I_n : x_k = 1\}$, i.e. m is not the candidate at stage n. Now let D* be the policy which is the same as D except for: (i) if the particular state σ is reached then the approach to applicant m is not made and (ii) if no other approaches in state σ are successful, then D* subsequently makes the same decisions as D would have assuming that the approach to m at stage n had been unsuccessful. It follows that, starting in state σ, the probability of employing the best applicant under D* is greater than or equal to that for D since the opportunity to employ an applicant who is certain not to be the best, has been removed. Also the probability distribution over all possible states for stages $\leqslant n$ is the same for D and D*, for each P on $\mathfrak{F}$. Clearly then D is inadmissible. There are only finite numbers of stages, states and policies, and thus by induction any policy which in any state approaches a non-candidate is inadmissible.

It is also obvious that at stage N, to approach the candidate is the only decision with a non-zero probability of success. Thus we have proved:

Proposition 2.1. Let δ_0, δ_1 be the decisions given by

δ_0 : interview the next applicant,

δ_1 : approach the current candidate,

where in Problems A and B δ_1 carries the understanding that if the approach is unsuccessful then the next applicant, provided there is one, will be interviewed. Then for each of Problems A,B and C any policy which does not choose either δ_0 or δ_1 at each stage $n < N$ and which does not choose δ_1 at stage N, is inadmissible. □

Clearly, all three problems can be formulated in terms of the Markov chain decision problem of section 2.1 with the decisions δ_0 and δ_1 as given above. All that remains for the solution of each problem is to define the state space in terms of Ω and the initial and transition probabilities and the expected immediate payoff in terms of P.

Problem A

For Problem A the state space is $S^A = \bigcup_{n=1}^{N} S_n^A$ where S_n^A is the set of stage n states

$$\sigma = \sigma(x_1, x_2, \ldots, x_n, J) , \quad x_k \in I_k , \quad J \subseteq I_{n-1} ,$$

given by

$$\sigma = \{\omega \in \Omega : X_k(0)(\omega) = x_k , \quad k \in I_n ; \quad Z_j(\omega) = 0 , \quad j \in J\} .$$

σ represents the state of being at stage n having observed relative ranks $x_1, x_2, \ldots, x_n$ and unsuccessfully approached the candidate at stage j for each $j \in J$. Clearly $S_m^A \cap S_n^A = \phi$ for $m \neq n$.

Since $S_1^A = \{\Omega\}$ the initial probability distribution is given by $f^A(\Omega) = 1$, $f^A(\sigma) = 0$, $\sigma \neq \Omega$.

For the decision δ_0 the transition probability function $f_0^A(\sigma, \cdot)$ for $\sigma \in S_n^A$, $n \in I_{N-1}$, is given by

$$f_0^A(\sigma, \sigma') = \begin{cases} P(\sigma')/P(\sigma) , & \sigma' = \sigma \cap \{X_{n+1}(0) = x\} \text{ for some } x \in I_{n+1} , \\ 0 , & \sigma' \text{ otherwise} . \end{cases}$$

Similarly the transition probabilities for decision δ_1 are given by

$$f_1^A(\sigma, \sigma') = \begin{cases} P(\sigma')/P(\sigma) , & \sigma' = \sigma \cap \{X_{n+1}(0) = x, Z_n = 0\} \text{ for some } x \in I_{n+1} , \\ 0 , & \sigma' \text{ otherwise} , \end{cases}$$

for $\sigma \in S_n^A$, $n \in I_{N-1}$.

The expected immediate utility for decision δ_1 in state σ is merely the probability the candidate is best and available. Hence

$$\overline{U}^A(\sigma) = P(\{\omega\in\Omega : Z_n(\omega) = 1 , \ \omega_{Y_{n(\omega)}}(0) = 1\}|\sigma) , \quad \sigma\in S_n^A , \ n\in I_N .$$

The maximum probability of employing the best applicant, V_A^N, is then given by (2.6) after solving (2.2) to (2.5) with $f_1(\sigma)U(\sigma)$ replaced by $\overline{U}(\sigma)$. Any policy Δ_A^N satisfying (2.7) to (2.9) will be optimal.

Problem B

For $n\in I_N$, let $\mathcal{X}_n = I_n \times \bigtimes_{k=1}^{n} \{0,1\}$. S_n^B, the set of stage n states for Problem B is the set of

$$\sigma = \sigma(x_1,x_2,\ldots,x_n) , \qquad x_k\in\mathcal{X}_k$$

given by

$$\sigma = \{\omega\in\Omega : X_1(\omega) = x_1 , \ X_2(\omega) = x_2,\ldots,X_n(\omega) = x_n\} .$$

The initial probability distribution is given by

$$f^B(\sigma) = \begin{cases} P(\sigma) , & \sigma\in S_1^B , \\ 0 , & \sigma\notin S_1^B , \end{cases}$$

and the transition probability function for the decision δ_0 is given by

$$f_0^B(\sigma,\sigma') = \begin{cases} P(\sigma')/P(\sigma) , & \sigma' = \sigma\cap\{X_{n+1} = x\} \text{ for some } x\in\mathcal{X}_{n+1} , \\ 0 , & \sigma' \text{ otherwise,} \end{cases}$$

for $\sigma\in S_n^B$, $n\in I_{N-1}$.

The availability of the candidate will be known before any decision is chosen. Clearly then the termination probabilities are either one or zero depending on whether the candidate is available or not. Let $\sigma = \sigma(x_1, x_2, \ldots, x_n)$ be a state in S_n^B, $n \in I_{N-1}$ and let Z_n be the availability of the candidate, then for all $\sigma' \in S^B$,

$$f_1^B(\sigma, \sigma') = \begin{cases} f_0^B(\sigma, \sigma'), & \sigma \subseteq \{Z_n = 0\}, \\ \\ 0, & \sigma \subseteq \{Z_n = 1\}. \end{cases}$$

Finally,

$$\overline{U}^B(\sigma) = P(\{\omega \in \Omega : Z_n(\omega) = 1, \omega_{Y_n(\omega)}(0) = 1\} | \sigma), \quad \sigma \in S_n^B, \; n \in I_N.$$

As in Problem A, the maximum probability of employing the best applicant, V_B^N, and an optimal policy Δ_B^N are given by (2.2) to (2.9).

<u>Problem C</u>

For $n \in I_N$, S_n^C is the set of all

$$\sigma = \sigma(x_1, x_2, \ldots, x_n), \qquad x_k \in I_k,$$

where

$$\sigma = \{\omega \in \Omega : X_1(0)(\omega) = x_1, X_2(0)(\omega) = x_2, \ldots, X_n(0)(\omega) = x_n\}.$$

The probability functions are determined by each P as follows.

1) $$f^C(\sigma) = \begin{cases} 1, & \sigma = \Omega, \\ \\ 0, & \sigma \text{ otherwise}, \end{cases}$$

2) for $\sigma \in S_n$, $n \in I_{N-1}$,

$$f_0^C(\sigma,\sigma') = \begin{cases} P(\sigma')/P(\sigma)\,, & \sigma' = \sigma \cap \{X_{n+1}(0) = x\} \text{ for some } x \in I_{n+1}\,, \\ 0\,, & \sigma' \text{ otherwise}\,, \end{cases}$$

and

$$f_1^C(\sigma,\sigma') = 0\,, \quad \sigma' \in S^C\,.$$

3) $$\bar{U}^C(\sigma) = P(\{\omega \in \Omega : Z_n(\omega) = 1,\ \omega_{Y_n(\omega)}(0) = 1\} | \sigma)\,, \quad \sigma \in S^C\,,$$

(with $f_1^C(\sigma) \equiv 1$) .

Finally, V_C^N is obtained by solving (2.2) to (2.6) and a policy Δ_C^N satisfying (2.7) to (2.9) is optimal.

Concluding Remarks

It can be shown for each distribution P, that $V_C^N \leqslant V_A^N \leqslant V_B^N$. This is fairly obvious since for Problem B more is known at each stage about the "true state of nature" ω than for Problem A. Similarly at least as much is known about ω for Problem A as is for Problem C, but more than one approach is permitted for Problem A.

While the method of solution of the problems has been given in theory, unless N is very small there are enormous practical difficulties in calculating the solution in any particular case. There are $n!\ 2^{n-1}$ states in S_n^A, $n!\ 2^{n(n+1)/2}$ states S_n^B and $n!$ states in S_n^C. Even for n = 10 these numbers are almost 2,000 million, about 10^{23} and more than 3½ million respectively.

Until now no assumptions about the form of the distribution P have been made. For the major part of the work of this thesis certain reasonable assumptions about P will be made which will make some analysis of the problems possible. These assumptions will have the effect of making the transition probabilities and termination payoffs the same for many of the states at each stage. This will enable us to treat the problems as Markov chain decision problems with vastly reduced state spaces.

CHAPTER III

PROBLEMS A AND B: A SIMPLIFIED CASE

For this chapter we shall make a number of assumptions about the distribution P on Ω, which will make possible some analysis of problems A and B. It will be shown that under these assumptions, problems A and B are equivalent.

1. SIMPLIFYING ASSUMPTIONS AND PRELIMINARIES

It is reasonable to assume that there is independence of availabilities between applicants. It will also be assumed that the availabilities of the applicants are independent of the order of presentation. Random order of presentation will also be assumed. Finally it will be assumed that unavailable applicants remain unavailable. Formally, we assume

Assumption 3.1. The distribution P on Ω is such that:

1) the vectors $\omega(0),\omega(1),\ldots,\omega(N)$ are mutually independent,

2) the marginal distribution of $\omega(0)$ has equal probability for all N! values of $\omega(0)$,

and 3) $\omega \in \Omega$ will have non-zero probability only if ω is such that for every $j \in I_N$ and every $n \in \{j,j+1,\ldots,N-1\}$

$$\omega_n(j) = 0 \Rightarrow \omega_{n+1}(j) = 0 .$$

Under Assumption 3.1, it is clear that P is uniquely determined by a lower triangular matrix π, called a matrix of availability probabilities where

$$\pi(n,j) = P(\{\omega \in \Omega : \omega_n(j) = 1\}) , \quad j \in I_n, \; n \in I_N .$$

Necessarily $\pi(n,j)$ will be non-increasing in n for each j, because of 3) in Assumption 3.1.

Problem A

It now follows that many of the past observations are immaterial when it comes to making the optimal decision at any particular stage. In fact under Assumption 3.1, only the position of the candidate and whether or not that person has been unsuccessfully approached is relevant. If the candidate has refused an earlier offer then her position is also irrelevant, since the only hope of employing the best applicant is that she has yet to be interviewed.

We can now formulate Problem A simplified by Assumption 3.1 in terms of the Markov chain decision problem in section 2.1.

The new state space will consist of pairs (n,j), $n \in I_N$ and $j \in \{0,1,\ldots,n\}$ where

1) for $j \in I_n$, (n,j) represents the state of being at stage n with the candidate, $Y_n = j$, having not been approached at or before stage n, and

2) $(n,0)$ represents the state of being at stage n with the candidate having been unsuccessfully approached at some stage at or before stage n.

Any policy can be represented by Δ, the subset of states in which the decision : approach the candidate (decision δ_1) is taken. It is pointless to consider policies which call for an approach in any of the $(n,0)$ states since unavailable applicants remain unavailable. Also from Assumption 3.1 it is clear that, for $j \in I_n$,

$$(3.1) \qquad P(\{Y_{n+1} = y\}|\{Y_n = j\}) = \begin{cases} 1/(n+1)\ , & y = n+1\ , \\ n/(n+1)\ , & y = j\ , \\ 0\ , & y \text{ otherwise,} \end{cases}$$

and consequently state (n,j) $j \in I_n$, is always followed by either state $(n+1,n+1)$ or state $(n+1,j)$ if δ_0 is taken; and by either termination or state $(n+1,0)$ or state $(n+1,n+1)$ if decision δ_1 is taken. Clearly then the state (n,j) $(j \in I_{n-1})$ can only be attained if the chain passed through states $(j,j),(j+1,j),\ldots,$ $(n-1,j)$ with decision δ_0 taken in each. Thus, the only states in any policy Δ, which can be attained are the states $(d(j),j)$, $j \in I_N$, where

$$d(j) = \min \{n \in I_N : (n,j) \in \Delta\}\ .$$

$d(j)$ exists for each j since $(N,j) \in \Delta$.

We have reduced the representation of a policy from a subset Δ to an N-tuple $d \in D_N$ where

$$D_N = \{(d(1),\ldots,d(N)) : j \leqslant d(j) \leqslant N,\ j \in I_N\}\ .$$

The policy represented by any $d \in D_N$ can be stated quite simply as follows:

> at stage n: approach the candidate (applicant Y_n) if and only if $d(Y_n) = n$, on all other occasions interview the next applicant (unless, of course, $n = N$).

The state space and the representation of policies have similar simplified forms in Problem B under Assumption 3.1.

<u>Problem B</u>

Again only the position of relatively best applicant and her availability (which in Problem B, is observed at each stage before making a decision) are relevant to making an optimal decision at any stage.

The new state space will again consist of pairs (n,j), $n \in I_N$, $j \in \{0,1,\ldots,n\}$ but these will represent slightly different situations to those for Problem A.

1) For $j \in I_n$, (n,j) represents the state of being at stage n, with the candidate $Y_n = j$ available, i.e. $Z_n = 1$.

2) $(n,0)$ represents the state of being at stage n, with the candidate unavailable, i.e. $Z_n = 0$.

As with problem A it will be pointless to include any of the states $(n,0)$ in a set of δ_1 states, Δ, since an approach in any one of them is certain to be unsuccessful. A policy Δ will consist of termination states only, since an approach in state (n,j), $j \geqslant 1$, is certain to be successful.

It follows from (3.1) that the state (n,j), $j \in I_{n-1}$, can only be attained by passing through the states (j,j), $(j+1,j)$, ..., $(n-1,j)$ without making an approach. Again the only relevant elements of a policy Δ are the "first" states in which each particular applicant is approached. Thus Δ also simplifies in Problem B to an N-tuple $d \in D_N$, where $d(j) = \min\{n \in I_N : (n,j) \in \Delta\}$, $j \in I_N$. However the policy that d represents for Problem B is different to the policy represented by d for Problem A. The statement of policy d for Problem B is:

> At stage n: stop and approach the candidate (applicant Y_n) if and only if she is available and $d(Y_n) = n$, on all other occasions interview the next applicant (unless, of course, $n = N$).

2. THE EQUIVALENCE OF PROBLEMS A AND B

Under Assumption 3.1 we have seen that for both problems a policy can be represented by an element of D_N. What we will now show is that the additional information in Problem B of knowing the availability of the candidate before making a decision gives no advantage.

For every policy represented by d, $d \in D_N$, for Problem A there corresponds a stopping variable $\tau^A : \Omega \to I_N$. There also corresponds a stopping variable $\tau^B : \Omega \to I_N$ for the policy represented by d in Problem B. For $n \in I_N$ and $j \in I_n$, the process for Problem A is in state (n,j) at stage n if and only if the process has not terminated earlier, $Y_n = j$ and $d(j) \geq n$. The process then terminates in state (n,j), $n \in I_{N-1}$, if and only if $d(j) = n$ and $Z_n = 1$. (The process always terminates in state (N,j), $0 \leqslant j \leqslant N$.) Thus τ^A is given by

$$\{\tau^A = 1\} = \{\omega \in \Omega : Y_1(\omega) = 1,\ d(1) = 1,\ Z_1(\omega) = 1\}$$

$$= \{\omega \in \Omega : d(Y_1(\omega)) = 1,\ Z_1(\omega) = 1\},$$

$$\{\tau^A \leqslant n\} = \{\tau^A \leqslant n-1\} \cup \{\omega \in \Omega : d(Y_n(\omega)) = n,\ Z_n(\omega) = 1\},$$

$$2 \leqslant n \leqslant N-1,$$

and finally

$$\{\tau^A \leqslant N\} = \Omega.$$

Now for Problem B the process is in state (n,j) at stage n if and only if it has not terminated before stage n and $Y_n = j$ and $Z_n = 1$. The process will then terminate in state (n,j), $n \in I_{N-1}$, if and only if $d(j) = n$. Thus τ^B is given by

$$\{\tau^B = 1\} = \{\omega \in \Omega : d(Y_1(\omega)) = 1,\ Z_1(\omega) = 1\}\ ,$$

$$\{\tau^B \leqslant n\} = \{\tau^B \leqslant n-1\} \cup \{\omega \in \Omega : d(Y_n(\omega)) = n,\ Z_n(\omega) = 1\}\ ,$$

$$2 \leqslant n \leqslant N-1\ ,$$

and finally

$$\{\tau^B \leqslant N\} = \Omega\ .$$

Hence $\tau^A \equiv \tau^B$ for each $d \in D_N$.

Given the distribution P, determined by the matrix of availability probabilities π, we denote by $V_\pi^A(d)$ and $V_\pi^B(d)$ the probabilities of employing the best applicant under the policies represented by d for Problems A and B respectively. Now $V_\pi^A(d)$ is the probability that the candidate at the time of termination is best and available. Hence

$$V_\pi^A(d) =$$

$$\sum_{n=1}^{N} \sum_{j=1}^{n} P(\{\omega : \tau^A(\omega) = n,\ Y_n(\omega) = j,\ \omega_j(0) = 1,\ Z_n(\omega) = 1\})\ ,$$

and the same expression with τ^A replaced by τ^B will hold for $V_\pi^B(d)$. Because of the identity of τ^A and τ^B, we have now proved:

<u>Theorem 3.1.</u> Under Assumption 3.1, Problems A and B are equivalent in the sense that for every matrix of availability probabilities π and for every policy $d \in D_N$,

$$V_\pi^A(d) = V_\pi^B(d)\ . \quad \square$$

For the remainder of this chapter we shall consider only Problem A as all the results will also be true for Problem B. Problem A has been chosen since the slight awkwardnesses which may arise for a state (n,j) for which $\pi(n,j) = 0$, are more easily resolved. Also Yang (1974) has treated a special case of Problem A. Subscripts and superscripts A will be dispensed with from now on.

Before setting up the equations from which an optimal policy can be derived we shall develop recursion formulae which compute $V_\pi(d)$ for any $d \in D_N$, given any matrix π.

In terms of the Markov chain decision problem of Chapter II, we have for Problem A that the initial state (1,1) is attained with certainty. The transition probabilities under decision δ_0 follow from (3.1) and are given by

$$f_0((n,j),\ (n+1,y)) = \begin{cases} 1/(n+1)\ , & y = n+1\ , \\ n/(n+1)\ , & y = j\ , \\ 0\ , & y \text{ otherwise}\ , \end{cases} \tag{3.2}$$

for $n \in I_{N-1}$ and $j \in \{0,1,\ldots,n\}$. Under δ_1 the transition probabilities are given by

$$f_1((n,j),\ (n+1,y)) = \begin{cases} (1-\pi(n,j))/(n+1)\ , & y = n+1\ , \\ n(1-\pi(n,j))/(n+1)\ , & y = 0\ , \\ 0\ , & y \text{ otherwise}\ , \end{cases} \tag{3.3}$$

for $n \in I_{N-1}$ and $j \in I_n$. Decision δ_1 is never taken in state (n,0). The expected immediate payoff function $\overline{U}$ is the probability that the candidate is the best applicant and an approach will be successful, as a function of each state. Now

$$P(\{\omega_j(0) = 1\} \mid \{Y_n = j\}) = n/N\ ,$$

and hence

$$\begin{aligned} \overline{U}(n,j) &= n\pi(n,j)/N\ , \quad n \in I_N\ , \quad j \in I_n\ , \\ \overline{U}(n,0) &= 0\ , \quad n \in I_N\ . \end{aligned} \tag{3.4}$$

Given a policy d and matrix of availability probabilities π, denote by $V_d(n,j)$, $n \in I_N$, $j \in \{0,1,\ldots,n\}$ the probability of employing the best applicant starting from state (n,j). (Clearly the state (n,j), where $n > d(j)$ is unattainable and V_d need not be defined for this state.) The recursion formulae for generating $V_\pi(d)$ are as follows.

