

Deflection of Light with the Equivalence Principle

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To my family with love.

Mum and Dad who always supported me,

my brother Jonathan,

and God, the strength of my life.

In the beginning God created the heavens and the earth. The earth was without form and void, and the darkness was on the face of the deep; and the Spirit of God was moving over the face of the waters. And God said, "let there be light"; and there was light.

Genesis 1 : 1

Abstract

A thorough treatment of the Strong Equivalence Principle is presented, demonstrating its failure in dealing with non-uniform gravitational fields. In particular, a calculation utilising the equivalence principle is shown to produce an incorrect rate of deflection of light. This calculation is used as a tool to investigate the nature of this deflection, and the meaning of the Strong Equivalence Principle.

Using a generalised metric for outside a static, spherically symmetric gravitational source, it is shown that the failure of the equivalence principle is geometric and not due to any particular choice of metric. When transformed into a displaced rectangular coordinate system, the generalised metric consists of both diagonal and off-diagonal elements. Only the diagonal elements are equivalent to a flat, uniformly accelerating frame. The off-diagonal elements produce non-zero elements in the Riemann Curvature Tensor and are thus attributed to curvature. Therefore, the Strong Equivalence principle is only valid in the weak field limit, where the components of the Riemann curvature tensor vanish. In this case the metric becomes flat, which is the equivalent of a uniform gravitational field.

Using the Schwarzschild metric in displaced rectangular coordinates, the effect of curvature on the rate of deflection of light are determined by tracing the effect of the off-diagonal elements. This calculation shows that only one-third of the deflection rate is due to acceleration in the local inertial frame, with the remaining two-thirds being the result of curvature. Because the rate of deflection is an infinitesimal quantity defined locally, this shows the effects of curvature are important even for local measurements.

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Contents

Figures	vi
1 Introduction	1
1.1 Overview of this Thesis	1
1.2 General References	2
2 Conceptual Foundations	5
2.1 The Galilean Equivalence Principle	5
2.2 Newtonian Gravity	6
2.3 Relativity	8
2.4 Einstein's General Theory of Relativity	12
2.5 Mathematical Notation and Computation	16
3 The Strong Equivalence Principle in a Generalised Gravitational Metric	21
3.1 Introduction and Overview	21
3.2 Derivation of Generalised Metric	22
3.3 Radial Motion in the Generalised Metric	25
3.4 Metric for a Uniformly Accelerating Frame	27
3.5 Generalised Metric in Displaced Rectangular Coordinates	30
3.6 Comparison of Line Elements	33
3.6.1 Result for Schwarzschild metric	35
3.6.2 Result for Conformal metric	36
3.7 Discussion	36
4 Rate of Light Deflection in French Coordinates	39
4.1 The Rate of Deflection of Light	39
4.2 Rate of Deflection of Light in French Metric	40
5 Deflection of Light with the Equivalence Principle	45
5.1 The Schwarzschild Metric in Displaced Rectangular Coordinates	45
5.2 Geodesic Equations with Curvature	49
5.3 Rate of Deflection with Curvature	51
5.4 Addendum: Off Diagonal Terms and Curvature	53
6 Conclusions	55
A Results from MACSYMA	57
A.1 Generalised Gravitational Metric	57
A.2 Uniformly Accelerating Flat Frame Metric	59
A.3 Generalised Gravitational Metric in Displaced Rectangular Coordinates	60
A.4 Schwarzschild Metric in Displaced Rectangular Coordinates	65
References	71

Figures

3.1	Displaced Rectangular Coordinates	28
4.1	Light Deflection Coordinates	40

Chapter 1

Introduction

Einstein's General Theory of Relativity has been successful by all measures that a theory can be successful. It revolutionised our understanding of the universe, by providing a new framework for physics in a unified spacetime, and revealing the very nature of gravitational phenomena. Through the experimental verification of the many extraordinary predictions it has made, the theory has become widely accepted. The exotic nature of the material covered by the theory has made it 'famous' in that it is *well known*, even if not *well understood!*

Although research into fundamental gravitational physics appears to be unfavoured at present, progress in physics is charging ahead so fast, and in so many directions, that the foundations need to be continually re-examined and tested. This work seeks to refine the *Strong Equivalence Principle* of General Relativity in manner which will remove the 'slag' encrusted upon it, and reveal the silver lining. Removing these ambiguities and misconceptions will entail demonstrating that common interpretations of this theory based on applying it to a real gravitational field are flawed. Einstein never intended that the Strong Equivalence Principle to be used in this way.

Any form of the Strong Equivalence Principle attempting to remove spacetime curvature, even infinitesimally, is shown to produce results that are inconsistent. This is demonstrated by the calculation of the rate of deflection of light due to gravity. The silver lining is that these results also enable a clear distinction to be made between the comparative effects of the acceleration of gravity and the tidal forces. This result can be used in calculations to separate contributions from 'curvature' and 'Newtonian acceleration'.

1.1 Overview of this Thesis

Chapter 1 This introduction.

Chapter 2 Basic background material on gravity and general relativity. Concepts are presented and discussed at a level that requires minimal background in physics, yet ensures that any reader will be familiar with the interpretation of these concepts as used in this thesis.

Chapter 3 Derivation of the general gravitational metric external to a static, spherically symmetric mass distribution, showing that it does not satisfy the Strong Equivalence Principle. When transformed into displaced rectangular coordinates, the generalised metric is shown to consist of two distinct parts: The diagonal elements are equivalent to the metric for a flat, uniformly accelerating frame, and the off-diagonal elements are attributable to curvature.

Chapter 4 Calculations of the rate of deflection of light in the Schwarzschild and French metrics, demonstrating the failure of the Strong Equivalence Principle, even for an infinitesimal region.

Chapter 5 Extension of calculations in the preceding chapters, demonstrating that space-time curvature is the cause of discrepancies calculating the rate of deflection of light with the Strong Equivalence Principle.

Chapter 6 Summary of results and conclusions that can be drawn from this thesis.

1.2 General References

It is a common misconception that General Relativity is a topic that is comprehensible only to a few intellectuals, and beyond the understanding of the vast majority of people. This was probably the opinion of a relativist, and the truth may well be that relativity is more beyond the *interest* of the vast majority of people! At any rate it is a misconception whichever way it is looked at, judging by the success of popular science books such as Steven Hawking's *A Brief History of Time* [11], which remained on best seller lists for over 200 weeks.

Many excellent books and papers were researched to provide a good general background for this thesis. Material from the post-graduate course in General Relativity here at the University of Canterbury was particularly helpful, especially the lecture notes of both Dr. W. R. Moreau and Prof. G. E. Stedman. The following is a brief description of additional material that an interested reader would find helpful in pursuing general relativity further.

For an account of relativity, the work of Albert Einstein himself is a natural place to start. Apart from his original papers on the subject he has written some very readable "popular expositions" intended for the people with little or no scientific background. One of these that is immensely useful for the physicist and non physicist alike is *Relativity, the Special and the General Theory; A clear explanation that anyone can understand* [9]. For the historical development of relativity and all his other work there is *Albert Einstein: Autobiographical notes* [10]. English translations of the original papers by Einstein and others developing relativity have been collected together in *The Principle of Relativity* [5].

Asimov's New Guide to Science [1] presents a good historical overview of how our understanding of the universe has developed, as well as explaining important theories and experimental results. Although this is very readable, the treatment of gravitation and relativity is somewhat brief. For a fuller introduction, both Weinberg [29] and Ohanian [24] provide good historical and non-mathematical material in their introductory chapters. They also both go on to give full mathematical accounts of general relativity, its consequences and applications

Readers wanting a non-mathematical account of relativity will find great value in Synge's book, *Talking About Relativity* [28]. This delightful book explains the most basic concepts (including the concept of a concept!), in a thoroughly entertaining manner that even those experienced in the subject should find enlightening. Einstein's popular exposition [9] is also suitable. A novel approach is taken by Stannard in his story-like book, *The time and space of Uncle Albert* [26].

General relativity is a mathematical theory, and for full appreciation the mathematics is unavoidable. Price [25] has written an excellent paper that gives a mathematical account of the development and consequences of the theory but without getting hindered by excessive detail. For more detailed mathematical accounts, the books by Ohanian [24] and Weinberg [29] are both excellent. Weinberg in particular gives a very good mathematical treatment that is both detailed in its coverage and clear in its explanations. Similarly *Relativity: the General Theory* [27] by Synge also excels in this respect, and he also presents his own powerful insight into relativity. Finally, the book *Gravitation* [20] by Misner, Thorn and Wheeler, is an authoritative modern textbook on general relativity. It gives a full treatment of most subjects relating to gravity and general relativity which makes. Although this makes the text somewhat large and cumbersome it is well set out making it an ideal reference book.

These are but a few of a great variety of texts that make the mysteries of gravity and general relativity accessible to readers, irrespective of their background.

Chapter 2

Conceptual Foundations

Gravitation is observable as a mutual attraction between two bodies determined by the *amount of matter* they contain. Theories of gravity such as Einstein's general theory of relativity, seek to provide an explanation of gravity, how it is generated and its effect on the motion of objects. Acceptable theories should seek to predict results that can be observed and measured, as well as provide an understanding of the basic nature of the phenomenon. Any theory of gravitation has a wealth of experimental data to measure up to. To be accepted above its peers a theory must predict results that distinguish it from all other theories. To date, Einstein's General Theory of Relativity has been entirely successful in meeting all experimental tests. It has predicted radically different results to its predecessors, which have been verified experimentally to increasing levels of accuracy as more precise measurements are made. There are still however, alternative theories of gravity that are not currently able to be distinguished from general relativity according to experimental observation. This would indicate that we have yet to achieve a complete understanding of the gravitational phenomena.

In order to understand gravity, in particular the theory of general relativity, it is first necessary to examine carefully the concepts that this theory is founded upon. These are presented through the following section in their historical context, justified through reasoning and examples. The reason for this is that some of these concepts have lost their intended meaning through repeated use, reinterpretation and sometimes misinterpretation. Some have lost their deeper meaning and more subtle implications altogether as a consequence of becoming cliché. For evidence of this, consider the phrase 'equivalence principle' which is used in so many different contexts that the words when used by themselves possess only a fraction of the intended meaning. In addition, some of the founding concepts of these theories may have simply been wrong. Newton went to a great deal of trouble at the start of *Principia* [22] to define all his concepts clearly, and so we shall follow his example. This is not to imply that the reader is expected to be ignorant of these concepts, but rather to emphasise the context they have been used through this thesis.

2.1 The Galilean Equivalence Principle

Simple observation must have revealed at least part of the nature of gravity to any individual curious enough. The first observations of gravity were the tendency of all objects to fall towards the earth. Galileo Galilei (1564-1642) was the probably first to experimentally record that the falling motion produced by gravity was an acceleration and not a constant velocity. This discovery revolutionised ballistics from a hit-and-miss affair to a science that could calculate the distance a cannon ball would travel. Galileo not only discovered and showed experimentally that gravity produced a downwards acceleration,

but that this acceleration was the same for all objects independent of the amount of matter they contain. This was the observation that would lead ultimately to the *principle of equivalence* in its various forms.

Galileo was remarkably innovative in his experimentation. Early in the seventeenth century there were no clocks measuring more accurately than by the hour, so he timed his experiments by measuring the amount of water dripping through a small hole, and used an inclined plane to slow the fall of the objects. Prior to Galileo, Simon Stevinus had shown that all objects take the same time to fall. Stevinus was the one who dropped different objects out of the tower to show they all hit the ground at the same time. However this was not widely accepted against the intuitive concept that heavier objects should fall faster! It was Galileo who actually measured this motion in terms of an acceleration, thus demonstrating to the scientific community the importance of backing up theory with rigorous experiment.

2.2 Newtonian Gravity

Although Galileo was a strong supporter of the Copernican world-view of a heliocentric universe with the planets, including the Earth, orbiting the sun, he did not relate this motion to gravitational phenomena. It was over a hundred years before Isaac Newton (1642-1727) was to make the that vital step in gravitational theory, when he formulated a universal law of gravitation that applied equally to apples and planets! His three laws of motion and the equivalence principle suggested by Galileo's experiments (and his own), led Newton to formulate a theory of a *universal gravitational force*. Although now shown to be too simplistic, his theory of gravity still remains the most widely used and remains the limit that all other theories must reproduce under the simple conditions considered by Newton.

Newton's first law of motion describes *inertia* as a fundamental property of all physical substance. Inertia is defined by this law to be the resistance of an object to a change in its state of motion.

Every body continues in its state of rest, or of uniform motion in a right [straight] line, unless it is compelled to change that state by forces impressed [acting] on it. [2, p.22]

An object in "uniform motion in a straight line" is moving at a constant velocity, that is neither changing speed nor direction of motion. It is important to understand that the first law also refers to an object at constant velocity, and not only to a stationary object that is at rest. In section 2.3 this will be expounded on in a discussion of the *relativity principle*.

The concept of a 'force' must be treated with some caution. To define a force in terms of the change of motion it produces leads to a circular argument in conjunction with the first law. Therefore, instead of producing a mathematical definition of force, consider a definition based hopefully on 'experience' of force. To change the state of motion of an object requires a physical 'push' or 'pull' that can be felt by the observer. This act of pushing or pulling is the transmission of 'force' to the object, and the magnitude of the force transmitted could be measured by, for instance, the pull on a spring. Although by no means a rigorous definition of force, the concept of 'chair' does not require a rigorous definition to enable us to sit in it!

Extending the first law of motion to describe the effect of the application of an external force on an objects state of motion leads to Newton's second law of motion.

The change in motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed. [2, p.23]

This is expressed mathematically through the well known equation *force equals mass times acceleration*, or

$$\mathbf{F} = m_I \mathbf{a}. \quad (2.1)$$

The mass term (m_I) is subscripted to note that it refers to *inertial mass*, distinguishing it from the concept of mass as related to gravity. Acceleration is the rate of change of velocity with respect to time, that is the deviation from uniform motion produced by the application of a force to this mass.

Newton's force law of gravity involves a *gravitational field* that, for each point in space, produces a force on an object according to its *gravitational charge*. The gravitational charge is mass, which will be labelled m_G to distinguish it from the inertial mass of Newton's second law. For Galileo's experiments, the gravitational force equation is then

$$\mathbf{F} = m_G \mathbf{g}, \quad (2.2)$$

where \mathbf{g} is the acceleration due to the gravitational field at the surface of the earth. Inserting this gravitational force into Eq. (2.1) and rearranging gives the acceleration an object experiences at the earths surface due to gravity,

$$\mathbf{a} = \left(\frac{m_G}{m_I} \right) \mathbf{g}. \quad (2.3)$$

Now the results of Galileo's experiment can be fully appreciated. In showing that all objects fall with the same acceleration, he fixed the ratio $\frac{m_G}{m_I}$ as a constant which identical for all objects irrespective of their composition.

So far, the effect of gravity on an objects motion has been discussed but nothing has been said about what generates gravity, the source of the gravitational field. For this Newton used his famous third law of motion.

To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. [2, p.23]

Thus the gravitational force exerted by the earth on a falling object, is matched by an equal force exerted by the falling object on the earth. Reasoning from this, Newton determined that gravity was a mutually attractive force between objects that was proportional to the product of their masses. Therefore, mass was also the source of the gravitational field.

To complete his theory of a universal gravitational force, Newton needed to determine how the strength of gravity varied with distance. The clues for this came from astronomy. Initially, Newton used observations of the moon to calculate the gravitational acceleration at the moons orbit. Knowing the radius of the moons orbit compared to the radius of the earth he was able to deduce that the gravitational force was proportional to the inverse square of the distance. However this required treating the earth as if all its mass

were at one point at the centre, which Newton later proved was valid for an inverse square law. Then Newton went on to show that this inverse square law was required to explain the laws of motion for the planets in the solar system as determined by Johannes Kepler (1571-1630).

Newton now had the first complete and universal theory of gravity, with the gravitational force exerted on mass m_1 at position \underline{r}_1 by m_2 located at \underline{r}_2 given by

$$\mathbf{F}_{12} = +G m_1 m_2 \frac{(\underline{r}_2 - \underline{r}_1)}{|\underline{r}_2 - \underline{r}_1|^3}. \quad (2.4)$$

The constant G was determined experimentally through observation of motion of the moon and planets within the solar system, but proposed to be universal. Indeed, subsequent results have shown this to be the case, for double stars orbiting each other, and even whole galaxies dancing in time to Newton's gravitational score.

Through the remainder of the seventeenth and eighteenth centuries, Newton's gravitational theory triumphantly matched increasingly precise measurements made by astronomers. Perhaps the most spectacular success of the theory was in 1846. John Adams and Urbain Le Verrier independently used irregularities in the orbit of Uranus to predict the existence and precise position of a new planet, to be named Neptune. However, at the same time, Le Verrier had used an irregularity in the orbit of Mercury to predict another planet, closer to the sun. This planet was never discovered, and in the following years there was no suitable explanation found for the orbit of Mercury in the context of the Newtonian theory. This enigma had to wait until Einstein's theory to be resolved.

Although ultimately superseded by Einstein's theory of general relativity, Newtonian gravity will always remain as the fundamental gravitational theory used as a measure of all others. In terms of understandability and ease of computation it is unmatched and still widely used. Only under extreme conditions of high velocities, large scales, or strong gravitational fields must an alternative gravitation theory be sought. For conditions prevalent in most of the universe it is sufficient to explain the gravitational interaction. It is a tribute to the ingenuity of Newton in making the most striking conceptual leaps in our understanding of gravity, that his theory should remain at the foundations of any modern gravitational theory.

2.3 Relativity

It is perhaps unfortunate, that relativity has almost become synonymous with the work of Albert Einstein (1879-1955), for although he is indeed the founder of modern relativity, there were theories of relativity in use by Newton and Galileo. The concept of relativity must be at the heart of any theory of how objects move, for any motion must be described with reference something else. To say a car is travelling at 50 km/h has meaning only because it is assumed the motion is in reference to the surface of the road. However, the surface of the earth is not at rest, but is rotating as the earth turns about its axis, and the whole earth is moving around the sun at 29.8 km/sec. It would be equally valid to describe the motion of the car as 29.8 km/sec, as long as it was noted that this velocity was *relative* to the sun! Relativity is certainly a concept that is widely used and appreciated outside of Einstein's theory.

Describing motion relative to another object such as the earth or the sun is satisfactory only if there is some way to compare these measurements. Consider two statements

of a car's velocity, one relative to the earth and one relative to the sun. How can it be determined if these two statements are in agreement? This would require a way of translating the car's motion relative to the earth into a motion relative to the sun. In the relativity of Galileo and Newton, such a translation is simple. The motion of the *car relative to the sun* was given by adding the motion of the *earth relative to the sun*, to the motion of the *car relative to the earth*.

When motion is described relative to a particular object, it is making reference to a *coordinate system* or *reference frame*. Any position on the surface of the earth requires two coordinates to distinguish it from all other positions because the surface of the earth is *two-dimensional*. These coordinates might be latitude and longitude, or a distance and direction from another defined point, such as "100km south of Christchurch". Both these coordinate systems are defined with reference to the earth and move with the surface of the earth, they are *rest frame* coordinates of the earth. There are similar coordinate systems defined for the rest frame of any other object, such as our moon, our sun, or our galaxy. To specify the position of a flying object or an orbiting object requires a third coordinate to describe the height of the object. In the universe according to Newton's laws of motion there are three spatial coordinates required to specify a position in a coordinate system at each instant of time.

The essence of relativity is that motion can be described equally validly in any coordinate system, and there are coordinate transformation laws that translate motion between coordinate systems. Now an important question arises. Are all coordinate systems equal, or are there some coordinate systems that possess properties which make them preferential above others? The short answer is no, not all coordinate systems are equal. There is a special class of coordinate systems called *inertial frames*. These are coordinate systems where Newton's three laws of motion (section 2.2) are always and everywhere true. What is perhaps more surprising is that there are coordinate systems in which Newton's laws do not hold, such as the class of accelerating frames.

Consider a coordinate system defined to coincide with a moving car. When the car accelerates, in that coordinate system an observer will feel a force pushing them into the car seat. This is not an external force, such as gravity, but merely a consequence of the fact that their reference frame (the car) is accelerating. This can also be applied to a reference system that is rotating, such as a roundabout. In such a coordinate system an observer feels a force called the centrifugal force which appears to push them outwards from the centre. Again this is not a true force but a consequence of the fact that a rotating frame is another form of accelerating coordinate system.

