DEGREE OF APPROXIMATION THEOREMS

FOR

APPROXIMATION WITH SIDE CONDITIONS

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ERRATA

for

R.K. Beatson, "Degree of Approximation Theorems for Approximation with Side Conditions", Ph. D. Thesis, University of Canterbury, Christchurch, New Zealand, 1977.

CHAPTER 2

Page	Line	Corrected Line
30	5	

$$\|\sin^{(j)}([\tfrac{\lambda}{8}]\ x) - p_\lambda^{(j)}(x)\|_{[-2,2]} \leq \frac{2^{\lambda-j+1}}{(\lambda-j+1)!}\ \|\sin^{(\lambda+1)}([\tfrac{\lambda}{8}]\ x)\|_{[-2,2]}$$

Page	Line	Corrected Line
30	7 - 9	there should be an extra factor of $(\frac{\lambda}{8})^{\kappa}$ on the right hand side of each of these lines.
30	12	$\|\sin^{(j)}([\frac{\lambda}{8}]\ x) - p_\lambda^{(j)}(x)\|_{[-2,2]} \leq C_2(3/4)^\lambda$

CHAPTER 3

Page	Line	Corrected Line
44	20	with $p_n(x) = \sum_{j=0}^{n} (f^{(j)}(r)(x-r)^j/j!)$
49	10	Since $p_n(x) \geqslant f(r(n))$ if $x \geqslant r(n)$ and n is sufficiently large, and
50	19	$x^* = (2/r)\ x - 1$, $g^*(x^*) = g(x)/(\|f^{(k)}\|_{[0,r]}(r/2)^k)$.

CHAPTER 4

Page	Line	
60	23	$\|L_n(f)^{(2-j)}\|_{1/4} \leq \bar{G}_k\ n\ \omega(f',n^{-1})\quad (j = 1,2)\ .\qquad (4.3.8')$
68	5 15	This part of the proof of Theorem 4.3 contained a logical error and should be replaced by the following twelve lines.

right hand side is non-negative. Lemma 4.1 implies that

$$\|g^{(j)}\|_{1/2} \leq C_{23} n^{k} \, \omega(h,n^{-1}) \ , \quad j = 0, \ldots, k-1 \ , \text{ if } h \in C[-\tfrac{1}{4},\tfrac{1}{4}] \ .$$

Hence using (4.3.16)

$$\|r\|_{1/4} \leq C_{24} n^{-k} \, \omega(h,n^{-1}) \ .$$

Let

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{24} \, n^{-k} \, \omega(h,n^{-1})$$

$p_n^{(k)}(x)$ is non negative on $[-\tfrac{1}{4},\tfrac{1}{4}]$ and by (4.3.31) p_n provides the first estimate of the theorem. Similarly when $h' \in C[-\tfrac{1}{4},\tfrac{1}{4}]$

$$\|g^{(j)}\|_{1/2} \leq C_{25} \, n^{k-1} \, \omega(h',n^{-1}), \quad \|r\|_{1/4} \leq C_{26} n^{-k-1} \, \omega(h',n^{-1})$$

and

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{26} n^{-k-1} \, \omega(h',n^{-1})$$

provides the second estimate of the theorem.

NOTE

It has been pointed out that section 4.4 repeats some work, as yet unsighted by the author, of D. Myers: "Comonotone and Co-convex Approximation", Ph.D. Thesis, Temple University 1975.

PREFACE

This thesis is a study of the degree of uniform linear approximation with side conditions. Many of the questions concerning best approximation with side conditions were settled before 1973. In contrast very few results concerning the degree of approximation with side conditions were known then (see for example [1], [2]). The present work has resulted from attempts to extend some of these previous results.

The unifying theme is the cost of imposing the constraints; that is the relationship between the degrees of approximation with and without constraints; both for particular classes of functions, and for individual functions. For example in Chapter 2 Jackson type theorems are obtained which imply that the orders of magnitude, of the degrees of approximation, of many classes of functions, are unaffected by the imposition of Hermite-Birkhoff interpolatory constraints. The degrees of approximation of an individual function with and without the constraints may however be of different orders of magnitude.

The side conditions considered fall into four categories namely: Lagrange interpolatory side conditions imposed on approximation from finite dimensional subspaces of $C(T)$, T compact Hausdorff; Hermite-Birkhoff interpolatory side conditions imposed on approximation by algebraic or trigonometric polynomials on finite intervals; the side condition "increasing to the right" imposed on approximation by algebraic polynomials on finite intervals (the results here are applied to rational approximation on $[0,\infty)$); and generalized monotonicity side conditions imposed on approximation by algebraic polynomials on finite intervals.

Jackson type estimates are obtained for the degree of approximation in each case. In addition, for the side conditions of an interpolatory type, best possible asymptotic bounds are found for the ratio of, the degree of approximation with side conditions, to, the degree of unconstrained approximation.

In the thesis as a whole all the proofs are very strongly of the constructive as opposed to the existence type, and most depend heavily on the properties of algebraic or trigonometric polynomials. Several are based on proofs of the Jackson theorems for unconstrained approximation.

REFERENCES

1. J.T. LEWIS, Approximation with Convex Constraints, *SIAM Review*, **15** (1973), 193-217.

2. G.D. TAYLOR, Uniform Approximation with Side Conditions, *in* "Approximation Theory", ed. G.G. Lorentz, Academic Press, New York and London, 1973, pp.495-503.

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Christchurch, December, 1977

CONTENTS

ABSTRACT

This thesis is a study of the degree of uniform linear approximation with side conditions.

The side conditions considered fall into four categories namely: Lagrange interpolatory side conditions imposed on approximation from finite dimensional subspaces of C(T),T compact Hausdorff; Hermite-Birkhoff interpolatory side conditions imposed on approximation by algebraic or trigonometric polynomials on finite intervals; the side condition "increasing to the right" imposed on approximation by algebraic polynomials on finite intervals (the results here are applied to rational approximation on $[0,\infty)$); and generalized monotonicity side conditions imposed on approximation by algebraic polynomials on finite intervals.

Jackson type estimates are obtained for the degree of approximation in each case. In addition, for the side conditions of an interpolatory type, best possible asymptotic bounds are found for the ratio of, the degree of approximation with side conditions, to, the degree of unconstrained approximation.