(3.5) $$V_d(N,0) = 0 ,$$

(3.6) $$V_d(n,0) = nV_d(n+1,0)/(n+1) + V_d(n+1,n+1)/(n+1), \quad n \in I_{N-1} .$$

Now for $j \in I_N$ such that $d(j) = N$ we have

(3.7) $$V_d(N,j) = \pi(N,j) .$$

For $n \in I_{N-1}$, $j \in I_n$ such that $n < d(j)$,

(3.8) $$V_d(n,j) = nV_d(n+1,j)/(n+1) + V_d(n+1,n+1)/(n+1) ,$$

and for $n \in I_{N-1}$, $j \in I_n$ such that $n = d(j)$,

$$\begin{aligned} V_d(n,j) = {} & \bar{U}(n,j) \\ & + n[1-\pi(n,j)]V_d(n+1,0)/(n+1) \\ & + [1-\pi(n,j)]V_d(n+1,n+1)/(n+1) . \end{aligned}$$

But from (3.4) and (3.6) we have that

(3.9) $$V_d(n,j) = n\pi(n,j)/N + (1-\pi(n,j))V_d(n,0), \quad n \in I_{N-1}, j \in I_n,\ d(j) = n .$$

Finally we have

(3.10) $$V_\pi(d) = V_d(1,1) .$$

This backward recursion can be simplified if we "standardise" V_d by setting

$$v_d(n,0) = NV_d(n,0)/n , \quad n \in I_N$$

and

$$v_d(n,j) = NV_d(n,j)/n - v_d(n,0) , \quad n \in I_N , \quad j \in I_n .$$

Substitution into (3.5) - (3.10) yields

(3.11) $$v_d(N,0) = 0 ,$$

(3.12) $$nv_d(n,0) = (n+1)v_d(n+1,0) + v_d(n+1,n+1) , \quad n \in I_{N-1} ,$$

(3.13) $$v_d(N,j) = \pi(N,j) , \quad j \in I_N , \quad d(j) = N ,$$

(3.14) $$v_d(n,j) = v_d(n+1,j) , \quad n \in I_{N-1} , \quad j \in I_n , \quad n < d(j) ,$$

(3.15) $$v_d(n,j) = \pi(n,j)[1 - v_d(n,0)] ,$$

$$n \in I_{N-1} , \quad j \in I_n , \quad n = d(j) ,$$

and finally

(3.16) $$V_\pi(d) = [v_d(1,0) + v_d(1,1)]/N .$$

Solving (3.11) and (3.12) gives

(3.17) $$v_d(n,0) = \frac{1}{n} \sum_{j=n+1}^{N} v_d(j,j) ,$$

adopting the convention that, for $a > b$, $\sum_a^b \equiv 0$. (3.13) to (3.16) then reduce to

(3.18) $$v_d(j,j) = \pi(d(j),j)[1 - v_d(d(j),0)] , \quad j \in I_N ,$$

and

$$V_\pi(d) = \frac{1}{N} \sum_{j=1}^{N} v_d(j,j) . \tag{3.19}$$

The number of recursion formulae for generating $V_\pi(d)$ have thus been reduced to three, namely (3.17), (3.18) and (3.19).

3. OPTIMAL POLICIES

Recursion formulae which give the solution of the simplified Problem A can be obtained from equations (2.2) to (2.6) for the Markov chain decision problem of Chapter II by substituting the transition probability functions and the payoff function as given by (3.2), (3.3) and (3.4).

We have that

$$V(N,0) = V_0(N,j) = 0 , \quad j \in I_N , \tag{3.20}$$

$$V(N,j) = V_1(N,j) = \pi(N,j) , \quad j \in I_N , \tag{3.21}$$

$$V(n,0) = nV(n+1,0)/(n+1) + V(n+1,n+1)/(n+1) , \quad n \in I_{N-1} , \tag{3.22}$$

$$V_0(n,j) = nV(n+1,j)/(n+1) + V(n+1,n+1)/(n+1) , \quad n \in I_{N-1} , \quad j \in I_n , \tag{3.23}$$

$$V_1(n,j) = \overline{U}(n,j) + n[1-\pi(n,j)]V(n+1,j)/(n+1) + [1-\pi(n,j)]V(n+1,n+1)/(n+1) , \quad n \in I_{N-1} , \quad j \in I_n ,$$

which using (3.4) and (3.23) gives

$$V_1(n,j) = n\pi(n,j)/N + [1-\pi(n,j)]V(n,0) , \quad n \in I_{N-1} , \quad j \in I_n , \tag{3.24}$$

and finally

$$(3.25) \qquad V(n,j) = \max\{V_0(n,j), V_1(n,j)\}, \quad n \in I_N, \quad j \in I_n.$$

Let V_π^N denote the maximum probability of employing the best applicant when the matrix of availability probabilities is π. By (2.6) and the fact that state (1,1) is certain to occur

$$(3.26) \qquad V_\pi^N = V(1,1).$$

In the original representation, an optimal policy Δ^N will be any policy such that

$$(n,j) \in \Delta^N \Rightarrow V_1(n,j) \geq V_0(n,j),$$

and

$$(n,j) \notin \Delta^N \Rightarrow V_1(n,j) \leq V_0(n,j).$$

It follows then that any policy $d \in D_N$ satisfying both

$$(3.27) \qquad V_1(d(j),j) \geq V_0(d(j),j), \quad j \in I_N,$$

and

$$(3.28) \qquad V_1(n,j) < V_0(n,j), \quad j \in I_N, \quad n \in \{j, j+1, \ldots, d(j)-1\},$$

is optimal. We shall denote by D_π^N the set of all policies $d \in D_N$ satisfying (3.27) and (3.28). That is, D_π^N is essentially the set of all policies which are optimal for the matrix of availability probabilities π. It can be seen that if there is an optimal policy d' not included in D_π^N then it will differ from a member of D_π^N only in the decisions taken in states which are unattainable under d'.

The recursion given by equations (3.20) to (3.26) is simplified and a useful characterisation of the set D_π^N is obtained if we "standardise" the quantities $V(n,0)$, $V(n,j)$, $V_0(n,j)$, $V_1(n,j)$, $n \in I_N$, $j \in I_n$, by setting

$$v(n,0) = NV(n,0)/n ,$$

$$v(n,j) = NV(n,j)/n - v(n,0) ,$$

$$u(n,j) = NV_1(n,j)/n - v(n,0) ,$$

and

$$w(n,j) = NV_0(n,j)/n - v(n,0) .$$

Substitution into (3.20) - (3.26) and rearrangement yields:

(3.29) $$v(N,0) = w(N,j) = 0 , \quad j \in I_N ,$$

(3.30) $$v(n,0) = [(n+1)v(n+1,0) + v(n+1,n+1)]/n , \quad n \in I_{N-1} ,$$

(3.31) $$u(n,j) = \pi(n,j)[1 - v(n,0)] , \quad n \in I_N , \quad j \in I_n ,$$

(3.32) $$w(n,j) = v(n+1,j) , \quad n \in I_{N-1} , \quad j \in I_n ,$$

(3.33) $$v(n,j) = \max \{u(n,j), w(n,j)\}, \quad n \in I_N , \quad j \in I_n ,$$

and

(3.34) $$V_\pi^N = [v(1,1) + v(1,0)]/N .$$

From (3.29), (3.32) and (3.33) we have that

(3.35) $$v(n,j) = \max_{k=n}^{N} u(n,j) , \quad n \in I_N , \quad j \in I_N ,$$

and solving (3.29) and (3.30) gives

(3.36) $$v(n,0) = \frac{1}{n} \sum_{j=n+1}^{N} v(j,j) , \quad n \in I_N ,$$
$$= \frac{1}{n} \sum_{j=n+1}^{N} \max_{k=j}^{N} u(k,j) .$$

Substitution into (3.34) then gives

$$(3.37)\qquad V_\pi^N = \frac{1}{N}\sum_{j=1}^{N} \max_{n=j}^{N} u(n,j) .$$

Finally it follows from (3.27) and (3.28) and the fact that $u(n,j) - w(n,j)$ has the same order relation to zero as $V_1(n,j) - V_0(n,j)$, that $d \in D_\pi^N$ if and only if

$$(3.38)\qquad u(d(j),j) = \max_{n=j}^{N} u(n,j) , \quad j \in I_N .$$

It should be noted for later reference that $v(n,0)$ is a non-negative non-increasing function. This is apparent from (3.36) and the fact that $v(j,j) \geqslant u(N,j) = \pi(N,j) \geqslant 0$.

Equations (3.31) and (3.35) to (3.38) give rise to a very simple algorithm for calculating from a given N and π, V_π^N and the optimal policy d given by

$$d(j) = \min \{n \geqslant j : u(n,j) = \max_{k=j}^{N} u(k,j)\} , \quad j \in I_N .$$

In this algorithm, which is written in FORTRAN language, the dummy variables M,J correspond to the n,j in the state (n,j). For M = n and J = j : $\pi(n,j)$ is read into P, U has the value $u(n,j)$, S has the value $\sum_{m=n+1}^{N} \max_{k=m}^{N} u(k,m)$, V(J) has the value $v(n,j)$. The final WRITE statement prints V_π^N and the components of d.

<u>Algorithm 3.1.</u>

```
  READ N
  S = 0.
  DO 1 J = 1,N
  ID(J) = N
1 V(J) = 0.
  DO 2 M = N,1,-1
  DO 3 J = 1,M
  READ P
4 U = P*(1.-S/M)
  IF(U.LT.V(J)) GO TO 3
  V(J) = U
  ID(J) = M
3 CONTINUE
2 S = S + V(M)
  VNP = S/N
  WRITE VNP, (ID(J), J = 1,N).
  END
```

Without knowing more about the structure of π, analysis of Problem A beyond the results which complete this section, would appear to be difficult.

Define the integer s_π for each matrix π, by

(3.39) $$s_\pi = \min \{n \in I_N : v(n,0) \leqslant 1\} .$$

<u>Theorem 3.2.</u> For each matrix of availability probabilities π there exists a policy $d \in D_\pi^N$ for which

$$d(j) \geqslant s_\pi , \quad j \in I_N .$$

<u>Proof.</u> $d(j) \geqslant j$ and thus we only have to consider $j < s_\pi$. For $n < s_\pi$ we have that $v(n,0) > 1$ and thus $u(n,j) \leqslant 0$ (with equality if and only if $\pi(n,j) = 0$). But $u(N,j) = \pi(N,j) \geqslant 0$ and hence

$$\max_{n=j}^{N} u(n,j) = \max_{n=s_\pi}^{N} u(n,j) .$$

The existence of $d \in D_\pi^N$ then follows by the characterisation (3.38). □

It is clear from the proof of Theorem 3.2 that it is never optimal to make an approach before stage s_π unless the approach is certain to be unsuccessful.

Some computing time could be saved by stopping the recursion at s_π. It follows from (3.37) and (3.36) that

(3.40) $$V_\pi^N = [nv(n,0) + \textstyle\sum_{j=1}^{n} v(j,j)]/N , \quad n \in I_N ,$$

and since $v(j,j) = v(s_\pi,j)$, $j \leqslant s_\pi$

$$V_\pi^N = [s_\pi v(s_\pi,0) + \textstyle\sum_{j=1}^{s_\pi} v(s_\pi,j)]/N .$$

It would be a simple task to modify Algorithm 3.1 to take this into account.

Let π, $\bar{\pi}$ be any two matrices of availability probabilities such that $\pi \leqslant \bar{\pi}$, that is, $\pi(n,j) \leqslant \bar{\pi}(n,j)$, $n \in I_N$, $j \in I_n$. In the final theorem in this section we will prove the intuitively obvious result that $V_{\pi}^{N} \leqslant V_{\bar{\pi}}^{N}$. It may seem reasonable to conjecture that $V_{\pi}(d) \leqslant V_{\bar{\pi}}(d)$ for every $d \in D_N$. This is not the case and indeed it is not always true for every policy in D_{π}^{N}. We give a counter-example to the conjecture.

Example 3.1. Let $N = 3$, $d = (1,3,3)$ and $\pi \equiv \frac{1}{2}$, $\bar{\pi} \equiv 1$. Now $\pi \leqslant \bar{\pi}$ but $V_{\pi}(d) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$ and $V_{\bar{\pi}}(d) = 1 \cdot \frac{1}{3} = \frac{1}{3}$. □

Theorem 3.3. Let π, $\bar{\pi}$ be any two matrices of availability probabilities such that $\pi \leqslant \bar{\pi}$. Then $V_{\pi}^{N} \leqslant V_{\bar{\pi}}^{N}$.

Proof. Let $\bar{u}, \bar{v}, \bar{w}$ be the quantities corresponding to $\bar{\pi}$ generated by equations (3.29) to (3.33). We shall show by backward induction that

$$v(n,n) + v(n,0) \leqslant \bar{v}(n,n) + \bar{v}(n,0)\,, \qquad n \in I_N\,,$$

from which it follows by (3.34) that $V_{\pi}^{N} \leqslant V_{\bar{\pi}}^{N}$.
First we have from (3.29) and (3.33) that

$$v(N,N) + v(N,0) = \pi(N,N) \leqslant \bar{\pi}(N,N) = \bar{v}(N,N) + \bar{v}(N,0)\,.$$

Let us now assume for some $k \in I_{N-1}$ that

$$(3.41) \qquad v(n,n) + v(n,0) \leqslant \bar{v}(n,n) + \bar{v}(n,0)\,, \qquad n \in \{k+1, k+2, \ldots, N\}\,.$$

Rearrangement of (3.29) gives

$$(3.42) \qquad v(n,0) - v(n+1,0) = [v(n+1,0) + v(n+1,n+1)]/n\,, \qquad n \in I_{N-1}\,,$$

and the same expression is also true for $\bar{v}$. By summing (3.42) over values from n to m - 1 and by using the induction hypothesis (3.41) it follows that

$$v(n,0) - v(m,0) \leq \bar{v}(n,0) - \bar{v}(m,0) , \quad k \leq n \leq m \leq N . \tag{3.43}$$

Now by Theorem 3.2 there exists a $d \in D_\pi^N$ for which $v(d(j),0) \leq 1$, $j \in I_N$. Thus

$$\begin{aligned} v(k,k) + v(k,0) &= u(d(k),k) + v(k,0) \\ &= \pi(d(k),k) + [1 - \pi(d(k),k)]v(d(k),0) \\ &\qquad + v(k,0) - v(d(k),0) \\ &\leq \bar{\pi}(d(k),k) + [1 - \bar{\pi}(d(k),k)]v(d(k),0) \\ &\qquad + v(k,0) - v(d(k),0) , \end{aligned}$$

since $v(d(k),0) \leq 1$ and $\pi \leq \bar{\pi}$. But from (3.43) with $n = d(k)$ and $m = N$ we have that $v(d(k),0) \leq \bar{v}(d(k),0)$. Also, with $n = k$ and $m = d(k)$, we have that $v(k,0) - v(d(k),0) \leq \bar{v}(k,0) - \bar{v}(d(k),0)$. Hence

$$\begin{aligned} v(k,k) + v(k,0) &\leq \bar{\pi}(d(k),k) + [1 - \bar{\pi}(d(k),k)]\bar{v}(d(k),0) \\ &\qquad + \bar{v}(k,0) - \bar{v}(d(k),0) \\ &= \bar{u}(d(k),k) + \bar{v}(k,0) \\ &\leq \bar{v}(k,k) + \bar{v}(k,0) , \end{aligned}$$

since $\bar{v}(k,k) \geq \bar{u}(d(k),k)$.

4. ELAPSED TIME DEPENDENT AVAILABILITIES: THE A(N,p)-PROBLEM

It is unlikely that the practitioner will know the value of all of the $N(N+1)/2$ elements with reasonable accuracy. It may be possible to assume that $\pi(n,j)$ is a function of $n-j$ alone. That is, the availability of an applicant depends only on the time elapsed since her interview. The practitioner would then only need to know the value of N probabilities. For the remaining sections of this chapter we shall consider Problem A under Assumption 3.1 and

Assumption 3.2. The distribution P on Ω is such that the matrix of availability probabilities, π, is given by

$$\pi(n,j) = p(n-j)\ , \quad n \in I_N\ , \quad j \in I_n\ ,$$

where $p = \{p(k)\}_{k=0}^{\infty}$ is a sequence of non-increasing non-negative numbers for which $0 < p(0) \leqslant 1$.

Even though only the first N terms need be known for the N applicant problem, we have hypothesised the existence of an infinite sequence p. This is because later we shall consider asymptotic properties as $N \to \infty$. The non-increasing property of p is necessary since under Assumption 3.1 unavailable applicants remain unavailable. The requirement that $p(0) > 0$ is made to avoid the trivial case where $\pi \equiv 0$.

We shall refer to any sequence p with the properties specified in Assumption 3.2 as a sequence of availability probabilities. Problem A under Assumptions 3.1 and 3.2 with N the total number of applicants and p the sequence of availability probabilities, will be referred to as the A(N,p)-problem.

Yang (1974) has considered the A(N,p)-problem but only for p where $p(0) = 1$. It is reasonable though, to allow for an unsuccessful approach at the time of the applicant's interview.

Changes in notation brought about by Assumption 3.2 will be the replacement of π by p. For example $V_\pi(d)$ for $d \in D_N$ will become $V_p(d)$ where π is determined by p.

Because of Assumption 3.2 the probability of a successful approach made in state (n,j) is the same as for that made in state (n+1,j+1). Also the candidate at stage n+1 is more likely to be the best applicant than the candidate at stage n. It therefore seems reasonable that if an approach is optimal in state (n,j) then an approach will be optimal in state (n+1,j+1). The following theorem proves this.

<u>Theorem 3.4.</u> For the A(N,p)-problem let u,w be the quantities as defined by (3.29) to (3.33). Then for each N,p,

$$u(n+1,j+1) - w(n+1,j+1) \geq u(n,j) - w(n,j) , \qquad n \in I_{N-1} , \quad j \in I_n .$$

<u>Proof.</u> Let $t(n,k) = w(n,n-k) - u(n,n-k)$, $n \in I_N$, $k \in \{0,1,\ldots,n-1\}$. It follows from (3.31) and (3.33) that

$$v(n,n-k) = t^+(n,k) + p(k)(1 - v(n,0)) ,$$

where x^+ denotes the positive part of x. Substitution into (3.29), (3.30) and (3.32) gives

$$(3.44) \qquad v(N,0) = 0 ,$$

$$(3.45) \qquad v(n,0) - v(n+1,0) = [(1-p(0))v(n+1,0) + p(0) + t^+(n+1,0)]/n , \qquad n \in I_{N-1} ,$$

$$t(N,k) = -p(k) , \quad k \in \{0,1,\ldots,N-1\} ,$$

and

$$(3.46)\qquad t(n,k) = p(k+1)[1 - v(n+1,0)] - p(k)[1 - v(n,0)] + t^{+}(n+1,k+1)\ ,\quad n \in I_{N-1}\ ,\quad k \in \{0,1,\ldots,n-1\}\ .$$

Recursion on (3.45) starting with (3.44) gives $v(n+1,0) \geq 0$ and hence $v(n,0) > v(n+1,0)$, since $p(0) > 0$. Consequently, for $n \in \{2,3,\ldots,N-1\}$,

$$\begin{aligned} v(n-1,0) - 2v(n,0) + v(n+2,0) &= [(1-p(0))v(n,0) + p(0) + t^{+}(n,0)]/(n-1) \\ &\quad - [(1-p(0))v(n+1,0) + p(0) + t^{+}(n+1,0)]/n \\ &> [t^{+}(n,0) - t^{+}(n+1,0)]/n\ . \end{aligned}$$

We complete the proof of the theorem by showing that $t(n,k) > t(n+1,k)$, $n \in I_{N-1}$, $k \in \{0,1,\ldots,n-1\}$.