What then makes a particular set of frames inertial? Newton's first law of motion provides part of the answer. Any coordinate system in "uniform motion in a straight line" with an inertial frame will also be inertial. Mathematically this can be expressed by the Galilean transformation equations from one inertial frame K to another K' moving with relative velocity v along the x axis,

$$\begin{aligned}x' &= x - vt \\t' &= t.\end{aligned}\tag{2.5}$$

These equations are not the most general equations for the Galilean transformation, the full transformation equations also allow for the second frame to be rotated by an arbitrary angle. This *rotated* coordinate system is not to be confused with a uniformly *rotating*

coordinate system, the later being non inertial as described previously. However, the above equations could also be used to show that if K was not an inertial frame, then K' would also not be inertial. What exactly is it then, that determines the difference between these inertial and non-inertial frames?

Newton concluded that there must exist an absolute space which defined the fundamental inertial frame. Quoting Newton from Weinberg's book.

Absolute space, in its own nature and with regard to anything external, always remains similar and unmovable. Relative space is some movable dimension or measure of absolute space, which our senses determine by its position with respect to other bodies, and is commonly taken for absolute space. [29, p.15]

This concept of an absolute space that existed independent of any physical object in the universe was debated long after Newton's death, with arguments on both sides largely philosophical.

Over two centuries after Newton's published his theory, Ernst Mach (1836-1916) rejected the Newtonian concept of an absolute space and replaced it with a new hypothesis known now as *Mach's Principle*.

The inertial forces should not be regarded as indicating motion in absolute space, but rather as indicating motion relative to the masses in the entire universe. [24, p260]

This means that inertial frames are those that are related to the centre-of-mass frame of the entire universe by the transformations of Galilean relativity. Mach's arguments [13] were later important to Einstein in his development of the special theory of relativity.

The principle of relativity as discussed so far has only required that Newton's laws of motion be valid in all inertial coordinate systems. It would be a natural generalisation to extend this principle to state that all the laws of physics should hold in any inertial frame, and this is what Einstein wanted to achieve in his *Special Theory of Relativity*. Unfortunately it was not possible to extend this generalisation into the laws of electromagnetism. In 1864, James Clark Maxwell(1831-1879) had developed a theory of electromagnetism [17] where electro-magnetic waves (light) travelled always at a constant velocity. That is the speed of light (c) in a vacuum was always constant.

Einstein commented in his autobiography how this guided him towards his special theory of relativity.

After ten years of reflection such a principle [special relativity] resulted from a paradox upon which I had already hit at the age of sixteen: If I pursue a beam of light with the velocity c (velocity of light in a vacuum), I should observe such a beam of light as an electromagnetic field at rest through spatially oscillating. There seems to be no such thing, however, neither on the basis of experience nor according to Maxwell's equations. From the very beginning it appeared to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer who, relative to the earth, was at rest. For how should the first observer know, or be able to determine, that he is in a state of fast uniform motion. [10, p.49]

All known examples of wave motion at the time required a medium through which to propagate. Therefore Maxwell proposed the "luminiferous ether" as the medium through

which light propagated in a vacuum. Maxwell's equations would then only be valid in coordinates at rest with respect to this "ether". This proposition was easy enough to verify experimentally since it was known that the earth was travelling at 29.8km/sec with respect to the sun, and considerably faster taking into account galactic motion. Using the Galilean transformation equations (Eq.2.5) with the relative motion of the Earth in the ether, light in this direction should be travelling slower with respect to the Earth. By simultaneously measuring the velocity of light both parallel and perpendicular to the direction of the earths orbital motion, there should be found a discrepancy in the two velocities. Several such experiments were carried out, the most well known being those of A.A. Michelson (1853-1931) and E.W. Morley [18]. It was perhaps one of the most surprising and unexpected series of experimental results of all time that no motion relative to the ether was ever detected.

Although there were a series of proposed explanations involving the earth 'dragging' the ether along in its orbit, none of these were particularly successful. Einstein came to the ultimate conclusion that the vacuum speed of light was a constant, independent of the motion of source or observer. No matter how fast you are travelling, you will always measure the speed of light as the same value.

... light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the source. [6, p.38]

This was one of the two fundamental postulates for Einstein's Special Theory of Relativity. The second was the *principle of relativity*.

If, relative to K , K' is a uniformly moving co-ordinate system devoid of rotation, then natural phenomena run their course with respect to K' according to exactly the same general laws as with respect to K . [9, p.13]

However, these two postulates appeared to contradict each other since the first required a constant speed of light and the second required it to change between inertial frames according to the Galilean transformation laws of Eq.2.5. This dilemma was finally resolved in Einstein's famous 1905 paper. [6]

What was needed was to reject the Galilean transformations, (Eq.2.5) in favour of new transformations that would allow the vacuum speed of light to remain constant in all inertial frames. Such transformation laws had already been derived by Hendrik Antoon Lorentz (1853-1928) as an explanation for the ether experiments. The Lorentz transformations from inertial frame K to a frame K' moving at relative velocity v along the x axis are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (2.6)$$

$$t' = \frac{t - \frac{v^2}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (2.7)$$

where c is the vacuum speed of light. Using these transformations to transform from one inertial frame to another, there is no incompatibility between the relativity principle and the law of constant propagation of light.

Einstein's Special Theory of Relativity added an appealing symmetry to the laws of physics. They do not distinguish between any coordinate systems in uniform relative

motion. That is, there is no preferred inertial frame that is more ‘at rest’ than others. In particular there is no ether with respect to which the vacuum speed of light is constant, rather this speed is a constant in all inertial frames independent of their state of motion. This provided a far more satisfactory explanation of the Michelson Morley experiment. In addition the theory made some predictions that were quite startling, such as a relationship between energy and mass,

$$E = mc^2. \quad (2.8)$$

This has become one of the most famous physics equations in history. All these predictions have now been experimentally verified with extraordinary precision, and Einstein’s special relativity is indispensable to our understanding of classical mechanics.

2.4 Einstein’s General Theory of Relativity

There were two things about special relativity that Einstein considered unsatisfactory. Firstly, Mach’s critique of the privileged place of the inertial frame in Galilean relativity was equally valid in Einstein’s theory. What is the special property of inertial frames that enables the laws of physics to distinguish them from non inertial frames? Secondly, Einstein had been unable to incorporate gravitational phenomena within the framework of special relativity. Both these dilemmas were resolved by his *General Theory of Relativity* [8].

It is important to distinguish between the relativity *principle* and Einstein’s *theory* of relativity. The relativity principle was only a postulate about a property of the laws of physics that seemed to be desirable, but was not necessarily true. Einstein’s Special Theory of Relativity was a justification that the relativity principle was indeed valid, and did not conflict with other physical laws such as the constancy of the velocity of light.

Einstein felt it was necessary to generalise the scope of the relativity principle from inertial frames to all reference frames, postulating the general relativity *principle*.

All bodies of reference K , K' , etc., are equivalent for the description of natural phenomena (formulation of the general laws of nature), whatever may be their state of motion [9, p61].

To justify this principle he proposed the General Theory of Relativity, which effectively challenged the Newtonian concepts of the very space that the laws of physics play in. There was no privileged coordinate systems as the coordinates themselves were just labels patched onto space in order to describe relative motion. Gravitational phenomena arises as a natural consequence of general relativity, whereas special relativity deliberately excludes the effects of gravity.

The link between Gravitation and relativity is first hinted at through the equivalence experiments of Galileo. Subsequent experiments have verified the equality of inertial and gravitational mass to great precision [30, Table 2.2, page 27], so Einstein postulated an exact equivalence as his Weak Equivalence Principle. This would make it impossible to determine whether the acceleration at a particular point was due to the coordinate system accelerating or the presence of a gravitational field. It is not just a matter of being unable to distinguish this through observation, Einstein said *inertia and gravity are the same in essence*.

An illustration of this is the following thought experiment. Inside a stationary elevator the force of gravity is transmitted through the floor as a sensation of weight. When the elevator accelerates as it moves up or down, that acceleration is felt through the floor as a sensation of increased or decreased weight, depending on whether the elevator is moving up or down. If the elevator was to be accelerated downward with the same acceleration as the gravitational acceleration, the observer would experience weightlessness just as if the elevator had been released and was in free fall. In a uniform gravitational field there would be no experiment within the elevator that could determine whether the elevator was stationary in the earth's gravitational field, or being uniformly accelerated in space with a negligible gravitational field. In fact the two cases are identical, and it would be possible to "generate a gravitational field" by transforming to an accelerating coordinate system. Alternatively, by constructing a gravitational field to counter the acceleration in a non inertial frame it is possible to turn it into an inertial frame. The concept of an inertial frame is therefore meaningless, and it is natural to extend the relativity principle to hold for all coordinate systems.

The 'elevator experiment' is a well known example of Einstein's *Strong Equivalence Principle*, and was used by Einstein [9] in his book *Relativity*. In essence the Strong Equivalence Principle is that a *uniform* gravitational field is locally equivalent or identical to uniform accelerating frame. This will be discussed further, and is the major subject of chapter 3.

However, gravitational fields are not uniform but fall off as the inverse square of the distance (Eq.(2.4)). Therefore an object near the floor of the lift will experience a slightly ($\sim 10^{-6}$ m/sec²) greater acceleration than an object near the roof. This difference in acceleration is known as the *tidal* effect, because it is responsible for the earth's tides. (Caused by a slight difference in the gravitational acceleration due to the moon on the near and far sides of the earth). Tidal distortions will always allow gravity to be distinguished from the effects of an accelerated reference frame and so provide a means of identifying a gravitational field.

General Relativity is a theory of *geometrodynamics*, a composition of the words geometry and dynamics. Geometry is a branch of mathematics that deals with the relationship between points in terms of distances and angles. Dynamics is a branch of physics that deals with motion and forces. Combining the two, general relativity is the relationship between the geometry of space and time and the motion of objects through space and time, a combination of physics and mathematics. The relationship is a mutual one, the geometry effects the motion of bodies, which in turn determine the geometry.

One of the first things that Einstein did with General Relativity was to discard the notion of treating the coordinates space and time as different entities, each requiring separate equations. This was due to a revolutionary theory by H. Minkowski that treated space and time on an equal footing.

Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union with the two will preserve an independent reality. [19]

Instead of a three dimensional space and a one dimensional time, there is only a 4-dimensional spacetime. All laws of physics should be expressed in a way that is consistent with this spacetime.

A requirement of Minkowski's theory was the universal constant c (speed of light in a vacuum), which enabled a link between the space and time coordinates. Multiplying time t by c , gives the time coordinate the dimension of length, the same unit as the spatial coordinates. From now on, the time coordinate will be taken to refer to ct and *spacetime* will refer to the four dimensional geometry of the universe. *Space*, on the other hand will be taken as a mathematical generalisation of an arbitrary geometry in any number of coordinates.

Geometry is concerned with relationships involving the angles and distances between points. An example of a familiar geometric statement is "*the sum of the interior angles in a triangle is equal to twice the angle of a right angle*". This is geometric because it is independent of the size or orientation of the triangle, to what precision or with what units the angles are measured. The same goes for another geometric statement, known as Pythagoras' theorem. "*In a right-angled triangle, the length of the hypotenuse squared is equal to the sum of the squares of the other two sides.*" All geometric statements such as these could be deduced from a small set of fundamental postulates that were set out by Euclid in the third century BC. These remained unchallenged as a definition of geometry for over 2000 years, and the geometry they describe is known as Euclidean geometry.

Consider a triangle drawn, not on a flat piece of paper but on the surface of a sphere, using the same reference coordinates of latitude and longitude as for the surface of the earth. Make the first point of this triangle at the north pole and let the first two sides be lines of longitude at right angles, extending down from the north pole to the equator which will compose the third side. All three lines are straight with reference to the surface of the sphere, and therefore compose a right angled triangle on the surface of a sphere. However careful consideration will reveal that the two angles formed on the equator will also be right angles, because all lines of longitude cross the equator at right angles. If all three interior angles of the triangle are right angles, the sum of them is obviously not equal two right angles as stated above as a requirement Euclidean geometry. Similarly, the sum of the squares of the sides can not obey Pythagoras' theorem because all three lines extend exactly $1/4$ the circumference of the sphere, are therefore the same length. The surface of a sphere is an example of a non-Euclidean geometry.

Geometry is a fundamental property of the space, completely independent of the coordinates, which are just labels. Mathematically, the geometry is determined for each point in a space by something called the *metric*. When combined with a set of coordinates, the metric can be used to give distances in a similar way to Pythagoras theorem in flat space. There is no way that the surface of a sphere can be coordinatised to make it obey the laws of Euclidean geometry. This is illustrated by the inability to produce a world map that accurately depicts the sizes and shapes of all the countries on it. Various different kinds of projection can be used, but all of them distort the true picture such as that given by a globe. The reason for this is the map represents a Euclidean geometry and the surface of a sphere is non-Euclidean. It is not possible use coordinate transformations to transform a non-Euclidean geometry into Euclidean geometry.

Euclidean spaces have a unique property of being *flat*, where all non-Euclidean spaces are *curved*. This property of curvature is a mathematical one, but is not too far removed from our conceptual understanding of curvature. Mathematically, curvature is determined at each point in a space by something called a Riemann curvature tensor, which will be defined in section 2.5. The surface of a sphere is an example of constant curvature because the curvature is the same at all points, however it is also possible to have curvature that

is dependent on position, such as when the surface of a trampoline is deformed by several weights. There are therefore an infinite number of possible non-Euclidean surfaces, and mathematically curvature can be extended to spaces of any number of dimensions.

In a flat space it is obvious what is meant by a *straight line*. However in a curved space the concept of the *straightest* line is somewhat vague and so will need a more rigorous definition. The straightest lines in curved space (or a straight line in flat space) are known as *geodesics*. These will be defined mathematically in section 2.5, but a suitable conceptual definition is that a geodesic is the shortest distance between two points. Note that there may be an infinite number of such lines, such as between the two poles of a sphere there may be an infinite number of such lines, all of which are equally valid geodesics.

Special relativity, and all other theories in physics had been made under the assumption that the geometry of spacetime was Euclidean. This assumption had not been made completely naively, as any substantial curvature in spacetime would have been noticeable in experiments. However, curvature of the surface of the earth only becomes obvious on maps that cover distances comparable to the radius of the earth, on much smaller scales such as street maps the effects of curvature are not noticeable. In the same way, spacetime curvature may not be noticeable over small scales, or where the curvature was slight.

All the pieces of the puzzle are now assembled and all that remains is to assemble them to get the full picture! There are a few key pieces that required the exceptional insight of Einstein to even think of joining them together. The key to the General Theory of Relativity is that gravity is a geometric phenomena. Motion in spacetime is always along "straight lines" (geodesics) unless acted on by an external force. Gravity is not an external force, it is a consequence of the fact that spacetime is not flat but curved. Freely falling objects, such as the earth, follow geodesics in spacetime. These geodesics appear curved to us because we still perceive motion in terms of space and time separately. The earth's orbit is only an ellipse in spatial coordinates, in spacetime it is a straight line.

Mass is the source of gravity, and therefore the presence of mass is the cause of spacetime curvature. Because of the relation between energy and mass given in Eq.(2.8) it is easy to show (see [25, p.304]) that the source of gravity is not just rest mass, but all forms of mass and energy (mass-energy). Mass-energy determines the curvature through a set of field equations which Einstein proposed, although his field equations are not unique. Using the field equations with a specified distribution of matter (such as a sphere), the metric, and hence the geometry is determined. This in turn determines the motion of objects, which travel along the geodesics. Moving matter changes the mass distribution and the field ... and so on it goes. The essence of Einstein's General Theory of Relativity in a single statement would be.

Matter tells space how to bend, space tells matter how to move!

One of the limitations imposed by special relativity is that nothing can travel faster than the speed of light, including gravitational effects. General relativity predicts that *gravitational waves* will be propagated at the vacuum speed of light. [25]

We conclude this section by summarising the essence of Einstein's General Theory of Relativity in a single statement. Free objects move along geodesics in a four dimensional spacetime that is curved by the presence of mass-energy.

2.5 Mathematical Notation and Computation

Several references have been made in the preceding sections to mathematical concepts and definitions. Mathematics in general relativity is unavoidable in a thesis such as this, and so a brief mathematical introduction will here be presented. For better and increasingly more detailed accounts read Price [25], Weinberg [29], or MTW [20]. One of the goals of Einstein in general relativity was to present a mathematical framework that would allow the laws of physics to be written *covariantly*. This means written using a four dimensional spacetime notation that is independent of the coordinatisation. A position in this notation is given by a vector with four components called a 4-vector. In order for measurements to be made on spacetime, some sort of coordination must be introduced. But the coordinates are just labels and have no geometrical significance. Changing coordinates does not effect any geometric properties of spacetime such as curvature, distance and angle measurements. For example, the two dimensional Euclidean plane is still the same regardless of whether it is labelled with rectangular or polar coordinates.

Coordinate labels in spacetime will be referred to by name or through a superscript, for example a four dimensional space time with $\mu = \{0, 1, 2, 3\}$ could be labelled

$$\{ct, x, y, z\} \equiv \{x^0, x^1, x^2, x^3\} \equiv x^\mu. \quad (2.9)$$

Convention being that x^0 will always refer to a time-like coordinate and $\{x^1, x^2, x^3\}$ to spatial coordinates. The time-like coordinate is multiplied by c , the vacuum speed of light which is the fundamental constant of Special Relativity. It is often advantageous to transform from one coordinate system to another in order to take advantage of the symmetry of a particular problem, and thus simplify the calculations. Coordinate transformations are an important component of Einstein's Strong Equivalence Principle, and this thesis. Measurements can only be compared when they are referenced to the same coordinate system.

In section 2.4 geometry was associated with an object called the *metric* that determined geometric properties such as distance and angles. Given two 4-vectors, the metric is defined at each point in spacetime to return the *line element* which is a generalisation of the scalar product. For example, the scalar product for a distance dS in two dimensional Euclidean space with coordinates $\{x, y\}$ is

$$(dS)^2 = (dx)^2 + (dy)^2, \quad (2.10)$$

coming directly from Pythagoras' Theorem with no $dx dy$ cross-terms. Generalising Eq. 2.10 for any two dimensional space with coordinates x^μ and $x^\nu = \{x^1, x^2\}$ gives

$$\begin{aligned} (dS)^2 &= g_{11}(dx^1)^2 + 2g_{12}(dx^1)(dx^2) + g_{22}(dx^2)^2 \\ &= g_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \quad (2.11)$$

where $g_{\mu\nu}$ is the *metric tensor*. Repeating an index both as a subscript and superscript implies summing over all values of that index according to the *Einstein summation convention* [8]. From this definition it can be seen that the metric is *symmetric* ($g_{\mu\nu} = g_{\nu\mu}$). This is because the line element is independent of the order of the mixed derivatives, $(dx^1)(dx^2) = (dx^2)(dx^1)$. In the case of the Euclidean space in (Eq. 2.10) it is clear that $g_{11} = g_{22} = 1$ and $g_{\mu\mu} = 0 (\mu \neq \nu)$.

For our four dimensional spacetime, with $x^\mu = \{x^0, x^1, x^2, x^3\}$, the metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}, \quad (2.12)$$

where the symmetry has been explicitly stated in the labelling. The line element for Eq.(2.12) can be calculated using Eq.(2.11), but would be quite long with the number of cross terms. However the line element can still be expressed $(dS)^2 = g_{\mu\nu}dx^\mu dx^\nu$ thus illustrating the convenience of covariant notation.

An important spacetime metric is the Minkowski metric

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.13)$$

which is the metric for flat spacetime in rectangular coordinates. The line element or 'distance' interval for Minkowski spacetime is the *proper length*

$$(dS)^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \quad (2.14)$$

This is said to be *spacelike*, *light-like* (or *null*), or *time-like* depending on whether $(dS)^2$ is positive, zero, negative, respectively. An example of a spacelike interval in spacetime would be a persons height at one particular instant, and a time-like interval might be the amount of time spent sitting on a chair. For time-like intervals the *proper time*, $(d\tau)$, is often used with $(dS)^2 = -c^2(d\tau)^2$.

The field equations link the spacetime geometry to the mass distribution as described in section 2.4. Einstein's field equations applied outside of a static, spherically symmetric source mass, yield the Schwarzschild solution. Using spherical polar coordinates, $\{ct, r, \theta, \phi\}$ (see fig.(3.1)), the Schwarzschild metric is

$$g_{\mu\nu} = \begin{bmatrix} -\left[1 - \frac{2m}{r}\right] & 0 & 0 & 0 \\ 0 & \left[1 - \frac{2m}{r}\right]^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}. \quad (2.15)$$

These coordinates use of the symmetry of the solution to simplify the metric. In the same coordinates the Schwarzschild line element is

$$dS^2 = -c^2 d\tau^2 = -\left[1 - \frac{2m}{r}\right] d(ct)^2 + \left[1 - \frac{2m}{r}\right]^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2.16)$$

The term m is the geometric mass, related to the gravitational mass of the source M , by

$$m = \frac{GM}{r^2}. \quad (2.17)$$

Using this it can be seen that g_{00} term has the same mass and distance relation as the Newtonian gravitational potential.