CHAPTER I

THE ASYMPTOTIC COST OF LAGRANGE INTERPOLATORY SIDE CONDITIONS IN THE SPACE C(T).

§1.1 SUMMARY

Estimates are obtained for the increase in approximation error when Lagrange interpolatory side conditions are imposed. Let ν index an increasing sequence of finite dimensional approximation subspaces in C(T), whose union N is dense in C(T). If T is compact Hausdorff then the degree of approximation with Lagrange (function value) interpolatory side conditions, $E_\nu(f,A)$, is related to the degree of unrestricted uniform approximation, $E_\nu(f)$, by the inequality

$$\limsup_{\nu \to \infty} E_\nu(f,A)/E_\nu(f) \leqslant 2 \quad , \quad \forall f \in C(T)\backslash N.$$

The constant 2 cannot be decreased in general. In particular it is best possible for uniform approximation of

(i) entire (therefore continuous) periodic functions by trigonometric polynomials;

(ii) entire (therefore continuous) functions on any closed finite interval [a,b] by algebraic polynomials.

The results generalize to the additional cost of Lagrange interpolatory side conditions placed on approximations already satisfying restricted range side conditions, which are non-binding on f at the interpolation nodes.

Return to the case where only side conditions of an interpolatory type occur. Specify the interpolatory side conditions via a set of point evaluation functionals in $C(T)^*$, the dual of C(T). If the triple (C(T), N, set of point evaluations) has the property SAIN then there is a ν_1, not depending on $f \in C(T)$, such that $E_\nu(f,A) \leqslant 2E_\nu(f) \quad , \quad \forall \nu \geqslant \nu_1$. Some of these results have appeared in Beatson [1].

§1.2 INTRODUCTION

Let T be compact Hausdorff, and let h, k belong to C(T), the space of continuous real valued functions on T, and satisfy

$$h(t) < k(t), \quad t \in T. \tag{1.2.1}$$

Define the set of functions

$$X = \{g \in C(T) : h(t) \leqslant g(t) \leqslant k(t) \text{ for all } t \in T\}.$$

For convenience the case of no constraints will be denoted by X = C(T).

Consider an increasing sequence of finite-dimensional linear subspaces $\{N_\nu\}_{\nu=1}^{\infty}$ of C(T), whose union N is dense in C(T), and the corresponding sequence of convex sets $M_\nu = N_\nu \cap X$ whose union M is clearly dense in X. Given a finite set $\{x_1^*,\ldots, x_\gamma^*\}$ of elements of C(T)*, the dual of C(T), and $f \in C(T)$, define the set of functions

$$A = \{g \in C(T) : x_i^*(g) = x_i^*(f),\ i = 1,\ldots, \gamma\}.$$

For each $\nu = 1,2,3,\ldots$ define $E_\nu = E_\nu(f)$ by

$$E_\nu(f) = \inf_{g \in M_\nu} \|f - g\| \tag{1.2.2}$$

where

$$\|f - g\| = \sup_{t \in T} |f(t) - g(t)|.$$

Similarly if $M_\nu \cap A$ is nonempty define

$$E_\nu(f,A) = \inf_{g \in (M_\nu \cap A)} \|f - g\|. \tag{1.2.3}$$

Clearly $E_\nu(f) \leqslant E_\nu(f,A)$ whenever the right member exists. It is natural to ask if the ratio $E_\nu(f,A)/E_\nu(f)$ has upper bounds. Paszkowski [6,7] first posed such questions, showing in [7];

<u>THEOREM 1.1</u>. *If* $X = C[a,b]$, $M_\nu = N_\nu$ *is the space of algebraic polynomials of degree not exceeding* ν *for* $\nu = 1,2,3,\ldots$, *and* $\{x_i^*\}_{i=1}^{\gamma} = \{f_{t_i}\}_{i=1}^{\gamma}$ *are point evaluations*

$$x_i^*(f) = f(t_i), \quad a \leqslant t_i \leqslant b, \quad i = 1,\ldots, \gamma,$$

then there is a number ν_1, *not depending on* f, *such that for all*

$f \in C[a,b]$ *and* $\nu \geq \nu_1$

$$E_\nu(f,A) \leq 2E_\nu(f).$$

More recently Johnson [4] has obtained theorems of a similar nature in a more general context. For the space $C(T)$ a general theorem of Johnson [4, Theorem 2.1] reduces to

THEOREM 1.2. *If* $X = C(T)$, *then given any* $x_1^*,\ldots,x_\gamma^* \in C(T)^*$, *there exists a constant* C *and a positive integer* ν_1, *not depending on* f, *such that for every* f *in* $C(T)$ *and* $\nu \geq \nu_1$, $E_\nu(f,A)$ *is defined and*

$$E_\nu(f,A) \leq CE_\nu(f). \tag{1.2.4}$$

He also shows (Johnson [4, Theorem 3.5])

THEOREM 1.3. *Suppose* $X = C(T)$ *and* $f \in C(T)$. *Suppose* $\{x_i^*\}_{i=1}^\gamma = \{f_{t_i}\}_{i=1}^\gamma$ *are point evaluations on* $C(T)$ *such that*

$$|f(t_i)| < \|f\|, \quad i = 1,\ldots,\gamma, \tag{1.2.5}$$

then there exist C *and* ν_1 *such that for every* $\nu \geq \nu_1$ *there is an* $m_\nu \in N_\nu$ *for which*

$$m_\nu(t_i) = f(t_i), \quad i = 1,\ldots,\gamma, \tag{1.2.6}$$

$$\|m_\nu\| \leq \|f\|, \tag{1.2.7}$$

$$\|f - m_\nu\| \leq CE_\nu(f). \tag{1.2.8}$$

§1.3 GENERALIZATIONS OF PASZKOWSKI'S RESULT

§1.3.1 *Upper bounds on* $E_\nu(f,A)/E_\nu(f)$ *when* $E_\nu(f)$ *denotes the degree of unrestricted approximation.*

The following theorem shows the unknown constant C of Theorem 1.2 can be asymptotically replaced by the constant 2 of Paszkowski's theorem when all the x_i^* are point evaluations. This constant is best possible.