Now, for each $k \in \{0,1,\ldots,N-2\}$, we have from (3.48) that

$$\begin{aligned} t(N-1,k) &= p(k+1) - p(k)[1 - v(N-1,0)] \\ &> -p(k) \\ &= t(N,k)\ . \end{aligned}$$

Let us now assume for some $n \in \{2,3,\ldots,N-1\}$ that $t(n,k) \geq t(n+1,k)$, $k \in \{0,1,\ldots,n-1\}$. It there follows that $t^{+}(n,0) \geq t^{+}(n+1,0)$ and hence that

$$v(n-1,0) - 2(n,0) + v(n+1,0) > 0\ .$$

But for each $k \in \{0,1,..,n-2\}$, we have from (3.46) that

$$t(n-1,k) - t(n,k) = p(k)[v(n-1,0) - v(n,0)] + t^{+}(n,k+1)$$
$$- p(k+1)[v(n,0) - v(n+1,0)] - t^{+}(n+1,k+1)$$
$$\geqslant p(k+1)[v(n-1,0) - 2v(n,0) + v(n+1,0)]$$
$$\geqslant 0 ,$$

since $p(k) \geqslant p(k+1) \geqslant 0$ and since the induction hypothesis implies that $t^{+}(n,k+1) \geqslant t^{+}(n+1,k+1)$. The proof is completed by induction on n backwards. □

It follows from this theorem that there exists a $d \in D_p^N$ such that

(3.47) $$d(j+1) \leqslant d(j) + 1 , \quad j \in I_{N-1} .$$

This is because if $d \in D_p^N$ then

$$u(d(j),j) \geqslant w(d(j),j),$$

which by Theorem 3.4 implies

$$u(d(j)+1,j+1) \geqslant w(d(j)+1,j+1) .$$

Hence there is a $d \in D_p^N$ such that (3.47) is true.

Theorem 3.4 will assist in the proof of several of the theorems later in this chapter.

As mentioned in Yang (1974) a simple policy (and one which is commonly used) is to make no approach before stage N. This is the policy represented by $d \equiv N$. The theorem which now follows includes Yang's result for the case $p(0) = 1$.

Theorem 3.5. The policy $d(j) = N$, $j \in I_N$ is optimal for the $A(N,p)$-problem if and only if p is such that

(3.48) $\quad p(k+1) \geq p(k)[N-1-p(0)]/(N-1)\,, \qquad k \in \{0,1,\ldots,N-2\}\,.$

In any case, the probability of employing the best applicant under policy d is

$$V_p(d) = \sum_{k=0}^{N-1} p(k)/N\,.$$

<u>Proof.</u> Now $d(j) = N$, $j \in I_N$ is optimal if and only if $u(n,j) \leq w(n,j)$, $n \in I_{N-1}, j \in I_n$, which by theorem 3.4 holds if and only if $u(N-1,j) \leq w(N-1,j)$, $j \in I_{N-1}$. But

$$\begin{aligned} u(N-1,j) - w(N-1,j) &= u(N-1,j) - u(N,j) \\ &= p(N-1-j)[1-v(N-1,0)] - p(N-j) \\ &= p(N-1-j)[1-p(0)/(N-1)] - p(N-j)\,, \end{aligned}$$

using (3.30). Hence the necessity and sufficiency of (3.48) is established. The expression for $V_p(d)$ follows immediately from (3.19) and the fact that $v_d(j,j) = p(N-j)$. □

It is worth noting that if $p(k) = 0$ for some $k \leq N-1$, then $d \equiv N$ is not optimal unless $N = 2$ and $p(0) = 1$.

In some of the work to come there will often arise expressions of the form

$$x = \min\,\{n \in I_N : \mathcal{P}(n) \text{ true}\}\,,$$

where $\mathcal{P}(n)$ is some proposition about n, for which $\mathcal{P}(N)$ is true. In most cases $\mathcal{P}(n)$ is monotone in the sense that $\mathcal{P}(n)$ true $\Rightarrow$ $\mathcal{P}(n+1)$ true, in which case the backward recursion for evaluating x can be stopped as soon as $\mathcal{P}(n)$ is found to be false, thus avoiding having to test $\mathcal{P}(n)$ for all $n \in I_N$. In some cases when $\mathcal{P}(n)$ may not be monotone we may instead be interested in the smallest value of n such that $\mathcal{P}(m)$ is true for all $m \geq n$.

In this case backward recursion can stop as soon as $\mathcal{P}(n)$ is false, even though $\mathcal{P}(n)$ may be true again for some smaller value of n. Formally, we define the <u>serial minimum</u> of a set of the form $\{n \in I_N : \mathcal{P}(n) \text{ true}\}$, where $\mathcal{P}(N)$ true, by

$$\text{sermin } \{n \in I_N : \mathcal{P}(n) \text{ true}\}$$
$$= \min \{n \in I_N : \mathcal{P}(m) \text{ true for } n \leqslant m \leqslant N\} .$$

Of course

$$\text{sermin } \{n \in I_N : \mathcal{P}(n) \text{ true}\}$$
$$= 1 + \max \{n \in I_N : \mathcal{P}(n) \text{ false}\}$$

provided the set on the R H S is non-empty. Clearly sermin $\geqslant$ min with equality when $\mathcal{P}(n)$ is monotone. We shall emphasize the case when $\mathcal{P}(n)$ is known to be monotone by denoting the serial minimum by MIN. For example (3.39) with π given by p will now be written

$$s_p = \text{MIN } \{n \in I_N : v(n,0) \leqslant 1\} . \tag{3.49}$$

We shall devote the remainder of this section to studying policies for which at most one approach is made to an applicant at a stage other than that of her interview.

Let E_N denote the set of policies $d \in D_N$ of the form

$$\begin{cases} d(j) = j , & j \in \{r, r+1, \ldots, N\} , \\ d(j) \geqslant r , & j \in I_{r-1} , \end{cases} \tag{3.50}$$

for some $r \in I_N$.

All policies $d \in E_N$ are policies which attempt recall at most once. This is clear since the approach to an applicant already passed over will be made only to the best of the first $r-1$ applicants at a stage $\geqslant r$. It turns out that there is an expression in closed form for $V_p(d)$ when $d \in E_N$. We shall need to refer the following functions.

For each positive integer N and each sequence of availability probabilities p, we define the function $g_p^N : I_N \to \mathbb{R}$ by

$$(3.51) \qquad g_p^N(n) = \begin{cases} \sum_{k=n}^{N-1} 1/k, & p(0) = 1, \\ \dfrac{p(0)}{1-p(0)} \left[-1 + \prod_{k=n}^{N-1} \left(1 + \dfrac{1-p(0)}{k}\right)\right], & 0 < p(0) < 1. \end{cases}$$

Theorem 3.6. Let d be any policy in E_N and let $r = \text{MIN}\,\{j \in I_N : d(j) = j\}$. Then for the A(N,p)-problem

$$(3.52) \qquad V_p(d) = \begin{cases} \{(r-1)g_p^N(r-1) + \sum_{j=1}^{r-1} p(d(j)-j)[1-g_p^N(d(j))]\}/N, & r > 1, \\ \{p(0) + (1-p(0))g_p^N(1)\}/N, & r = 1. \end{cases}$$

Proof. From (3.15) we have that

$$v_d(n+1,n+1) = p(0)[1-v_d(n+1,0)], \qquad n \geqslant r-1,$$

which on substitution into (3.12) gives

$$v_d(n,0) = [n+1-p(0)]v_d(n+1,0)/n + p(0)/n, \qquad n \geqslant r-1.$$

The solution of this equation subject to $v_d(N,0) = 0$ is $v_d(n,0) = g_p^N(n)$, $n \geqslant r-1$. But for $j \in I_N$ $d(j) \geqslant r$ and hence $v_d(j,j) = p(d(j)-j)[1-g_p^N(d(j))]$. Now for $r > 1$ we have from (3.17) and (3.19) that

$$V_p(d) = \{v_d(r-1,0) + \sum_{j=1}^{r-1} v_d(j,j)\}/N$$

and (3.52) for $r > 1$ follows by substituting for v_d in this expression. The result for $r = 1$ is obtained by substituting in (3.16). □

<u>Corollary</u>. If d is of the form

$$(3.53) \qquad d(j) = \begin{cases} j, & j \in \{r, r+1, \ldots, N\}, \\ N, & j \in I_{r-1}, \end{cases}$$

for some $r \in I_N$ then for the A(N,p)-problem

$$(3.54) \qquad V_p(d) = \begin{cases} \{(r-1)g_p^N(r-1) + \sum_{j=1}^{r-1} p(N-j)\}/N, & r > 1, \\ \{p(0) + [1 - p(0)]g_p^N(1)\}/N, & r = 1. \end{cases} \quad \square$$

For each N and p we denote

$$(3.55) \qquad r_p = \text{MIN}\ \{j \in I_N : u(j,j) = \max_{n=j}^{N} u(n,j)\}.$$

(MIN rather than sermin because of Theorem 3.4.) Clearly if there is an optimal policy in E_N for A(N,p)-problem then the value of r in the form (3.50) will be r_p. It would be convenient to have a simple sufficient condition on N,p for the optimal policy to belong to E_N. This would mean that the recursion given in Algorithm 3.1 could be stopped as soon as r_p was calculated and could result in some saving in computing time since $s_p \leq r_p$. $(0 \leq u(r_p,r_p) = p(0)[1 - v(r_p,0)]$ $\Rightarrow v(r_p,0) \leq 1 \Rightarrow s_p \leq r_p$.)

For each sequence of availability probabilities define the sequence $q = \{q(k)\}_{k=1}^{\infty}$ by

$$(3.56) \qquad q(k) = \begin{cases} p(k)/p(k-1), & p(k-1) > 0, \\ 1, & p(k-1) = 0. \end{cases}$$

$q(k)$ is the conditional probability that an applicant is available k stages after her interview given that she was available one stage previously. We have that

$$p(k) = p(0) \prod_{i=1}^{k} q(i) , \quad k \in I .$$

Theorem 3.7. If p is such that

$$q(k) \leqslant q(k+1) , \quad k \in I_{N-2} ,$$

then for the $A(N,p)$-problem there exists an optimal policy in E_N.

Proof. By the definition of r_p there will exist an optimal policy d such that $d(r_p - 1) \geqslant r_p$ and $d(j) = j$, $j \geqslant r_p$. Let m denote $r_p - 1$, and thus $d(m) > m$. Hence

$$\begin{aligned} 0 &> u(m,m) - u(d(m),m) \\ &= p(0)[1 - v(m,0)] - p(d(m) - m)[1 - v(d(m),0)] \\ &= p(0)\{1 - v(m,0) - [1 - v(d(m),0] \prod_{k=1}^{d(m)-m} q(k)\} . \end{aligned}$$

Consequently

$$1 - v(m,0) > [1 - v(d(m),0)] \prod_{k=1}^{d(m)-m} q(k) .$$

But for any $j < m$

$$\begin{aligned} u(m,j) - w(m,j) &= u(m,j) - \max_{n=m+1}^{N} u(n,j) \\ &\leqslant u(m,j) - u(d(m),j) \\ &= p(m-j)\{1 - v(m,0) - [1 - v(d(m),0)] \prod_{k=m-j+1}^{d(m)-j} q(k)\} \\ &\leqslant 0 , \end{aligned}$$

since $v(d(m),0) \leqslant 1$ (Theorem 3.2) and $q(k) \leqslant q(k+m-j)$. We have now established that

$$u(r_p - 1, j) \leqslant w(r_p - 1, j), \qquad j \in I_{r_p - 1},$$

and it follows from Theorem 3.4 that

$$u(n,j) \leqslant w(n,j), \qquad n \in I_{r_p - 1}, \qquad j \in I_n .$$

Hence there is an optimal policy d for which $d(j) \geqslant r_p$, $j \in I_{r_p - 1}$, and the theorem is proved. □

The condition that q be non-decreasing is quite weak, since we shall show in the section 3.5 that if q is constant then the policy with $d(j) = r_p$, $j \in I_{r_p}$ is optimal.

Optimal policies of the form (3.53) will crop up in the later work. We shall denote by E_N^* the set of policies $d \in D_N$ which have the form:

$$d(j) = \begin{cases} j, & j \in \{r, r+1, \dots, N\}, \\ N, & j \in I_{r-1}, \end{cases}$$

for a fixed $r \in I_N$. Such a policy can be stated:

make no approach before stage r; from stage r onwards approach a candidate only at the time of interview; if stage N is reached, approach the candidate no matter when she was interviewed.

If there is an optimal policy in E_N^* then only r has to be evaluated. This will of course be r_p as given in (3.55). But we have that

$$u(j,j) \begin{Bmatrix} \geqslant \\ \\ < \end{Bmatrix} u(N,j) \qquad \begin{matrix} j \geqslant r_p, \\ \\ j = r_p - 1, \end{matrix}$$

and hence

$$r_p = \text{sermin}\ \{j \in I_N : g_p^N(j) \leqslant 1 - p(N-j)/p(0)\}\ ,$$

since $u(j,j) = p(0)[1 - g_p^N(j)]$ for $j \geqslant r_p - 1$. The maximum probability of employing the best applicant is then given by (3.54) with $r = r_p$. Again it would be useful to have a simple sufficient condition on p for there to be an optimal policy in E_N^*. If q(k) was increasing at a sufficiently large rate it seems likely that there would be an optimal policy in E_N^*. It is possible to show that if p is such that

$$(3.57) \qquad (3 - p(0))(1 - q(k+1)) \leqslant q(k+1)(1 - q(k))\ , \qquad k \in I_{N-2}\ ,$$

then there is an optimal policy in E_N^*. However this condition does not appear to be of great practical use. Indeed, for the only example in the next section to which the condition is applicable, the optimal policy was found by a simple and direct method. On the other hand a more satisfactory condition than (3.57) does not seem to exist.

The proof of the sufficiency of condition (3.57) is quite long and involved and will be omitted.

5. FAMILIES OF SEQUENCES OF AVAILABILITY PROBABILITIES

In this section we shall investigate properties of optimal policies for the A(N,p)-problem when the first N terms of p belong to various parametric families of sequences.

The Geometric Family

A sequence of availability probabilities p belongs to the geometric family with parameters α and β, $(0 < \alpha \leqslant 1\ ,\ 0 < \beta < 1)$ for the A(N,p)-problem if p satisfies

$$p(k) = \alpha\beta^{k}, \quad k \in \{0,1,\ldots,N-1\}.$$

This family has strong appeal in practical situations since the probability an applicant becomes unavailable between successive stages is a constant, $1-\beta$. The optimal policy for the case $\alpha = 1$ was obtained independently in Yang (1974).

Theorem 3.8. If for the A(N,p)-problem, p belongs to the geometric family with parameters α,β then the policy

$$(3.58) \qquad d(j) = \begin{cases} j, & j \in \{r, r+1, \ldots, N\}, \\ r, & j \in I_{r-1}, \end{cases}$$

is optimal and

$$V_p^N = \begin{cases} \{(r-1)g_p^N(r-1) + \alpha[1-g_p^N(r)](1-\beta^r)/(1-\beta)\}/N, & r > 1, \\ \{\alpha + (1-\alpha)g_p^N(1)\}/N, & r = 1, \end{cases}$$

where

$$(3.59) \qquad r = \max\{n \in I_N : 1 - g_p^N(n-1) < \beta[1 - g_p^N(n)]\},$$

and g_p^N is given by (3.51) with $p(0) = \alpha$.

Proof. Let r_p be as defined by (3.55), and thus

$$u(r_p, r_p) = \max_{n=r_p}^{N} u(n, r_p).$$

For $j \in I_{r_p - 1}$,

$$\begin{aligned} u(r_p, j) &= \beta^{r_p - j}\, u(r_p, r_p) \\ &= \beta^{r_p - j} \max_{n=r_p}^{N} u(n, r_p) \end{aligned}$$

$$= \max_{n=r_p}^{N} u(n,j) .$$

Thus there is an optimal policy with $d(j) \leqslant r_p$. But $q(k) = \beta$, $k \in I_{N-1}$ and hence by theorem 3.7 there is an optimal policy d such that $d(j) = j$, $j \geqslant r_p$ and $d(j) \geqslant r_p$, $j \leqslant r_p$. We have shown that the policy given by (3.58), with $r = r_p$, is optimal. Now from the definition of r_p we have that

$$u(n,n) \left\{ \begin{matrix} \geqslant \\ \\ < \end{matrix} \right\} u(n+1,n) \qquad \begin{matrix} , & n \geqslant r_p . \\ \\ , & n = r_p - 1 , \end{matrix}$$

and hence

$$1 - g_p^N(n) \left\{ \begin{matrix} \geqslant \\ \\ < \end{matrix} \right\} \beta[1 - g_p^N(n+1)] \qquad \begin{matrix} , & n \geqslant r_p , \\ \\ , & n = r_p - 1 , \end{matrix}$$

which gives expression (3.59). The Expression for V_p^N follows by substitution into expression (3.52) in Theorem 3.6.

The optimal policy given by (3.58) can be stated simply as follows:

> make no approach before stage r; at stage r approach the candidate irrespective of the stage when she was interviewed; after stage r approach any candidate at the time of her interview.

The Step Family

This family is important because of its asymptotic properties.

A sequence of availability probabilities p belongs to the step family with parameters α, β, K, $(0 \leqslant \beta < \alpha \leqslant 1,\ K \in I_N)$ for the A(N,p)-problem if p satisfies

$$p(k) = \begin{cases} \alpha , & k \in \{0,1,\ldots,K-1\} , \\ \beta , & k \in \{K,K+1,\ldots,N-1\} , \end{cases}$$

It follows that

$$u(n,j) = \begin{cases} \alpha[1 - v(n,0)] , & j \leq n \leq j + K - 1 , \\ \beta[1 - v(n,0)] , & j + K \leq n \leq N . \end{cases}$$

Now v(n,0) is decreasing in n and hence

$$\max_{n=j}^{N} u(n,j) = \begin{cases} \alpha , & j \geq N - K + 1 , \\ \max \{\beta, \alpha[1 - v(j + K - 1,0)]\} , & 1 \leq j \leq N - K , \end{cases}$$

$$(3.60) \qquad = \begin{cases} \alpha , & N - K + 1 \leq j \leq N , \\ \alpha[1 - v(j + K - 1,0)] , & t^N - K + 1 \leq j \leq N - K , \\ \beta , & 1 \leq j \leq t^N - K . \end{cases}$$

$$= \begin{cases} u(j + K - 1,j) , & t^N - K + 1 \leq j \leq N - K , \\ u(N,j) , & j \text{ otherwise} , \end{cases}$$

where

$$(3.61) \qquad t^N = \text{MIN} \{n \in \{K,K+1,\ldots,N\} : \alpha[1 - v(n,0)] \geq \beta\} .$$

It then follows that the policy d given by

$$d(j) = \begin{cases} j + K - 1 , & t^N - K + 1 \leq j \leq N - K , \\ N , & j \text{ otherwise} , \end{cases}$$

is optimal.