General relativity states that objects moving only under the influence of gravity follow geodesics. The *geodesic equations* are a generalisation of Newton's second law of motion and describe the motion of objects using an affine parameter such as proper time τ . In general coordinates, $x^\rho = \{x^0, x^1, x^2, x^3\}$, the geodesic equation can be written

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (2.18)$$

remembering the Einstein summation convention. The term $\Gamma^\rho_{\mu\nu}$ is called a *connection coefficient* or *Christoffel symbol*. Recognising $\frac{d^2 x^\rho}{d\tau^2}$ in Eq.(2.18) as an acceleration of sorts, and equating this with the acceleration in Eq.(2.3), it can be seen that the connection coefficients represent Newton's gravitational field. Since the gravitational field is really just the curvature of spacetime, it is not surprising that the connection coefficients can be defined from the derivative of metric

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\kappa\rho} (g_{\nu\kappa,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}). \quad (2.19)$$

The comma in $g_{\nu\kappa,\mu}$ is a simple notation for the partial derivative so $g_{\nu\kappa,\mu} = \frac{\partial g_{\nu\kappa}}{\partial x^\mu}$. The metric with raised indices $g^{\mu\nu}$ is equal to the metric inverse of $g_{\mu\nu}$, but because the metric is diagonal $\{g^{\mu\nu}\} = \{g_{\mu\nu}\}^{-1}$, ie. just take scalar inverse of each element.

As an example calculation, consider the geodesic equation in the radial coordinate $x^1 = r$ with the Schwarzschild metric (Eq. 2.15). The connection coefficients simplify to

$$\begin{aligned} \Gamma^1_{\mu\nu} &= \frac{1}{2} g^{\kappa 1} (g_{\nu\kappa,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}) \\ &= \frac{1}{2} g^{11} (g_{\nu 1,\mu} + g_{1\mu,\nu} - g_{\mu\nu,1}). \end{aligned} \quad (2.20)$$

Remembering again the summation convention gives $\kappa = 1$ from the $g^{\kappa 1}$ term, since the metric is diagonal and all other combinations would be zero. Substituting the metric elements from Eq.(2.15) into Eq.(2.20) and simplifying to calculate the first connection coefficient gives,

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2} g^{11} (g_{01,0} + g_{10,0} - g_{00,1}) \\ &= \frac{1}{2} (g_{11})^{-1} \frac{d}{dr} (-g_{00}) \\ &= \frac{1}{2} \left(1 - \frac{2m}{r}\right) \left(\frac{2m}{r^2}\right) \\ &= \frac{m}{r^3} (r - 2m). \end{aligned} \quad (2.21)$$

Similar calculations give the other connection coefficients for the radial geodesic equation. All the non-zero connection coefficients together are

$$\begin{aligned} \Gamma^1_{00} &= \frac{m}{r^3} (r - 2m) \\ \Gamma^1_{11} &= -\frac{m}{r(r - 2m)} \\ \Gamma^1_{22} &= -(r - 2m) \\ \Gamma^1_{33} &= -(r - 2m) \sin^2(\theta). \end{aligned} \quad (2.22)$$

Summing all the terms for Eq.(2.20) and substituting in the values of the connection coefficients from Eq.(2.22), and the geodesic equation in r becomes

$$\begin{aligned}
0 &= \frac{d^2r}{d\tau^2} + \Gamma^1_{00} \left(\frac{d(ct)}{d\tau} \right)^2 + \Gamma^1_{11} \left(\frac{dr}{d\tau} \right)^2 + \Gamma^1_{22} \left(\frac{d\theta}{d\tau} \right)^2 + \Gamma^1_{33} \left(\frac{d\phi}{d\tau} \right)^2 \\
&= \frac{d^2r}{d\tau^2} + \frac{m}{r^3} (r - 2m) \left(\frac{d(ct)}{d\tau} \right)^2 - \frac{m}{r(r - 2m)} \left(\frac{dr}{d\tau} \right)^2 \\
&\quad - (r - 2m) \left(\frac{d\theta}{d\tau} \right)^2 - (r - 2m) \sin^2(\theta) \left(\frac{d\phi}{d\tau} \right)^2.
\end{aligned} \tag{2.23}$$

In order to compute the radial acceleration consider only radial motion by setting $d\theta = 0$ and $d\phi = 0$. Substituting these into Eq.(2.23) and rearranging for the radial acceleration,

$$\begin{aligned}
\frac{d^2r}{d\tau^2} &= -(r - 2m) \frac{m}{r^3} \left(\frac{d(ct)}{d\tau} \right)^2 + \frac{m}{r(r - 2m)} \left(\frac{dr}{d\tau} \right)^2 \\
&= \frac{m}{r^2} \left[- \left[1 - \frac{2m}{r} \right] \left(\frac{d(ct)}{d\tau} \right)^2 + \left[1 - \frac{2m}{r} \right]^{-1} \left(\frac{dr}{d\tau} \right)^2 \right].
\end{aligned} \tag{2.24}$$

With $d\theta = 0$ and $d\phi = 0$ in the Schwarzschild line element for a time like interval, Eq.(2.16) divided by $d\tau^2$, becomes

$$-c^2 = - \left[1 - \frac{2m}{r} \right] \left(\frac{d(ct)}{d\tau} \right)^2 + \left[1 - \frac{2m}{r} \right]^{-1} \left(\frac{dr}{d\tau} \right)^2. \tag{2.25}$$

Substituting Eq.(2.25) and Eq.(2.17) into Eq.(2.24) the radial acceleration takes the familiar form,

$$\frac{d^2r}{d\tau^2} = -\frac{m}{r^2} c^2 = -\frac{GM}{r^2} \tag{2.26}$$

of the acceleration in Newtonian gravity. This is to be expected as one of the requirements Einstein placed on his field equations was that they gave the Newtonian gravitational potential and acceleration as a limit, as should any gravitational theory.

Chapter 3

The Strong Equivalence Principle in a Generalised Gravitational Metric

3.1 Introduction and Overview

Einstein's original formulation of his Strong Equivalence Principle,[8] propose an equivalence between a uniformly accelerating frame, and a uniform gravitational field in a stationary frame. However, in metric theories of gravity such as the general theory of relativity, the gravitational field is a consequence of spacetime curvature, indicated by a non zero Riemann curvature Tensor. A uniform gravitational field has a zero Riemann curvature and so is not a physically realisable gravitational field under the above definition. Desloge [4] derives a metric for a uniform gravitation field to test the Strong Equivalence Principle, but his metric does not satisfy Einstein's field equations. The strong equivalence principle is therefore often restated as a local equivalence between any gravitational field and a uniformly accelerating frame, such as in the following from Weinberg [29, p.68].

At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in un-accelerated Cartesian coordinate systems in the absence of gravitation.

This form of the Strong Equivalence Principle was tested for the Schwarzschild line element by Moreau et al. [21]. Their results demonstrated that although the diagonal terms in the line element correspond to those of a uniformly accelerating frame, there are additional off diagonal terms as a consequence of the curvature. In this chapter, we extend their results to a generalised metric external to a static, spherically symmetric gravitational source. Although this work closely follows their own derivation, using a generalised metric is important in showing the results are a geometric effect which independent of both the coordinate system and choice of metric.

This general form of metric includes the Schwarzschild metric, as well as the metric in the conformal gravity proposed by Mannheim [16, 15], but excludes the metric proposed by Desloge [4]. The results of Moreau et al. are confirmed in this general metric, indicating that the presence of a real gravitational field is distinguishable from a uniformly accelerating frame due to the effects of curvature even at the infinitesimal level. However it is interesting to note that the Strong Equivalence Principle maintained by Einstein, pertaining only to a uniform gravitational field with zero curvature, does satisfy this test. It is clear that Einstein understood that the equivalence principle was not to be used to transform away arbitrary gravitational fields, as he states in the following.[23, p. 9]

One can also invert the previous consideration. Let the system K' , formed with the gravitational field considered above [homogeneous], be the original

one. Then one can introduce a new reference system K , accelerated with respect to K' , with respect to which (isolated) masses (in the region considered) move uniformly in a straight line. But one may *not* go on and say: if K' is a reference system provided with an arbitrary gravitational field, then it is always possible to find a reference system K , in relationship to which isolated bodies move uniformly in a straight line, i.e., in relation to which no gravitational field exists...

In this chapter, a metric external to a static, spherically symmetric source of gravity is derived, and the geodesic equations computed. The geodesic equations are solved to give the radial acceleration at an arbitrary point in this gravitational field, and the acceleration is used to derive a uniformly accelerating frame in French coordinates. In order to compare the general gravitational metric with the French metric, the gravitational metric is transformed into the displaced rectangular coordinate system used for the French metric. Comparing the two metrics in accordance with the Strong Equivalence Principle, confirms the results obtained by Moreau et al. [21].

3.2 Derivation of Generalised Metric

The Generalised Metric here is restricted to a static, spherically symmetric gravitational source. Many of the real gravitational sources such as the earth and sun satisfy these symmetry requirements, making this a useful restriction for a generalised metric. Following in part the derivations of the Schwarzschild metric given in Price, [25] and Ohanian, [24], most terms in the metric are eliminated by considering the symmetry of the gravitational source. The Einstein field equations can be then used to calculate the remaining terms, which leads directly to the Schwarzschild metric. However it is noted by Mannheim [15] that Einstein's field equations are not the only possible choice of field equations for a metric theory of gravity. To avoid restricting these calculations to any particular field equation, the metric will be left in its most general form fitting the symmetry requirements as given. Any results derived in this metric will be independent of the field equations and are a consequence of the symmetry of the geometry and geodesic motion.

A metric tensor for a four dimensional space can be written

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}, \quad (3.1)$$

using the fact that a metric tensor must be symmetric ($g_{\mu\nu} = g_{\nu\mu}$). Each element of the metric will be a general function of the coordinates that is usually determined from the mass distribution by solving field equations. The coordinates have not yet been specified, but will be chosen in such a way to make the geometry of the metric appear in its simplest form. A natural choice is the spherical polar coordinates $\{T, r, \theta, \phi\}$ with the origin at the centre of mass. T is some time-like coordinate multiplied by the vacuum speed of light. Using these coordinates in Eq.(2.11), the line element is

$$\begin{aligned} dS^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -g_{00} dT^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2 \\ &\quad + 2(g_{01} dT dr + g_{02} dT d\theta + g_{03} dT d\phi + g_{12} dr d\theta + g_{13} dr d\phi + g_{23} d\theta d\phi). \end{aligned} \quad (3.2)$$

However, what *time* is measured by dT and what *distance* is measured by dr ? The full geometric meaning of these coordinates is not defined in the line element, as the scale for each coordinate is not specified. Appropriate scales can be chosen for each coordinate to simplify the elements of $g_{\mu\nu}$ as much as possible.

Spherical symmetry requires that the line element be unchanged by spatial rotations about the origin. In polar coordinates, this refers to an infinite set of two dimensional spherical surfaces that are coordinatised by θ and ϕ , with r and T parametrising each surface. The only $d\theta$ and $d\phi$ terms remaining from Eq.(3.2) that are spherically symmetric are given by the line element on the surface of a sphere of radius R ,

$$dS_{sphere}^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.3)$$

Now the arbitrary scale on the radial coordinate r can be fixed by setting $R = r$, which geometrically fixes r as the radial coordinate that makes the area of each spherical surface equal to $4\pi r^2$. This is called the Schwarzschild radial coordinate, and gives us the metric elements

$$g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta. \quad (3.4)$$

Any terms in Eq.(3.2) that are first order in $d\theta$ and $d\phi$ must be zero, as these terms would change sign, distinguishing between directions of increasing and decreasing θ or ϕ . This geometric distinction would violate the spherical symmetry. Therefore the following metric elements must be zero;

$$g_{02} = 0, \quad g_{03} = 0, \quad g_{12} = 0, \quad g_{13} = 0, \quad g_{23} = 0. \quad (3.5)$$

Requiring the metric to be static implies that not only must the metric coefficients be time independent, but also that they remain unchanged by time reversal. Under a time reversal the $dTdr$ term would change sign, therefore we have

$$g_{01} = 0. \quad (3.6)$$

Now substituting Eqs.(3.4), (3.5), and (3.6) into Eq.(3.2) the line element becomes

$$\begin{aligned} dS^2 &= -g_{00}dT^2 + g_{11}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \\ &= -A(r) dT^2 + B(r) dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \end{aligned} \quad (3.7)$$

where $A(r)$ and $B(r)$ are functions only of r because of the spherical symmetry and time independence. Determining the functions $A(r)$ and $B(r)$ requires solving the field equations, which specify exactly how the source mass effects spacetime geometry and thus generates gravity. However, keeping the metric *independent* of the choice of field equations, the line element in Eq.(3.2) gives the general metric tensor

$$g_{\mu\nu} = \begin{bmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}. \quad (3.8)$$

There is one more restriction that can be used to simplify the metric further. When Moreau et al. [21] derived an accelerating frame to test the strong equivalence principle, they required that the acceleration was independent of the proper velocity. Imposing

a similar constraint here will require that the radial acceleration, $\frac{d^2r}{d\tau^2}$ is independent of the proper radial velocity $\frac{dr}{d\tau}$ for purely radial motion. The radial geodesic equation for the metric Eq.(3.8), was calculated using MACSYMA, and rearranged to give the proper acceleration

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2B} \frac{dA}{dr} \left(\frac{dT}{d\tau}\right)^2 - \frac{1}{2B} \frac{dB}{dr} \left(\frac{dr}{d\tau}\right)^2 - \frac{r}{B} \left(\frac{d\theta}{d\tau}\right)^2 - \frac{r}{B} \sin^2 \theta \left(\frac{d\phi}{d\tau}\right)^2, \quad (3.9)$$

where τ is the proper time. Restricting the geodesic equation to radial motion gives $\frac{d\theta}{d\tau} = 0$ and $\frac{d\phi}{d\tau} = 0$, and Eq.(3.9) becomes

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2B} \frac{dA}{dr} \left(\frac{dT}{d\tau}\right)^2 - \frac{1}{2B} \frac{dB}{dr} \left(\frac{dr}{d\tau}\right)^2. \quad (3.10)$$

The acceleration is required to be independent of radial velocity, however simply setting $\frac{1}{2B} \frac{dB}{dr} = 0$ will not achieve this since $\frac{dT}{d\tau}$ is also related to $\frac{dr}{d\tau}$ by the line element. Neglecting the terms in $\frac{d\theta}{d\tau}$ and $\frac{d\phi}{d\tau}$ in Eq.3.2, the line element is

$$dS^2 = -c^2 d\tau^2 = -A(r) dT^2 + B(r) dr^2. \quad (3.11)$$

Dividing Eq.(3.11) by $A(r)d\tau^2$ and rearranging gives

$$\left(\frac{dT}{d\tau}\right)^2 = \frac{c^2}{A} + \frac{B}{A} \left(\frac{dr}{d\tau}\right)^2. \quad (3.12)$$

To get the radial acceleration, Eq.(3.12) is substituted into Eq.(3.10) and rearranged to show the velocity Dependant terms;

$$\begin{aligned} \frac{d^2r}{d\tau^2} &= -\frac{1}{2B} \frac{dA}{dr} \left[\frac{c^2}{A} + \frac{B}{A} \left(\frac{dr}{d\tau}\right)^2 \right] - \frac{1}{2B} \frac{dB}{dr} \left(\frac{dr}{d\tau}\right)^2 \\ &= -\frac{c^2}{2AB} \frac{dA}{dr} - \left[\frac{1}{2A} \frac{dA}{dr} + \frac{1}{2B} \frac{dB}{dr} \right] \left(\frac{dr}{d\tau}\right)^2. \end{aligned} \quad (3.13)$$

Now the radial acceleration can be made independent of radial velocity by requiring that the coefficient of the $\left(\frac{dr}{d\tau}\right)^2$ term in Eq.(3.13) is equal to zero. This gives the differential equation

$$\frac{1}{2A} \frac{dA}{dr} + \frac{1}{2B} \frac{dB}{dr} = 0. \quad (3.14)$$

Multiplying Eq.(3.14) by $2 dr$ and rearranging gives

$$\frac{1}{B} dB = -\frac{1}{A} dA. \quad (3.15)$$

Eq.(3.15) can be integrated to get a solution

$$\ln B = -\ln A + k' \quad (3.16)$$

with constant of integration k' . Taking the exponential of Eq.(3.16) with $k = e^{k'}$ gives

$$B(r) = \frac{k}{A(r)} = \frac{1}{A(r)}. \quad (3.17)$$

The constant k is arbitrary, but can be absorbed into the definition of the time-like coordinate through the function $A(r)$. Making the transformation $\mathcal{A}(r) \rightarrow kA(r)$ correspond to transforming the time-like coordinate $T \rightarrow t$. For simplicity the function \mathcal{A} will be relabelled back to A , then substituting Eq.(3.17) into Eq.(3.2) gives the general line element

$$dS^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (3.18)$$

Similarly the general metric tensor will be

$$g_{\mu\nu} = \begin{bmatrix} -A(r) & 0 & 0 & 0 \\ 0 & \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (3.19)$$

for the gravitational field external to a static, spherically symmetric gravitational source. For the rest of this thesis, the metric of Eq.(3.19) will be implied by the term *generalised metric*.

3.3 Radial Motion in the Generalised Metric

One of the requirements used to derive the generalised metric was that the radial acceleration was independent of velocity. It was assumed that this could be done by considering only motion in a radial direction, that is setting

$$d\theta = 0 \quad \text{and} \quad d\phi = 0. \quad (3.20)$$

This assumption is justified by showing that all the geodesic equations have solutions under these conditions. The radial geodesic equation is used to compute the radial acceleration, and in the case of the Schwarzschild metric this is shown to reduce to the Newtonian acceleration, but with proper time in place of coordinate time.

MACSYMA was used to compute the geodesic equations in the generalised metric, Eq.(3.19), with the results given in appendix A.1. For purely radial motion, substituting Eq.(3.20) into these geodesic equations gives;

$$\frac{d^2 t}{d\tau^2} = -\frac{1}{A} \frac{dA}{dr} \frac{dr}{d\tau} \frac{dt}{d\tau}, \quad (3.21)$$

$$\frac{d^2 r}{d\tau^2} = -\frac{A}{2} \frac{dA}{dr} \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{2A} \frac{dA}{dr} \left(\frac{dr}{d\tau} \right)^2, \quad (3.22)$$

$$\frac{d^2 \theta}{d\tau^2} = 0, \quad (3.23)$$

$$\frac{d^2 \phi}{d\tau^2} = 0. \quad (3.24)$$

Only equations, Eq.(3.23) and Eq.(3.24), refer to θ and ϕ coordinate and these are trivially solved with solutions $\frac{d\theta}{d\tau} = 0$ and $\frac{d\phi}{d\tau} = 0$. This is consistent with the initial assumption of purely radial motion in Eq.(3.20).

To solve the geodesic equation in time, Eq.(3.21) can be rearranged and then simplified using the product rule for differentiation,

$$\begin{aligned}
0 &= \frac{d^2t}{d\tau^2} + \frac{1}{A} \frac{dA}{dr} \frac{dr}{d\tau} \frac{dt}{d\tau} \\
&= A \frac{d}{d\tau} \left(\frac{dt}{d\tau} \right) + \frac{dA}{d\tau} \frac{dt}{d\tau} \\
&= \frac{d}{d\tau} \left(A \frac{dt}{d\tau} \right).
\end{aligned} \tag{3.25}$$

This has the solution $A \frac{dt}{d\tau} = k$ where k is a constant if integration. Taking the line element Eq.(3.18) divided by $d\tau^2$, and substituting in Eq.(3.20) to restrict this to radial intervals gives

$$-c^2 = -A \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{A} \left(\frac{dr}{d\tau} \right)^2. \tag{3.26}$$

Then inserting the solution of Eq.(3.25) for $\frac{dt}{d\tau}$ in Eq.(3.26) and simplifying

$$\begin{aligned}
-c^2 &= -A \left(\frac{k}{A} \right)^2 + \frac{1}{A} \left(\frac{dr}{d\tau} \right)^2 \\
&= \frac{1}{A} \left[-k^2 + \left(\frac{dr}{d\tau} \right)^2 \right].
\end{aligned} \tag{3.27}$$

Rearranging Eq.(3.27) to get

$$\frac{dr}{d\tau} = \sqrt{-A c^2 + k^2}, \tag{3.28}$$

and then differentiating to get the radial acceleration

$$\begin{aligned}
\frac{d^2r}{d\tau^2} &= -\frac{1}{2} \sqrt{-A c^2 + k^2}^{-1} c^2 \frac{dA}{d\tau} \\
&= -\frac{c^2}{2} \sqrt{-A c^2 + k^2}^{-1} \frac{dA}{dr} \frac{dr}{d\tau} \\
&= -\frac{c^2}{2} \frac{dA}{dr}.
\end{aligned} \tag{3.29}$$

This needs to be consistent with the results of the radial geodesic equation. Simplifying Eq.(3.22) gives

$$\begin{aligned}
0 &= \frac{d^2r}{d\tau^2} + \frac{A}{2} \frac{dA}{dr} \left(\frac{dt}{d\tau} \right)^2 - \frac{1}{2A} \frac{dA}{dr} \left(\frac{dr}{d\tau} \right)^2 \\
&= \frac{d^2r}{d\tau^2} - \frac{1}{2} \frac{dA}{dr} \left[-A \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{A} \left(\frac{dr}{d\tau} \right)^2 \right] \\
&= \frac{d^2r}{d\tau^2} - \frac{1}{2} \frac{dA}{dr} [-c^2].
\end{aligned} \tag{3.30}$$

The last step comes from inserting the line element, Eq.(3.26) . Thus the radial acceleration is

$$\frac{d^2r}{d\tau^2} = -\frac{c^2}{2} \frac{dA}{dr} \tag{3.31}$$

which agrees with Eq.(3.29). Therefore it has been demonstrated that radial motion is a solution that is consistent with the full set of geodesic equations for the generalised metric.