THEOREM 1.4. *If* $X = C(T)$, *and* $\{x_i^*\}_{i=1}^\gamma = \{f_{t_i}\}_{i=1}^\gamma$ *are point* *evaluations on* C(T), *then there exist* ν_1 *and a sequence* $\{\delta_\nu\}_{\nu=\nu_1}^{\infty}$, *not depending on* f, *such that for any* $f \in C(T)$, $E_\nu(f,A)$ *is defined for* $\nu \geq \nu_1$ *and*

$$E_\nu(f,A) \leq (2 + \delta_\nu)E_\nu(f), \qquad \nu \geq \nu_1, \tag{1.3.1}$$

where

$$\lim_{\nu\to\infty} \delta_\nu = 0.$$

The constant 2 *in the inequality above cannot be decreased.*

<u>*Proof of inequality (1.3.1)*</u>. First we construct some "bump functions" essential to the proof. An approximation satisfying the side conditions will be constructed by perturbing the best approximation using approximations to these "bump functions".

By the Hausdorff property of T we can find disjoint open sets $B_1, \ldots, B_\gamma$ containing $t_1, \ldots, t_\gamma$, respectively. $T \setminus B_j$ is closed $j = 1, \ldots, \gamma$ and so also are the singletons $\{t_j\}$. Since compact Hausdorff implies normal the Urysohn theorem (see, e.g., Dugundji [3]) guarantees the existence of functions f_j, $j = 1, \ldots, \gamma$ such that

$$f_j(t_j) = 1 \tag{1.3.2}$$

$$0 \leq f_j(t) \leq 1, \qquad t \in B_j, \tag{1.3.3}$$

$$f_j(t) = 0, \qquad t \in T \setminus B_j. \tag{1.3.4}$$

Consider the following theorem of Yamabe [9].

<u>THEOREM 1.5</u>. *Let* M *be a dense convex subset of a real normed linear space* X *and let* $x_1^*, \ldots, x_\gamma^* \in X^*$. *Then for each* $f \in X$ *and each* $\varepsilon > 0$ *there exists a* $g \in M$ *such that* $\|f - g\| < \varepsilon$ *and*

$$x_i^*(g) = x_i^*(f), \qquad i = 1, \ldots, \gamma.$$

By this theorem there exist functions in N arbitrarily close to f_j which interpolate to f_j at the points t_i, $i = 1, \ldots, \gamma$. Using also the finite dimensionality of the N_ν, there exists a ν_1 such that for $\nu \geq \nu_1$ there exist best approximations $q_{\nu j}$ from N to f_j with side conditions

$$q_{\nu j}(t_i) = f_j(t_i) = \delta_{ij}, \quad i = 1,\ldots, \gamma, \quad j = 1,\ldots, \gamma,$$

where δ_{ij} is the Kronecker delta, and also if

$$\delta_\nu = \max_{j=1,\ldots,\gamma} \|q_{\nu j} - f_j\|$$

then

$$\delta_\nu \to 0 \quad \text{as } \nu \to \infty.$$

Let $q_{\nu 0}$ be a best approximation to f from N_ν. For $\nu \geqslant \nu_1$ define

$$q_\nu = q_{\nu 0} + \sum_{j=1}^{\gamma} (f(t_j) - q_{\nu 0}(t_j))\, q_{\nu j}.$$

Then q_ν interpolates f at the points t_j, $j = 1,\ldots, \gamma$ and

$$\|q_\nu - f\| \leqslant \|q_{\nu 0} - f\| + \left\| \sum_{j=1}^{\gamma} (f(t_j) - q_{\nu 0}(t_j))\, q_{\nu j}\right\|.$$

Since the B_j are disjoint and

$$|q_{\nu j}(t)| \leqslant \begin{cases} 1 + \delta_\nu, & t \in B_j \\ \delta_\nu, & t \in T\backslash B_j \end{cases} \quad j = 1,\ldots, \gamma,$$

the second term on the right-hand side is bounded by

$$E_\nu(f)(1 + \gamma\delta_\nu).$$

This completes the proof of inequality (1.3.1).

Proof that the constant 2 of inequality (1.3.1) cannot be decreased.

A trigonometric approximation problem is constructed with $\limsup_{\nu\to\infty} E_\nu(f,A)/E_\nu(f) = 2$. f is chosen so that: the residual of best uniform approximation $(f-h_n)(x)$ oscillates between extreme points much faster than is possible for a polynomial of degree n; and $(f-h_n)(0) = E_n(f)$. Perturbing h_n to satisfy an interpolation condition at zero will then asymptotically increase the error in the approximation by E_n.

Let T be the unit circle ; $X = C(T)$; N_ν be the space of trigonometric polynomials of degree not exceeding ν ; and $A = \{g \in C(T) : g(0) = f(0)\}$ where $f \in C(T)$ will be chosen later.

Consider the sequence of functions $\{g_i(\theta) = \cos(3^i\theta)\}_{i=1}^{\infty}$.

Since

$$g_i(\theta) = (-1)^n \quad \text{when } \theta = \frac{n\pi}{3^i} \quad ; \quad n = 0, \pm 1, \pm 2, \ldots \quad ,$$

g_i has 2.3^i extremes on T. Also g_k, $k \geqslant i$ has all the extremes of g_i with the same sign as g_i. Let $\sum_{i=1}^{\infty} a_i$ be some convergent series of positive real numbers, and define

$$f(\theta) = \sum_{i=1}^{\infty} a_i \, g_{2^i}(\theta).$$

Consider the residual of best uniform approximation, to f from N_ν. This residual is characterized by the existence of a set of $2\nu+2$ points, its value at each such point being equal in magnitude to its norm but opposite in sign to its value at the two adjacent such points. Hence the best uniform approximation to f from $N_{3(2^i)}$ is

$$h_i(\theta) = \sum_{k=1}^{i} a_k \, g_{2^k}(\theta)$$

with

$$(f-h_i) = r_i = \sum_{k=i+1}^{\infty} a_k \, g_{2^k} \tag{1.3.5}$$

and

$$\| f - h_i \| = E_{3(2^i)} = \sum_{k=i+1}^{\infty} a_k \ .$$

Let t_i be any function in $N_{3(2^i)} \cap A$.

i.e. $t_i \in N_{3(2^i)}$ and $t_i(0) = f(0)$. Then

$$\| f - t_i \| = \| (f-h_i) - (t_i - h_i) \| = \| r_i - p_i \| \ , \tag{1.3.6}$$

where p_i, the perturbation of the best approximation, is

$$p_i = t_i - h_i .$$

The argument now proceeds using that

$$p_i(0) = r_i(0) = \| r_i \| \text{ while } r_i\left(\frac{\pi}{3(2^{i+1})}\right) = - \| r_i \| ;$$

and that the slope of $p_i(\theta)$ is related to its norm by Bernstein's inequality. We treat two cases.