Unless $K = 1$ then there is no optimal policy in E_N. If $K = 1$ then there is an optimal policy in E_N^* and clearly $t^N = r_p$. Hence we have proved the following theorem, using the Corollary of Theorem 3.6 and the fact that $v(n,0) = g_p^N(n)$, $n \geqslant t^N - 1$.

Theorem 3.9. If for the A(N,p)-problem, p is of the form

$$p(k) = \begin{cases} \alpha, & k = 0, \\ \beta, & k \in I_{N-1}, \end{cases}$$

for fixed α, β, $0 \leqslant \beta < \alpha \leqslant 1$, then the policy

$$d(j) = \begin{cases} j, & t^N \leqslant j \leqslant N, \\ N, & 1 \leqslant j \leqslant t^N - 1, \end{cases}$$

is optimal and

$$\text{(3.62)} \qquad v_p^N = \begin{cases} (t^N - 1)\{g_p^N(t^N - 1) + \beta\}/N, & t^N > 1, \\ \{\alpha + (1 - \alpha) g_p^N(1)\}/N, & t^N = 1, \end{cases}$$

where

$$\text{(3.63)} \qquad t^N = \text{MIN}\ \{n \in I_N : g_p^N(n) \leqslant 1 - \beta/\alpha\},$$

and $g_p^N(n)$ is given by (3.51) with $p(0) = \alpha$. □

MIN appears in (3.63) instead of sermin since $g_p^N(n)$ is decreasing in n.

For the case $K > 1$ there is no relatively simple closed form for v_p^N in terms of t^N. However a simple recursion can be given which generates v_p^N and t^N.

First we extend $v(n,0)$ by setting

$$(3.64) \qquad v(n,0) = 0 , \quad N \leqslant n \leqslant N+K-1 ,$$

and it follows from (3.60) that for $t^N - K \leqslant n \leqslant N-1$,

$$v(n+1,n+1) = \alpha[1 - v(n+K,0)] .$$

Hence by (3.30)

$$(3.65) \qquad v(n,0) = \{(n+1)v(n+1,0) + \alpha[1 - v(n+K,0)]\}/n ,$$

$$t^N - K \leqslant n \leqslant N-1 .$$

Starting with (3.64) $v(n,0)$, for all $n \geqslant t^N - K$, is evaluated recursively using (3.65) and t^N is evaluated at the same time by means of

$$(3.66) \qquad t^N = \text{MIN } \{n \in \{K, K+1, \ldots, N\} : v(n,0) \leqslant 1 - \beta/\alpha\} .$$

V_p^N is then calculated using the expression

$$(3.67) \qquad V_p^N = \begin{cases} (t^N - K)\{v(t^N - K,0) + \beta\}/N , & t^N > K , \\ \\ \{v(1,0) + \alpha[1 - v(K,0)]\}/N , & t^N = K , \end{cases}$$

which follows from (3.40) with $n = t^N - K$ and $v(k,k) = \beta$ $k \leqslant t^N - K$ for the case $t^N > K$; and with $n = 1$ and $v(1,1) = \alpha[1 - v(K,0)]$ for the case $t^N = K$.

The form of the optimal policies for the step family is very much what one would expect. Clearly an approach before stage N would only be considered if the candidate had been interviewed $K-1$ stages previously.

If K/N is sufficiently large it is possible to give closed form expressions for V_p^N and t^N. It can be shown that if

(3.68) $$K \geqslant N(\alpha-\beta)/(\alpha^2+\alpha-\beta) ,$$

then

(3.69) $$t^N = \max\{K, N - \lfloor N(\alpha-\beta)/(\alpha^2+\alpha-\beta)\rfloor\} ,$$

(3.70) $$V_p^N = \alpha+\alpha^2 - [(\alpha^2+\alpha-\beta)t^N - (\alpha-\beta)K]/N - \alpha^2 \sum_{k=t^N}^{N-1} 1/k ,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

The result is not important enough to warrant presenting the proof which is quite long and detailed.

Problems which are equivalent to the A(N,p)-problem for particular step sequences p, have been considered in the literature. The classical best choice problem is equivalent to taking $\alpha = 1$, $\beta = 0$, $K = 1$. Yang (1974) gave the solution for the sub-family $\alpha = 1$, $0 < \beta < 1$, $K = 1$. Smith (1975) solved a problem equivalent to the sub-family with $0 < \alpha < 1$, $\beta = 0$, $K = 1$. The problem with p belonging to the sub-family $\alpha = 1$, $\beta = 0$, $K \geqslant 2$, is treated in Smith and Deely (1975). It was shown in that article that if $K \geqslant N/2$ then $t^N = K$ and

$$V_p^N = 2 - K/N - \sum_{k=K}^{N-1} 1/k ,$$

which are the values given by substituting $\alpha = 1$ and $\beta = 0$ in (3.68) to (3.70).

The Linear Family

A sequence of availability probabilities p belongs to the linear family with parameters α, β, $(0 < \alpha \leqslant 1 ,\ \ 0 \leqslant \beta \leqslant 1/(N-1))$ for the A(N,p)-problem if p satisfies

$$p(k) = \alpha(1-\beta k) , \quad k \in \{0,1,\ldots,N-1\} .$$

Now

$$q(k) = 1 - \beta/(1+\beta-\beta k) , \quad k \in I_{N-1} ,$$

which is decreasing (unless $\beta = 0$) . Consider the case $N = 3$, $\alpha = 1$, $\beta = 2/5$. We have

$$v(3,0) = 0 , \quad u(3,j) = (1/5,\underline{3/5},\underline{1}) ,$$

$$v(2,0) = 1/2 , \quad u(2,j) = (\underline{3/10},1/2,\cdot) ,$$

$$v(1,0) = 8/5 , \quad u(1,j) = (-3/5,\cdot,\cdot) .$$

$V_p^3 = 19/30$, and the only optimal policy is $d = (2,3,3)$. Clearly then, there does not exist an optimal policy in E_N for every linear p. On the other hand it can be shown that the only optimal policy for the case $\alpha = 1/2$, $\beta = 1/3$ with $N = 3$ is $d = (2,2,3)$ which is in E_3 .

If β is sufficiently small then the policy $d \equiv N$ will be optimal. By Theorem 3.5 we have

$$\begin{aligned} d \equiv N \text{ is optimal} \\ &\Leftrightarrow q(k) \geqslant 1 - \alpha/(N-1) , \quad k \in I_{N-1} \\ &\Leftrightarrow q(N-1) \geqslant 1 - \alpha/(N-1) \\ &\Leftrightarrow \beta/[1+\beta-\beta(N-1)] \leqslant \alpha/(N-1) \\ &\Leftrightarrow \beta \leqslant \alpha/[N-1+\alpha(N-2)] . \end{aligned}$$

<u>The Hyperbolic Family</u>

A sequence of availability probabilities p belongs to the <u>hyperbolic family</u> with parameters α,β,γ, $(0 < \alpha \leqslant 1,\ \beta \geqslant 0,\ \gamma \geqslant 0)$

for the A(N,p)-problem if p satisfies

$$p(k) = \alpha/(1+\beta k)^{\gamma}, \quad k \in \{0,1,\ldots,N-1\}.$$

Now

$$q(k) = [1-\beta/(1+\beta k)]^{\gamma}, \quad k \in I_{N-1}.$$

q is non-decreasing so it follows by Theorem 3.7 that there is an optimal policy in E_N for every hyperbolic sequence p.

While for some hyperbolic sequences there are optimal policies in E_N^* this is not true for every case. Take the case $p(k) = 1/2(1+k)$ with $N = 3$.

$$v(3,0) = 0 \qquad u(3,j) = (1/6, 1/4, \underline{1/2})$$

$$v(2,0) = 1/4 \qquad u(2,j) = (\underline{3/16}, \underline{3/8}, \cdot)$$

$$v(1,0) = 7/8 \qquad u(1,j) = (1/16, \cdot, \cdot)$$

Thus $V_p^3 = 17/48$ and the only optimal policy is $d = (2,2,3) \notin E_3^*$.

By Theorem 3.5 we have that

$d \equiv N$ is optimal

$$\Leftrightarrow q(k) \geq 1-\alpha/(N-1), \quad k \in I_{N-1},$$

$$\Leftrightarrow \beta \leq [1-\alpha/(N-1)]^{-1/\gamma} - 1.$$

If this inequality is not satisfied then any particular case can be solved by means of Algorithm 3.1, perhaps with the modification mentioned after Theorem 3.6.

6. ASYMPTOTIC PROPERTIES OF THE A(N,p)-PROBLEM

It is well-known that for the classical best choice problem $\lim_{N\to\infty} V^N = e^{-1}$, where V^N is the maximum probability of employing the best applicant. There is of course, no optimal policy when there is an infinite number of applicants. However it can be shown the sequence of policies $\{Q_N\}_{N=1}^{\infty}$ where Q_N is described by:

from stage $\lceil N/e \rceil$ onwards employ at time of her interview the first applicant who is a candidate,

(where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x), is asymptotically optimal in the sense that the probability of employing the best applicant among a total of N applicants when using policy Q_N, tends to e^{-1} as $N \to \infty$. The policy Q_N is optimal for many values of N and gives a probability of employing the best which is close to V^N when Q_N is not optimal.

For the A(N,p)-problem the questions which naturally arise are whether $\lim_{N\to\infty} V_p^N$ exists for various sequences p, and whether there is a simple sequence of asymptotically optimal policies.

Every sequence of availability probabilities, p, is non-increasing and non-negative. Consequently every p has a limit and we shall denote

$$p(\infty) = \lim_{k\to\infty} p(k) .$$

It seems likely then that $\lim_{N\to\infty} V_p^N$ exists for all p. By approximating the recursion equations (3.30) to (3.34) by differential equations it has been possible to show heuristically that $\lim_{N\to\infty} V_p^N = \nu(p(0),p(\infty))$ for every sequence p, where ν is defined by (3.71) and (3.72). It may be possible to show that the differential equation technique is rigorous, as Mucci (1973a and b) was able to do for the Secretary

Problem. Instead we shall prove the existence of the limit by the more direct approach of bounding the solution of the recursion equations and then taking the limit.

First we shall evaluate $\lim V_p^N$ for every p which is a member of the step family for the A(N,p)-problem for all N.

The functions ρ and ν defined on the set $\{(x,y) : 0 < x \leqslant 1,\ 0 \leqslant y \leqslant x\}$ by

$$(3.71) \qquad \rho(x,y) = \begin{cases} e^{y-1}, & x = 1, \\ \left(\dfrac{x^2}{x - y + xy}\right)^{1/(1-x)}, & 0 < x < 1, \end{cases}$$

and

$$(3.72) \qquad \nu(x,y) = (1 + y - y/x)\,\rho(x,y),$$

will be referred to in our derivations. We shall also make use of the bounds for g_p^N given by

$$(3.73) \qquad \log N - \log n \leqslant g_p^N(n) \leqslant \log(N-1) - \log(n-1), \qquad p(0) = 1$$

and

$$(3.74) \qquad \frac{p(0)}{1-p(0)}\left(\left[\frac{N}{n}\right]^{1-p(0)} - 1\right) \leqslant g_p^N(n) \leqslant \frac{p(0)}{1-p(0)}\left(\left[\frac{N-1}{n-1}\right]^{1-p(0)} - 1\right),$$

$$0 < p(0) < 1,$$

which follow respectively from the inequalities

$$\log(k+1) - \log k \leqslant 1/k \leqslant \log k - \log(k-1),$$

and

$$\left[\frac{k+1}{k}\right]^{1-p(0)} \leqslant \frac{k+1-p(0)}{k} \leqslant \left[\frac{k}{k-1}\right]^{1-p(0)}.$$

Theorem 3.10. Let p be any sequence of availability probabilities of the form

$$p(k) = \begin{cases} \alpha , & k \in \{0,1,\ldots,K-1\} , \\ \beta , & k \in \{K,K+1,\ldots\} , \end{cases}$$

where K is a fixed positive integer and $0 \leqslant \beta < \alpha \leqslant 1$. Then

$$\lim_{N\to\infty} V_p^N = \nu(\alpha,\beta) ,$$

where ν is defined by (3.71) and (3.72).

Proof. For $N \geqslant K$, p belongs to the step family for the A(N,p)-problem. Now from (3.64) and (3.65) we have that

$$nv(n,0) - (n+1)v(n+1,0) \leqslant \alpha \leqslant 1 , \qquad t^N - K \leqslant n \leqslant N-1 ,$$

where t^N is given by (3.66).

It follows for $n \geqslant t^N - K$, that

$$(n+1)v(n+1,0) - (n+K)v(n+K,0) \leqslant K-1 ,$$

and

$$(n+1)v(n+1,0) \leqslant N-(n+1) .$$

Consequently,

$$\begin{aligned} 1-v(n+K,0) &\leqslant 1-v(n+1,0) + (K-1)(1+v(n+1,0))/(n+K) \\ &\leqslant 1-v(n+1,0) + N(K-1)/(n+K)(n+1) \\ &\leqslant 1-v(n+1,0) + C_N , \end{aligned}$$

where

$$C_N = N(K-1)/t^N(t^N-K+1) .$$

Also $1 - v(n+1,0) \leqslant 1 - v(n+K,0)$ and hence from (3.65) we have for $n \geqslant t^N - K$, that

$$\frac{\alpha}{n} \leqslant v(n,0) - \left(1 + \frac{1-\alpha}{n}\right) v(n+1,0) \leqslant \frac{\alpha}{n}(1 + C_N) .$$

Starting with $v(N,0) = 0$, it can be seen that

$$g_p^N(n) \leqslant v(n,0) \leqslant (1 + C_N)\, g_p^N(n) , \quad t^N - K \leqslant n \leqslant N ,$$

where g_p^N is given by (3.51) with $p(0) = \alpha$. But by (3.66)

$$g_p^N(t^N) \leqslant 1 - \beta/\alpha < (1 + C_N)\, g_p^N(t^N - 1) ,$$

and it follows from (3.73) and (3.74) that

$$\log[N/t^N] \leqslant 1 - \beta < (1 + C_N) \log[(N-1)/(t^N-2)] , \quad \alpha = 1 , \tag{3.75}$$

and

$$\frac{\alpha}{1-\alpha}\left(\left[\frac{N}{t^N}\right]^{1-\alpha} - 1\right) \leqslant 1 - \beta/\alpha \tag{3.76}$$

$$< \frac{\alpha(1 + C_N)}{1-\alpha}\left(\left[\frac{N-1}{t^N-2}\right]^{1-\alpha} - 1\right) , \quad 0 < \alpha < 1 .$$

From the left hand inequalities of (3.75) and (3.76) it follows that $t^N/N \geqslant \rho(\alpha,\beta) > 0$ for all N, which implies that $t^N \to \infty$ and hence $C_N \to 0$ as $N \to \infty$. As a consequence then of (3.75) and (3.76) we have that

$$\lim_{N\to\infty} (t^N - K)/N = \lim_{N\to\infty} t^N/N = \rho(\alpha,\beta) , \tag{3.77}$$

and that

$$\lim_{N\to\infty} v(t^N - K,0) = \lim_{N\to\infty} g_p^N(t^N - K)$$

$$= \lim_{N\to\infty} g_p^N(t^N)$$

$$= 1 - \beta/\alpha .$$

It now follows from (3.67) that

$$\lim_{N\to\infty} V_p^N = \lim_{N\to\infty} (t^N - K)[\,v(t^N - K, 0) + \beta]/N$$

$$= \nu(\alpha,\beta)\ . \qquad \square$$

We can now prove the main asymptotic result.

Theorem 3.11. For every sequence of availability probabilities, p,

$$\lim_{N\to\infty} V_p^N = \nu(p(0), p(\infty))\ ,$$

where ν is defined by (3.71) and (3.72).

Proof. First we shall consider sequences p for which $p(0) = p(\infty)$. Therefore $p \equiv p(0)$ and clearly $d \equiv N$ is optimal for all N. Hence $V_p^N = p(0) = \nu(p(0), p(0))$ for all N.

Let us now assume that $p(\infty) < p(0)$. For every ε, $0 < \varepsilon < p(0) - p(\infty)$, there exists a positive integer $K = K(p,\varepsilon)$ such that

$$p(K) \leqslant p(\infty) + \varepsilon < p(K-1)\ .$$

For each p and each ε, then, we can define two sequences $\underline{p}$ and $\overline{p}$, which belong to the step family for every $N \geqslant K$, by

$$\underline{p}(k) = \begin{cases} p(0)\ , & k = 0\ , \\ p(\infty)\ , & k \geqslant 1\ , \end{cases}$$

and

$$\overline{p}(k) = \begin{cases} p(0)\ , & 0 \leqslant k \leqslant K-1\ , \\ p(\infty) + \varepsilon\ , & k \geqslant K\ . \end{cases}$$

Clearly $\underline{p}(k) \leqslant p(k) \leqslant p(k)$ and hence by Theorem 3.3 we have that

$V_{\underline{p}}^N \leqslant V_p^N \leqslant V_{\overline{p}}^N$, for every N. But Theorem 3.10 implies that $V_{\underline{p}}^N \to \nu(\underline{p}(0),\underline{p}(1))$ and $V_{\overline{p}}^N \to \nu(\overline{p}(0),p(K))$ as $N \to \infty$ and hence

$$\nu(p(0),p(\infty)) \leqslant \lim_{N\to\infty} V_p^N \leqslant \nu(p(0),p(\infty)+\varepsilon) .$$

Finally, ε is arbitrary and ν is a continuous function and thus

$$\lim_{N\to\infty} V_p^N = \nu(p(0),p(\infty)) . \quad \square$$

Because the asymptote of V_p^N has a relatively simple form it seems likely that there will be a sequence of simple policies which is asymptotically optimal. The optimal policies for the step sequence with K = 1 inspired the choice of the particular sequence in the Theorem which now follows.