The generalised metric can be reduced to the Schwarzschild metric by setting $A = \left(1 - \frac{2m}{r}\right)$, where $m = \frac{GM}{c^2}$. Substituting this into Eq.(3.31), gives

$$\frac{d^2r}{d\tau^2} = -\frac{m}{r^2}c^2 = -\frac{GM}{r^2} \quad (3.32)$$

which is the Newtonian gravitational acceleration as should be expected.

3.4 Metric for a Uniformly Accelerating Frame

Using the radial acceleration in the generalised gravitational metric of §3.2, the metric for a corresponding uniformly accelerated frame at $r = R$ is derived. This frame is most conveniently coordinatised using the *displaced rectangular* spacelike coordinates shown in Fig.(3.1) and the usual time-like coordinate. Acceleration in the uniformly accelerating frame, must equal the radial acceleration in the generalised metric at $r = R$. Evaluating Eq.(3.31) at R then gives $\frac{d^2z}{d\tau^2}$, the acceleration in the accelerating frame.

$$\frac{d^2z}{d\tau^2} = -\frac{c^2}{2} \left[\frac{dA}{dr} \right]_{r=R} \quad (3.33)$$

A general metric tensor for a flat frame with uniform acceleration, is given by Moreau et al. [21] as

$$g_{\mu\nu} = \begin{bmatrix} -\alpha(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta(z) \end{bmatrix} \quad (3.34)$$

where α and β are general functions of z which determine the acceleration. The line element for this metric is

$$-c^2 d\tau^2 = -\alpha(z) (cdt)^2 + dx^2 + dy^2 + \beta(z) dz^2. \quad (3.35)$$

The functions α and β can be determined by imposing the conditions that the metric be flat and that the acceleration is given by Eq.(3.33).

Since a uniformly accelerating frame must be flat, it is required that all components of Riemann curvature tensor are zero. In appendix §A.2 the two, non zero Riemann curvature tensor components are given as calculated by MACSYMA. Equating R^3_{003} and R^0_{303} to zero, both give

$$0 = \beta \left(\frac{d\alpha}{dz} \right)^2 + \alpha \frac{d\beta}{dz} \frac{d\alpha}{dz} - 2\beta \alpha \frac{d^2\alpha}{dz^2}. \quad (3.36)$$

By solving this equation, restrictions can be placed on the functions α and β for the Riemann curvature tensor to be flat. Solutions of α and β corresponding to the French

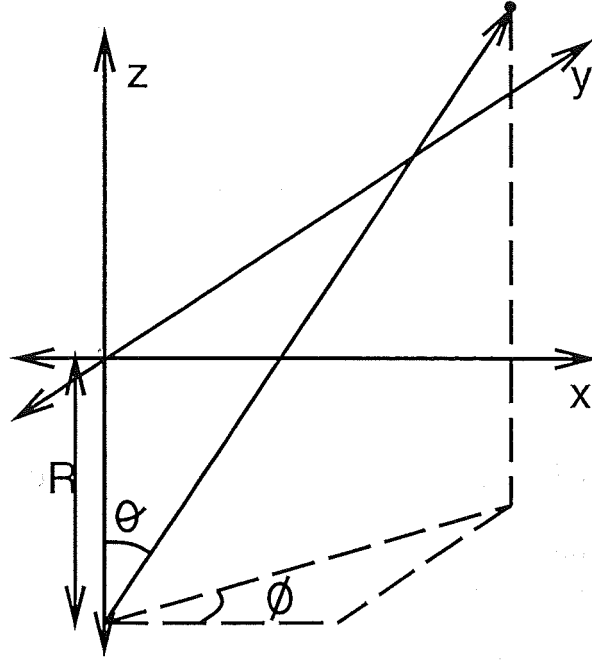


Figure 3.1: Transforming from spherical polar coordinates to displaced rectangular coordinates. The centre of mass is the origin of the spherical polar coordinates and R shows the displacement to the rectangular coordinates.

and Fermi coordinates are flat and therefore fulfill this requirement. Rearranging and simplifying Eq.(3.36) becomes

$$\begin{aligned}
 0 &= 2\alpha\beta \frac{d^2\alpha}{dz^2} - \frac{d\alpha}{dz} \left(\beta \frac{d\alpha}{dz} + \alpha \frac{d\beta}{dz} \right) \\
 &= \alpha\beta \frac{dz}{d\alpha} \frac{d}{dz} \left(\frac{d\alpha}{dz} \right)^2 - \frac{d\alpha}{dz} \frac{d}{dz} (\alpha\beta) \\
 &= \alpha\beta \frac{d}{dz} \left(\frac{d\alpha}{dz} \right)^2 - \left(\frac{d\alpha}{dz} \right)^2 \frac{d}{dz} (\alpha\beta). \tag{3.37}
 \end{aligned}$$

Making the substitutions $U = \left(\frac{d\alpha}{dz} \right)^2$ and $V = \alpha\beta$ into Eq.(3.37) and dividing by V^2 gives

$$0 = \frac{V \frac{dU}{dz} - U \frac{dV}{dz}}{V^2}. \tag{3.38}$$

Now the right hand side of Eq.(3.38) is easily recognised from the quotient rule of calculus,

$$d \left(\frac{U}{V} \right) = \frac{V \frac{dU}{dz} - U \frac{dV}{dz}}{V^2}. \tag{3.39}$$

Using the quotient rule to equate Eq.(3.39) with Eq.(3.38) then substituting back in the values for U and V gives

$$\begin{aligned}
 0 &= \frac{d}{dz} \left[\frac{U}{V} \right] \\
 &= \frac{d}{dz} \left[\frac{1}{\alpha\beta} \left(\frac{d\alpha}{dz} \right)^2 \right]. \tag{3.40}
 \end{aligned}$$

This has the general solution $\frac{1}{\alpha\beta} \left(\frac{d\alpha}{dz}\right)^2 = k$ for a constant of integration k . Can this be used to place any restrictions on the functions α or β ?

There are at least two solutions for a flat accelerating frame. Setting $\beta = \frac{1}{\alpha}$ will give a solution if $\frac{d\alpha}{dz} = k$, such as the French coordinates $\alpha = 1 + \frac{2mz}{R^2}$. Alternatively with $\beta = 1$ and $\alpha = (1+z)^2$ gives Fermi coordinates. Using MACSYMA, these equations were integrated to give the solution

$$\alpha = \frac{k}{4} \left[\int_{z_0}^z \sqrt{\beta(z')} dz' \right]^2. \quad (3.41)$$

However this does not yield a simple form of a general solution.

Similarly to the method used in §3.2, the functions α and β in line element Eq.(3.35) can be determined by requiring that the acceleration be velocity independent. According to the Strong Equivalence principle this acceleration can then be equated to the radial acceleration in the gravitational frame, given in Eq.(3.31). Using the metric for a uniformly accelerating frame, the geodesic equations calculated by MACSYMA are listed in appendix A.2. Rearranging the geodesic equation in z to give the acceleration

$$\frac{d^2z}{d\tau^2} = \frac{-1}{2\beta} \frac{d\beta}{dz} \left(\frac{dz}{d\tau}\right)^2 - \frac{1}{2\beta} \frac{d\alpha}{dz} \left(\frac{dt}{d\tau}\right)^2. \quad (3.42)$$

Since the acceleration is in the z direction, dy and dx must both be constant. Considering motion only in the z axis by setting $dx = 0 = dy$, the line element for a uniformly accelerating frame, Eq.(3.35) becomes

$$-c^2 = -\alpha c^2 \left(\frac{dt}{d\tau}\right)^2 + \beta \left(\frac{dz}{d\tau}\right)^2. \quad (3.43)$$

Rearranging terms and dividing by α

$$c^2 \left(\frac{dt}{d\tau}\right)^2 = \frac{c^2}{\alpha} + \frac{\beta}{\alpha} \left(\frac{dz}{d\tau}\right)^2. \quad (3.44)$$

Substituting 3.44 into Eq.(3.42) gives the acceleration

$$\begin{aligned} \frac{d^2z}{d\tau^2} &= -\frac{1}{2\beta} \frac{d\beta}{dz} \left(\frac{dz}{d\tau}\right)^2 - \frac{1}{2\beta} \frac{d\alpha}{dz} \left[\frac{c^2}{\alpha} + \frac{\beta}{\alpha} \left(\frac{dz}{d\tau}\right)^2 \right] \\ &= -\frac{1}{2} \left[\frac{1}{\beta} \frac{d\beta}{dz} + \frac{1}{\alpha} \frac{d\alpha}{dz} \right] \left(\frac{dz}{d\tau}\right)^2 - \frac{c^2}{2\alpha\beta} \frac{d\alpha}{dz}, \end{aligned} \quad (3.45)$$

which clearly has a velocity dependent term in $\frac{dz}{d\tau}$. To make the acceleration independent of velocity, the coefficient of $\frac{dz}{d\tau}$ must be zero. This gives the differential equation

$$\frac{1}{\beta} \frac{d\beta}{dz} + \frac{1}{\alpha} \frac{d\alpha}{dz} = 0. \quad (3.46)$$

Integrating this with a constant of integration k yields the solution

$$\beta = \frac{k}{\alpha}. \quad (3.47)$$

By defining the time-like coordinate accordingly (as in §3.2) the constant k can be unity and the general metric tensor for a uniformly accelerating frame becomes

$$g_{\mu\nu} = \begin{bmatrix} -\alpha(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha(z)} \end{bmatrix}. \quad (3.48)$$

Substituting $\beta = \frac{1}{\alpha}$ into Eq.(3.45) to make the acceleration in the uniformly accelerating frame independent of velocity gives

$$\frac{d^2z}{d\tau^2} = -\frac{c^2}{2} \frac{d\alpha}{dz}. \quad (3.49)$$

At the start of this section, the acceleration in the uniformly accelerating frame was defined to be the corresponding radial acceleration in the generalised gravitational metric, Eq.(3.33). Equating this with Eq.(3.49)

$$\frac{d^2z}{d\tau^2} = -\frac{c^2}{2} \left[\frac{dA}{dr} \right]_{r=R} = -\frac{c^2}{2} \frac{d\alpha}{dz}, \quad (3.50)$$

then simplifying gives

$$\frac{d\alpha}{dz} = \left[\frac{dA}{dr} \right]_{r=R}. \quad (3.51)$$

Integrating Eq.(3.51) with integration constant k gives the solution

$$\alpha = z \left[\frac{dA}{dr} \right]_{r=R} + k. \quad (3.52)$$

The line element for the general accelerating frame metric of Eq.(3.48) is then

$$-c^2 d\tau^2 = -\alpha (cdt)^2 + dx^2 + dy^2 + \frac{1}{\alpha} dz^2, \quad (3.53)$$

with α defined by Eq.(3.52). This can be used to test the Strong Equivalence Principle, because it has been derived from the radial acceleration in the generalised metric in §3.2.

3.5 Generalised Metric in Displaced Rectangular Coordinates

The generalised metric external to a static, spherically symmetric mass distribution as derived in §3.2, uses a spherical polar coordinate system with the origin at the centre of mass. However the uniformly accelerating frame in §3.4 used rectangular coordinates displaced to a distance R from the centre of mass as given in Fig. (3.1). To enable a comparison with the accelerating frame metric, the line element for the general gravitational metric,

$$dS^2 = -A c^2 dt^2 + \frac{1}{A} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (3.54)$$

will be transformed into the displaced rectangular coordinates. This is achieved by generalising the transformations of Moreau et al. [21] used for the Schwarzschild case.

General transformation equations from polar to displaced rectangular coordinates are

$$\begin{aligned}x &= r \sin \theta \cos \phi, \\y &= r \sin \theta \sin \phi, \\z &= r \cos \theta - R, \\ct &= ct.\end{aligned}\tag{3.55}$$

The constant R is the displacement in the z axis of the origin in the uniformly accelerating frame from the centre of mass. Differentiating the spatial components of each of the transformations in Eq.(3.55) provides the differentials

$$\begin{aligned}dx &= \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi, \\dy &= \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi, \\dz &= \cos \theta dr - r \sin \theta d\theta.\end{aligned}\tag{3.56}$$

Summing the squares of the differentials in Eq.(3.56) and simplifying gives

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.\tag{3.57}$$

This just the line element transformation from rectangular to polar coordinates in three dimensional Euclidean (flat) space. Subtracting dr^2 from both sides of Eq.(3.57) and substituting for $r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$ in Eq.(3.54), the line element becomes

$$dS^2 = -A(r) c^2 dt^2 + \left(\frac{1}{A(r)} - 1 \right) dr^2 + dx^2 + dy^2 + dz^2.\tag{3.58}$$

The dr^2 term and the function $A(r)$ in Eq.(3.58) still require transforming into the displaced rectangular coordinates. Transforming the dr^2 term requires solving Eqs. (3.55) for r

$$\begin{aligned}r &= \sqrt{x^2 + y^2 + (z + R)^2} \\ &= R \sqrt{1 + \frac{2z}{R} + \frac{x^2 + y^2 + z^2}{R^2}}.\end{aligned}\tag{3.59}$$

Differentiating Eq.(3.59) and then simplifying,

$$\begin{aligned}dr &= R \left(\sqrt{1 + \frac{2z}{R} + \frac{x^2 + y^2 + z^2}{R^2}} \right)^{-1} \frac{1}{2} \left(\frac{2x}{R^2} dx + \frac{2y}{R^2} dy + 2 \frac{(z + R)}{R^2} dz \right) \\ &= \left(\sqrt{1 + \frac{2z}{R} + \frac{x^2 + y^2 + z^2}{R^2}} \right)^{-1} \frac{(x dx + y dy + (z + R) dz)}{R}.\end{aligned}\tag{3.60}$$

Finally, squaring Eq.(3.60) gives

$$\begin{aligned}dr^2 &= \frac{(x dx + y dy + (z + R) dz)^2}{\left(1 + \frac{2z}{R} + \frac{x^2 + y^2 + z^2}{R^2} \right) R^2} \\ &= \frac{\left(x^2 dx^2 + y^2 dy^2 + (z + R)^2 dz^2 + 2xy dx dy + 2x(z + R) dx dz + 2y(z + R) dy dz \right)}{\left(1 + \frac{2z}{R} + \frac{x^2 + y^2 + z^2}{R^2} \right) R^2}.\end{aligned}\tag{3.61}$$

This can be substituted into the line element, Eq.(3.58) to give the line element in displaced rectangular coordinates.

Because the strong equivalence principle for an arbitrary gravitational field is defined only as a local equivalence, it will only be necessary to test this for a small region at the origin of the displaced rectangular coordinates. A small region will be taken to mean that $\left|\frac{z}{R}\right|$, $\left|\frac{y}{R}\right|$ and $\left|\frac{x}{R}\right| \ll 1$, allowing the line element to be restricted to first order in these terms. It will not be necessary to make any sort of weak field approximation and place any first order restriction on $\frac{m}{R}$. Discarding terms higher than first order in $\frac{z}{R}$, $\frac{y}{R}$ and $\frac{x}{R}$ from Eq. (3.61) gives

$$\begin{aligned} dr^2 &= \frac{(z+R)^2 dz^2 + 2x(z+R) dx dz + 2y(z+R) dy dz}{\left[1 + \frac{2z}{R}\right] R^2} \\ &= \frac{(2zR + R^2) dz^2 + 2xR dx dz + 2yR dy dz}{R^2 + 2zR} \\ &= dz^2 + \frac{2x dx dz + 2y dy dz}{R + 2z}. \end{aligned} \quad (3.62)$$

Substituting Eq. (3.62) into Eq.(3.58), the line element becomes

$$\begin{aligned} dS^2 &= -A(r) c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &\quad + \left(\frac{1}{A(r)} - 1\right) \left(dz^2 + \frac{2x dx dz + 2y dy dz}{R + 2z}\right) \\ &= -A(r) c^2 dt^2 + dx^2 + dy^2 + \frac{1}{A(r)} dz^2 + \left(\frac{1}{A(r)} - 1\right) \frac{2x dx dz + 2y dy dz}{R + 2z}. \end{aligned} \quad (3.63)$$

This line element still requires the arbitrary function $A(r)$ to be transformed into rectangular coordinates. Using the locality approximation to simplify Eq.(3.59) to first order in $\frac{z}{R}$, $\frac{y}{R}$ and $\frac{x}{R}$ gives

$$r = R \sqrt{1 + \frac{2z}{R}}. \quad (3.64)$$

Substituting Eq.(3.64) for all occurrences of r will transform $A(r)$ into $a(z)$, a general function of z

$$A(r) \rightarrow a(z) : a(z) = A\left(r \rightarrow R \sqrt{1 + \frac{2z}{R}}\right). \quad (3.65)$$

Using the transformation of Eq.(3.65) for the line element Eq.(3.63) gives the line element for general gravitational metric in displaced rectangular coordinates

$$dS^2 = -a(z) c^2 dt^2 + dx^2 + dy^2 + \frac{1}{a(z)} dz^2 + \left(\frac{1}{a(z)} - 1\right) \frac{2x dx dz + 2y dy dz}{R + 2z}. \quad (3.66)$$

In general this line element is not diagonal unless x and y are both zero, preventing a complete equivalence with the uniformly accelerating frame metric, Eq.(3.48) which is diagonal. Using the line element, Eq.(3.66) and labelling the off diagonal elements with

a dummy variable Q gives the metric

$$g_{\mu\nu} = \begin{bmatrix} -a(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{Q\left(\frac{1}{a(z)}-1\right)x}{R+2z} \\ 0 & 0 & 1 & \frac{Q\left(\frac{1}{a(z)}-1\right)y}{R+2z} \\ 0 & \frac{Q\left(\frac{1}{a(z)}-1\right)x}{R+2z} & \frac{Q\left(\frac{1}{a(z)}-1\right)y}{R+2z} & \frac{1}{a(z)} \end{bmatrix}, \quad (3.67)$$

in displaced rectangular coordinates $\{ct, x, y, z\}$. By tagging the off diagonal elements with a dummy parameter Q , it is possible to trace the effect of these elements on the Riemann curvature tensor. Removing the off diagonal elements by setting $Q = 0$ in Eq.(3.67) makes it equivalent to the general metric for a uniformly accelerating frame, Eq.(3.48). Since the uniformly accelerating frame is flat this metric will also be flat.

MACSYMA was used to compute the Riemann Curvature tensor components for Eq.(3.67) and the results are given in Appendix A.3. By setting $Q = 0$ to remove the off-diagonal elements all components of the Riemann curvature tensor will also be zero and hence there would be no curvature. The diagonal elements of the metric are attributable to the curvature of the metric as the factor Q appears in each component of the Riemann tensor. This demonstrates that the curvature is still present even with the locality approximation of the strong equivalence principle and should in principle be detectable even over infinitesimal distances.

3.6 Comparison of Line Elements

The metric for a uniformly accelerating frame has been derived that, by the Strong Equivalence Principle, should be locally equivalent to the generalised metric derived in §3.2. To test this the gravitational metric has been transformed into the same displaced rectangular coordinate system as the accelerating frame, using only the locality approximation. Because these two metrics are in the same coordinate system, they can be compared term by term as a test of the strong equivalence principle.