Case 1. $\| p_i \| \geqslant 3E_{3(2^i)}$.

Then

$$\|r_i - p_i\| \geq \|p_i\| - \|r_i\| \geq 2E_{3^{(2^i)}} . \tag{1.3.7}$$

Case 2. $\|p_i\| \leq 3E_{3^{(2^i)}}$.

Then using Bernstein's inequality

$$p_i\left(\frac{\pi}{3^{(2^{i+1})}}\right) = E_{3^{(2^i)}} + 0\left(3E_{3^{(2^i)}} \cdot 3^{(2^i)} \cdot \frac{\pi}{3^{(2^{i+1})}}\right)$$

$$= E_{3^{(2^i)}}\left(1 + 0\left(\frac{1}{3^{(2^i)}}\right)\right) \quad \text{as } i \to \infty;$$

and since $r_i\left(\frac{\pi}{3^{(2^{i+1})}}\right) = - E_{3^{(2^i)}}$ we find

$$\|r_i - p_i\| \geq \left|(r_i - p_i)\left(\frac{\pi}{3^{(2^{i+1})}}\right)\right| = 2E_{3^{(2^i)}}(1 + o(1)). \tag{1.3.8}$$

By (1.3.6), (1.3.7), (1.3.8)

$$\limsup_{i \to \infty}(E_{3^{(2^i)}}(f,A)/E_{3^{(2^i)}}(f)) \geq 2. \qquad //$$

Remarks. The second part of the proof above requires only that each a_i be positive and that the series $\sum_{i=1}^{\infty} a_i$ converges. Hence there is no requirement that f be "non-smooth"; suitable choice of the a_i will in fact make f entire.

The example above may be transformed to show the constant 2 of Theorem 1.4 is also best possible for approximation by algebraic polynomials on closed finite intervals. Simply use the standard transformation between, even 2π periodic functions and functions defined on $[-1,1]$,

$$x = \cos\theta \quad , \quad f^*(x) = f(\theta).$$

If t is any trigonometric polynomial satisfying the interpolation condition its even part $\tilde{t}$ will also satisfy it. Hence standard arguments establish $E_\nu^*(f^*,A^*) = E_\nu(f,A)$; where $E_\nu^*(.)$ denotes the degree of approximation by algebraic polynomials of degree not exceeding ν, and $A^* = \{g \in C[-1,1] : f^*(1) = g(1)\}$. Since also $E_\nu^*(f^*) = E_\nu(f)$;

$$\limsup_{\nu\to\infty}(E_\nu^*(f^*,A^*)/E_\nu^*(f^*)) = \limsup_{\nu\to\infty}(E_\nu(f,A)/E_\nu(f)) \geqslant 2. \qquad //$$

§1.3.2 *Upper bounds on* $E_\nu(f,A)/E_\nu(f)$ *when* $E_\nu(f)$ *denotes the degree of a certain type of restricted range approximation.*

Let T,h,k, etc. be as in §1.2. Then

THEOREM 1.6. *If* $\{x_i^*\}_{i=1}^{\gamma} = \{f_{t_i}\}_{i=1}^{\gamma}$ *are point evaluations on* C(T) *and* $f \in X\setminus M$ *satisfies*

$$h(t_i) < f(t_i) < k(t_i), \quad i = 1,\ldots,\gamma,$$

then there exists a ν_1 *such that* $E_\nu(f,A)$ *is defined for* $\nu \geqslant \nu_1$ *and*

$$\limsup_{\nu\to\infty}(E_\nu(f,A)/E_\nu(f)) \leqslant 2. \qquad (1.3.9)$$

The constant 2 *in the inequality above cannot be decreased.*

Proof of inequality (1.3.9). Firstly we need some lemmas.

LEMMA 1.7. *Let* X, f, *and* $f(t_i)$, $i = 1,\ldots,\gamma$, *be as in the statement of Theorem 1.6. Then for each* $\varepsilon > 0$ *there exists* $g \in M$ *satisfying*

$$g(t_i) = f(t_i), \quad i = 1,\ldots,\gamma, \qquad (1.3.10)$$

$$\|g - f\| < \varepsilon. \qquad (1.3.11)$$

Proof. There exists $\varepsilon_0 > 0$ such that if $\varepsilon_0 \geqslant \varepsilon > 0$ the function

$$f_\varepsilon(t) = \begin{cases} k(t) - \varepsilon & \text{if } f(t) > k(t) - \varepsilon, \\ h(t) + \varepsilon & \text{if } f(t) < h(t) + \varepsilon, \\ f(t) & \text{otherwise,} \end{cases}$$

is continuous. Also there exists ε_1, $\varepsilon_0 \geqslant \varepsilon_1 > 0$ such that for $\varepsilon_1 \geqslant \varepsilon > 0$

$$f_\varepsilon(t) = f(t_i), \quad i = 1,\ldots,\gamma,$$

For such an ε, by Yamabe's theorem (Theorem 1.5) applied to C(T) and N there exists $g \in N$ such that

$$g(t_i) = f_\varepsilon(t_i),\ i = 1,\ldots,\gamma, \quad \text{and} \quad \|f_\varepsilon - g\| < \varepsilon.$$

Thus

$$g \in M, g(t_i) = f(t_i),\ i = 1,\ldots,\gamma \quad \text{and} \quad \|f - g\| < 2\varepsilon.$$

The result follows. //

Construct disjoint open sets $B_1, \ldots, B_\gamma$ containing $t_1, \ldots, t_\gamma$ respectively; and corresponding functions $f_j \in C(T)$, satisfying

$$f_j(t_j) = 1; \quad 0 \leqslant f_j(t) \leqslant 1,\ t \in B_j; \quad f_j(t) = 0,\ t \in T\backslash B_j\ ;$$

as in the first part of the proof of Theorem 1.4.