<u>Theorem 3.12</u>. Let $\{d_N\}_{N=1}^{\infty}$ be a sequence of policies given by

$$d_N(j) = \begin{cases} j, & j \in \{r_N, r_N+1, \ldots, N\}, \\ N, & j \in I_{r_N - 1}, \end{cases}$$

where

(3.78) $$r_N = [\![\ulcorner N\rho(p(0),p(N-1))\urcorner]\!] ,$$

then the sequence d_N, N = 1,2,..., is asymptotically optimal for the A(N,p)-problem in the sense that, for every p

$$\lim_{N\to\infty} V_p(d_N) = \nu(p(0),p(\infty)) .$$

<u>Proof</u>: There is an N_0 such that $r_N > 1$ for every $N > N_0$. Thus, from (3.52) in the statement of Theorem 3.6 we have

$$V_p(d_N) \geqslant (r_N - 1)\left[g_p^N(r^N - 1) + p(\infty)\right]/N ,$$

since $p(k) \geqslant p(\infty)$. Now ρ is continuous and thus $(r_N - 1)/N \to \rho(p(0),p(\infty))$ as $N \to \infty$. It then follows from the bounds for g_p^N, (3.73) and (3.74), that $g_p^N(r_N - 1) \to 1 - p(0)/p(\infty)$ and hence that

$$\lim_{N\to\infty} V_p(d_N) \geqslant \nu(p(0),p(\infty)) .$$

But $V_p(d_N) \leqslant V_p^N \to \nu(p(0),p(\infty))$, and the theorem is proved. □

For the sequence $\{d_N\}$ the convergence of $V_p^N - V_p(d_N)$ to zero is not uniform over all p. Consider the step sequence p_N with $K = N - 1$, $\alpha = 1$, $\beta = 0$.

$$V_{p_N}^N \geqslant \frac{1}{N} \sum_{k=0}^{N-1} p(k) = 1 - 1/N ,$$

and hence $V_{p_N}^N \to 1$ as $N \to \infty$. But $r_N = \lceil\!\lceil N/e \rceil\!\rceil$ and thus by (3.54) for $r_N \geqslant 2$

$$V_{p_N}(d_N) = \{(r_N - 1) g_{p_N}^N (r_N - 1) + r_N - 2\}/N$$

$$\to 2/e , \qquad N \to \infty .$$

Clearly there is an ε, $0 < \varepsilon < 1 - 2/e$, and an N_ε such that $V_{p_N}^N - V_{p_N}(d_N) > \varepsilon$ for every $N > N_\varepsilon$, and hence the convergence is not uniform. However it would appear that the convergence of $V_p^N - V_p(d_N)$ to zero would be uniform over a class of uniformly convergent sequences of availability probabilities.

It may be possible to find a sequence of policies which is uniformly asymptotically optimal over all p, perhaps by approximating by differential equations the recursion equations for generating the optimal policy. By their very nature the individual policies in an asymptotically optimal sequence are likely to be very similar to the corresponding optimal policies. The simplicity of Algorithm 3.1

for generating an optimal policy reduces the practical value of such a sequence.

Let $\{a_N\}_{N=1}^{\infty}$ be any sequence of non-negative integers which diverges to ∞. If, in expression (3.78), $p(N-1)$ is replaced by $p(a_N)$, then the sequence of policies given in Theorem 3.12 will still be asymptotically optimal. The difference between V_p^N and $V_p(d_N)$ may be reduced by a suitable choice of a_N. For example take a_N to be an integer in $\{0,1,\ldots,N-1\}$ which maximises $V_p(d_N)$. The sequence of policies so formed would obviously be asymptotically optimal.

7. CONCLUDING REMARKS

In Chapter III it has been necessary to make certain assumptions in order to carry out further analysis of Problems A and B. Assumption 3.2 was chosen because it was the most reasonable of several possibilities. As seen in Section 3.6 this leads to some interesting asymptotic properties.

A reasonable alternative to Assumption 3.2 which is worth considering, is to assume that the matrix of availability probabilities, π is given by $\pi(n,j) = \tilde{\omega}(n)$, $n \in I_N$, $j \in I_n$, where $\tilde{\omega} = \{\tilde{\omega}(n)\}_{n=1}^{\infty}$ is also a non-increasing non-negative sequence with $0 < \tilde{\omega}(1) < 1$. This might apply in Example 1.1.B, the house-buying example, where the list of houses is accurate at the start of inspections. It immediately follows from (3.31) that the number of the candidate is not relevant to the optimal decision at each stage. An optimal policy $d \in D_N$ can be calculated by means of Algorithm 3.1, although a simpler representation of a policy is the set $T = \{d(j) : j \in I_N\}$ where an approach is made to the candidate at stage n (irrespective of her number) if and only if $n \in T$. Obviously, equations (3.29) to (3.34) can be simplified since there is no dependence upon j for $j \geq 1$. It would be possible to find conditions on $\tilde{\omega}$ in order that an optimal policy T is of the form $\{r, r+1, \ldots, N\}$. The asymptotic properties are relatively uninteresting. It can easily be shown that $\lim_{N\to\infty} V_{\tilde{\omega}}^{N} = \lim_{n\to\infty} \tilde{\omega}(n)$ and that $T_N = \{N\}$, $N = 1,2,\ldots$, is a sequence of asymptotically optimal policies.

Perhaps one of the more unreasonable parts of Assumption 3.1 is part 3) - that unavailable applicants remain unavailable. If this were relaxed then Problems A and B may no longer be equivalent. An alternative which has considerable appeal, is to assume that the vectors of availabilities, $\omega(j) = (\omega_j(j), \omega_{j+1}(j), \ldots, \omega_N(j))$, $j \in I_N$, are homogeneous Markov chains with the same distributions. For an optimal decision at stage n in Problem A, it would be necessary to know the number of the current candidate if no approach to her had been made at or before stage n, otherwise it would be necessary to know how far back the latest unsuccessful approach to the current candidate was made. Thus there would only be 2n stage n states, and it would appear that some fruitful analysis of the problem could be made. For Problem B the number of the current candidate would be irrelevant, only her availability would need to be known in order to make the optimal decision at any stage. Thus for Problem B there would only be 2 possible states at each stage. (This fact could have been used to establish very quickly the form of the optimal policy for the geometric family in Section 3.5.)

Problems A and B could be turned into games by allowing an opponent to try and minimise the probability of employing the best either by choosing π subject to some constraints, or by choosing the distribution on the order of the applicants.

CHAPTER IV

PROBLEM C: A SIMPLIFIED CASE

In this chapter we consider the problem of employing the best applicant when only one approach is permitted, under certain assumptions about the distribution, P. Even though these assumptions are much weaker than those made in Chapter III, many of the results will be analogous to those for the simplified Problem A.

1. SIMPLIFYING ASSUMPTIONS AND PRELIMINARIES

In addition to the assumptions for Problem C given in Chapter I we make:

Assumption 4.1. The distribution P on Ω is such that for each permutation of $I_N, \underline{x} = (x_1, \ldots, x_N)$,

1) the conditional distribution of $\omega_n(j)$ given $\omega(0) = \underline{x}$ is dependent upon x_j alone, and

2) $P(\{\omega \in \Omega : \omega(0) = \underline{x}\}) = 1/N!$.

The above assumption states that the availability of any applicant at any stage may depend upon the applicant's ability, and that the order of interview of applicants is random. We have not assumed either that availabilities are independent between applicants or that unavailable applicants remain unavailable. Such assumptions are unnecessary since only the availability of one particular applicant at one particular stage is ever observed.

We shall set up the solution of Problem C in terms of the Markov chain decision problem in Section 2.1. To establish a suitably small state space, we shall have to proceed more carefully than we did in Section 3.1, because there are no independence assumptions about the vector components of ω.

Suppose the situation is at stage n and relative ranks $X_1(0) = x_1$, $X_2(0) = x_2, \ldots, X_n(0) = x_n$ have been observed. Suppose also that $Y_n = j$, that is $\max\{k \in I_n : x_k = 1\} = j$. Then the expected immediate payoff from making an approach to the candidate is simply the probability that applicant j is both best and available. This is given by

$$(4.1) \qquad P(\{\omega_j(0) = 1, \omega_n(j) = 1\} | \{X_1(0) = x_1, \ldots, X_n(0) = x_n\})$$
$$= P(\{\omega_n(j) = 1\} | \{\omega_j(0) = 1, X_1(0) = x_1, \ldots, X_n(0) = x_n\})$$
$$\times P(\{\omega_j(0) = 1\} | \{X_1(0) = x_1, \ldots, X_n(0) = x_n\})$$
$$= P(\{\omega_n(j) = 1\} | \{\omega_j(0) = 1\}) \; P(\{\omega_j(0) = 1\} | \{Y_n = j\}) ,$$

since this last line follows from both parts of Assumption 4.1. It also follows from part 2) of Assumption 4.1 that

$$(4.2) \qquad P(\{\omega_j(0) = 1\} | \{Y_n = j\}) = n/N , \qquad 1 \leqslant j \leqslant n \leqslant N ,$$

and that

$$(4.3) \qquad P(\{Y_{n+1} = y\} | \{X_1(0) = x_1, \ldots, X_n(0) = x_n\})$$
$$= P(\{Y_{n+1} = y\} | \{Y_n = j\})$$
$$= \begin{cases} 1/(n+1) , & y = n+1 , \\ \\ n/(n+1) , & y = j . \end{cases}$$

It is clear from (4.1) to (4.3) that in order to make an optimal decision at any stage only the position of the candidate need be known.

The state space for the simplified Problem C is the set of pairs (n,j), $n \in I_N$, $j \in I_n$, where the state (n,j) represents the situation of being at stage n with applicant number j being the candidate.

Any policy can be represented by a set of stopping states Δ. To reach the state (n,j), $n > j$, it is clear from (4.3) that the chain must have passed through the states $(j,j),(j+1,j),\dots,(n-1,j)$ without stopping. As with Problem A the only attainable states in Δ are the states $(d(j),j)$ where $d(j) = \min\{n \in I_N : (n,j) \in \Delta\}$ (assuming that $(N,j) \in \Delta$ for each j). As for the simplified Problems A and B we can represent each policy by an N-tuple $d \in D_N$, the statement of the policy being:

at stage n: stop and approach the candidate, Y_n,
if and only if $d(Y_n) = j$.

Corresponding to each $d \in D_N$ there is a stopping variable $\tau^C : \Omega \to I_N$ given by

$$\{\tau^C = 1\} = \{\omega \in \Omega : d(Y_1(\omega)) = 1\}$$

and

$$\{\tau^C \leqslant n\} = \{\omega \in \Omega : d(Y_n(\omega)) = n\} \cup \{\tau^C \leqslant n-1\},$$

$$2 \leqslant n \leqslant N.$$

To complete the formulation of the simplified Problem C in terms of the Markov chain decision problem, we shall need to specify the initial and transition probabilities as well as the expected immediate payoff. These are deduced directly from (4.1), (4.2) and (4.3). The initial state with probability 1 is (1,1). Under the no approach decision the transition probability function is given by

$$f_0((n,j),(n+1,y)) = \begin{cases} 1/(n+1)\ , & y = n+1\ , \\ n/(n+1)\ , & y = j\ , \\ 0\ , & y \text{ otherwise}, \end{cases} \tag{4.4}$$

for $n \in I_{N-1}$, $j \in I_n$. The process terminates under the decision to approach and the expected payoff is, from (4.1) and (4.2), given by

$$\bar{U}(n,j) = n\pi(n,j)/N\ , \qquad n \in I_N,\ j \in I_n\ , \tag{4.5}$$

where $\pi(n,j)$ denotes $P(\{\omega_n(j) = 1\} | \{\omega_j(\underline{0}) = 1\})$, and is the probability that applicant j is available at stage n given that applicant j is the best. The lower triangular matrix, π, will be referred to as a <u>matrix of availability probabilities for the best applicant</u>.

We shall denote by $V_\pi(d)$ the probability of employing the best applicant using policy $d \in D_N$ for the simplified Problem C with N applicants and π, the matrix of availability probabilities for the best applicant. The superscript C will be added when there might otherwise be confusion. We shall also denote the maximum probability of employing the best for the simplified Problem C, by V_π^N.

<u>Theorem 4.1</u>. Let d be any policy in D_N and let $\pi, \bar{\pi}$ be two matrices of availability probabilities for the best applicant.

1) If $\pi \leqslant \bar{\pi}$ then $V_\pi(d) \leqslant V_{\bar{\pi}}(d)$.

2) If $\pi = \alpha\bar{\pi}$ where α is a positive constant, then $V_\pi(d) = \alpha V_{\bar{\pi}}(d)$.

<u>Proof</u>. Now

$$V_\pi(d) = E[\bar{U}(\tau^C, Y_{\tau^C})]$$

$$(4.6) \qquad = E[\tau^C \pi(\tau^C, Y_{\tau^C})]/N ,$$

where the expectation is with respect to the joint distribution of $\tau^C, Y_1, \ldots, Y_N$ and hence does not depend upon π. Both results follow because of the linearity of the expectation operation. □

Corollary.

1) If $\pi \leqslant \bar{\pi}$ then $V_\pi^N \leqslant V_{\bar{\pi}}^N$.

2) If $\pi = \alpha\bar{\pi}$ then $V_\pi^N = \alpha V_{\bar{\pi}}^N$. Also d^N is optimal for π if and only if it is optimal for $\bar{\pi}$.

Proof.

1) $V_\pi^N = \max_{d \in D_N} V_\pi(d)$

$\leqslant \max_{d \in D_N} V_{\bar{\pi}}(d) = V_{\bar{\pi}}^N$.

2) d^N is optimal for $\bar{\pi}$ if and only if

$$V_{\bar{\pi}}(d^N) = \max_{d \in D_N} V_{\bar{\pi}}(d)$$

$$\Longleftrightarrow \qquad \alpha V_{\bar{\pi}}(d^N) = \alpha\left[\max_{d \in D_N} V_{\bar{\pi}}(d)\right]$$

$$\Longleftrightarrow \qquad V_\pi(d^N) = \max_{d \in D_N} V_\pi(d) ,$$

if and only if d^N is optimal for π. Also

$$V_\pi^N = V_\pi(d^N) = \alpha V_{\bar{\pi}}(d^N) = V_{\bar{\pi}}^N . \qquad □$$

An equivalent result for Problems A and B is not true since the distributions of τ^A and τ^B depend upon the matrix of availability probabilities.

For Problem C the matrix of availability probabilities for the best applicant, π, only appears in the payoff function. This means that the elements of π are open to a wider interpretation than merely availability probabilities. Because the problem is unchanged for a positive scale change in the elements of π, one reasonable alternative is that any approach is certain to be successful and $\pi(n,j)$ is instead a discount factor for employing applicant j at stage n given that applicant j is best.

In similar manner to that at the end of section 3.2 we shall now obtain simple recursion formulaé for deriving $V_\pi(d)$ from a given policy d and matrix π.

Let $V_d(n,j)$, for all $j \in I_N$ and all $n \leqslant d(j)$, denote the probability of employing the best applicant starting from state (n,j) and using policy d.

From (4.4) and (4.5) we have for $j \in I_N$ that

(4.7) $$V_d(n,j) = n\pi(n,j)/N\ , \qquad n = d(j)\ ,$$

and

(4.8) $$V_d(n,j) = nV_d(n+1,j)/(n+1) + V_d(n+1,n+1)/(n+1)\ , \qquad j \leqslant n < d(j)\ .$$

Also

(4.9) $$V_\pi(d) = V_d(1,1)\ .$$

Formulae very similar to (3.17), (3.18) and (3.19) can be derived if we rearrange (4.7), (4.8) and (4.9). Let $h_d(n)$, $n \in I_N$, be defined recursively by the following two equations ($h_d(n)$ is equivalent to $v_d(n,0)$ for the simplified Problem A.)

$$h_d(N) = 0\ ,$$

and for $n \in I_{N-1}$,

$$h_d(n) = h_d(n+1) + NV_d(n+1,n+1)/n(n+1) .$$

We now denote

$$v_d(n,j) = NV_d(n,j)/n - h_d(n) ,$$

and substitute into (4.7), (4.8) and (4.9). The equations obtained are

$$v_d(n,j) = \pi(n,j) - h_d(n) , \qquad d(j) = n ,$$

$$v_d(n,j) = v_d(n+1,j) , \qquad j \leq n < d(j) ,$$

and

(4.10) $$V_\pi(d) = [v_d(1,1) + h_d(1)]/N ,$$

together with

(4.11) $$nh_d(n) = (n+1)h_d(n+1) + v_d(n+1,n+1) , \qquad n \in I_{N-1} .$$

The solution then is

(4.12) $$h_d(n) = \frac{1}{n} \sum_{j=n+1}^{N} v_d(j,j) ,$$

(4.13) $$v_d(j,j) = \pi(d(j),j) - h_d(d(j)) ,$$

and

(4.14) $$V_\pi(d) = \frac{1}{N} \sum_{j=1}^{N} v_d(j,j) .$$

Equations (4.12), (4.13) and (4.14) are more complicated than (4.7), (4.8) and (4.9), however they simplify the proof of the next theorem.

Theorem 4.2. Let π be a matrix of availability probabilities for the simplified Problem A and let d be a policy. If $\pi(d(j),j) = 1$ for every $j \in I_N$ such that $d(j) < N$, then $V_\pi^C(d) = V_\pi^A(d)$, and furthermore, if d is optimal for the simplified Problem A then it is optimal for the simplified Problem C.

Proof. Putting in the superscripts A,C where appropriate and using induction, it can be seen by comparing (3.17) and (3.18) with (4.12) and (4.13), that $v_d(n,0) = h_d(n)$ and that for $j \in I_N$,

$$v_d^A(j,j) = \begin{cases} \pi(N,j), & d(j) = N, \\ 1 - h_d(d(j)), & j \leqslant d(j) < N, \end{cases}$$

$$= v_d^C(j,j).$$

Thus by (3.19) and (4.14) we have that $V_\pi^A(d) = V_\pi^C(d)$.
Now for every policy $d' \in D_N$ it can be seen from (3.7), (3.9) and (4.7) that

$$v_{d'}^A(d'(j),j) \geqslant v_{d'}^C(d'(j),j), \qquad j \in I_N$$

since $v_{d'}^A(n,0) \geqslant 0$. It then follows by induction using (3.8) and (4.8) that $v_{d'}^A(1,1) \geqslant v_{d'}^C(1,1)$. Thus we have that $V_\pi^A(d') \geqslant V_\pi^C(d')$ for all $d' \in D_N$. Finally if d is optimal for the simplified problem A then

$$V_\pi^C(d) = V_\pi^A(d) = \max_{d' \in D_N} V_\pi^A(d') \geqslant \max_{d' \in D_N} V_\pi^C(d'),$$

and hence d is optimal for the simplified Problem C. □

The above theorem is not really surprising since for Problem A, d and π are such that only one approach is made. This is because any approach made before stage N is certain to be successful.

2. OPTIMAL POLICIES

The recursion formulae for generating the maximum probability of employing the best applicant, V_π^N, and an optimal policy for the simplified Problem C given a matrix π, will now be derived. These follow from the formulae (2.2) to (2.6) for the Markov chain decision problem in Section 2.1, by substituting from (4.4) and (4.5).

(4.15) $$V_1(n,j) = n\pi(n,j)/N\,, \quad n \in I_N\,, \quad j \in I_n\,,$$

(4.16) $$V_0(n,j) = nV(n+1,j)/(n+1) + V(n+1,n+1)/(n+1)\,, \quad n \in I_{N-1}\,, \quad j \in I_n\,,$$

(4.17) $$V(N,j) = V_1(N,j)\,, \quad j \in I_N\,,$$

(4.18) $$V(n,j) = \max\{V_0(n,j)\,,\ V_1(n,j)\}\,, \quad n \in I_{N-1}\,, \quad j \in I_n\,.$$

Then

(4.19) $$V_\pi^N = V(1,1)$$

and any policy d satisfying both

(4.20) $$V_1(d(j),j) \geq V_0(d(j),j)\,,$$

and

(4.21) $$V_1(n,j) \leq V_0(n,j)\,, \quad j \leq n < d(j)\,,$$

for each $j \in I_N$, is optimal (by the same argument as used in Section 3.3). Denote by D_π^N the set of all such policies, then D_π^N is essentially the set of all optimal policies for Problem C with matrix π.