Using Eq.(3.53), the line element for a uniformly accelerating frame is

$$dS^2 = -\alpha(z) c^2 dt^2 + dx^2 + dy^2 + \frac{1}{\alpha(z)} dz^2, \quad (3.68)$$

and similarly from Eq.(3.66) the gravitational line element in displaced rectangular coordinates is

$$dS^2 = -a(z) c^2 dt^2 + dx^2 + dy^2 + \frac{1}{a(z)} dz^2 + \left(\frac{1}{a(z)} - 1\right) \frac{2x dx dz + 2y dy dz}{R + 2z}. \quad (3.69)$$

As noted in §3.5, Eq.(3.69) contains off-diagonal elements that are a consequence of curvature not present in Eq.(3.68) which is flat, so the two metrics can not be identical. However the diagonal components of Eq.(3.69) and Eq.(3.68) are equivalent if $\alpha(z) = a(z)$, where $\alpha(z)$ comes from Eq.(3.52)

$$\alpha(z) = z \left[\frac{dA}{dr} \right]_{r=R} + k, \quad (3.70)$$

and $a(z)$ from Eq.(3.65)

$$a(z) = A \left(r \rightarrow R \sqrt{1 + \frac{2z}{R}} \right). \quad (3.71)$$

Both these functions depend on $A(r)$, the arbitrary function from the general gravitational metric in polar coordinates. An arbitrary function such as $A(r)$ can be expanded into a power series in the dependent variable. To express $A(r)$ as a power series in r , define

$$A(r) = \sum_{n=-\infty}^{\infty} C_{(n)} r^n \quad (3.72)$$

where $C_{(n)}$ is the constant coefficient of the n^{th} power of r . For example, in the Schwarzschild metric $A = 1 - \frac{2m}{r}$ as a power series expansion has the coefficients $C_0 = 1$, $C_{-1} = \frac{2m}{R}$ and all other $C_{(n)}$ are zero.

The general power series expansion of $A(r)$ will be used to determine the function $a(z)$ for the general metric in displaced rectangular coordinates. This can be achieved by substituting Eq.(3.64) for r in the power series expansion of $A(r)$, from Eq.(3.72). Each term r^n transforms as

$$\begin{aligned} r^n &= \left[R \sqrt{1 + \frac{2z}{R}} \right]^n \\ &= R^n \left[1 + \frac{2z}{R} \right]^{\frac{n}{2}}. \end{aligned} \quad (3.73)$$

Because $\frac{z}{R} \ll 1$ according to the locality approximation it is valid to use the binomial expansion

$$\left[1 + \frac{2z}{R} \right]^{\frac{n}{2}} = \left[1 + \frac{n}{2} \frac{2z}{R} + O\left(\frac{z}{R}\right)^2 + \dots \right]. \quad (3.74)$$

Restricting the binomial expansion Eq.(3.74) to first order in $\frac{z}{R}$, and using this to simplify Eq.(3.73) gives

$$\begin{aligned} r^n &= R^n \left[1 + \frac{n}{2} \frac{2z}{R} \right] \\ &= R^n + n z R^{n-1}. \end{aligned} \quad (3.75)$$

This can then be substituted into Eq.(3.75) into Eq.(3.72) transforms $A(r) \rightarrow a(z)$ as required,

$$\begin{aligned} a(z) &= \sum_{n=-\infty}^{\infty} C_{(n)} \left[R^n + n z R^{n-1} \right] \\ &= z \sum_{n=-\infty}^{\infty} C_{(n)} n R^{n-1} + \sum_{n=-\infty}^{\infty} C_{(n)} R^n \\ &= z \left[\frac{dA}{dr} \right]_{r=R} + A(R). \end{aligned} \quad (3.76)$$

Comparing $\alpha(z)$ in Eq.(3.70) with $a(z)$ in Eq.(3.76), the two functions are identical if $k = A(R)$, remembering R is a constant. Substituting these constants back into Eq.(3.70) gives

$$\alpha(z) = a(z) = z \left[\frac{dA}{dr} \right]_{r=R} + A(R), \quad (3.77)$$

showing that the diagonal elements of the transformed gravitational metric are identical to the metric for a linearly accelerating frame. Consequently, the general metric for a static and spherically symmetric gravitational field transformed into a displaced rectangular coordinate system, contains diagonal terms equivalent to a uniformly accelerating reference frame and off-diagonal elements responsible for curvature and geodesic deviation.

The general equivalence of these diagonal terms is not computed for the Schwarzschild and the Conformal metrics as an example.

3.6.1 Result for Schwarzschild metric

In the Schwarzschild metric, the line element is

$$dS^2 = - \left(1 - \frac{2m}{r}\right) c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (3.78)$$

and therefore

$$A(r) = 1 - \frac{2m}{r}. \quad (3.79)$$

Transforming $A(r)$ into $a(z)$ using Eq.(3.64),

$$\begin{aligned} a(z) &= 1 - 2m \left[R \sqrt{1 + \frac{2z}{R}} \right]^{-1} \\ &= 1 - \frac{2m}{R} \left[1 + \frac{2z}{R} \right]^{-\frac{1}{2}}. \end{aligned} \quad (3.80)$$

Applying the binomial approximation to Eq.(3.80) and expanding gives

$$\begin{aligned} a(z) &\simeq 1 - \frac{2m}{R} \left[1 - \frac{1}{2} \frac{2z}{R} \right] \\ &\simeq 1 - \frac{2m}{R} + \frac{2m}{R^2} z. \end{aligned} \quad (3.81)$$

Therefore the Schwarzschild line element in displaced rectangular coordinates as

$$\begin{aligned} dS^2 &= - \left(1 - \frac{2m}{R} + \frac{2m}{R^2} z\right) c^2 dt^2 + dx^2 + dy^2 + \left(1 - \frac{2m}{R} + \frac{2m}{R^2} z\right)^{-1} dz^2 \\ &\quad + \left(\frac{1}{1 - \frac{2m}{R} + \frac{2m}{R^2} z} - 1 \right) \frac{2x dx dz + 2y dy dz}{R + 2z}. \end{aligned} \quad (3.82)$$

After simplifying the off diagonal terms, this result agrees with Moreau et al. [21]. For the accelerating frame, substituting Eq.(3.79) into Eq.(3.70) gives

$$\begin{aligned} \alpha(z) &= z \left[\frac{dA}{dr} \right]_{r=R} + A(R) \\ &= \frac{2mz}{R^2} + 1 - \frac{2m}{R}, \end{aligned} \quad (3.83)$$

making the diagonal terms in the two metrics identical.

3.6.2 Result for Conformal metric

A conformal gravitational metric is given by Mannheim[16] in polar coordinates which has the line element

$$dS^2 = - \left(1 - \frac{2m}{r}\right) c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (3.84)$$

where

$$A(r) = 1 - \frac{2\beta - 3\beta^2\gamma}{r} + \gamma r - 3\beta\gamma - kr^2. \quad (3.85)$$

Rearranging Eq.(3.85) to make $A(r)$ a power series

$$A(r) = (2\beta + 3\beta^2\gamma) r^{-1} + (1 - 3\beta\gamma) r^0 + \gamma r^1 - k r^2.$$

Taking the derivative of this and substituting into Eq.(3.70) to determine α for the accelerating frame

$$\begin{aligned} \alpha(z) &= z \left[\frac{dA}{dr} \right]_{r=R} + A(R) \\ &= z \left[-\frac{2\beta + 3\beta^2\gamma}{R^2} + \gamma - 2kR \right] + 1 + \frac{-2\beta + 3\beta^2\gamma}{R} + \gamma R - 3\beta\gamma - kR^2. \end{aligned} \quad (3.86)$$

To calculate $a(z)$ for the metric in displaced rectangular coordinates, transform $A(r) \rightarrow a(z)$ using the substitution $r^n \simeq R^n + n z R^{n-1}$ from Eq.(3.75)

$$\begin{aligned} a(z) &= (2\beta + 3\beta^2\gamma) (R^{-1} - zR^{-2}) + (1 - 3\beta\gamma) + \gamma (R + z) - k (R^2 + 2zR) \\ &= \left(-\frac{(2\beta + 3\beta^2\gamma)}{R^2} + \gamma - 2kR \right) z + \frac{2\beta + 3\beta^2\gamma}{R} + 1 - 3\beta\gamma + \gamma R - kR^2. \end{aligned} \quad (3.87)$$

Again we have $\alpha = a$, and the diagonal elements of the two metrics agree.

3.7 Discussion

The most general form of metric for a static, spherically symmetric gravitational field has been derived, independent of any field equations in general relativity. With this metric, the radial acceleration at a point P of distance R from the centre of mass was calculated and used to derive the metric for an accelerating frame. By transforming the general gravitational metric into the same displaced rectangular coordinates as the accelerating frame, it has been possible to test the Strong Equivalence Principle by directly comparing elements of the two metrics. In transforming the metric into displaced rectangular coordinates, the only approximation made were that $\frac{x}{R}$, $\frac{y}{R}$ and $\frac{z}{R}$ terms were only taken to first order. This approximation was made in accordance with the Strong Equivalence Principle being defined only for sufficiently small distances about the origin of the accelerating frame.

Comparison of the two metrics found that they were not identical. Although the gravitational metric was found to have diagonal elements corresponding to the acceleration metric, there were also additional off-diagonal elements not present in the acceleration

metric. This means there is not a complete equivalence between an accelerating frame and a gravitational field, even locally over small distances.

Because the off-diagonal terms are proportional to x and y , the Strong Equivalence Principle would be true for purely radial motion where x and y both remain zero. This would be the case for Gravitational red-shift calculations, which could be performed using an accelerating frame such as the French Metric. There would be a maximal violation of the equivalence principle for motion in x and y with $z = 0$.

By computing the Riemann curvature tensor for gravitational metric in displaced rectangular coordinates, it was shown that the off-diagonal elements were entirely responsible for the curvature and geodesic deviation. Coordinate transforms can not effect the curvature of a space, so the presence of curvature in the transformed metric was to be expected. The fact that a uniformly accelerating frame is rigid and has no curvature was always an indication that the two metrics could not be equivalent. These results show that curvature is still present, no matter what restrictions are placed on the size of the interval.

It has been demonstrated conclusively, that the standard formulation of the Strong Equivalence Principle can not be relied upon to provide a rigid, accelerating frame in order to simplify calculations in general relativity. However, the transformed metric can be used in place of the equivalence principle to show the comparative effects of curvature terms, and acceleration terms in the metric.

The strong equivalence principle (SEP) is a generalisation of the weak equivalence principle (WEP) to include all forms of energy and momentum. It states that in a sufficiently small region of spacetime, the laws of physics are indistinguishable from those in a freely falling frame in a gravitational field. This principle is a cornerstone of general relativity and is essential for understanding the behavior of matter and energy in curved spacetime.

In the context of a generalised gravitational metric, the SEP is often expressed in terms of the local inertial frame. This frame is defined by a set of coordinates in which the metric tensor is locally flat, and the Christoffel symbols vanish. In such a frame, the equations of motion for particles and fields reduce to their special relativistic counterparts, demonstrating the local validity of special relativity.

The SEP has profound implications for the structure of spacetime and the nature of gravity. It implies that gravity is not a force in the traditional sense, but rather a manifestation of the curvature of spacetime. This view of gravity is central to Einstein's theory of general relativity and has been extensively tested through various experiments and observations.

In a generalised gravitational metric, the SEP is often used to derive the Einstein field equations, which relate the curvature of spacetime to the distribution of matter and energy. The principle ensures that the local physics is consistent with special relativity, while the global structure of spacetime is determined by the presence of mass and energy.

The SEP is also a key element in the development of alternative theories of gravity, such as scalar-tensor theories and modified gravity models. These theories often attempt to modify the Einstein field equations while maintaining the local validity of the SEP. The principle serves as a guiding constraint in the construction of these alternative models.

In summary, the strong equivalence principle is a fundamental principle of physics that underpins general relativity. It states that the laws of physics are locally indistinguishable from those in a freely falling frame. This principle is essential for understanding the behavior of matter and energy in curved spacetime and is a key element in the derivation of the Einstein field equations.

Chapter 4

Rate of Light Deflection in French Coordinates

The solar eclipse of 29th May 1919 marked a great occasion for Einstein's General Relativity Theory, which four years earlier had predicted that light was deflected by the presence of a gravitational field. Two expeditions were sent to the Islands of Sobral (off the coast of Brazil) and Principe (in the gulf of Guinea) to look for apparent shifts in the positions of stars passing close to the limb of the sun caused by this gravitational deflection. The deflection predicted by Einstein was only 1.7" for a light ray passing the limb of the sun, and would only be observed during a solar eclipse and with precise measurements. Observations from the two expeditions gave results in agreement with Einstein's predictions, and despite large uncertainties in the experimental results, were responsible for the subsequent rapid acceptance and popularisation of the theory.

Einstein's results for light deflection in his paper on General Relativity[8] give the angle of deflection as

$$\alpha = \frac{4m}{R}, \quad (4.1)$$

where R is the radial distance from the centre of mass at the point of closest approach. The quantity $m = GM/c^2$ is the *geometric mass*, where M is the mass of the source, G is Newton's gravitational constant and c is the vacuum speed of light. This result is in sharp contrast to his predictions five years earlier in a 1911 paper [7] of

$$\alpha = \frac{2m}{R}, \quad (4.2)$$

a value of exactly one half that given by general relativity. In calculating his 1911 result Einstein proposed a local equivalence between a stationary frame in a gravitational field, and a uniformly accelerating frame in the absence of a gravitational field. He used this equivalence to calculate the deflection of light due to gravity by considering the acceleration of light using the uniformly accelerating frame. When he later formulated general relativity, Einstein had realised that the Strong Equivalence Principle could only be applied to a uniform gravitational field.

4.1 The Rate of Deflection of Light

One of the propositions for extending the equivalence principle to arbitrary gravitational fields was the infinitesimal formulation. This states that there is an equivalence between a non-uniform gravitational field and a uniformly accelerating frame for infinitesimal displacements. In this chapter we consider the quantity $\frac{d\alpha}{ds}$, the rate of change of deflection angle α with respect to spatial displacement ds . The deflection angle α and ds are defined in displaced rectangular coordinates according to Fig.(4.1).

A simple treatment of the deflection of light using the equivalence principle is given by Comer and Lathrop[3]. Consider a photon travelling with velocity $v_x = c$ in the x

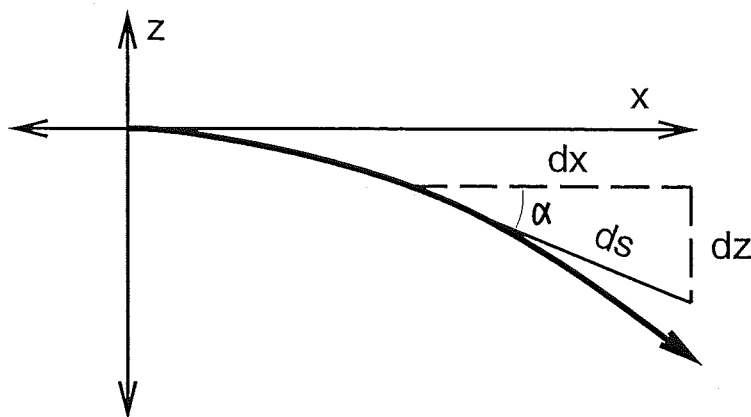


Figure 4.1: Definition of deflection angle in the French coordinate system. The curved path is a light-like geodesic.

direction, perpendicular to a uniform gravitational field $g = \frac{GM}{R^2} = \frac{m c^2}{R^2}$ directed in the y direction. The photon will experience an acceleration g and gain a velocity $v_y = g t$. If the photon travels only a short distance s , then v_x will not change significantly and the travel time will be $t \simeq s/c$. The deflection angle α will be given by

$$\alpha \simeq \frac{v_y}{v_x} \simeq \frac{g s/c}{c} = \frac{g s}{c^2}. \quad (4.3)$$

Comer and Lathrop go on to calculate the total deflection of light and get the same value as Einstein's 1911 calculation in the equivalence principle. Using the deflection angle in Eq.(4.3) the rate of deflection will be given by

$$\frac{d\alpha}{ds} = \frac{g}{c^2} = \frac{m}{R^2}. \quad (4.4)$$

Wood [31] calculates the rate of deflection for light in the Schwarzschild metric at the point of closest approach and obtains

$$\left[\frac{d\alpha}{ds} \right]_{\phi=0} = \frac{3m}{R^2}. \quad (4.5)$$

This is in disagreement with Eq.(4.4). Wood then goes on to calculate the rate of deflection using wavefront relativity in both the Fermi and French uniformly accelerating frames. In both cases the rate of deflection is given by Eq.(4.4) and not the true value as given in Eq.(4.5).

In the following section the rate of deflection of light is derived using a full geodesic analysis in the French metric. This will determine conclusively if the above discrepancies are indeed a consequence of a failure of the Strong Equivalence Principle in non uniform gravitational fields, even for infinitesimal distance intervals.

4.2 Rate of Deflection of Light in French Metric

Using the Strong Equivalence Principle to set up a uniformly accelerating reference frame equivalent to the Schwarzschild metric, gives the French metric. This is done in §3.6.1 and

is also given by Moreau et al.[21]. In the displaced rectangular coordinates, the French line element is

$$dS^2 = -Adt^2 + dx^2 + dy^2 + A^{-1}dz^2, \quad (4.6)$$

with

$$A = 1 - \frac{2m}{R} + \frac{2m}{R^2}z, \quad (4.7)$$

from Eq.(3.83). R is the displacement of the origin in the z axis and the m is the *geometric mass*, which is related to the mass of the gravitational source M by $m = \frac{GM}{c^2}$.

Light travels along null (light-like) geodesics which are characterised by the line element $dS^2 = 0$. Motion is governed by the geodesic equations which will be solved in order to determine the light path in the French metric and hence the curvature. The following are the geodesic equations for the French metric as generated by MACSYMA.

$$0 = \frac{d^2x}{dP^2}, \quad (4.8)$$

$$0 = \frac{d^2y}{dP^2}, \quad (4.9)$$

$$0 = \frac{d^2z}{dP^2} - \frac{1}{2A} \frac{dA}{dz} \left(\frac{dz}{dP} \right)^2 + \frac{A}{2} \frac{dA}{dz} \left(\frac{dt}{dP} \right)^2, \quad (4.10)$$

$$0 = \frac{d^2t}{dP^2} + \frac{1}{A} \frac{dA}{dz} \frac{dt}{dP} \frac{dz}{dP}. \quad (4.11)$$

In these equations P is an affine parameter that is related to the proper time by a linear transformation $\tau = c_1P + c_2$. Calculating the light deflection in the $x - z$ plane, we will restrict motion to $y = 0$. This automatically satisfies the geodesic equation in y , Eq.(4.9). Integrating Eq.(4.8) gives the solution

$$\frac{dx}{dP} = K \quad (4.12)$$

for an arbitrary integration constant K .

Multiplying Eq.(4.11) by $\frac{dP}{dt}$ and simplifying gives,

$$0 = \frac{dP}{dt} \frac{d}{dP} \left(\frac{dt}{dP} \right) + \frac{1}{A} \frac{dA}{dP} = \frac{d}{dP} \left[\ln \frac{dt}{dP} + \ln A \right]. \quad (4.13)$$

This has the solution, $\ln \left[A \frac{dt}{dP} \right] = \text{const}$, or more usefully the integration constant is taken up in the definition of the affine parameter P giving

$$\frac{dt}{dP} = \frac{1}{A}. \quad (4.14)$$

Similarly multiplying Eq.(4.10) by $\frac{2}{A} \frac{dz}{dP}$, substituting in Eq.(4.14) and then rearranging,

$$\begin{aligned} 0 &= \frac{2}{A} \frac{dz}{dP} \frac{d}{dP} \left(\frac{dz}{dP} \right) - \frac{1}{A^2} \frac{dz}{dP} \frac{dA}{dz} \left(\frac{dz}{dP} \right)^2 + \frac{dz}{dP} \frac{dA}{dz} \left(\frac{dt}{dP} \right)^2 \\ &= \frac{2}{A} \frac{dz}{dP} \frac{d}{dP} \left(\frac{dz}{dP} \right) - \frac{1}{A^2} \frac{dA}{dP} \left(\frac{dz}{dP} \right)^2 + \frac{dA}{dP} \left(\frac{1}{A} \right)^2 \\ &= \frac{d}{dP} \left[\frac{1}{A} \left(\frac{dz}{dP} \right)^2 - \frac{1}{A} \right]. \end{aligned} \quad (4.15)$$

This has a solution

$$\frac{1}{A} \left(\frac{dz}{dP} \right)^2 - \frac{1}{A} = E, \quad (4.16)$$

where E is an integration constant.

In considering the deflection of a ray of light, the path followed will be a light-like geodesic characterised by an interval $dS^2 = 0$. Inserting the affine parameter dP^2 defined above, and the solutions of the geodesic equations Eqs.(4.12,4.16, & 4.14) into the French metric for light-like intervals gives

$$\begin{aligned} 0 &= -A \left(\frac{dt}{dP} \right)^2 + \left(\frac{dx}{dP} \right)^2 + \left(\frac{dy}{dP} \right)^2 + A^{-1} \left(\frac{dz}{dP} \right)^2 \\ &= -A \left(\frac{1}{A} \right)^2 + K^2 + A^{-1} \left(\frac{dz}{dP} \right)^2 \\ &= E + K^2. \end{aligned} \quad (4.17)$$

It follows that for a light-like interval we have $K^2 = -E$.