Given $f \in X$ satisfying the conditions of Theorem 1.6 define for each $j = 1, \ldots, \gamma$, constants

$$a_j^+ = (k(t_j) - f(t_j))/2, \quad a_j^- = (h(t_j) - f(t_j))/2,$$

and continuous functions

$$f_j^+(t) = \min[(a_j^+ f_j + f)(t), k(t)],$$
$$f_j^-(t) = \max[(a_j^- f_j + f)(t), h(t)].$$

By Lemma 1.7, and the finite dimensionality of the N_ν, there exists a ν_1 such that for $\nu \geqslant \nu_1$ there exist best approximations from $M_\nu, p^+_{\nu j}, p^-_{\nu j}$ of f_j^+, f_j^-, respectively, satisfying

$$p^+_{\nu j}(t_i) = f_j^+(t_i), \quad i = 1, \ldots, \gamma,$$

$$p^-_{\nu j}(t_i) = f_j^-(t_i), \quad i = 1, \ldots, \gamma,$$

and the normalized maximum error in these approximations

$$\delta_\nu = \max_{j=1,\ldots,\gamma} \max(\|p^+_{\nu j} - f_j^+\|, \|p^-_{\nu j} - f_j^-\|) / \min_{i=1,\ldots,\gamma} \min(|a_i^+|, |a_i^-|)$$

converges to zero as ν goes to infinity. Let $p_{\nu 0}$ be a best approximation to f from M_ν. Define

$$\lambda^+_{\nu j} = \max(0, (f(t_j) - p_{\nu 0}(t_j))/a_j^+),$$

$$\lambda^-_{\nu j} = \max(0, (f(t_j) - p_{\nu 0}(t_j))/a_j^-). \qquad (1.3.12)$$

We note that $\lambda^+_{\nu j}$, $\lambda^-_{\nu j}$ are both nonnegative and at least one is zero.

Define

$$p_{\nu j} = p^+_{\nu j}, \quad a_j = a_j^+, \quad \text{if } \lambda^+_{\nu j} > 0,$$

$$p_{\nu j} = p^-_{\nu j}, \quad a_j = a_j^-, \quad \text{if } \lambda^-_{\nu j} > 0,$$

$$p_{\nu j} = p^+_{\nu j}, \quad a_j = a_j^-, \quad \text{if } \lambda^+_{\nu j} = \lambda^-_{\nu j} = 0,$$

and

$$\lambda_{\nu j} = \lambda^+_{\nu j} + \lambda^-_{\nu j}, \quad j = 1,\ldots, \gamma.$$

We choose $\nu_2 \geqslant \nu_1$ so large that $\lambda_{\nu j}$ is less than 1 for $j = 1,\ldots, \gamma$ and $\nu \geqslant \nu_2$. Then

<u>LEMMA 1.8.</u> *Let* $\lambda_{\nu j}, p_{\nu j}, \nu_2$ *be defined as above. Then for all* $\nu \geqslant \nu_2$ *there exist*

$$\theta_i = \theta_i(\nu), \quad i = 0,\ldots, \gamma$$

such that

$$\theta_0 > 0; \quad \theta_i \geqslant 0, \quad i = 1,\ldots, \gamma, \tag{1.3.13}$$

$$\sum_{i=0}^{\gamma} \theta_i = 1, \tag{1.3.14}$$

$$\left(\sum_{i=0}^{\gamma} \theta_i p_{\nu i}\right)(t_j) = f(t_j), \quad j = 1,\ldots, \gamma, \tag{1.3.15}$$

$$\theta_i(\nu) \leqslant (1 + \varepsilon_\nu)\lambda_{\nu i}, \quad i = 1,\ldots, \gamma, \tag{1.3.16}$$

where

$$\varepsilon_\nu \to 0 \quad \text{as } \nu \to \infty. \tag{1.3.17}$$

<u>*Proof*</u>. The existence of $\{\theta_i\}_{i=0}^{\gamma}$ satisfying (1.3.13)-(1.3.15) can be established by induction.

Induction basis. Take $\theta_{00} = 1$.

Induction step. Given $\theta_{s0},\ldots, \theta_{ss} \geqslant 0$ such that

$$\theta_{s0} > 0; \quad \theta_{si} \geqslant 0, \ i = 1,\ldots, s; \ \sum_{0}^{s} \theta_{si} = 1, \tag{1.3.18}$$

$$\sum_{i=0}^{s} \theta_{si} p_{\nu i}(t_j) = f(t_j), \quad j = 1,\ldots, s, \tag{1.3.19}$$

for $s = \gamma_1$, $0 \leqslant \gamma_1 < \gamma$, we prove the existence of $\theta_{s+1,0},\ldots, \theta_{s+1,s+1}$ satisfying (1.3.18) and (1.3.19) for $s = \gamma_1 + 1$. Take

$$p(\alpha) = (1 - \alpha)\left(\sum_{0}^{s} \theta_{si} p_{\nu i}(t_{s+1})\right) + \alpha p_{\nu,s+1}(t_{s+1}) - f(t_{s+1}).$$

If $p(0) = 0$, take $\theta_{s+1,i} = \theta_{si}$, $i \leqslant s$; $\theta_{s+1,s+1} = 0$. If $p(0) \neq 0$, then by the choice of $p_{\nu,s+1}$ and since $\lambda_{\nu j} < 1$, $p(0)$ lies on one side of 0 and $p(1)$ on the other. Hence by linearity of the function $p(\alpha)$ there is a unique

α, $0 < \alpha < 1$ such that $p(\alpha) = 0$. Taking

$$\theta_{s+1,i} = \begin{cases} (1 - \alpha)\ \theta_{si}, & i = 0,\ldots, s, \\ \alpha & i = s + 1, \end{cases}$$

we have the induction step.

It remains to show (1.3.16) and (1.3.17).