Again we simplify the above equations by "standardising" the quantities V, V_0, V_1. First though we define $h(n)$, $n \in I_N$, recursively by

(4.22) $$h(N) = 0 ,$$

and

(4.23) $$h(n) = h(n+1) + NV(n+1,n+1)/n(n+1) , \quad n \in I_{N-1} .$$

(The role of h(n) is equivalent to that of v(n,0) in the simplified Problem A.) Denote

$$u(n,j) = NV_1(n,j)/n - h(n) ,$$

$$v(n,j) = NV(n,j)/n - h(n) ,$$

and

$$w(n,j) = NV_0(n,j)/n - h(n) ,$$

for $n \in I_N$, $j \in I_n$. Equation (4.23) becomes

(4.24) $$nh(n) = (n+1)h(n+1) + v(n+1,n+1) , \quad n \in I_{N-1} .$$

Equations (4.15) - (4.19) become

(4.25) $$u(n,j) = \pi(n,j) - h(n) , \quad n \in I_N , \quad j \in I_n ,$$

(4.26) $$w(n,j) = v(n+1,j) , \quad n \in I_{N-1} , \quad j \in I_n ,$$

(4.27) $$v(N,j) = \pi(N,j) , \quad j \in I_N ,$$

(4.28) $$v(n,j) = \max \{u(n,j), w(n,j)\} , \quad n \in I_{N-1} , \quad j \in I_n ,$$

and

(4.29) $$V_\pi^N = [v(1,1) + h(1)]/N .$$

Solving these gives

(4.30) $$v(n,j) = \max_{k=n}^{N} u(k,j) , \quad n \in I_N , \quad j \in I_n ,$$

(4.31) $$h(n) = \frac{1}{n} \sum_{j=n+1}^{N} \max_{k=j}^{N} u(k,j) , \quad n \in I_N ,$$

and

(4.32) $$V_\pi^N = \frac{1}{N} \sum_{j=1}^{N} \max_{n=j}^{N} u(n,j) ,$$

where $u(n,j)$ is given by (4.25). Because $u(n,j) - w(n,j)$ has the same order relationship to zero as $V_1(n,j) - V_0(n,j)$, we have that $d \in D_\pi^N$ if and only if

(4.33) $$u(d(j),j) = \max_{n=j}^{N} u(n,j) , \quad j \in I_N .$$

Formulae (4.30), (4.31) and (4.32) are identical to formulae (3.35), (3.36) and (3.37) respectively. The recursions for Problem A and Problem C only differ in the formulae for $u(n,j)$ - (3.31) and (4.25) respectively. Hence the algorithm for generating V_π^N and an optimal policy for the simplified Problem C will only differ from Algorithm 3.1 in that statement number 4 is replaced by the statement

U = P - S/M.

We shall refer to this modified algorithm as Algorithm 4.1.

Further analysis of the simplified Problem C without any additional assumptions about the structure of π, is unlikely to be rewarding.

3. ELAPSED TIME DEPENDENT AVAILABILITIES: THE C(N,p) - PROBLEM

For the remaining sections of this chapter we shall assume that $\pi(n,j)$ depends on $n - j$ alone. Formally, we shall consider the simplified Problem C under Assumption 4.1 and

Assumption 4.2. The distribution P on Ω is such that the matrix of availability probabilities for the best applicant, π, is given by

$$\pi(n,j) = p(n-j) , \quad n \in I_N , \quad j \in I_n ,$$

where $p = \{p(k)\}_{k=0}^{\infty}$ is a sequence of real numbers, $0 \leqslant p(k) \leqslant 1$.

The sequence p will be referred to as a sequence of availability probabilities for the best applicant. Problem C under Assumptions 4.1 and 4.2 with a total of N applicants and sequence of availability probabilities for the best applicant given by p, shall be referred to as the C(N,p)-problem. As was the case for the A(N,p)-problem we are assuming there is an infinite sequence p since we shall later examine some asymptotic properties. However, there is no monotonicity assumption about p for the C(N,p)-problem.

Where appropriate, the notation will be modified by substituting p for π.

Many of the theorems in this section parallel those in Section 3.4. The next theorem corresponds to Theorem 3.4 and it shows that if an approach is optimal in state (n,j) then an approach is optimal in state $(n+1,j+1)$.

Theorem 4.3. For the C(N,p)-problem let u,w be the quantities as given by (4.22) and (4.24) to (4.28). Then for each N,p

$$u(n+1,j+1) - w(n+1,j+1) \geqslant u(n,j) - w(n,j) , \qquad n \in I_{N-1} , \quad j \in I_n .$$

Proof. Define $t(n,k) = w(n,n-k) - u(n,n-k)$ for $n \in I_N$ and $k \in \{0,1,\ldots,n-1\}$. It follows by (4.28) that

$$v(n,n-k) = u(n,n-k) + t^+(n,n-k) , \quad 0 \leqslant k \leqslant n-1 ,$$

and hence by (4.23) and (4.25) we have that

$$\begin{aligned} h(n) - h(n+1) &= [h(n+1) + v(n+1,n+1)]/n \\ &= [p(0) + t^+(n+1,0)]/n , \quad n \in I_{N-1} . \end{aligned}$$

Thus

$$(4.34)\qquad h(n-1) - 2h(n) + h(n+1) \geq [t^+(n,0) - t^+(n+1,0)]/n .$$

We now prove the theorem by showing by backward induction that $t(n,k)$ is non-increasing in n for each k. Firstly, for $k \in \{0,1,\ldots,N-2\}$ we have that

$$\begin{aligned} t(N-1,k) - t(N,k) &= w(N-1,N-1-k) - u(N-1,N-1-k) + u(N,N-k) \\ &= u(N,N-1-k) - u(N-1,N-1-k) + u(N,N-k) \\ &= p(k+1) + h(N-1) \geq 0 . \end{aligned}$$

Now let us assume for some $n \in \{2,3,\ldots,N-1\}$ that $t(n,k) \geq t(n+1,k)$ for each $k \in \{0,1,\ldots,n-1\}$. For every $k \in \{0,1,\ldots,n-2\}$ we have that

$$\begin{aligned} t(n,k) &= v(n+1,n-k) - u(n,n-k) \\ &= u(n+1,n-k) + t^+(n+1,k+1) - u(n,n-k) \\ &= p(k+1) - p(k) + h(n) - h(n+1) + t^+(n+1,k+1), \end{aligned}$$

and hence

$$\begin{aligned} t(n-1,k) - t(n,k) &= h(n-1) - 2h(n) + h(n+1) \\ &\qquad + t^+(n,k+1) - t^+(n+1,k+1) . \end{aligned}$$

But by the induction hypothesis $t(n,k+1) \geq t(n+1,k+1)$ and hence $t^+(n,k+1) \geq t^+(n+1,k+1)$. Similarly $t^+(n,0) \geq t^+(n+1,0)$ and then by (4.34) we have that $h(n-1) - 2h(n) + h(n+1) \geq 0$. Thus we have shown that $t(n-1,k) \geq t(n,k)$ for each $k \in \{0,1,\ldots,n-2\}$ and the proof of the theorem is complete. □

We can now give a necessary and sufficient condition on p for the policy $d \equiv N$ to be optimal. Obviously condition (3.48) in Theorem

3.5 is sufficient (even if p is not monotone and hence not a sequence for the A(N,p)-problem) since the conditions of Theorem 4.2 are satisfied. However (3.48) is not a necessary condition.

Theorem 4.4. The policy $d \equiv N$ is optimal for the C(N,p)-problem if and only if p is such that

$$p(k) - p(k+1) \leqslant p(0)/(N-1) , \quad k \in \{0,1,\dots,N-2\} .$$

In any case, $V_p(d) = \frac{1}{N} \sum_{k=0}^{N-1} p(k)$.

Proof. d is optimal if and only if

$$V_1(n,j) \leqslant V_0(n,j) , \quad n \in I_{N-1} , \quad j \in I_n ,$$

$$\Longleftrightarrow \quad u(n,j) \leqslant w(n,j) , \quad n \in I_{N-1} , \quad j \in I_n ,$$

which, by Theorem 4.3 is true if and only if

$$u(N-1,j) \leqslant w(N-1,j) , \quad j \in I_{N-1} .$$

But $w(N-1,j) = u(N,j) = p(N-j)$ and by (4.31) $h(N-1) = u(N,N)/(N-1) = p(0)/(N-1)$. Thus d is optimal if and only if

$$p(N-1-j) - p(0)/(N-1) \leqslant p(N-j) , \quad j \in I_{N-1} ,$$

and the necessity and sufficiency of the condition is established. The expression for $V_p(d)$ follows by substitution for $v_d(j,j) = p(N-j)$ into (4.14) . □

By way of Theorem 4.3 we have proved the existence of an optimal policy d which satisfies

$$d(j+1) \leqslant d(j) + 1 , \quad j \in I_{N-1} .$$

(See the argument following Theorem 3.4.). Now if we denote

(4.35) $\qquad r_p = \mathrm{MIN}\ \{n \in I_N : u(n,n) = \max_{k=n}^{N} u(k,n)\}$

we see that there is an optimal policy d for which $d(j) = j,\ r_p \leqslant j \leqslant N$. This follows by Theorem 4.3 which also ensures that r_p is defined by MIN rather than sermin. It often turns out that also $d(j) \geqslant r_p,\ j \in I_{r_p - 1}$, for an optimal policy, that is, there is an optimal policy in E_N. As for the A(N,p)-problem there is a simple closed expression for $V_p(d)$ for any $d \in E_N$. First though we shall define the function g_N on I_N by

$$g_N(n) = \sum_{k=n}^{N-1} 1/k .$$

If $p(0) = 1$, then $g_N \equiv g_p^N$ where g_p^N is given by (3.51).

It should be noted that

(4.36) $\qquad h(n) = p(0)g_N(n),\quad r_p - 1 \leqslant n \leqslant N.$

This is because, by (4.24) and the fact that

$$v(n+1,n+1) = u(n+1,n+1) = p(0) - h(n+1),\quad n \geqslant r_p - 1,$$

we have that

$$h(n) - h(n+1) = p(0)/n,\quad n \geqslant r_p - 1.$$

The solution of this equation subject to $h(N) = 0$ is given by (4.36).

As might be expected the proof of the next theorem closely follows the proof of Theorem 3.6.

Theorem 4.5. Let d be any policy in E_N and let $r = \mathrm{MIN}\ \{j \in I_N : d(j) = j\}$. Then for the C(N,p)-problem

(4.37) $$V_p(d) = \begin{cases} \{p(0)(r-1)g_N(r-1) \\ \quad + \sum_{j=1}^{r-1} [p(d(j)-j) - p(0)g_N(d(j))]\}/N, & r > 1, \\ p(0)/N, & r = 1. \end{cases}$$

Proof. By (4.13) we have that

$$V_d(n+1,n+1) = p(0) - h_d(n+1), \quad n \geqslant r-1,$$

and hence by (4.11), $h_d(n) - h_d(n+1) = p(0)/n$, for $n \geqslant r-1$. The solution with boundary condition $h_d(N) = 0$ gives $h_d(n) = p(0)g_N(n)$, $r-1 \leqslant n \leqslant N$. For $j \in I_{r-1}$, $d(j) \geqslant r$ and hence $v_d(j,j) = p(d(j)-j) - p(0)g_N(d(j))$. But for $r > 1$ we have from (4.12) and (4.14) that

$$V_p(d) = \{(r-1)h_d(r-1) + \textstyle\sum_{j=1}^{r-1} v_d(j,j)\}/N$$

and expression (4.37) then follows. For $r = 1$ expression (4.37) follows directly from (4.10). □

Corollary. If the policy d is also in E_N^*, that is $d(j) = N$, $j \in I_{r-1}$, then for the C(N,p)-problem

$$(4.38) \qquad V_p(d) = \begin{cases} \{p(0)(r-1)g_N(r-1) + \sum_{j=1}^{r-1} p(N-j)\}/N, & r > 1, \\ p(0)/N, & r = 1. \end{cases}$$

Proof. (4.38) follows from (4.37) since $d(j) = N$, $j \leqslant r$. □

If there is an optimal policy for the C(N,p)-problem in E_N, then there is an optimal policy d satisfying $d(j) = j$, $r_p \leqslant j \leqslant N$ and $d(j) \geqslant r_p$, $1 \leqslant j \leqslant r_p - 1$, where r_p is defined by (4.35). V_p^N would then be given by (4.37) with $r = r_p$.

In the next theorem it will be shown that if p is convex then there is an optimal policy in E_N. Again, this theorem parallels Theorem 3.7 and in fact the sufficient condition given there is equivalent to $\log p$ being convex.

Theorem 4.6. If, for the C(N,p)-problem, p is such that

$$p(k-1) - 2p(k) + p(k+1) \geqslant 0\,, \qquad k \in I_{N-2}\,,$$

then there is an optimal policy in E_N.

<u>Proof</u>. Let r_p be as defined by (4.35). Thus there exists an $m, r_p \leqslant m \leqslant N$ such that

$$u(r_p - 1\,,\, r_p - 1) < u(m, r_p - 1)\,.$$

Now for every $j \in I_{r_p - 1}$

$$\begin{aligned} u(r_p - 1, j) &- w(r_p - 1, j) \\ &= u(r_p - 1, j) - \max_{k=r_p}^{N} u(k,j) \\ &\leqslant u(r_p - 1, j) - u(m,j) \\ &= p(r_p - j - 1) - p(m-j) + h(m) - h(r_p - 1) \\ &\leqslant p(0) - p(m - r_p + 1) + h(m) - h(r_p - 1) \\ &= u(r_p - 1, r_p - 1) - u(m, r_p - 1) < 0\,, \end{aligned}$$

since p is convex. It then follows by Theorem 4.3 that $u(n,j) < w(n,j)\,,\; j \in I_{r_p - 1}\,,\; j \leqslant n \leqslant r_p - 1\,,$ and hence every optimal policy d must satisfy $d(j) \geqslant r_p\,,\; j \in I_{r_p - 1}$. But we have already shown by Theorem 4.3 that there is an optimal policy d for which $d(j) = j,\; r_p \leqslant j \leqslant N\,,$ and thus the theorem is proved. □

As for the A(N,p)-problem there may be some savings in computing time from knowing that there is an optimal policy in E_N. Nevertheless r_p and the first $r_p - 1$ components of the policy have to be calculated before the policy is known.

If on the other hand it can be shown that there is an optimal policy in E_N^* then the C(N,p)-problem is virtually solved. Only r_p has to be evaluated and there is usually a simple formula for computing it. It turns out that there is a useful sufficient condition on p for the existence of an optimal policy in E_N^*. In order to derive this condition we shall require the following lemma.

Lemma 4.1. For the C(N,p)-problem let r_p be as given by (4.35). If p is such that

$$(4.39) \qquad u(N-1, r_p - 1) \leqslant u(N, r_p - 1) \Rightarrow u(N-1, j) \leqslant u(N, j) , \quad j \in I_{r_p - 1} ,$$

and

$$(4.40) \qquad u(n, r_p - 1) > u(n+1, r_p - 1) \Rightarrow u(n-1, r_p - 1) > u(n, r_p - 1) , \quad n \in \{r_p, r_p + 1, \ldots, N-1\} ,$$

then the policy

$$d(j) = \begin{cases} j , & r_p \leqslant j \leqslant N , \\ N , & 1 \leqslant j \leqslant r_p - 1 , \end{cases}$$

is optimal.

Proof. It follows from the definition of r_p that

$$u(d(j), j) = u(j, j) = \max_{n=j}^{N} u(n, j) \qquad j \geqslant r_p .$$

Now all that has to be shown is that $u(N,j) = \max_{n=j}^{N} u(n,j)$ for $j \in I_{r_p - 1}$. Let m denote $r_p - 1$. Condition (4.40) implies that $u(n,m) - u(n+1,m)$ changes sign at most once as n goes from m to $N-1$, and then only from positive to non-positive. Thus

$$\max_{n=m}^{N} u(n,m) = \max \{u(m,m) , u(N,m)\} ,$$

and the definition of r_p then requires that $\max_{n=m}^{N} u(n,m) = u(N,m)$.

Consequently

$$(4.41) \qquad u(n,m) \leqslant u(N,m) = w(n,m) , \quad n \in \{m, m+1, \ldots, N-1\} .$$

It now follows by condition (4.39) that

$$(4.42) \qquad u(N-1,j) \leqslant u(N,j) = w(N-1,j) , \quad j \in I_m .$$

Thus, using Theorem 4.3, conditions (4.41) and (4.42) imply

$$u(n,j) \leqslant w(n,j) , \quad j \in I_m , \quad n \in \{j, j+1, \ldots, N-1\}$$

which in turn implies that $u(N,j) = \max_{n=j}^{N} u(n,j)$ for all $j \in I_m$ and the Lemma is proved. □

Theorem 4.7. If, for the C(N,p)-problem, p is such that

$$(4.43) \qquad p(k+1) - 2p(k) + p(k-1) \geqslant 0 , \quad k \in I_{N-2} ,$$

and

$$(4.44) \qquad p(k+1) - 2p(k) + p(k-1)$$
$$\geqslant [p(k-1) - p(k)][p(k) - p(k+1)]/p(0) ,$$
$$k \in \{i \in I_{N-2} : p(0) \geqslant p(i) > p(i+1)\} ,$$

then the policy

$$d(j) = \begin{cases} j , & j \in \{r_p, r_p + 1, \ldots, N\} , \\ N , & j \in I_{r_p - 1} , \end{cases}$$

is optimal. Furthermore r_p is given by

$$(4.45) \qquad r_p = \text{sermin} \{n \in I_N : p(0) \textstyle\sum_{k=n}^{N-1} 1/k \leqslant p(0) - p(N-n)\} .$$

Proof. First we shall show that d is optimal by showing that conditions (4.43) and (4.44) imply conditions (4.39) and (4.40) respectively in Lemma 4.1. Then we shall show that r_p is given by expression (4.45).

As before denote $r_p - 1$ by m. For $j \in I_{N-2}$,

$$u(N-1,j+1) - u(N,j+1) - u(N-1,j) + u(N,j)$$

$$= p(N-j-2) - 2p(N-j-1) + p(N-j)$$

$$\geqslant 0 ,$$

and hence (4.43) implies (4.39) no matter what the value of r_p is. The definition of r_p implies that $u(r_p,r_p) \geqslant u(n,r_p)$, $r_p + 1 \leqslant n < N$, and hence $p(0) - h(r_p) \geqslant p(n-r_p) - h(n)$, which in turn gives

$$(4.46) \qquad p(0) \geqslant p(k) , \quad k \in I_{N-r_p} ,$$

since h(n) is non-increasing in n. Let us now assume that for some $n \in \{r_p, r_p + 1, \ldots, N-1\}$

$$u(n,m) > u(n+1,m) .$$

It then follows that

$$(4.47) \qquad p(n-m) - p(n-m+1) > p(0)/n .$$

(4.47) implies that $p(n-m) > p(n-m+1)$ and since $n-m \leqslant N-r_p$, we have by (4.46) that $p(n-m) \leqslant p(0)$. Now using (4.44) we have that

$$p(n-m+1) - 2p(n-m) + p(n-m-1)$$

$$\geqslant [p(n-m) - p(n-m+1)][p(n-m-1) - p(n-m)]/p(0) .$$

Division of both sides of this inequality by $[p(n-m) - p(n-m+1)]/p(0)$ and rearrangement of the resulting expression yields

$$p(n-m-1) - p(n-m) \geqslant \frac{p(0)}{p(0)/[p(n-m) - p(n-m+1)] - 1},$$

since $p(0) > p(n-m) - p(n-m+1)$. But it then follows using (4.47) that

$$p(n-m-1) - p(n-m) > p(0)/(n-1),$$

and thus $u(n-1,m) - u(n,m) > 0$. Finally, the form of the optimal policy requires that r_p must satisfy

$$\max_{n=j}^{N} u(n,j) = \begin{cases} u(j,j), & r_p \leqslant j \leqslant N, \\ u(N,j), & j = r_p - 1, \end{cases}$$

$$= \begin{cases} p(0) - p(0) \sum_{k=j}^{N-1} 1/k, & r_p \leqslant j \leqslant N, \\ p(N-j), & j = r_p - 1, \end{cases}$$

and thus r_p is given by (4.45). □

<u>Corollary</u>. If p is such that $p(0) > 0$, $p(k)$ is non-increasing for $k \in \{0,1,\dots,N-1\}$ and

(4.48) $$p(k-1)p(k+1) \geqslant [p(k)]^2, \quad k \in I_{N-2},$$

then the policy

$$d(j) = \begin{cases} j, & r_p \leqslant j \leqslant N, \\ N, & 1 \leqslant j \leqslant r_p - 1, \end{cases}$$

is optimal for the C(N,p)-problem.