From Fig.(4.1) it can be seen that the angle of deflection α can be defined close to the origin by $\alpha \simeq \tan \alpha = \frac{dz}{dx}$. Using Eq.(4.12) and Eq.(4.16)

$$\frac{dz}{dx} = \frac{dx}{dP} \frac{dP}{dz} = \frac{\sqrt{A \cdot E + 1}}{K} = \sqrt{-A - \frac{1}{E}}. \quad (4.18)$$

Substituting Eq.(4.7) for the function A in Eq.(4.18) gives

$$\frac{dz}{dx} = \sqrt{-1 + \frac{2m}{R} - \frac{2m}{R^2}z - \frac{1}{E}}. \quad (4.19)$$

Noting that at the origin in Fig.(4.1), the light path is tangential to the x -axis therefore $\left[\frac{dz}{dx} \right]_{z=0} = 0$. Evaluating at the origin and solving for E , gives

$$\begin{aligned} \left[\frac{dz}{dx} \right]_{z=0} &= 0 = \sqrt{-\left(1 - \frac{2m}{R}\right) - \frac{1}{E}} \\ &\Rightarrow \frac{1}{E} = -\left(1 - \frac{2m}{R}\right). \end{aligned} \quad (4.20)$$

Substituting Eq.(4.20) back into Eq.(4.19) gives the general expression for α ,

$$\alpha \simeq \frac{dz}{dx} = \sqrt{\frac{-2m}{R^2}z}. \quad (4.21)$$

This is consistent with the light-like geodesic depicted in Fig.4.1. Rearranging Eq.4.21 and integrating to solve for z

$$\begin{aligned} \int \frac{1}{\sqrt{z}} dz &= \int \sqrt{\frac{-2m}{R^2}} dx \\ &\Rightarrow z = \frac{-m}{2R^2} x^2, \end{aligned} \quad (4.22)$$

where the constant of integration is zero by virtue of the path crossing the origin requiring ($x = z = 0$).

Since $dS^2 = 0$ for a light-like interval, a new purely spatial interval is defined $ds^2 = dx^2 + dz^2$ and

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dz}{dx}\right)^2}. \quad (4.23)$$

Using the chain rule to calculate the rate of deflection gives

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dx} \frac{dx}{ds} = \frac{d^2z}{dx^2} \left(\frac{ds}{dx}\right)^{-1} = \frac{-m}{R^2} \left(\sqrt{1 + \left(\frac{dz}{dx}\right)^2}\right)^{-1}, \quad (4.24)$$

where Eq.(4.22) has been differentiated twice to calculate $\frac{d^2z}{dx^2}$. Evaluating this expression at the origin, where $\frac{dz}{dx} = 0$, the deflection of light per unit distance in the French metric is

$$\left[\frac{d\alpha}{ds}\right]_{x=0} = \frac{-m}{R^2}. \quad (4.25)$$

According to the Strong Equivalence Principle this should agree with the equivalent calculation for the Schwarzschild metric given by Eq.(4.5)

$$\left[\frac{d\alpha}{ds}\right]_{\phi=0} = \frac{-3m}{R^2}. \quad (4.26)$$

However it is obvious that these two results do not agree, there is a difference of a factor of three, which distinguishes between the two supposedly equivalent frames. The calculation of the rate of deflection of light appears to directly violate the Strong Equivalence Principle. Even for infinitesimal displacements a non-uniform gravitational field is not completely equivalent to a uniformly accelerating reference frame.

Einstein's erroneous calculation of the total deflection of light by the sun in his 1911 paper [7] shows the Strong Equivalence Principle for real gravitational fields does not hold with non-infinitesimal calculations. However, many formulations of the equivalence principle such as in Weinberg [29, p.98], still maintain that it holds for infinitesimal regions even in real gravitational fields. This result for the rate of deflection of light, an infinitesimal quantity, shows such formulations of the equivalence principle are wrong. In the following chapter, this non-equivalence of the equivalence principle is determined to be due to the presence curvature in a real gravitational field that is present even infinitesimally

Chapter 5

Deflection of Light with the Equivalence Principle

The Strong Equivalence Principle was shown to be flawed in chapter 4, even for an infinitesimal region. Calculating the rate of deflection of light due to a gravitational source produces different results in the Schwarzschild and French Metrics. However, since the French metric defines a uniformly accelerating frame in the Schwarzschild space, the Strong Equivalence Principle states that the rate of deflection in both frames should agree. Also, in § 3.6.1 we have shown that transforming the Schwarzschild metric into displaced rectangular coordinates gives a metric that is similar to the French metric, but contains additional off diagonal elements relating to curvature. In this chapter we calculate the rate of deflection of light using this transformed Schwarzschild metric and find that the previous discrepancy is due to the off diagonal terms.

By tagging the off diagonal terms in the transformed Schwarzschild metric, we trace their contribution to the rate of deflection of light. The result is that the rate of deflection is determined only in one-third part by the local inertial acceleration, and in two-thirds part by the off diagonal elements.

Calculating the rate of deflection of light per unit propagation distance in the French coordinates we had

$$\left[\frac{d\alpha}{ds} \right]_{x=0} = \frac{-m}{R^2}, \quad (5.1)$$

where α is defined as the angle between the light path and the x-axis and ds is a space-like interval. However, according to results obtained by Wood [31], the same calculation in the standard Schwarzschild coordinates gives

$$\left[\frac{d\alpha}{ds} \right]_{x=0} = \frac{-3m}{R^2}. \quad (5.2)$$

The calculated rate of deflection is an infinitesimal quantity and should be the same in both coordinates if there was a local equivalence as suggested by the Strong Equivalence Principle.

5.1 The Schwarzschild Metric in Displaced Rectangular Coordinates

Using the general metric for displaced rectangular coordinates, Eq.(3.67)

$$g_{\mu\nu} = \begin{bmatrix} -a(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{Q\left(\frac{1}{a(z)}-1\right)x}{R+2z} \\ 0 & 0 & 1 & \frac{Q\left(\frac{1}{a(z)}-1\right)y}{R+2z} \\ 0 & \frac{Q\left(\frac{1}{a(z)}-1\right)x}{R+2z} & \frac{Q\left(\frac{1}{a(z)}-1\right)y}{R+2z} & \frac{1}{a(z)} \end{bmatrix}, \quad (5.3)$$

the Schwarzschild metric is given using Eq.(3.81) by setting

$$a(z) = 1 - \frac{2m}{R} + \frac{2mz}{R^2}. \quad (5.4)$$

Geometric mass m is related to the mass M of the gravitational source by $m \equiv \frac{GM}{c^2}$ and G is Newton's gravitational constant. Using the dimensionless parameter Q to tag the off-diagonal elements will enable the effect of these terms to be traced through subsequent calculations. Setting $Q = 1$ will give results for the full Schwarzschild metric with curvature, or $Q = 0$ removes the curvature terms and reproduces the results in the French metric. Calculating the rate of deflection of light in terms of this parameter Q will show the relative contributions of inertial acceleration and curvature.

Simplifying the off-diagonal elements of Eq.(3.67) for the Schwarzschild metric leads to the same form of metric found by Moreau et al. [21]. Rearranging g_{13} from Eq.(5.3) and substituting in $(1 - a)$ from Eq.(5.4) gives

$$\begin{aligned} g_{13} &= \frac{\left(\frac{1}{a} - 1\right) xQ}{R + 2z} \\ &= \frac{xQ (1 - a)}{a (R + 2z)} \\ &= \frac{xQ \left(\frac{2m}{R} - \frac{2mz}{R^2}\right)}{a (R + 2z)} \\ &= \frac{2mxQ \left(1 - \frac{z}{R}\right)}{aR^2 \left(1 + \frac{2z}{R}\right)}. \end{aligned} \quad (5.5)$$

Using a first order binomial expansion, $\left(1 + \frac{2z}{R}\right)^{-1} \simeq \left(1 - \frac{2z}{R}\right)$ and Eq.(5.5) to first order in $\frac{z}{R}$, simplifies to

$$\begin{aligned} g_{13} &= \frac{2mxQ}{aR^2} \left(1 - \frac{z}{R}\right) \left(1 - \frac{2z}{R}\right) \\ &= \frac{2mxQ}{aR^2} - \frac{6mxzQ}{aR^3} + \frac{4mxz^2Q}{aR^4} \\ &\simeq \frac{2mxQ}{aR^2}. \end{aligned} \quad (5.6)$$

The last two terms for Eq.(5.6) were discarded because they contain the second order expression $\frac{z^2}{R^2}$. Repeating this for g_{23} gives a similar result with y substituted for x , and the metric can be written

$$g_{\mu\nu} = \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{2mxQ}{aR^2} \\ 0 & 0 & 1 & \frac{2myQ}{aR^2} \\ 0 & \frac{2mxQ}{aR^2} & \frac{2myQ}{aR^2} & \frac{1}{a} \end{bmatrix}. \quad (5.7)$$

This is exactly the Schwarzschild metric in displaced rectangular coordinates as obtained by Moreau et al. [21].

The metric can also be more conveniently represented as an infinitesimal interval by the line element

$$-c^2 d\tau^2 = -a(z)d(ct)^2 + dx^2 + dy^2 + \frac{1}{a(z)}dz^2 + \frac{4mQ}{a(z)R^2}(x dx dz + y dy dz). \quad (5.8)$$

The factor c in the time coordinate will be implicitly included by defining $ct \rightarrow t$, to simplify the equations. MACSYMA was used to compute the geodesic equations in the metric of Eq.(5.7). From appendix §A.4; the geodesic equation in t is

$$0 = \frac{d^2t}{dP^2} + \frac{1}{a} \frac{da}{dz} \frac{dt}{dP} \frac{dz}{dP}, \quad (5.9)$$

the geodesic equation in x is

$$\begin{aligned} 0 = & (4m^2x^2Q^2 + 4m^2y^2Q^2 - aR^4) \frac{d^2x}{dP^2} + a \frac{da}{dz} mR^2 x Q \left(\frac{dt}{dP} \right)^2 \\ & + 4m^2xQ^2 \left(\frac{dx}{dP} \right)^2 + 4m^2xQ^2 \left(\frac{dy}{dP} \right)^2 + \frac{1}{a} \frac{da}{dz} mR^2 x Q \left(\frac{dz}{dP} \right)^2, \end{aligned} \quad (5.10)$$

the geodesic equation in y is

$$\begin{aligned} 0 = & (4m^2x^2Q^2 + 4m^2y^2Q^2 - aR^4) \frac{d^2y}{dP^2} + a \frac{da}{dz} mR^2 y Q \left(\frac{dt}{dP} \right)^2 \\ & + 4m^2yQ^2 \left(\frac{dx}{dP} \right)^2 + 4m^2yQ^2 \left(\frac{dy}{dP} \right)^2 + \frac{1}{a} \frac{da}{dz} mR^2 y Q \left(\frac{dz}{dP} \right)^2, \end{aligned} \quad (5.11)$$

and the geodesic equation in z is

$$\begin{aligned} 0 = & (4m^2x^2Q^2 + 4m^2y^2Q^2 - aR^4) \frac{d^2z}{dP^2} - \frac{1}{2} a^2 \frac{da}{dz} R^4 \left(\frac{dt}{dP} \right)^2 - 2amR^2Q \left(\frac{dx}{dP} \right)^2 \\ & - 2amR^2 \left(\frac{dy}{dP} \right)^2 - \frac{1}{2a} \frac{da}{dz} (8m^2x^2Q^2 + 8m^2y^2Q^2 - aR^4) \left(\frac{dz}{dP} \right)^2. \end{aligned} \quad (5.12)$$

In these equations P is an affine parameter related to the proper time τ by the linear relation $\tau = AP + B$ for constants A and B . With $Q = 0$, Eqs.(5.9)-(5.12) reduce to the geodesic equations in the French metric given by Eqs.(4.11, 4.8, 4.9, 3.42).

Dividing the geodesic equation in the time coordinate, Eq.(5.9) through by $\frac{dt}{dP}$ and simplifying gives

$$\begin{aligned} 0 &= \frac{1}{\frac{dt}{dP}} \left[\frac{d}{dP} \left(\frac{dt}{dP} \right) + \frac{1}{a} \frac{da}{dz} \frac{dt}{dP} \frac{dz}{dP} \right] \\ &= \frac{1}{\frac{dt}{dP}} \frac{d}{dP} \left(\frac{dt}{dP} \right) + \frac{1}{a} \frac{da}{dP} \\ &= \frac{d}{dP} \left(\ln \frac{dt}{dP} + \ln a \right) \\ &= \frac{d}{dP} \left(\ln a \frac{dt}{dP} \right). \end{aligned} \quad (5.13)$$

This can then be integrated with respect to P to give $a \frac{dt}{dP} = \text{const}$, and the constant absorbed into the normalisation of the affine parameter P so

$$\frac{dt}{dP} = \frac{1}{a}. \quad (5.14)$$

It is sufficient to consider motion in two spatial dimensions, x and z . Setting $y = 0$, makes $\frac{dy}{dP} = 0$ and $\frac{d^2y}{dP^2} = 0$, providing a trivial solution to the geodesic equation in y , Eq.(5.11). Taking $y = 0$, and substituting $\frac{dt}{dP}$ from Eq.(5.14) into Eqs. (5.10) simplifies the geodesic equation for x to

$$\begin{aligned} 0 = & (4m^2x^2Q^2 - aR^4) \frac{d^2x}{dP^2} + \frac{1}{a} \frac{da}{dz} mR^2xQ \\ & + 4m^2xQ^2 \left(\frac{dx}{dP} \right)^2 + \frac{1}{a} \frac{da}{dz} mR^2xQ \left(\frac{dz}{dP} \right)^2. \end{aligned} \quad (5.15)$$

Similarly (5.12), the geodesic equation in z simplifies to

$$\begin{aligned} 0 = & (4m^2x^2Q^2 - aR^4) \frac{d^2z}{dP^2} - \frac{1}{2} \frac{da}{dz} R^4 - 2amR^2Q \left(\frac{dx}{dP} \right)^2 \\ & - \frac{1}{2a} \frac{da}{dz} (8m^2x^2Q^2 - aR^4) \left(\frac{dz}{dP} \right)^2. \end{aligned} \quad (5.16)$$

These equations can be solved for various boundary conditions and define all geodesic motion in the x, z plane.

Light rays travel along a restricted class of geodesics called null geodesics that are derived by equating the line element to zero. Setting $-c^2d\tau^2 = 0$, and $y = 0$ for motion to the x, z plane, the line element, Eq.(5.8) becomes

$$0 = -a(z)dt^2 + dx^2 + \frac{1}{a(z)}dz^2 + \frac{4mQ}{a(z)R^2}x dx dz. \quad (5.17)$$

Dividing Eq.(5.17) through by dP^2 , then substituting $\frac{dt}{dP} = \frac{1}{a}$ from Eq.(5.14) gives

$$0 = -\frac{1}{a} + \left(\frac{dx}{dP} \right)^2 + \frac{1}{a} \left(\frac{dz}{dP} \right)^2 + \frac{4mxQ}{aR^2} \frac{dx}{dP} \frac{dz}{dP}. \quad (5.18)$$

Rearranging Eq.(5.18), and multiplying through by a gives

$$1 = a \left(\frac{dx}{dP} \right)^2 + \frac{4mxQ}{R^2} \frac{dx}{dP} \frac{dz}{dP} + \left(\frac{dz}{dP} \right)^2. \quad (5.19)$$

This will be used to restrict the geodesic equations to null geodesics in order to calculate the light deflection.

Setting $Q = 0$ in Eqs.(5.15, 5.16, 5.19), reduces these equations to their French metric equivalent. Calculating the rate of change of deflection angle α , will give the same result as Eq.(4.25) in the French metric

$$\left[\frac{d\alpha}{ds} \right]_{x=0} = -\frac{m}{R^2}. \quad (5.20)$$

5.2 Geodesic Equations with Curvature

We now re-introduce curvature into the geodesic equations by setting $Q \neq 0$ in Eqs.(5.15, 5.16, 5.19) To solve these geodesic equations, first introduce new dimensionless coordinates with the following transformations,

$$\mu = \frac{m}{R}, \quad \xi = \frac{z}{R}, \quad \zeta = \frac{x}{R}, \quad \lambda = \frac{P}{R}, \quad (5.21)$$

and with these new coordinates we can derive the following quantities:

$$\begin{aligned} \frac{d}{dP} &= \frac{1}{R} \frac{d}{d\lambda}, \\ a &= 1 - \frac{2m}{R} + \frac{2mz}{R^2} = 1 - 2\mu + 2\mu\xi, \\ \frac{da}{dz} &= \frac{da}{d\xi} \frac{d\xi}{dz} = 2\mu \frac{1}{R} = \frac{2\mu}{R}, \\ \frac{dz}{dP} &= \frac{1}{R} \frac{d(R\xi)}{d\lambda} = \frac{d\xi}{d\lambda}, \quad \frac{d^2z}{dP^2} = \frac{d}{d\lambda} \frac{dz}{dP} = \frac{1}{R} \frac{d^2\xi}{d\lambda^2}, \\ \frac{dx}{dP} &= \frac{1}{R} \frac{d(R\zeta)}{d\lambda} = \frac{d\zeta}{d\lambda}, \quad \frac{d^2x}{dP^2} = \frac{d}{d\lambda} \frac{dx}{dP} = \frac{1}{R} \frac{d^2\zeta}{d\lambda^2}. \end{aligned} \quad (5.22)$$

Using the above equations to transform equation 5.15, the geodesic equation in ζ becomes

$$\begin{aligned} 0 &= (4\mu^2 R^4 \zeta^2 Q^2 - aR^4) \frac{1}{R} \frac{d^2\zeta}{d\lambda^2} + \frac{1}{a} \frac{2\mu}{R} R^4 \mu \zeta Q \\ &\quad + 4R^2 \mu^2 R \zeta Q^2 \left(\frac{d\zeta}{d\lambda} \right)^2 + \frac{1}{a} \frac{2\mu}{R} R^4 \mu \zeta Q \left(\frac{d\xi}{d\lambda} \right)^2. \end{aligned} \quad (5.23)$$

Simplifying this and multiplying through by aR^3 gives

$$0 = a(4\mu^2 \zeta^2 Q^2 - a) \frac{d^2\zeta}{d\lambda^2} + 2\mu^2 \zeta Q + 4a \mu^2 \zeta Q^2 \left(\frac{d\zeta}{d\lambda} \right)^2 + 2\mu^2 \zeta Q \left(\frac{d\xi}{d\lambda} \right)^2. \quad (5.24)$$

Repeating the above transformation for Eq.(5.16) the geodesic equation in ξ becomes

$$\begin{aligned} 0 &= (4\mu^2 R^4 \zeta^2 Q^2 - aR^4) \frac{1}{R} \frac{d^2\xi}{d\lambda^2} - \frac{1}{2} \frac{2\mu}{R} R^4 - 2a \mu R^3 Q \left(\frac{d\zeta}{d\lambda} \right)^2 \\ &\quad - \frac{1}{2a} \frac{2m}{R} (8R^2 \mu^2 R^2 x^2 Q^2 - aR^4) \left(\frac{d\xi}{d\lambda} \right)^2, \end{aligned} \quad (5.25)$$

then simplifying,

$$0 = a(4\mu^2 \zeta^2 Q^2 - a) \frac{d^2\xi}{d\lambda^2} - a\mu - 2a^2 \mu Q \left(\frac{d\zeta}{d\lambda} \right)^2 - \mu (8\mu^2 \zeta^2 Q^2 - a) \left(\frac{d\xi}{d\lambda} \right)^2. \quad (5.26)$$