We note that on entering the inductive step we deal with a function

$$\sum_{i=0}^{s} \theta_{si} p_{\nu i}$$

whose value at t_{s+1} lies on the line segment joining $p_{\nu_0}(t_{s+1})$ and $f(t_{s+1})$. This shows that each $\theta_{s+1,s+1}$ is less than or equal to θ'_{s+1} where θ'_{s+1} is chosen so that $(1 - \theta'_{s+1})p_{\nu_0} + \theta'_{s+1}p_{\nu,s+1}$ interpolates to f at t_{s+1}. Since the θ_{si}, $i = 0,\ldots, s$, decrease towards the $\theta_i = \theta_i(\nu)$ as s increases; it follows that

$$0 \leqslant \theta_i(\nu) \leqslant \theta'_i, \quad i = 1,\ldots, \gamma. \tag{1.3.20}$$

Now if $\lambda_{\nu i}$ is zero then so is θ'_i and from (1.3.20), (1.3.16) holds. If $\lambda_{\nu i}$ is nonzero so is θ'_i and

$$\lambda_{\nu i} = \frac{(f - p_{\nu_0})(t_i)}{(p_{\nu i} - f)(t_i)}, \qquad \theta'_i = \frac{(f - p_{\nu_0})(t_i)}{(p_{\nu i} - p_{\nu_0})(t_i)}.$$

Thus

$$\frac{\theta'_i}{\lambda_{\nu i}} = \frac{(p_{\nu i} - f)(t_i)}{(p_{\nu i} - p_{\nu_0})(t_i)} \to 1 \quad \text{as } \nu \to \infty \text{ through } \nu \text{ such that } \lambda_{\nu i} \neq 0.$$

This proves (1.3.16) and (1.3.17). //

From the above lemma and the convexity of M_ν there exists $(\nu \geqslant \nu_2)$

$$p_\nu{}^* = \sum_{i=0}^{\gamma} \theta_i(\nu)\ p_{\nu i}$$

in M_ν which interpolates to f at t_j, $j = 1,\ldots, \gamma$.

Write $|p_\nu{}^*(t) - f(t)| \leqslant \sum_{i=0}^{\gamma} \theta_i(\nu) |\ p_{\nu i}(t) - f(t)|$.

Using the estimate of the last lemma, namely

$$0 \leqslant \theta_i(\nu) < (1 + \varepsilon_\nu)\ \lambda_{\nu i}$$

where $\varepsilon_\nu \to 0$ as $\nu \to \infty$, the estimate

$$|p_{\nu i}(t) - f(t)| \leqslant |a_i|(|f_i(t)| + \delta_\nu)$$

$$\leqslant \begin{cases} |a_i|\delta_\nu & t \in T\backslash B_i, \\ |a_i|(1 + \delta_\nu) & t \in B_i, \end{cases} \qquad i = 1,\dots, \gamma;$$

and the estimates

$$\lambda_{\nu i}|a_i| \leqslant E_\nu(f), \quad i = 1,\dots, \gamma,$$

we obtain

$$|p^*_\nu(t) - f(t)| \leqslant \begin{cases} E_\nu + (1 + \varepsilon_\nu)(1 + \delta_\nu)E_\nu \\ \quad + (\gamma - 1)(1 + \varepsilon_\nu)\delta_\nu E_\nu, & \text{if } t \in \bigcup_{i=1}^{\gamma} B_i \\ E_\nu + \gamma(1 + \varepsilon_\nu)\delta_\nu E_\nu, & \text{if } t \in T\backslash\left(\bigcup_{i=1}^{\gamma} B_i\right) \end{cases}$$

and writing $\delta'_\nu = \gamma(1 + \varepsilon_\nu)\delta_\nu + \varepsilon_\nu$

$$\|p^*_\nu(t) - f(t)\| \leqslant (2 + \delta'_\nu)E_\nu.$$

This concludes the proof of the first part of Theorem 1.6.

Proof of the best possible nature of the constant 2 in inequality (1.3.9.). As in Theorem 1.4 the proof is to construct a trigonometric approximation problem with

$$\limsup_{\nu \to \infty} E_\nu(f,A)/E_\nu(f) = 2.$$

Note however that $E_\nu(f,A)$, $E_\nu(f)$ now have a different meaning. Since Theorem 1.6 does not apply to the example of Theorem 1.4, that example will be modified. Consider $p(x) = 0.2\,x^5 - 0.75\,x^3 + 0.5\,x$, this algebraic polynomial has a derivative

$$p'(x) = x^4 - 2.25\,x^2 + 0.5 = (x^2 - 0.25)(x^2 - 2)$$

with zeros at $x = \pm\, 0.5$ or $x = \pm\, \sqrt{2}$. Also $p(1) = -\,0.05 = -\,p(-1)$ while $p(0.5) = 0.1625 = -\,p(-0.5)$. Therefore the maximum modulus of $p(x)$ on $[-1,1]$ is 0.1625. Hence if $\sum_{i=1}^{\infty} a_i$ is any convergent series of positive numbers, and $g_{2^i}(\theta) = \cos(3^{(2^i)}\theta)$, the function

$$f(\theta) = -(0.2)(\cos\theta)^5 + 0.75(\cos\theta)^3 - 0.5\cos\theta + \sum_{i=1}^{\infty} a_i g_{2^i}(\theta)$$

will satisfy

$$\|f\| = .1625 + \sum_{i=1}^{\infty} a_i = f(\tfrac{4\pi}{3}) > f(0) > f(\tfrac{\pi}{3}) = -\|f\|.$$

Let

$$X = \{g \in C(T) : -\|f\| \leqslant g(t) \leqslant \|f\| \text{ for all } t \in T\},$$

$$A = \{g \in C(T) : g(0) = f(0)\},$$

and

$$N_\nu = \{\text{trigonometric polynomials of degree} \leqslant \nu\}.$$

Theorem 1.6 applies to this new example. The proof that $\limsup_{\nu\to\infty} E_\nu(f,A)/E_\nu(f) \geqslant 2$ imitates the proof in Theorem 1.4 from this point. //

Remarks. Again, as in Theorem 1.4, f can be chosen very smooth; and the example can be transformed to show that the constant 2 of Theorem 1.6 is best possible for approximation by algebraic polynomials on closed finite intervals.

By [4, Theorem 4.1], in which we may take the constant as 2, there follows

COROLLARY 1.9. *If* $X = C(T)$, $f \in C(T)\backslash N$, *and* $|f(t_i)| < \|f\|$ *for* $i = 1,\ldots,\gamma$ *then there exist a* ν_1 *and a sequence* $\{g_\nu\}_{\nu=\nu_1}^{\infty}$ *of* $g_\nu \in N_\nu$ *satisfying* $g_\nu(t_i) = f(t_i)$ *for* $i = 1,\ldots,\gamma$, $\|g_\nu\| \leqslant \|f\|$ *and* $\limsup_{\nu\to\infty}(\|f - g_\nu\|/E_\nu(f)) \leqslant 4$.