Proof. Now

$$[p(k-1) - p(k)][p(k) - p(k+1)]$$
$$= p(k)[p(k-1) - p(k) + p(k+1)] - p(k-1)p(k+1) .$$

Hence by (4.48) we have, for $k \in I_{N-2}$, that

$$0 \leqslant [p(k-1) - p(k)][p(k) - p(k+1)]/p(0)$$
$$\leqslant \frac{p(k)}{p(0)} [p(k-1) - 2p(k) + p(k+1)]$$
$$\leqslant p(k-1) - 2p(k) + p(k+1) ,$$

since p(k) is non-increasing. Thus (4.43) and (4.44) hold and the Corollary is proved. □

The above Theorem and Corollary will be used in the next section to show that there are optimal policies in E_N^* for some of the families considered there.

In the latter part of this section we have considered sequences p, for which there is an optimal policy in E_N^*. Obviously there are many sequences for which this is not so, but attempts at further analysis in such cases has not proved fruitful. However Algorithm 4.1 can be used to solve the C(N,p)-problem for any particular case.

4. FAMILIES OF SEQUENCES OF AVAILABILITY PROBABILITIES

In parallel with Section 3.5 we will consider a number of parametric families of sequences p. Some of these families could reasonably apply in practical situations. Others will be considered because of their usefulness in proving some of the asymptotic results in Section 4.5.

The Geometric Family

A sequence of availability probabilities for the best applicant, p, belongs to the geometric family with parameters α,β,γ, $(0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < 1$, $0 \leqslant \gamma \leqslant 1)$ for the C(N,p)-problem if p satisfies

$$(4.49) \qquad p(k) = (\alpha - \gamma)\beta^k + \gamma , \qquad k \in \{0,1,\ldots,N-1\} .$$

This family is particularly important since it arises when the availabilities of an applicant form a Markov chain. Let ϕ,ψ be the transition probabilities for going from being available, unavailable respectively to be being available at the next stage. The corresponding sequence p must satisfy

$$p(k+1) = \phi p(k) + \psi(1-p(k)) ,$$

which with boundary condition $p(0) = \alpha$ has solution (4.49) where $\beta = \phi - \psi$ and $\gamma = \psi/(1-\beta)$. The restriction $\beta \geqslant 0$ is reasonable and is likely to be true in most practical situations.

Theorem 4.8. If, for the C(N,p)-problem, p belongs to the geometric family with parameters α,β,γ, then the policy

$$d(j) = \begin{cases} j , & j \in \{r, r+1, \ldots, N\} , \\ N , & j \in I_{r-1} , \end{cases}$$

is optimal, where

$$(4.50) \qquad r = \text{sermin} \{n \in I_N : \alpha \textstyle\sum_{k=n}^{N-1} 1/k \leqslant (\alpha-\gamma)(1-\beta^{N-n})\} .$$

Furthermore

$$(4.51) \qquad V_p^N = \begin{cases} \alpha/N, & r = 1, \\ \{\alpha(r-1)\sum_{k=r-1}^{N-1} 1/k + \gamma(r-1) \\ \quad + (\alpha-\gamma)\beta^{N-r+1}(1-\beta^{r-1})/(1-\beta)\}/N, & r > 1. \end{cases}$$

Proof: For proving that d is optimal there are 2 cases.

Case (i): $\alpha \leqslant \gamma$. p is non-decreasing and by Theorem 4.3, $d \equiv N$ is an optimal policy. For this case (4.39) gives $r = N$.

Case (ii): $\alpha > \gamma$. $p(0) > 0$ and $p(k)$ is non-increasing. For $k \in I_{N-2}$,

$$\begin{aligned} p(k-1)p(k+1) &= (\alpha-\gamma)^2\beta^{2k} + \gamma(\alpha-\gamma)\beta^{k-1}(1+\beta^2) + \gamma^2 \\ &\geqslant [(\alpha-\gamma)\beta^k + \gamma]^2 \\ &= [p(k)]^2, \end{aligned}$$

since $1+\beta^2 \geqslant 2\beta$. Hence by the Corollary to Theorem 4.7, d is optimal and expression (4.50) follows from (4.45).

Finally the expression (4.51) follows in both cases by Theorem 4.5. □

In the definition of the geometric family β was a non-negative parameter. If β were to have a negative value $p(k)$ would oscillate about the asymptote γ, and the above result would not be true in general. For example, take $N = 3$, $\beta = -\frac{1}{2}$, $\alpha = 0$, and $\gamma = \frac{1}{2}$, hence $p(0) = 0$, $p(1) = 3/4$, $p(2) = 3/8$. It is easy to show that $d = (2,3,3)$ is the only optimal policy and this is not even in E_3.

The Step Family

A sequence of availability probabilities for the best applicant, p, belongs to the step family with parameters α, β, K, $(0 \leqslant \beta < \alpha \leqslant 1,\ K \in I_N)$ for the $C(N,p)$-problem if p satisfies

$$p(k) = \begin{cases} \alpha\,, & k \in \{0,1,\ldots,K-1\}\,, \\ \beta\,, & k \in \{K,K+1,\ldots,N-1\}\,. \end{cases}$$

If β/α is close to 1 then it seems reasonable that $d \equiv N$ is optimal. In fact if $\beta \geqslant \alpha(N-2)/(N-1)$ then for $k \in I_{N-1}$

$$p(k-1) - p(k) \leqslant \alpha - \beta \leqslant \alpha/(N-1) = p(0)/(N-1)\,,$$

and by Theorem 4.4, $d \equiv N$ is optimal and $V_p^N = \{\alpha K + \beta(N-K)\}/N$.

For $\beta < \alpha(N-2)/(N-1)$, the general form of the optimal policy is the same simple form as for the step family for the A(N,p)-problem. However, unless K = 1, there is not a simple closed expression for V_p^N.

Let $p'(k) = p(k)/\alpha$. Now p' is a step sequence, parameters $1,\beta/\alpha,K$, for $A(N,p')$-problem. In equations (3.64), (3.65) and (3.66) the replacement of $v(n,0)$ by $h'(n)$, α by 1 and β by β/α, gives

$$h'(n) = 0\,, \qquad N \leqslant n \leqslant N+K-1\,, \tag{4.52}$$

$$h'(n) = \{(n+1)h'(n+1) + 1 - h'(n+K,0)\}/n\,, \qquad t^N - K \leqslant n \leqslant N-1\,, \tag{4.53}$$

where

$$t^N = \mathrm{MIN}\{n \in \{K,K+1,\ldots,N\} : \ h'(n) \leqslant 1 - \beta/\alpha\}\,, \tag{4.54}$$

and it then follows by Theorem 4.2 (with π given by p') that the policy

$$d(j) \begin{cases} j+K-1\,, & t^N - K + 1 \leqslant j \leqslant N-K\,, \\ N\,, & j \text{ otherwise}\,, \end{cases}$$

is optimal for the $C(N,p')$-problem. It also follows that $V_{p'}^{C}(d) = V_{p'}^{A}(d)$. Finally by Theorem 4.1 and Corollary we have that d is optimal for the $C(N,p)$-problem and that $V_p^N = \alpha V_{p'}^{C}(d)$. Thus by (3.67) (again with v replaced by h', α replaced by 1 and β replaced by β/α) we have that

$$V_p^N = \begin{cases} (t^N - K)\{\alpha h'(t^N - K) + \beta\}/N, & t^N > K, \\ \alpha\{h'(1) + 1 - h'(K)\}/N, & t^N = K. \end{cases} \tag{4.55}$$

Closed expressions for t^N and V_p^N when $K = 1$ follow from Theorem 3.9, using a similar argument. We have

$$t^N = \text{MIN}\ \{n \in I_N : \quad g_N(n) \leqslant 1 - \beta/\alpha\}$$

and

$$V_p^N = \begin{cases} (t^N - 1)\{\alpha g_N(t^N - 1) + \beta\}/N, & t^N > 1, \\ \alpha/N, & t^N = 1, \end{cases}$$

since $g_{p'}^N \equiv g_N$.

The case $K = 1$, $\alpha = 1$, $\beta = 0$ is equivalent to the classical best choice problem.

The Linear Family

A sequence of availability probabilities for the best applicant, p belongs to the linear family, parameters α, β, $(0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant 1/(N-1))$, for the $C(N,p)$-problem if p satisfies

$$p(k) = \alpha(1 - \beta k), \quad k \in \{0, 1, \ldots, N-1\}.$$

The restriction that $\beta \leqslant 1/(N-1)$ is necessary for $p(k) \geqslant 0$.

However it makes the solution of the C(N,p)-problem trivial since for $k \in I_{N-1}$

$$p(k-1) - p(k) = \alpha\beta \leqslant \alpha/(N-1) = p(0)/(N-1) .$$

Therefore, $d \equiv N$ is an optimal policy and $V_p^N = \alpha\{1 - \beta(N-1)/2\}$, by Theorem 4.4.

The Hyperbolic Family

A sequence of availability probabilities for the best applicant p, belongs to the hyperbolic family with parameters α, β, γ, $(0 < \alpha \leqslant 1,\ 0 \leqslant \beta,\ 0 \leqslant \gamma)$, for the C(N,p)-problem if p satisfies

$$p(k) = \alpha/(1+\beta k)^{\gamma}, \qquad k \in \{0,1,\dots,N-1\} .$$

Now $p(0) > 0$, $p(k)$ is non-increasing and for $k \in I_{N-2}$

$$\begin{aligned} p(k-1)p(k+1) &= \alpha^2[(1+\beta k)^2 - \beta^2]^{-\gamma} \\ &\geqslant \alpha^2(1+\beta k)^{-2\gamma} \\ &= [p(k)]^2 . \end{aligned}$$

It follows from Theorem 4.7 and Corollary that the policy

$$d(j) = \begin{cases} N, & r \leqslant j \leqslant N, \\ j, & 1 \leqslant j \leqslant r-1, \end{cases}$$

is optimal, where

$$r = \text{sermin}\ \{n \in I_N : \textstyle\sum_{k=n}^{N-1} 1/k \leqslant 1 - [1 + \beta(N-n)]^{-\gamma}\} .$$

An expression for V_p^N is obtained by substituting for p(k) in (4.38).

<u>The Spike Family</u>

The final family which we shall consider in detail is, for want of a better name, called the spike family. It is unlikely to occur in practical situations but the asymptotic properties are used in some of the theorems in the next section.

A sequence of availability probabilities for the best applicant, p, belong to the <u>spike family</u>, with parameters α,β,K,L, $(0 \leqslant \beta < \alpha \leqslant 1,\ K,L \in I_N,\ K \leqslant L)$, for the C(N,p)-problem if p satisfies

$$p(k) = \begin{cases} \alpha, & k = K-1, \\ \beta, & L \leqslant k \leqslant N-1, \\ 0, & k \text{ otherwise.} \end{cases}$$

It is clear since p(k) for $k \geqslant K$ is non-decreasing and since h(n) is non-increasing that for $j \in I_N$,

$$\max_{n=j}^{N} u(n,j) = \begin{cases} u(N,j), & N-K+2 \leqslant j \leqslant N, \\ \max\{u(j+K-1,j),\ u(N,j)\}, & 1 \leqslant j \leqslant N-K+1, \end{cases}$$

(4.56)
$$= \begin{cases} 0, & N-K+2 \leqslant j \leqslant N, \\ \alpha - h(j+K-1), & s^N - K+1 \leqslant j \leqslant N-K+1, \\ 0, & N-L+1 \leqslant j \leqslant s^N - K, \\ \beta, & 1 \leqslant j \leqslant N-L, \end{cases}$$

where

(4.57) $$s^N = \text{MIN}\ \{n \in \{K,K+1,\ldots,N\} : \alpha - h(n) \geqslant p(N-n+K-1)\}.$$

(MIN rather than sermin since $\alpha - h(n)$ is non-decreasing and $p(N-n+K-1)$ is non-increasing as n goes from K to N-1.) It follows then that the policy

$$d(j) = \begin{cases} j+K-1, & s^N - K + 1 \leqslant j \leqslant N - K + 1, \\ N, & j \text{ otherwise}, \end{cases}$$

is optimal.

Let $p^*(k) = p(k)/\alpha$. By Theorem 4.1 and Corollary, d is optimal for the C(N,p*)-problem and $V_p^N = \alpha V_{p^*}^N$. Now p* is spike parameters $1,\beta/\alpha,K,L$, and so by replacing h by h*, α by 1, and β by β/α, in (4.56), and then substituting into (4.22) and (4.24), we get

(4.58) $$h^*(n) = 0, \quad N-K+1 \leqslant n \leqslant N,$$

(4.59) $$h^*(n) = \{(n+1)h^*(n+1) + 1 - h^*(n+K)\}/n, \quad s^N - K \leqslant n \leqslant N-K,$$

$$h^*(n) = (n+1)h^*(n+1)/n, \quad N-L \leqslant n \leqslant N-K-1,$$

and

$$h^*(n) = \{(n+1)h^*(n+1) + \beta/\alpha\}/n, \quad 1 \leqslant n \leqslant N-L.$$

Also by (4.57) we have

(4.60) $$s^N = \text{MIN}\ \{n \in \{K,K+1,\ldots,N\} : h^*(n) \leqslant 1 - p(N-n+K-1)/\alpha\}.$$

Finally V_p^N is given by

(4.61) $$V_p^N = \begin{cases} \alpha\{h^*(1) + 1 - h^*(K)\}/N, & s^N = K, \\ \{\alpha(s^N - K)h^*(s^N - K) + \sum_{j=1}^{s^N-K} p(N-j)\}/N, & s^N > K, \end{cases}$$

which follows from (4.29) when $s^N = K$, and from (4.30) and (4.31) when $s^N > K$.

s^N and V_p^N can be evaluated by first solving (4.58) and (4.59) recursively for h*(n), and then by using (4.60) and (4.61).

We shall need the following Lemma in the next section.

Lemma 4.2. Let α, β be fixed numbers, $0 \leqslant \beta < \alpha \leqslant 1$, let K,L,N be fixed positive integers, $N \geqslant L \geqslant K$, $N \geqslant 2K+1$, let p'' be a step sequence with parameters α, β, K for the $C(N-K+1, p'')$-problem, and let p be a spike sequence with parameters α, β, K, L for the $C(N,p)$-problem. If $N - t^{N-K+1} \geqslant L - K$, then

$$(4.62) \qquad V_p^N = [1 - (K-1)/N] \, V_{p''}^{N-K+1},$$

where t^{N-K+1} is defined by (4.52), (4.53) and (4.54) with N replaced by $N-K+1$.

Proof. By comparing (4.58) and (4.59) with (4.52) and (4.53) (in which N is replaced by $N-K+1$), it is obvious that $h'(n) = h^*(n)$ for all n such that $\max\{s^N, t^{N-K+1}\} - K \leqslant n \leqslant N$, where s^N is given by (4.60). We shall now show that $s^N = t^{N-K+1}$ and (4.62) will then follow from (4.55) and (4.61) and the fact that $p(N-j) = \beta$, $1 \leqslant j \leqslant t^{N-K+1} - K$, since $t^{N-K+1} \leqslant N+K-L$.
Let us now suppose that $s^N > t^{N-K+1}$. Thus $s^N > K$, $h'(s^N - 1) = h^*(s^N - 1)$ and hence by (4.60) we have that $h'(s^N - 1) > 1 - \beta/\alpha$, since $p(N - s^N + 1 + K - 1) \leqslant \beta$. But the definition of t^{N-K+1} would then imply that $s^N \leqslant t^{N-K+1}$, which is a contradiction. Therefore we have shown that $s^N \leqslant t^{N-K+1}$.
Finally it follows from (4.54) that

$$h^*(t^{N-K+1} - 1) > 1 - p(N - t^{N-K+1} + K)/\alpha,$$

since $p(N - t^{N-K+1} + K) = \beta$ and $h^*(n) = h'(n)$ for $n = t^{N-K+1} - 1$. It then follows by (4.60) that $s^N \geqslant t^{N-K+1}$, and the Lemma is proved. □

Other Families

The only families of sequences considered in detail have been non-increasing, with the exception of the spike family. It is quite possible that sequences which have at least one local maximum will be of practical interest. However, analysis of optimal policies is unlikely to be straightforward for such families. It is clear though, that it is never optimal to approach the candidate in state (n,j) when $p(n-j) < p(n-j+1)$ since $u(n,j) < u(n+1,j)$.

5. ASYMPTOTIC PROPERTIES OF THE C(N,p)-PROBLEM

When N is so large that calculation of an optimal policy and v_p^N for the C(N,p)-problem by means of Algorithm 4.1 is not practicable, consideration of the asymptotic properties of the C(N,p)-problem become important. In this section we shall investigate the convergence of v_p^N and the existence of asymptotically optimal sequences of policies.

Because there is no monotonicity restriction on p, the asymptotic properties of the C(N,p)-problem are more interesting than for the A(N,p)-problem. We shall show that v_p^N converges for every convergent sequence p by first finding the limit for step and spike sequences.

For each sequence of availability probabilities for the best applicant, p, and any non-negative integers m,n, $m < n$, the following notational identities will be used:

$$p_m^n = \sup_{k=m}^{n-1} p(k) \, ,$$

$$p_0 = \sup_{k=0}^{\infty} p(k) \, ,$$

and

$$p^\infty = \lim\sup p = \lim_{m\to\infty} p_m^\infty \, .$$

In the case of convergent sequences we shall also use p^{∞} to denote $\lim_{k\to\infty} p(k)$.