The transformation of the Schwarzschild metric into the displaced rectangular coordinates in chapter 3 was only defined for a small region about the origin. The metric

in these coordinates neglected second order and higher terms in $\frac{x}{R}$, $\frac{y}{R}$ and $\frac{z}{R}$ [21]. This approximation can be carried into the dimensionless coordinate system without effecting the calculation because the rate of deflection is an infinitesimal quantity about the origin. In addition, the geodesic equations will be restricted to a weak field approximation, that is $\mu = \frac{m}{R} \ll 1$. Neglecting terms $O(\zeta)^2$, $O(\xi)^2$, $O(\mu)^2$ and $O(\mu\xi)$, the geodesic equation (5.24) in ζ becomes

$$0 = -a^2 \frac{d^2\zeta}{d\lambda^2} = -(1 - 4\mu) \frac{d^2\zeta}{d\lambda^2}, \quad (5.27)$$

after substituting $a(z) \equiv 1 - 2\mu + 2\mu\xi$. Similarly the geodesic equation (5.26) in ξ becomes

$$\begin{aligned} 0 &= -a^2 \frac{d^2\xi}{d\lambda^2} - a\mu - 2a^2\mu Q \left(\frac{d\zeta}{d\lambda} \right)^2 + a\mu \left(\frac{d\xi}{d\lambda} \right)^2 \\ &= -(1 - 4\mu) \frac{d^2\xi}{d\lambda^2} - \mu - 2\mu Q \left(\frac{d\zeta}{d\lambda} \right)^2 + \mu \left(\frac{d\xi}{d\lambda} \right)^2, \end{aligned} \quad (5.28)$$

and the line element Eq.(5.19) becomes

$$1 = (1 - 2\mu) \left(\frac{d\zeta}{d\lambda} \right)^2 + 4\mu\zeta Q \frac{d\zeta}{d\lambda} \frac{d\xi}{d\lambda} + \left(\frac{d\xi}{d\lambda} \right)^2. \quad (5.29)$$

The geodesic equations, Eqs.(5.27, 5.28) can now be solved using Eq.(5.29) for the null geodesic constraint, and appropriate boundary conditions. The boundary conditions used are at $\lambda = 0$,

$$\xi = 0, \quad \zeta = 0, \quad \frac{d\zeta}{d\lambda} = 0. \quad (5.30)$$

Because μ is not a function of ζ , the solution to Eq.(5.27) is $\zeta = C_1\lambda + C_2$. To satisfy the boundary conditions (Eq.5.30) at $\lambda = 0$, we have $C_2 = 0$ and therefore

$$\begin{aligned} \zeta &= C_1\lambda \\ \frac{d\zeta}{d\lambda} &= C_1. \end{aligned} \quad (5.31)$$

The geodesic equation in ξ (Eq.5.28) is harder to solve as it is non linear, but the $\left(\frac{d\xi}{d\lambda}\right)^2$ term can be eliminated by constraining the geodesic equations to light-like geodesics. Substituting Eq.(5.31) into the line element Eq.(5.29)

$$1 = (1 - 2\mu)(C_1)^2 + 4\mu\zeta QC_1 \frac{d\xi}{d\lambda} + \left(\frac{d\xi}{d\lambda} \right)^2. \quad (5.32)$$

Rearranging this slightly,

$$\left(\frac{d\xi}{d\lambda} \right)^2 = 1 - (1 - 2\mu)(C_1)^2 - 4\mu\zeta QC_1 \frac{d\xi}{d\lambda}, \quad (5.33)$$

Then multiplying by μ and neglecting all terms higher than first order gives

$$\mu \left(\frac{d\xi}{d\lambda} \right)^2 = \mu - \mu(C_1)^2. \quad (5.34)$$

Now substituting Eq.(5.31) and Eq.(5.34) into the geodesic equation in ξ (Eq.5.28),

$$\begin{aligned} 0 &= -(1-4\mu) \frac{d^2\xi}{d\lambda^2} - \mu - 2\mu Q (C_1)^2 + \mu - \mu (C_1)^2 \\ &= -(1-4\mu) \frac{d^2\xi}{d\lambda^2} - \mu (2Q+1) (C_1)^2. \end{aligned} \quad (5.35)$$

Integrating and solving this for ξ gives

$$\begin{aligned} \xi &= -\frac{1}{2} \frac{\mu (2Q+1)}{(1-4\mu)} (C_1)^2 \lambda^2 + K_1 \lambda + K_2 \\ &= -\frac{1}{2} \frac{\mu (2Q+1)}{(1-4\mu)} (C_1)^2 \lambda^2. \end{aligned} \quad (5.36)$$

Where $K_1 = K_2 = 0$ from the boundary conditions in equation 5.30 when $\lambda = 0$. Now to express ξ as a function of ζ , we substitute in $\zeta = C_1 \lambda$ from Eq.(5.31),

$$\xi = -\frac{1}{2} \frac{\mu (2Q+1)}{(1-4\mu)} \zeta^2. \quad (5.37)$$

Using a Taylor expansion for $(1-4\mu)^{-1}$ and again keeping only first order terms in μ , Eq.(5.37) can be further simplified,

$$\begin{aligned} \xi &= -\frac{1}{2} \mu (1-4\mu)^{-1} (2Q+1) \zeta^2 \\ &= -\frac{1}{2} \mu (1+4\mu+16\mu^2+\dots) (2Q+1) \zeta^2 \\ &= -\frac{1}{2} \mu (2Q+1) \zeta^2. \end{aligned} \quad (5.38)$$

This is valid for a weak field approximation close to the origin of the displaced rectangular coordinate system.

5.3 Rate of Deflection with Curvature

It is now possible to show the effect of curvature on the rate of deflection of light in the Schwarzschild metric. Eq.(5.38), which defines a light-like geodesic, is valid for a weak field approximation close to the origin of the displaced rectangular coordinate system, so can be used to calculate the rate of deflection. Using the transformations in equation (5.21) to write Eq.(5.38) back in x and z coordinates gives

$$z = -\frac{1}{2} \frac{m}{R^2} (2Q+1) x^2. \quad (5.39)$$

Calculating the rate of deflection of light then proceeds in a similar way to the French metric. The angle of deflection α at a point is the angle between the x axis and the tangent to the light-like geodesic at that point, as defined in Figure 4.1. From this we have $\alpha \simeq \tan \alpha = \frac{dz}{dx}$, which we calculate from equation 5.39,

$$\alpha \simeq \frac{dz}{dx} = -\frac{m}{R^2} (2Q+1) x. \quad (5.40)$$

The rate of deflection is calculated with the spatial parameter $ds = \sqrt{dx^2 + dz^2}$ defined in Figure (4.1). Dividing this through by dx we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dz}{dx}\right)^2}. \quad (5.41)$$

Inverting this and using a Taylor series approximation,

$$\begin{aligned} \frac{dx}{ds} &= \left(1 + \left(\frac{dz}{dx}\right)^2\right)^{-\frac{1}{2}} \\ &\simeq 1 - \frac{1}{2} \left(\frac{dz}{dx}\right)^2 + \dots \end{aligned} \quad (5.42)$$

which is valid since $\frac{dz}{dx} \propto \frac{m}{R^2} \ll 1$ in the weak field approximation. Then substituting in equation (5.40) and again neglecting terms higher than first order in $\frac{m}{R}$,

$$\begin{aligned} \frac{dx}{ds} &\simeq 1 - \frac{1}{2} \left(-\frac{m}{R^2} (2Q + 1) x\right)^2 + \dots \\ &\simeq 1 - \frac{1}{2} \frac{m^2}{R^4} (2Q + 1)^2 x^2 + \dots \\ &\simeq 1. \end{aligned} \quad (5.43)$$

Differentiating equation (5.40) with respect to x ,

$$\frac{d\alpha}{dx} \simeq -\frac{m}{R^2} (2Q + 1). \quad (5.44)$$

The rate of deflection of light $\frac{d\alpha}{ds}$ can be calculated by multiplying Eqs.(5.44) and(5.43) according to the chain rule,

$$\begin{aligned} \frac{d\alpha}{ds} &= \frac{d\alpha}{dx} \frac{dx}{ds} \\ &\simeq -\frac{m}{R^2} (2Q + 1). \end{aligned} \quad (5.45)$$

This clearly shows the comparative effects of the diagonal and off-diagonal (tagged by Q) elements, to the rate of deflection of light. The diagonal elements only account for one-third of the deflection rate, with the remaining two-thirds coming from the off diagonal elements. According the Strong Equivalence Principle, only the diagonal terms are equivalent to the metric for the uniformly accelerating frame, which produces incorrect results. The effect of curvature in any real gravitational field can not be ignored, even locally for infinitesimal calculations, and there can be no Strong Equivalence Principle in such a metric.

Using Eq.(5.45) for the rate of deflection of light, the comparative effects of the inertial acceleration and curvature are apparent. Setting $Q = 1$ for the full Schwarzschild metric with curvature gives the rate of light deflection as

$$\frac{d\alpha}{ds} \simeq -\frac{3m}{R^2}, \quad (5.46)$$

which reproduces the result of Wood[31]. However, by removing the curvature through setting $Q = 0$, the result is

$$\frac{d\alpha}{ds} \simeq -\frac{m}{R^2}. \quad (5.47)$$

This is identical to the result obtained in chapter 4 for the French coordinate system.

5.4 Addendum: Off Diagonal Terms and Curvature

It was noted in chapter 3, that the off diagonal elements of the transformed Schwarzschild metric are exclusively related to curvature. However, Mannheim [14] has recently pointed out that the Riemann tensor elements are of order $(m/R)^2$ and to order (m/R) the metric in the displaced coordinates is flat. Consequently from the analysis in § 5.3, one cannot rigorously claim that the contribution to the rate of deflection from the off-diagonal elements of the metric is due to curvature because the solution of the null-geodesic equations was in a first order approximation in (m/R) . The $Q \neq 0$ contribution is simply an effect of the off-diagonal elements of the metric that remain in the weak-field limit. On the other hand, there is reason to believe that the result of Eq.(5.45) is valid in general, even though it was obtained in a weak-field approximation. This is simply because for $Q = 1$ it agrees with the general result calculated from the full Schwarzschild metric, Eq.(4.5).

Chapter 6

Conclusions

Throughout the course of this thesis we have been concerned with the meaning and application of the Strong Equivalence Principle (SEP) in general relativity. In essence, the principle can be stated

(a): *There exists a complete physical equivalence between a stationary frame in a **uniform** gravitational field, and a uniformly accelerating frame in the absence of a gravitational field.*

The context of the SEP is not an arbitrary gravitational field, but limited to a uniform gravitational field. Through the historical development of relativity, Einstein used the Strong Equivalence Principle to show that his *special relativity* was a ‘special’ case of *general relativity* for flat spacetime. However, a common miss-understanding of the Strong Equivalence Principle reformulates it for an local region in an arbitrary gravitational field

(b): *There is a complete physical equivalence locally, between a stationary frame in a gravitational field, and a uniformly accelerating frame in the absence of a gravitational field.*

Einstein’s first application of the SEP was to calculate the deflection of light by the sun which is a non-uniform gravitational field. This was the first example of trying to extend his SEP to non-uniform gravitational fields, however the result was incorrect, predicting only half the correct value later given by general relativity. Subsequent examples of calculations based on equivalence principle in non-uniform gravitational fields have also failed to produce the correct results, yet it has not been completely understood why that should be so. This thesis has demonstrated why the standard formulation of the SEP (a), can not be extended to arbitrary gravitational fields and thus provides a deeper understanding of why attempts to do so fail to produce the correct results.

A generalised metric was derived corresponding to the gravitational field external to a static, spherically symmetric mass source. This derivation was based only on geometrical considerations and is not dependent on the field equations, so is valid in any four dimensional metric theory of gravity. By transforming the generalised gravitational metric into to the displaced rectangular coordinates of the supposedly equivalent uniformly accelerating frame, the terms of the two metrics could be compared. The result of this was that the generalised metric was composed not only of diagonal terms corresponding to a uniformly accelerating frame, but also had off-diagonal terms. These off diagonal elements were demonstrated to be related to curvature.

In accordance with the locality requirement in the SEP (b), the transformed metric was only calculated to first order in displacements about the origin, however no weak field approximation was made. When a weak field approximation is made, the components Riemann curvature tensor all become equal to zero, making the metric flat. This indicates

that under a weak field approximation the transformed metric could be further transformed into the Minkowski metric and so would be completely equivalent to a uniformly accelerating frame which is also flat. The Strong Equivalence Principle then can be extended to a non-uniform arbitrary gravitational field using the weak field approximation and the locality approximation and we propose

The Strong Equivalence Principle: A stationary frame in the local, weak field limit of an arbitrary gravitational field has a complete physical equivalence to a uniformly accelerating frame in the absence of any gravitational field.

Without making a weak field approximation, transforming the gravitational metric into displaced rectangular coordinates makes it possible to compare the relative contributions of the ‘curvature’ and ‘acceleration’ effects of gravity in a calculation. This shows clearly that gravity can be decomposed into the acceleration that is our every day perception of gravity, and tidal effects that are not always so apparent. By tracking the effect of the off-diagonal elements of the metric tensor through a ‘tagging’ parameter Q , the curvature can be artificially removed in calculations to show its effect on the result.

As a test of the Strong Equivalence Principle (b), the rate of deflection of light in French coordinates for a uniformly accelerated frame was calculated. By using the rate of deflection we have a quantity that can be defined at a particular point, in accordance with the locality approximation. This is in contrast with Einstein’s calculation of the total deflection which is non-local. However, this rate of deflection in French coordinates was only one-third of the value obtained using the Schwarzschild metric in direct violation of the Strong Equivalence Principle (b).

Transforming the Schwarzschild metric into displaced rectangular coordinates, the comparative effect of curvature on the rate of deflection of light was determined. It was shown acceleration and curvature contributed to the rate of deflection of light according to the ratio 1:2. This is the first known example of a calculation making proper use of the Strong Equivalence Principle in a non-uniform gravitational field.

This thesis has presented a full and detailed treatment of Einstein’s Strong Equivalence Principle in general relativity. A new formulation of the equivalence principle was presented that extends it to non-uniform gravitational fields. This equivalence principle was shown to hold in the generalised metric for any static, spherically symmetric gravitational field. Calculations of the rate of deflection of light using the standard formulation of the Strong Equivalence Principle (b) show that this principle does not hold in the Schwarzschild metric. However, the results using our new formulation of the equivalence principle do hold in the Schwarzschild metric and in addition provide information on the effect of curvature on light deflection.

Appendix A

Results from MACSYMA

The following sections give the Connection Coefficients, Riemann curvature tensor components, and geodesic equations for metrics referred to in this thesis. These results were compiled using the component tensor package of MACSYMA 6.2. In all cases only the non zero components are recorded and the following symmetries are assumed.

Christoffel Symbols	$\Gamma_{ab}^c = \Gamma_{ba}^c$
Riemann Tensor	$R_{ijk}^m = -R_{ikj}^m$

A.1 Generalised Gravitational Metric

Metric in coordinates $\{0, 1, 2, 3\} = \{ct, r, \theta, \phi\}$

$$g_{\mu\nu} = \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & \frac{1}{A} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

Non zero Connection Coefficients

$$\Gamma_{01}^0 = \frac{1}{2A} \frac{dA}{dr}$$

$$\Gamma_{00}^1 = \frac{A}{2} \frac{dA}{dr}$$

$$\Gamma_{11}^1 = -\frac{1}{2A} \frac{dA}{dr}$$

$$\Gamma_{22}^1 = -Ar$$

$$\Gamma_{33}^1 = -Ar \sin^2 \theta$$

$$\Gamma_{12}^2 = \frac{1}{r}$$

$$\Gamma_{33}^2 = -\cos \theta \sin \theta$$

$$\Gamma_{13}^3 = \frac{1}{r}$$

$$\Gamma_{23}^3 = \frac{\cos \theta}{\sin \theta}$$

Non zero Riemann Tensor components

$$R^0_{001} = -\frac{A}{2} \frac{d^2 A}{dr^2}$$

$$R^0_{101} = -\frac{1}{2A} \frac{d^2 A}{dr^2}$$

$$R^0_{202} = -\frac{r}{2} \frac{dA}{dr}$$

$$R^0_{303} = -\frac{r}{2} \sin^2 \theta \frac{dA}{dr}$$

$$R^1_{212} = -\frac{r}{2} \frac{dA}{dr}$$

$$R^1_{313} = -\frac{r}{2} \sin^2 \theta \frac{dA}{dr}$$

$$R^2_{002} = -\frac{A}{2r} \frac{dA}{dr}$$

$$R^2_{112} = \frac{1}{2Ar} \frac{dA}{dr}$$

$$R^2_{323} = -(A-1) \sin^2 \theta$$

$$R^3_{003} = -\frac{A}{2r} \frac{dA}{dr}$$

$$R^3_{113} = \frac{1}{2Ar} \frac{dA}{dr}$$

$$R^3_{223} = A-1$$

Geodesic equations

$$\frac{d^2 ct}{d\tau^2} = -\frac{1}{A}$$

$$\frac{d^2 r}{d\tau^2} = -\frac{1}{2} A \frac{dA}{dr} \left(\frac{dct}{d\tau}\right)^2 + \frac{1}{2A} \frac{dA}{dr} \left(\frac{dr}{d\tau}\right)^2 + Ar \left(\frac{d\theta}{d\tau}\right)^2 + Ar \sin^2 \theta \left(\frac{d\phi}{d\tau}\right)^2$$

$$\frac{d^2 \theta}{d\tau^2} = \left(\frac{d\phi}{d\tau}\right)^2 \cos \theta \sin \theta - \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau}$$

$$\frac{d^2 \phi}{d\tau^2} = -\frac{2}{r} \frac{d\phi}{d\tau} \frac{dr}{d\tau} - 2 \frac{\cos \theta}{\sin \theta} \frac{d\theta}{d\tau} \frac{d\phi}{d\tau}$$

A.2 Uniformly Accelerating Flat Frame Metric

Metric in coordinates $\{0, 1, 2, 3\} = \{ct, x, y, z\}$.

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

Non zero Connection Coefficients

$$\Gamma^3_{00} = \frac{1}{2\beta} \frac{d\alpha}{dz}$$

$$\Gamma^0_{03} = \frac{1}{2\alpha} \frac{d\alpha}{dz}$$

$$\Gamma^3_{33} = \frac{1}{2\beta} \frac{d\beta}{dz}$$

Non zero Riemann Tensor components

$$R^0_{303} = \frac{\alpha \frac{d\alpha}{dz} \frac{d\beta}{dz} - 2\alpha \frac{d^2\alpha}{dz^2} \beta + \left(\frac{d\alpha}{dz}\right)^2 \beta}{4\alpha^2 \beta}$$

$$R^3_{003} = \frac{\alpha \frac{d\alpha}{dz} \frac{d\beta}{dz} - 2\alpha \frac{d^2\alpha}{dz^2} \beta + \left(\frac{d\alpha}{dz}\right)^2 \beta}{4\alpha \beta^2}$$

Geodesic equations

$$\frac{d^2 ct}{d\tau^2} = -\frac{1}{\alpha} \frac{d\alpha}{dz} \frac{dct}{d\tau} \frac{dz}{d\tau}$$

$$\frac{d^2 x}{d\tau^2} = 0$$

$$\frac{d^2 y}{d\tau^2} = 0$$

$$\frac{d^2 z}{d\tau^2} = -\frac{\frac{d\alpha}{dz} \left(\frac{dct}{d\tau}\right)^2 + \frac{d\beta}{dz} \left(\frac{dz}{d\tau}\right)^2}{2\beta}$$

A.3 Generalised Gravitational Metric in Displaced Rectangular Coordinates

Metric in coordinates $\{0, 1, 2, 3\} = \{ct, x, y, z\}$

$$g_{\mu\nu} = \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{(\frac{1}{a}-1)xQ}{2z+R} \\ 0 & 0 & 1 & \frac{(\frac{1}{a}-1)yQ}{2z+R} \\ 0 & \frac{(\frac{1}{a}-1)xQ}{2z+R} & \frac{(\frac{1}{a}-1)yQ}{2z+R} & \frac{1}{a} \end{bmatrix}$$

Non zero Connection Coefficients

$$\Gamma^0_{03} = \frac{1}{2a} \frac{da}{dz}$$

$$\Gamma^1_{00} = -\frac{(a-1)a \frac{da}{dz} x (2z+R) Q}{2(a-1)^2 (y^2+x^2) Q^2 - 2a(2z+R)^2}$$

$$\Gamma^1_{11} = \frac{(a-1)^2 x Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a(2z+R)^2}$$

$$\Gamma^1_{22} = \frac{(a-1)^2 x Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a(2z+R)^2}$$

$$\Gamma^1_{33} = \frac{x \left(2a \frac{da}{dz} z + 2 \frac{da}{dz} z + a \frac{da}{dz} R + \frac{da}{dz} R - 4a^2 + 4a \right) Q}{2(a-1)^2 a (y^2+x^2) Q^2 - 2a^2 (2z+R)^2}$$

$$\Gamma^2_{00} = -\frac{(a-1)a \frac{da}{dz} y (2z+R) Q}{2(a-1)^2 (y^2+x^2) Q^2 - 2a(2z+R)^2}$$

$$\Gamma^2_{11} = \frac{(a-1)^2 y Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a(2z+R)^2}$$

$$\Gamma^2_{22} = \frac{(a-1)^2 y Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a(2z+R)^2}$$

$$\Gamma^2_{33} = \frac{y \left(2a \frac{da}{dz} z + 2 \frac{da}{dz} z + a \frac{da}{dz} R + \frac{da}{dz} R - 4a^2 + 4a \right) Q}{2(a-1)^2 a (y^2+x^2) Q^2 - 2a^2 (2z+R)^2}$$

$$\Gamma^3_{00} = -\frac{a^2 \frac{da}{dz} (2z+R)^2}{2(a-1)^2 (y^2+x^2) Q^2 - 2a(2z+R)^2}$$

$$\Gamma^3_{11} = \frac{(a-1)a(2z+R)Q}{(a-1)^2 (y^2+x^2) Q^2 - a(2z+R)^2}$$

$$\Gamma_{22}^3 = \frac{(a-1) a (2z+R) Q}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2}$$

$$\Gamma_{33}^3 = \frac{2(a-1)(y^2+x^2) \left(2 \frac{da}{dz} z + \frac{da}{dz} R - 2a^2 + 2a\right) Q^2 + a \frac{da}{dz} (2z+R)^3}{2(a-1)^2 a (y^2+x^2) (2z+R) Q^2 - 2a^2 (2z+R)^3}$$