§1.3.3 *SAIN and the bound* $E_\nu(f,A)/E_\nu(f) \leqslant 2$ *for all* $\nu \geqslant \nu_1$ *when* $E_\nu(f)$ *denotes the degree of unrestricted approximation.*

Let N be a subspace of C(T), and $\{\delta_{t_1}, \ , \delta_{t_\gamma}\}$ be a finite set of point evaluations in the dual space $C(T)^*$. Following Deutsch and Morris [2] make the following definition: *The triple* $(C(T), N, \{\delta_{t_1},\ldots,\delta_{t_\gamma}\})$ *will be said to have the property* SAIN *(simultaneous approximation and* *interpolation which is norm preserving) provided* that the following condition is satisfied:

For each $f \in C(T)$ *and each* $\varepsilon > 0$, *there exists a* $g \in N$ *such that* $\|f - g\| < \varepsilon$, $\delta_{t_i}(f) = \delta_{t_i}(g)$ $(i = 1,\ldots,\gamma)$, *and* $\|f\| = \|g\|$.

Then

THEOREM 1.10. *Let T be compact Hausdorff,* $t_1,\ldots, t_\gamma$ *be distinct points of T and* $\{N_\nu\}_{\nu=1}^{\infty}$ *be an increasing sequence of subspaces of* $C(T)$ *whose union* N *is dense in* $C(T)$. *If the triple* $(C(T), N, \{\delta_{t_1},\ldots, \delta_{t_\gamma}\})$ *has the property SAIN then there exists a positive integer* ν_1, *such that for arbitrary real numbers* $y_1,\ldots, y_\gamma$ *there exists an* n *in* N_{ν_1} *satisfying*

(i) $n(t_i) = y_i$, $i = 1,\ldots, \gamma$,

and

(ii) $\|n\| = \max_{i = 1,\ldots,\gamma} |y_i|$.

Proof. Fix the positive integer γ, and $t_1,\ldots, t_\gamma$. Consider the set of 2^γ points in $\mathbf{R}^\gamma$

$\{\underline{x} = (\alpha_1,\alpha_2,\alpha_3, \ldots, \alpha_\gamma) : \alpha_i = -1 \text{ or } 1 ; \quad i = 1,\ldots, \gamma\}$.

Denote these points by $\underline{x}_1,\ldots, \underline{x}_{2^\gamma}$ and the j^{th} component of $\underline{x}_i$ by x_{ij}. Clearly the convex hull, or set of all finite convex linear combinations of $\{\underline{x}_i : i = 1,\ldots, 2^\gamma\}$, is the set

$\mathcal{H} = \{\underline{x} = (y_1,\ldots, y_\gamma) : |y_i| \leq 1 ; \quad i = 1,\ldots, \gamma\}$.

Corresponding to each $\underline{x}_i$ Urysohn's lemma implies the existence of a function $f_i(t) \in C(T)$ such that

(i) $f_i(t_j) = x_{ij}$, $j = 1,\ldots, \gamma$.

(ii) $\|f_i\| = 1$.

By the SAIN property there exists a ν_1 and $p_1, p_2,\ldots, p_{2^\gamma}$ in N_{ν_1} such that

(i) $p_i(t_j) = x_{ij}$, $i = 1,\ldots, 2^\gamma$, $j = 1,\ldots, \gamma$,

and

(ii) $\|p_i\| = 1$, $i = 1,\ldots, 2^\gamma$.

Consider any γ real numbers $y_1, y_2, \ldots, y_\gamma$. Either all the y_j are zero or at least one is non zero. In the first case the theorem is obvious; otherwise, since N_{ν_1} is closed under scalar multiplication, assume without loss of generality

$$\max\{|y_1|, |y_2|, \ldots, |y_\gamma|\} = 1.$$

Then $\underline{y} = (y_1, \ldots, y_\gamma)$ lies in $\mathcal{H}$. i.e. There exists $\theta_i \geqslant 0$; $i = 1, \ldots, 2^\gamma$; with $\sum_{i=1}^{2^\gamma} \theta_i = 1$ and $\sum_{i=1}^{2^\gamma} \theta_i \underline{x}_i = \underline{y}$. Hence

$$p(t) = \sum_{i=1}^{2^\gamma} \theta_i p_i(t)$$

belongs to N_{ν_1} and satisfies

$$p(t_j) = y_j, \quad j = 1, \ldots, \gamma,$$
$$\|p\| = 1 ,$$

as required. //

<u>THEOREM 1.11</u>. *If* $X = C(T)$ *and the triple* $(C(T), N, \{\delta_{t_1}, \ldots, \delta_{t_\gamma}\})$ *has the property* SAIN *then there exists an* ν_1 *not depending on* f, *such that for any* $f \in C(T)$ $E_\nu(f,A)$ *is defined and*

$$E_\nu(f,A) \leqslant 2E_\nu(f) \quad , \quad \forall \nu \geqslant \nu_1$$

Proof. Let ν_1 be the ν_1 of Theorem 1.10. Fix $\nu \geqslant \nu_1$. Let p_ν be a best uniform approximation to f from N_ν. Let $y_i = (f-p_\nu)(t_i)$ $i = 1, \ldots, \gamma$, and $n \in N_{\nu_1}$ be the function with $n(t_i) = y_i$, $\|n\| = \max_{i=1,\ldots,\gamma} |y_i|$; existing by virtue of Theorem 1.10. Then $p_\nu^* = p_\nu + n$ belongs to $A \cap N_\nu$ and

$$\|f - p_\nu^*\| \leqslant \|f - p_\nu\| + \|n\| \leqslant 2E_\nu(f).$$ //

Remark. Deutsch and Morris [2] discuss which triples have the property SAIN showing for example [2, Theorem 4.1]; that with C(T), T compact Hausdorff, and a finite number of point evaluations; it is sufficient for N to be a dense subalgebra of C(T). Hence in particular Theorem 1.11 applies to approximation by trigonometric polynomials.

§1.4 REMARKS ON THE CONSTANT 2 AND OTHER COST OF SIDE CONDITION THEOREMS.