Theorem 4.9. Let α,β be any fixed real numbers, $0 \leqslant \alpha < \beta \leqslant 1$, and let K,L be any fixed positive integers, $K \leqslant L$. If the sequence of availability probabilities for the best applicant, p, is given by either

$$(4.63) \qquad p(k) = \begin{cases} \alpha , & k \in \{0,1,\ldots,K-1\} , \\ \beta , & k \in \{K,K+1,\ldots\} , \end{cases}$$

or

$$(4.64) \qquad p(k) = \begin{cases} \alpha , & k = K-1 , \\ \beta , & k \in \{L,L+1,\ldots\} , \\ 0 , & k \text{ otherwise} , \end{cases}$$

then

$$(4.65) \qquad \lim_{N\to\infty} V_p^N = \alpha \exp(\beta/\alpha - 1) .$$

Proof. (i) Assume p is given by (4.63). Let p' be the sequence given by $p'(k) = p(k)/\alpha$. Thus p' is a step sequence with parameters $1,\beta/\alpha,K$, for every $N \geqslant K$ for both the $C(N,p')$-problem and the $A(N,p')$-problem. Now for every $N \geqslant K$ it follows from section 3.5 that the optimal policy for the $A(N,p')$-problem is

$$(4.66) \qquad d^N(j) = \begin{cases} j+K-1 , & t^N - K + 1 \leqslant j \leqslant N-K , \\ N , & j \text{ otherwise} , \end{cases}$$

where t^N is given by (3.61). It then follows by Theorem 4.2 that d^N is optimal, and $V_{p'}^N$ is identical for both the $A(N,p')$-problem and the

$C(N,p')$-problem. Thus by Theorem 3.10 we have that

$$\lim_{N\to\infty} V_{p'}^N = \nu(1,\beta/\alpha) = \exp(\beta/\alpha - 1) ,$$

and expression (4.65) then follows since $V_p^N = \alpha V_{p'}^N$ by Theorem 4.1 and Corollary. It should also be noted that from (3.77) in the proof of Theorem 3.10 that

(4.67) $$\lim_{N\to\infty} t^N/N = \rho(1,\beta/\alpha) = \exp(\beta/\alpha - 1) .$$

(ii) Now assume p is given by (4.64). Let p'' denote the sequence given by (4.63). For every $N \geq K$ the policy d^N given by (4.66) is optimal for the $C(N,p'')$-problem, where t^N has the property (4.67). It therefore follows that

$$\lim_{N\to\infty} t^{N-K+1}/N = \lim_{N\to\infty} t^N/N$$

$$= \exp(\beta/\alpha - 1) < 1 ,$$

and hence $N - t^{N-K+1} \geq L - K$ for every N sufficiently large. But by Lemma 4.2 we have that

$$V_p^N = [1 - (K-1)/N] V_{p''}^{N-K+1}$$

and thus $\lim_{N\to\infty} V_p^N = \lim_{N\to\infty} V_{p''}^{N-K+1}$. Expression (4.65) then follows for p given by (4.64) since it has already been proved for p given by (4.63). □

Theorem 4.10. For every convergent sequence of availability probabilities for the best applicant, p,

(4.68) $$\lim_{N\to\infty} V_p^N = p_0 \exp(p_0/p^{\infty} - 1) .$$

Proof. There are two cases to consider.

Case (i). Suppose $p_0 = p^{\infty}$, then for each ε, $0 < \varepsilon < p_0$ there is an integer K such that $p(k) \geq p_0 - \varepsilon$ for all $k \geq K$. Define $\underline{p}$ and $\overline{p}$ by

$\overline{p}(k) \equiv p_0$ and

$$\underline{p}(k) = \begin{cases} 0\,, & 0 \leqslant k \leqslant K-1\,, \\ p_0 - \varepsilon\,, & k \geqslant K\,. \end{cases}$$

Clearly $\underline{p} \leqslant p \leqslant \overline{p}$ and hence $V_{\underline{p}}^N \leqslant V_p^N \leqslant V_{\overline{p}}^N$ for all N by Theorem 4.1 and Corollary. It then follows by Theorem 4.4 that $V_{\overline{p}}^N = p_0$ and for $N \geqslant K$ that $V_{\underline{p}}^N = (1 - K/N)(p_0 - \varepsilon)$. Thus

$$p_0 - \varepsilon \leqslant \lim_{N\to\infty} V_p^N \leqslant p_0\,,$$

and (4.68) with $p_0 = p^\infty$ follows since ε is arbitrary.

Case (ii). Suppose now that $p_0 > p^\infty$. Then for each ε, $0 < \varepsilon < p_0 - p^\infty$, there is an integer L such that $|p(k) - p^\infty| < \varepsilon$ for all $k \geqslant L$. Let K denote the largest integer k such that $p(k-1) = p_0$, and it follows that $K \leqslant L$. Now define $\underline{p}$ and $\overline{p}$ by

$$\overline{p}(k) = \begin{cases} p_0\,, & 0 \leqslant k \leqslant L-1\,, \\ p^\infty + \varepsilon\,, & k \geqslant L\,, \end{cases}$$

and

$$\underline{p}(k) = \begin{cases} p_0\,, & k = K-1\,, \\ p^\infty - \varepsilon\,, & k \geqslant L\,, \\ 0\,, & k \text{ otherwise.} \end{cases}$$

Again we have $\underline{p} \leqslant p \leqslant \overline{p}$ and hence $V_{\underline{p}}^N \leqslant V_p^N \leqslant V_{\overline{p}}^N$. Theorem 4.9 then gives

$$p_0 \exp([p^\infty - \varepsilon]/p_0 - 1) \leqslant \lim_{N\to\infty} V_p^N$$
$$\leqslant p_0 \exp([p^\infty + \varepsilon]/p_0 - 1)\,,$$

and (4.68) follows since ε is arbitrary and the exponential function is continuous. □

The condition that p be convergent, while being sufficient for $\lim_{N\to\infty} V_p^N$ to exist, is by no means necessary. Since p is bounded non-convergent sequences oscillate infinitely in some sense. Consider the following example.

Example 4.1. Let p be the sequence given by

$$p(k) = \begin{cases} 1\,, & k \text{ even}\,, \\ 0\,, & k \text{ odd}\,, \end{cases}$$

For the C(N,p)-problem it is clear that

$\max_{n=j}^{N} u(n,j) = \max\{u(N-1,j), u(N,j)\}$, where $u(n,j) = p(n-j) - h(n)$.

Now $h(N-1) = 1/(N-1)$ and hence

$$d^N(j) = \begin{cases} N\,, & N-j \text{ even}\,, \\ N-1\,, & N-j \text{ odd}, \end{cases}$$

is optimal provided $N \geqslant 2$. In this case

$$V_p^N = \frac{1}{N} \sum_{j=1}^{N} u(d^N(j), j) = 1 - [\![N-2]\!]/N(N-1)$$

where $[\![x]\!]$ denotes the integer part of x. Therefore $\lim_{N\to\infty} V_p^N = 1$, and we have V_p^N convergent whereas p is not. □

However it can be shown by the following counter-example that V_p^N does not converge for every p.

Example 4.2. Consider the C(N,p)-problem for the sequence p given by $p(0) = 1$ and

$$p(k) = \begin{cases} 0\,, & (2i-1)! \leqslant k \leqslant (2i)! - 1\,, \\ 1\,, & (2i)! \leqslant k \leqslant (2i+1)! - 1\,, \end{cases}$$

for each $i \in I$. Since $d \equiv (2i+1)!$ is sub-optimal for the $C((2i+1)!,p)$-problem

$$V_p^{(2i+1)!} \geqslant \frac{1}{(2i+1)!} \sum_{k=0}^{(2i+1)!-1} p(k) \geqslant \frac{1}{(2i+1)!} \sum_{k=(2i)!}^{(2i+1)!-1} p(k)$$

$$= 1 - 1/(2i+1)\,.$$

Hence $\lim_{i\to\infty} V_p^{(2i+1)!} = 1$ and thus

$$\limsup_{N\to\infty} V_p^N = 1\,.$$

Now consider the $C((2i)!,p)$-problem. For each $i \in I$ define the sequence ${}_ip$ by

$${}_ip(k) = \begin{cases} 1\,, & 0 \leqslant k \leqslant (2i-1)! - 1\,, \\ 0\,, & k \geqslant (2i-1)! - 1\,. \end{cases}$$

Now ${}_ip$ is a step sequence with parameters $1,0,(2i-1)!$, for both the $C((2i)!,{}_ip)$-problem and the $A((2i)!,{}_ip)$-problem. From section 3.5 it can be seen that the policy d^i given by

$$d^i(j) = \begin{cases} j + (2i-1)! - 1\,, & t^i - (2i-1)! + 1 \leqslant k \leqslant (2i)! - (2i-1)!\,, \\ (2i)!\,, & k \text{ otherwise}\,, \end{cases}$$

is optimal for the $A((2i)!,{}_ip)$-problem, where t^i is given by (3.66) (with the appropriate values of K,N,α,β). But by Theorem 4.2, d^i is optimal for both the A and C problems and $V_{{}_ip}^{(2i)!}$ is also the same for both. If we now follow the proof of Theorem 3.10, it can be seen that

$$\log[(2i)!/t^i] \leqslant 1 \leqslant (1+c_i)\log[((2i)!-1)/(t^i-2)] ,$$

where

$$c_i = (2i)![(2i-1)!-1]/t^i[t^i-(2i-1)!+1] .$$

Hence $t^i/(2i)! \geqslant e^{-1}$ and thus $c_i \leqslant (2i-e^{-1})^{-1}$. Thus $c_i \to 0$ as $i \to \infty$ and hence

$$\lim_{i\to\infty} \frac{t^i-(2i-1)!}{(2i)!} = \lim_{i\to\infty} \frac{t^i}{(2i)!} = e^{-1} .$$

The final argument in the proof of Theorem 3.10 then gives, (even though $K = (2i-1)!$ is not fixed),

$$\lim_{i\to\infty} V_{{}_ip}^{(2i)!} = \nu(1,0) = e^{-1} .$$

But $p(k) \leqslant {}_ip(k)$ for $k \leqslant (2i)!-1$ and hence

$$\lim_{i\to\infty} V_p^{(2i)!} \leqslant \lim_{i\to\infty} V_{{}_ip}^{(2i)!}$$

and we have shown that $\liminf_{N\to\infty} V_p^N \leqslant e^{-1}$. Now for p^* given by $p^*(0) = 1$ and $p^*(k) = 0$, $k \in I$, it is known that $V_{p^*}^N \geqslant e^{-1}$ since the $C(N,p^*)$-problem is equivalent to the classical best choice problem. Thus we have shown that

$$\liminf_{N\to\infty} V_p^N = e^{-1} ,$$

since $V_p^N \geqslant V_{p^*}^N$. □

Because $e^{-1} \leqslant V_{p'}^N \leqslant 1$ for all N and for every sequence p' for which $p'(0) = 1$, Example 4.2 is, in a sense, a worst possible example of a sequence p for which V_p^N is not convergent.

An interesting and apparently difficult problem, is that of finding a characterisation of sequences of availability probabilities for the best applicant, for which V_p^N is convergent. We shall not consider the problem here apart from making the remark that, without going into the details, it would appear that V_p^N converges for any divergent sequence p in which there are bounded numbers of terms between successive local maxima.

In practice it is unlikely that one would meet the situation where p is not convergent. There may of course be a philosophical problem with the existence of an infinite sequence p, although this would not be so if a Bayesian viewpoint is taken.

Asymptotically optimal sequences of policies are of value if the individual policies have simple forms. There are many sequences of asymptotically optimal policies when p is convergent, but the policies considered in our final theorem would appear to be of the simplest form possible. We shall need the following lemma.

<u>Lemma 4.3</u>. Let p be a sequence of availability probabilities for the best applicant and let $K, r \in I_N$ be such that $K \leqslant r$ and $p(K-1) = p_0^N$. Let d be the policy given by

$$d(j) = \begin{cases} j+K-1, & r-K+1 \leqslant j \leqslant N-K, \\ N, & j \text{ otherwise.} \end{cases}$$

Then for the C(N,p)-problem

$$V_p(d) \geqslant \frac{1}{N} \sum_{k=0}^{N-1} p(k) - \left(\frac{N}{r} + \frac{r}{N} - 2\right) p_0^N, \tag{4.69}$$

and provided $r > K$,

$$V_p(d) \geqslant \frac{1}{N} \left\{(r-K)p_0^N \log\left(\frac{N-K}{r-K}\right) + \sum_{k=N-r+K}^{N-1} p(k)\right\}. \tag{4.70}$$

<u>Proof</u>. By (4.11) and (4.13),

$$nh_d(n) = (n+1)h_d(n+1) + p(d(n+1)-n-1) - h_d(d(n+1)) ,$$

and since $0 \leqslant h_d(d(n+1)) \leqslant h_d(n+1)$ and $p_0^N \geqslant p(d(n+1)-n-1)$, we have that

$$(4.71) \qquad nh_d(n+1) + p(d(n+1)-n-1) \leqslant nh_d(n) \leqslant (n+1)h_d(n+1) + p_0^N .$$

Summing of the righthand inequality from n to N - 1 gives

$$h_d(n) \leqslant (N-n)p_0^N/n ,$$

and hence

$$(4.72) \qquad \sum_{n=r}^{N-1} h_d(n) \leqslant (N-r)^2 p_0^N/r .$$

Now from the lefthand inequality of (4.71) we have that

$$h_d(n) - h_d(n+1) \geqslant p(d(n+1)-n-1)/n ,$$

which when summed from n to N - K - 1 gives

$$(4.73) \qquad h_d(n) \geqslant p_0^N \sum_{k=n}^{N-K-1} 1/k + h_d(N-K)$$

$$\geqslant p_0^N \log((N-K)/n) , \qquad r-K \leqslant n \leqslant N-K ,$$

since $p(d(n+1)-n-1) = p(K-1) = p_0^N$ for $r-K+1 \leqslant n+1 \leqslant N-K$.

From (4.14) we have that

$$NV_p(d) = \sum_{j=1}^{N} \{p(d(j)-j) - h_d(d(j))\}$$

$$\geqslant \sum_{j=1}^{N} p(N-j) - \sum_{n=r}^{N-1} h_d(n) ,$$

since $h_d(N) = 0$ and $p(K-1) = p_0^N \geqslant p(N-j)$. (4.69) then follows on substitution of (4.72).

Finally if $r > K$ we have by (4.12) and (4.14) that

$$NV_p(d) = (r-K)h_d(r-K) + \sum_{j=1}^{r-K} p(N-j) ,$$

since $h_d(d(j)) = h_d(N) = 0$ for $1 \leq j \leq r - K$. Relation (4.70) then follows using the inequality (4.73). □

Theorem 4.11. Let p be a sequence of availability probabilities and for N = 1,2,... let

$$K_N = \max \{k \in I_N : p(k-1) = p_0^N\},$$

and

$$r_N = \max \{K_N, \lceil N \exp(p(N-1)/p_0^N - 1) \rceil\},$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. If p is convergent then the sequence of policies d_N, N = 1,2,..., given by

$$d_N(j) = \begin{cases} j + K_N - 1, & r_N - K_N + 1 \leq j \leq N - K_N, \\ N, & j \text{ otherwise,} \end{cases}$$

is asymptotically optimal.

Proof. $V_p(d_N) \leq V_p^N \to p_0 \exp(p^\infty/p_0 - 1)$ by Theorem 4.8 and therefore it will be sufficient to prove that

$$(4.74) \qquad \liminf_{N\to\infty} V_p(d_N) \geq p_0 \exp(p^\infty/p_0 - 1).$$

There are two cases to consider.

First, suppose $p_0 = p^\infty$, then $r_N/N \to 1$, and hence by (4.69) (with r replaced by r_N)

$$\liminf_{N\to\infty} V_p(d_N) \geq \liminf_{N\to\infty} \sum_{k=0}^{N-1} p(k)/N$$
$$= p_0 = p_0 \exp(p^\infty/p_0 - 1).$$

Second, suppose $p_0 < p^\infty$, then there is a $K \in I_N$ such that $p_0 = p(K-1)$

and $p(k-1) < p_0$ for $k > K$. It also follows that $K_N \nearrow K$ and that $r_N/N \to \exp(p^\infty/p_0 - 1)$ since $p(N-1) \to p^\infty$ and $p_0^N \to p_0$. Furthermore $0 < \exp(p^\infty/p_0 - 1) < 1$ and consequently for each $\varepsilon > 0$ there exists an N_ε such that

$$r_N > K_N , \quad N \geqslant N_\varepsilon ,$$

and

$$p(k) \geqslant p^\infty - \varepsilon , \quad k \geqslant N_\varepsilon - r_N + K_N .$$

Thus, by (4.70) (with r replaced by r_N and K replaced by K_N) and for $N \geqslant N_\varepsilon$ it can be seen that

$$V_p(d_N) \geqslant \frac{r_N - K_N}{N} p_0^N \log \left(\frac{N - K_N}{r_N - K_N}\right) + \frac{r_N - K_N}{N} (p^\infty - \varepsilon) .$$

Hence

$$\liminf_{N\to\infty} V_p(d_N) \geqslant (p_0 - \varepsilon) \exp(p^\infty/p_0 - 1) ,$$

since $(r_N - K_N)/N \to \exp(p^\infty/p_0 - 1)$, $\log((N - K_N)/(r_N - K_N)) \to 1 - p^\infty/p_0$, and $p_0^N \to p_0$. But ε is arbitrary and thus (4.74) is established. □

It can be shown that the sequence of policies $\{d_N\}_{N=1}^\infty$, as defined in Theorem 4.11, is not asymptotically optimal in some cases when V_p^N is convergent but p is not convergent. In Example 4.1 $K_N = N$ if N is odd and $K_N = N - 1$ if N is even. Also $r_n = K_N$ and the policy d_N is given by $d_N(j) = N$, $2 \leqslant j \leqslant N$ and $d_N(1) = K_N$. It then follows that $\lim_{N\to\infty} V_p(d_N) = 1/2$, whereas we had shown that $\lim_{N\to\infty} V_p^N = 1$. It may be possible to find a sequence of policies $\{d'_N\}_{N=1}^\infty$ which is asymptotically optimal in the sense that $V_p^N - V_p(d'_N) \to 0$, for every p (including those p for which V_p^N does not converge). However such a sequence is likely to be very complicated and therefore of doubtful value.

As was the case with the A(N,p)-problem the convergence of $V_p^N - V_p(d_N)$ to zero is not uniform over all convergent p. However the example used at the end of Section 3.6 cannot be used here without a small modification. Instead we shall consider p'_N given by

$$p'_N(k) = \begin{cases} 1\,, & k = 0\,, \\ (N-1)/N\,, & 1 \leqslant k \leqslant N-2\,, \\ 0\,, & k \geqslant N-1\,. \end{cases}$$

Now

$$V^N_{p'_N} \geqslant \frac{1}{N} \sum_{k=0}^{N-1} p(k) \geqslant (N-1)^2/N^2\,,$$

and hence $V^N_{p'_N} \to 1$ as $N \to \infty$. For the $C(N,p'_N)$-problem under policy d_N we have $K_N = 1$ and $r_N = \lceil N/e \rceil$. But by the Corollary to Theorem 4.5

$$V_{p'_N}(d_N) = \{(r_N - 1)g_N(r_N - 1) + (N-1)(r_N - 2)/N\}/N\,,$$

provided $r_N \geqslant 2$. Now $r_N/N \to e^{-1}$ and hence $g_N(r_N - 1) = \sum_{k=r_N-1}^{N-1} 1/k \to 1$. Thus

$$\lim_{N\to\infty}\left\{V^N_{p'_N} - V_{p'_N}(d_N)\right\} = 1 - 2e^{-1}\,,$$

and consequently convergence is not uniform over all convergent sequences p.

The remarks made at the end of Section 3.6 also apply here and it seems unlikely that a uniformly asymptotically optimal sequence of policies will have a simple closed form.

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NOTATION INDEX

Number refers to page where symbol is defined.