Non zero Riemann Tensor components

$$R_{101}^0 = \frac{(a-1) \frac{da}{dz} (2z+R) Q}{2(a-1)^2 (y^2+x^2) Q^2 - 2a (2z+R)^2}$$

$$R_{202}^0 = \frac{(a-1) \frac{da}{dz} (2z+R) Q}{2(a-1)^2 (y^2+x^2) Q^2 - 2a (2z+R)^2}$$

$$R_{303}^0 = \frac{\left(\begin{array}{c} -2(a-1)^2 a \frac{d^2a}{dz^2} (y^2+x^2) (2z+R) Q^2 \\ + (a-1)(a+1) \left(\frac{da}{dz}\right)^2 (y^2+x^2) (2z+R) Q^2 \\ -4(a-1)^2 a \frac{da}{dz} (y^2+x^2) Q^2 \\ +2a^2 \frac{d^2a}{dz^2} (2z+R)^3 \end{array} \right)}{4(a-1)^2 a^2 (y^2+x^2) (2z+R) Q^2 - 4a^3 (2z+R)^3}$$

$$R_{1001}^1 = \frac{(a-1)^3 a \frac{da}{dz} y^2 (2z+R) Q^3 - (a-1) a^2 \frac{da}{dz} (2z+R)^3 Q}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4}$$

$$R_{1002}^1 = \frac{(a-1)^3 a \frac{da}{dz} x y (2z+R) Q^3}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4}$$

$$R_{1003}^1 = \frac{\left(\begin{array}{c} 2(a-1)^3 a \frac{d^2a}{dz^2} x (y^2+x^2) (2z+R) Q^3 \\ - (a-1)^2 (a+1) \left(\frac{da}{dz}\right)^2 x (y^2+x^2) (2z+R) Q^3 \\ +4(a-1)^3 a \frac{da}{dz} x (y^2+x^2) Q^3 \\ -2(a-1) a^2 \frac{d^2a}{dz^2} x (2z+R)^3 Q \end{array} \right)}{4(a-1)^4 (y^2+x^2)^2 Q^4 - 8(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 4a^2 (2z+R)^4}$$

$$R_{1112}^1 = \frac{(a-1)^4 x y Q^4}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4}$$

$$R_{1113}^1 = \frac{(a-1)(a+1) \frac{da}{dz} x (2z+R)^2 Q^2 - 4(a-1)^2 a x (2z+R) Q^2}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4}$$

$$R_{1212}^1 = \frac{(a-1)^4 y^2 Q^4 - (a-1)^2 a (2z+R)^2 Q^2}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4}$$

$$R_{1223}^1 = \frac{(a-1)(a+1) \frac{da}{dz} x (2z+R)^2 Q^2 - 4(a-1)^2 a x (2z+R) Q^2}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4}$$

$$\begin{aligned}
R^1_{313} &= \frac{\left((a-1)^2 (a+1) \frac{da}{dz} y^2 (2z+R) Q^3 - 4 (a-1)^3 a y^2 Q^3 \right)}{2 (a-1)^4 a (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a^2 (y^2+x^2) (2z+R)^2 Q^2 + 2 a^3 (2z+R)^4} \\
R^1_{323} &= \frac{4 (a-1)^3 a x y Q^3 - (a-1)^2 (a+1) \frac{da}{dz} x y (2z+R) Q^3}{2 (a-1)^4 a (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a^2 (y^2+x^2) (2z+R)^2 Q^2 + 2 a^3 (2z+R)^4} \\
R^2_{001} &= - \frac{(a-1)^3 a \frac{da}{dz} x y (2z+R) Q^3}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4} \\
R^2_{002} &= \frac{(a-1)^3 a \frac{da}{dz} x^2 (2z+R) Q^3 - (a-1) a^2 \frac{da}{dz} (2z+R)^3 Q}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4} \\
R^2_{003} &= \frac{\left(\begin{array}{l} 2 (a-1)^3 a \frac{d^2a}{dz^2} y (y^2+x^2) (2z+R) Q^3 \\ - (a-1)^2 (a+1) \left(\frac{da}{dz} \right)^2 y (y^2+x^2) (2z+R) Q^3 \\ + 4 (a-1)^3 a \frac{da}{dz} y (y^2+x^2) Q^3 \\ - 2 (a-1) a^2 \frac{d^2a}{dz^2} y (2z+R)^3 Q \end{array} \right)}{4 (a-1)^4 (y^2+x^2)^2 Q^4 - 8 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 4 a^2 (2z+R)^4} \\
R^2_{112} &= \frac{(a-1)^2 a (2z+R)^2 Q^2 - (a-1)^4 x^2 Q^4}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4} \\
R^2_{113} &= \frac{(a-1) (a+1) \frac{da}{dz} y (2z+R)^2 Q^2 - 4 (a-1)^2 a y (2z+R) Q^2}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4} \\
R^2_{212} &= - \frac{(a-1)^4 x y Q^4}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4} \\
R^2_{223} &= \frac{(a-1) (a+1) \frac{da}{dz} y (2z+R)^2 Q^2 - 4 (a-1)^2 a y (2z+R) Q^2}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4} \\
R^2_{313} &= \frac{4 (a-1)^3 a x y Q^3 - (a-1)^2 (a+1) \frac{da}{dz} x y (2z+R) Q^3}{2 (a-1)^4 a (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a^2 (y^2+x^2) (2z+R)^2 Q^2 + 2 a^3 (2z+R)^4} \\
R^2_{323} &= \frac{\left(\begin{array}{l} (a-1)^2 (a+1) \frac{da}{dz} x^2 (2z+R) Q^3 - 4 (a-1)^3 a x^2 Q^3 \\ - a (a+1) \frac{da}{dz} (2z+R)^3 Q + 4 (a-1) a^2 (2z+R)^2 Q \end{array} \right)}{2 (a-1)^4 a (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a^2 (y^2+x^2) (2z+R)^2 Q^2 + 2 a^3 (2z+R)^4} \\
R^3_{001} &= - \frac{(a-1)^2 a^2 \frac{da}{dz} x (2z+R)^2 Q^2}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4} \\
R^3_{002} &= - \frac{(a-1)^2 a^2 \frac{da}{dz} y (2z+R)^2 Q^2}{2 (a-1)^4 (y^2+x^2)^2 Q^4 - 4 (a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2 a^2 (2z+R)^4}
\end{aligned}$$

$$\begin{aligned}
R^3_{003} &= \frac{\left(\begin{array}{c} 2(a-1)^2 a^2 \frac{d^2 a}{dz^2} (y^2+x^2) (2z+R)^2 Q^2 \\ -(a-1)a(a+1) \left(\frac{da}{dz}\right)^2 (y^2+x^2) (2z+R)^2 Q^2 \\ +4(a-1)^2 a^2 \frac{da}{dz} (y^2+x^2) (2z+R) Q^2 \\ -2a^3 \frac{d^2 a}{dz^2} (2z+R)^4 \end{array} \right)}{4(a-1)^4 (y^2+x^2)^2 Q^4 - 8(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 4a^2 (2z+R)^4} \\
R^3_{112} &= \frac{(a-1)^3 a y (2z+R) Q^3}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4} \\
R^3_{113} &= \frac{a(a+1) \frac{da}{dz} (2z+R)^3 Q - 4(a-1)a^2 (2z+R)^2 Q}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4} \\
R^3_{212} &= -\frac{(a-1)^3 a x (2z+R) Q^3}{(a-1)^4 (y^2+x^2)^2 Q^4 - 2(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + a^2 (2z+R)^4} \\
R^3_{223} &= \frac{a(a+1) \frac{da}{dz} (2z+R)^3 Q - 4(a-1)a^2 (2z+R)^2 Q}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4} \\
R^3_{313} &= \frac{4(a-1)^2 a x (2z+R) Q^2 - (a-1)(a+1) \frac{da}{dz} x (2z+R)^2 Q^2}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4} \\
R^3_{323} &= \frac{4(a-1)^2 a y (2z+R) Q^2 - (a-1)(a+1) \frac{da}{dz} y (2z+R)^2 Q^2}{2(a-1)^4 (y^2+x^2)^2 Q^4 - 4(a-1)^2 a (y^2+x^2) (2z+R)^2 Q^2 + 2a^2 (2z+R)^4}
\end{aligned}$$

Geodesic equation in t

$$\frac{d^2 ct}{d\tau^2} = -\frac{1}{a} \frac{da}{dz} \frac{dct}{d\tau} \frac{dz}{d\tau}$$

Geodesic equation in x

$$\begin{aligned} \frac{d^2 x}{d\tau^2} = & \frac{(a-1) a \frac{da}{dz} \left(\frac{dct}{d\tau}\right)^2 x (2z+R) Q}{2 (a-1)^2 (y^2+x^2) Q^2 - 2a (2z+R)^2} \\ & - \frac{x \left(2a \frac{da}{dz} z + 2 \frac{da}{dz} z + a \frac{da}{dz} R + \frac{da}{dz} R - 4a^2 + 4a\right) \left(\frac{dz}{d\tau}\right)^2 Q}{2 (a-1)^2 a (y^2+x^2) Q^2 - 2a^2 (2z+R)^2} \\ & - \frac{(a-1)^2 x \left(\frac{dx}{d\tau}\right)^2 Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} - \frac{(a-1)^2 x \left(\frac{dy}{d\tau}\right)^2 Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} \end{aligned}$$

Geodesic equation in y

$$\begin{aligned} \frac{d^2 y}{d\tau^2} = & \frac{(a-1) a \frac{da}{dz} \left(\frac{dct}{d\tau}\right)^2 y (2z+R) Q}{2 (a-1)^2 (y^2+x^2) Q^2 - 2a (2z+R)^2} \\ & - \frac{y \left(2a \frac{da}{dz} z + 2 \frac{da}{dz} z + a \frac{da}{dz} R + \frac{da}{dz} R - 4a^2 + 4a\right) \left(\frac{dz}{d\tau}\right)^2 Q}{2 (a-1)^2 a (y^2+x^2) Q^2 - 2a^2 (2z+R)^2} \\ & - \frac{(a-1)^2 \left(\frac{dx}{d\tau}\right)^2 y Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} - \frac{(a-1)^2 y \left(\frac{dy}{d\tau}\right)^2 Q^2}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} \end{aligned}$$

Geodesic equation in z

$$\begin{aligned} \frac{d^2 z}{d\tau^2} = & \frac{a^2 \frac{da}{dz} \left(\frac{dct}{d\tau}\right)^2 (2z+R)^2}{2 (a-1)^2 (y^2+x^2) Q^2 - 2a (2z+R)^2} \\ & - \frac{2 (a-1) (y^2+x^2) \left(2 \frac{da}{dz} z + \frac{da}{dz} R - 2a^2 + 2a\right) \left(\frac{dz}{d\tau}\right)^2 Q^2 + a \frac{da}{dz} (2z+R)^3 \left(\frac{dz}{d\tau}\right)^2}{2 (a-1)^2 a (y^2+x^2) (2z+R) Q^2 - 2a^2 (2z+R)^3} \\ & - \frac{(a-1) a \left(\frac{dy}{d\tau}\right)^2 (2z+R) Q}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} - \frac{(a-1) a \left(\frac{dx}{d\tau}\right)^2 (2z+R) Q}{(a-1)^2 (y^2+x^2) Q^2 - a (2z+R)^2} \end{aligned}$$

A.4 Schwarzschild Metric in Displaced Rectangular Coordinates

Metric in coordinates $\{0, 1, 2, 3\} = \{ct, x, y, z\}$

$$g_{\mu\nu} = \begin{bmatrix} -\frac{a}{R^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{2mxQ}{a} \\ 0 & 0 & 1 & \frac{2myQ}{a} \\ 0 & \frac{2mxQ}{a} & \frac{2myQ}{a} & \frac{a}{R^2} \end{bmatrix}$$

$$a = 2mz + R^2 - 2mR$$

Fully Expanded Metric

$$g_{\mu\nu} = \begin{bmatrix} -\frac{2mz}{R^2} + \frac{2m}{R} - 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{2mxQ}{R^2 \left(\frac{2mz}{R^2} - \frac{2m}{R} + 1 \right)} \\ 0 & 0 & 1 & \frac{2myQ}{R^2 \left(\frac{2mz}{R^2} - \frac{2m}{R} + 1 \right)} \\ 0 & \frac{2mxQ}{R^2 \left(\frac{2mz}{R^2} - \frac{2m}{R} + 1 \right)} & \frac{2myQ}{R^2 \left(\frac{2mz}{R^2} - \frac{2m}{R} + 1 \right)} & \frac{1}{\frac{2mz}{R^2} - \frac{2m}{R} + 1} \end{bmatrix}$$

Non zero Connection Coefficients

$$\Gamma^1_{00} = \frac{2m^2 x (2mz + R^2 - 2mR) Q}{R^2 (4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3)}$$

$$\Gamma^2_{00} = \frac{2m^2 y (2mz + R^2 - 2mR) Q}{R^2 (4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3)}$$

$$\Gamma^3_{00} = -\frac{m (2mz + R^2 - 2mR)^2}{R^2 (4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3)}$$

$$\Gamma^0_{03} = \frac{m}{2mz + R^2 - 2mR}$$

$$\Gamma^1_{11} = \frac{4m^2 x Q^2}{4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3}$$

$$\Gamma^2_{11} = \frac{4m^2 y Q^2}{4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3}$$

$$\Gamma^3_{11} = -\frac{2m (2mz + R^2 - 2mR) Q}{4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3}$$

$$\Gamma^1_{22} = \frac{4m^2 x Q^2}{4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3}$$

$$\Gamma^2_{22} = \frac{4m^2 y Q^2}{4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3}$$

$$\begin{aligned}\Gamma_{22}^3 &= -\frac{2m(2mz + R^2 - 2mR)Q}{4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3} \\ \Gamma_{33}^1 &= \frac{2m^2R^2xQ}{(2mz + R^2 - 2mR)(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)} \\ \Gamma_{33}^2 &= \frac{2m^2R^2yQ}{(2mz + R^2 - 2mR)(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)} \\ \Gamma_{33}^3 &= -\frac{m(8m^2y^2Q^2 + 8m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)}{(2mz + R^2 - 2mR)(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)}\end{aligned}$$

Non zero Riemann Tensor components

$$\begin{aligned}R_{101}^0 &= -\frac{2m^2Q}{4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3} \\ R_{202}^0 &= -\frac{2m^2Q}{4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3} \\ R_{303}^0 &= -\frac{4m^4(y^2 + x^2)Q^2}{(2mz + R^2 - 2mR)^2(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)} \\ R_{001}^1 &= -\frac{2m^2(2mz + R^2 - 2mR)Q(4m^2y^2Q^2 - 2mR^2z - R^4 + 2mR^3)}{R^2(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{002}^1 &= \frac{8m^4xy(2mz + R^2 - 2mR)Q^3}{R^2(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{003}^1 &= -\frac{8m^5x(y^2 + x^2)Q^3}{R^2(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{112}^1 &= \frac{16m^4xyQ^4}{(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{113}^1 &= -\frac{4m^3R^2xQ^2}{(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{212}^1 &= \frac{4m^2Q^2(4m^2y^2Q^2 - 2mR^2z - R^4 + 2mR^3)}{(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2} \\ R_{223}^1 &= -\frac{4m^3R^2xQ^2}{(4m^2y^2Q^2 + 4m^2x^2Q^2 - 2mR^2z - R^4 + 2mR^3)^2}\end{aligned}$$

$$\begin{aligned}
R^1_{313} &= \frac{2 m^2 R^2 Q (4 m^2 y^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)}{(2 m z + R^2 - 2 m R) (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^1_{323} &= - \frac{8 m^4 R^2 x y Q^3}{(2 m z + R^2 - 2 m R) (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{001} &= \frac{8 m^4 x y (2 m z + R^2 - 2 m R) Q^3}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{002} &= - \frac{2 m^2 (2 m z + R^2 - 2 m R) Q (4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{003} &= - \frac{8 m^5 y (y^2 + x^2) Q^3}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{112} &= - \frac{4 m^2 Q^2 (4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{113} &= - \frac{4 m^3 R^2 y Q^2}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{212} &= - \frac{16 m^4 x y Q^4}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{223} &= - \frac{4 m^3 R^2 y Q^2}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{313} &= - \frac{8 m^4 R^2 x y Q^3}{(2 m z + R^2 - 2 m R) (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^2_{323} &= \frac{2 m^2 R^2 Q (4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)}{(2 m z + R^2 - 2 m R) (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^3_{001} &= - \frac{4 m^3 x (2 m z + R^2 - 2 m R)^2 Q^2}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^3_{002} &= - \frac{4 m^3 y (2 m z + R^2 - 2 m R)^2 Q^2}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^3_{003} &= \frac{4 m^4 (y^2 + x^2) (2 m z + R^2 - 2 m R) Q^2}{R^2 (4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2} \\
R^3_{112} &= - \frac{8 m^3 y (2 m z + R^2 - 2 m R) Q^3}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}
\end{aligned}$$

$$R_{113}^3 = \frac{2 m^2 R^2 (2 m z + R^2 - 2 m R) Q}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}$$

$$R_{212}^3 = \frac{8 m^3 x (2 m z + R^2 - 2 m R) Q^3}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}$$

$$R_{223}^3 = \frac{2 m^2 R^2 (2 m z + R^2 - 2 m R) Q}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}$$

$$R_{313}^3 = \frac{4 m^3 R^2 x Q^2}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}$$

$$R_{323}^3 = \frac{4 m^3 R^1 y Q^2}{(4 m^2 y^2 Q^2 + 4 m^2 x^2 Q^2 - 2 m R^2 z - R^4 + 2 m R^3)^2}$$

Geodesic equation in t

$$0 = \frac{2 \frac{dct}{d\tau} m \frac{dz}{d\tau}}{2mz + R^2 - 2mR} + \frac{d^2ct}{d\tau^2}$$

Geodesic equation in x

$$\begin{aligned} 0 = & \frac{d^2x}{d\tau^2} \left(4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3 \right) \\ & + 4m^2 x \left(\frac{dy}{d\tau} \right)^2 Q^2 + 4m^2 x \left(\frac{dx}{d\tau} \right)^2 Q^2 + \frac{2m^2 R^2 x \left(\frac{dz}{d\tau} \right)^2 Q}{2mz + R^2 - 2mR} \\ & + \frac{2 \left(\frac{dct}{d\tau} \right)^2 m^2 x (2mz + R^2 - 2mR) Q}{R^2} \end{aligned}$$

Geodesic equation in y

$$\begin{aligned} 0 = & \frac{d^2y}{d\tau^2} \left(4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3 \right) \\ & + 4m^2 y \left(\frac{dy}{d\tau} \right)^2 Q^2 + 4m^2 \left(\frac{dx}{d\tau} \right)^2 y Q^2 + \frac{2m^2 R^2 y \left(\frac{dz}{d\tau} \right)^2 Q}{2mz + R^2 - 2mR} \\ & + \frac{2 \left(\frac{dct}{d\tau} \right)^2 m^2 y (2mz + R^2 - 2mR) Q}{R^2} \end{aligned}$$

Geodesic equation in z

$$\begin{aligned} 0 = & - \frac{m \left(\frac{dz}{d\tau} \right)^2 (8m^2 y^2 Q^2 + 8m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3)}{2mz + R^2 - 2mR} \\ & + \frac{d^2z}{d\tau^2} \left(4m^2 y^2 Q^2 + 4m^2 x^2 Q^2 - 2mR^2 z - R^4 + 2mR^3 \right) \\ & - 2m \left(\frac{dy}{d\tau} \right)^2 (2mz + R^2 - 2mR) Q - 2m \left(\frac{dx}{d\tau} \right)^2 (2mz + R^2 - 2mR) Q \\ & - \frac{\left(\frac{dct}{d\tau} \right)^2 m (2mz + R^2 - 2mR)^2}{R^2} \end{aligned}$$

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