It is interesting to note that the constant 2 plays, in several other cost of restricted range side condition theorems, a role similar to that which it plays in Theorems 1.4, 1.6.

For example: let $X = C[a,b]$,

$$A = \{g \in C[a,b] : g(t) \geqslant f(t) \quad \forall t \in [a,b]\},$$

and each N_ν be a Haar subspace of $C[a,b]$. A glance at the alternation theorem for one sided uniform approximation (see e.g. Lewis [5, Theorem 3.3]) shows that when the function: $e_0(t) = 1$, $\forall t \in [a,b]$, is in N_{ν_1} then $E_\nu(f,A) = 2E_\nu(f)$ for all $\nu \geqslant \nu_1$. If $e_0 \notin N$ then at least it can be arbitrarily well approximated by functions in N, and we can conclude

$$E_\nu(f,A) \leqslant (2 + \delta_\nu)E_\nu(f)$$

where $\{\delta_\nu > 0\}$ does not depend on $f \in C[a,b]$ and $\lim_{\nu\to\infty} \delta_\nu = 0$.

As a second example consider the cost of the side condition $\|\text{approximation}\| \leqslant \|f\|$. Since we assume N_ν is finite dimensional a best approximation to $f \in C(T)$ from N_ν exists. Hence Johnson [4, Theorem 4.1] becomes

<u>THEOREM 1.12</u>. *For each* f *in* C(T) *there is an integer* ν_1 *such that for every* $\nu \geqslant \nu_1$ *there exists* $n_\nu \in N_\nu$ *with* $\|n_\nu\| = \|f\|$ *and*

$$\|f - n_\nu\| \leqslant 2E_\nu(f).$$

Thus with $A = \{g \in C(T) : \|g\| \leqslant \|f\|\}$ it follows from Johnson's theorem

<u>COROLLARY 1.13</u>. *For each* f *in* C(T) *there is an integer* ν_1 *such that* $E_\nu(f,A) \leqslant 2E_\nu(f)$ *for all* $\nu \geqslant \nu_1$.

A similar corollary will hold for $E_\nu(f,B)$ with $B = \{g \in C(T) : \|g\| \geqslant \|f\|\}$.

The constant 2 of Corollary 1.13 is in fact best possible in that there exists a trigonometric approximation problem with

$$\limsup_{\nu \to \infty} \frac{E_\nu(f,A)}{E_\nu(f)} = 2.$$

As in the second part of Theorem 1.4 the proof consists of constructing a function f as far from satisfying the side condition as possible; and such that the residual of best approximation $(f-h_n)$ alternates much faster than can any polynomial of degree n. Unfortunately constructing the function f involves considerably more algebraic detail than did the corresponding part of Theorem 1.4.

Let T be the unit circle; $X = C(T)$; N_ν be the space of trigonometric polynomials of degree not exceeding ν; and $A = \{g \in C(T) : \|g\| \leqslant \|f\|\}$ where $f \in C(T)$ will be chosen later. Define the function

$$\operatorname{sgn}(x) = \begin{cases} 1 , & \text{if } x \text{ is positive,} \\ 0 , & \text{if } x = 0 , \\ -1 , & \text{if } x \text{ is negative.} \end{cases}$$

Define $c(i) = 3^{(2^i)}$, and recall from the proof of Theorem 1.4 that $\cos(c(k)\theta)$, $k > i$ has all the extremes of $\cos(c(i)\theta)$ and takes the same value as $\cos(c(i)\theta)$ at such points.

Consider a function

$$f(\theta) = \sum_{j=1}^{\infty} (-1)^j a_j \cos(c(j)\theta) \tag{1.4.1}$$

where $\{a_j > 0\}_{j=1}^{\infty}$ is to be chosen as follows:

Let $a_1 = 1$. If $a_1, \ldots, a_{i-1}$; $i \geqslant 2$ have been chosen then $a_i > 0$ is chosen so

$$\max\{a_i, \|a_i \cos''(c(i)\theta)\|\} = a_i(c(i))^2 \leqslant \frac{1}{10} a_{i-1} . \tag{1.4.2}$$

(1.4.2) implies $\sum_{j=2}^{\infty} a_j(c(j))^2 \leqslant \frac{1}{9}$; hence the uniform convergence of the two series

$$r_i(\theta) = \sum_{j=i+1}^{\infty} (-1)^j a_j \cos(c(j)\theta) \tag{1.4.3}$$

$$, i = 0,1,\ldots \quad ;$$

$$r_i''(\theta) = \sum_{j=i+1}^{\infty} (-1)^{j+1} a_j(c(j))^2\cos(c(j)\theta)$$

is guaranteed by the Weierstrass M test. This (see e.g. Widder [8, p305]) implies that $r_i(\theta)$, $r_i''(\theta)$ are given by the series above and continuous. Note $f(\theta) = r_0(\theta)$.

Consider

$$r_i''(\theta) = (-1)^{i+1} a_{i+1} \cos''(c(i+1)\theta) + \sum_{j=i+2}^{\infty} (-1)^j a_j \cos''(c(j)\theta).$$

Since the sum $\sum_{j=i+2}^{\infty}..$ on the righthand side above has norm not exceeding $a_{i+1}/9$ in modulus,

$$\mathrm{sgn}(r_i''(\theta)) = (-1)^{i+1} \mathrm{sgn} \cos''(c(i+1)\theta) \tag{1.4.4}$$

whenever $|\cos''(c(i+1)\theta)| \geqslant 1$; equivalently whenever

$$|\cos(c(i+1)\theta)| \geqslant c(i+1)^{-2} . \tag{1.4.5}$$

$r_i'(\theta) = 0$ at least at all the points where $\cos'(c(i+1)\theta) = 0$. Let θ^* be such a point then $r_i'(\theta) = \int_{\theta^*}^{\theta} r_i''(y)dy$. Hence (1.4.4), (1.4.5) above imply

$$\mathrm{sgn}(r_i'(\theta)) = (-1)^{i+1} \mathrm{sgn}(\cos'(c(i+1)\theta)), \tag{1.4.6}$$

whenever $|\cos(c(i+1)\theta)| \geqslant c(i+1)^{-2}$.

Also

$$\|r_i\| \geqslant a_{i+1} - \sum_{j=i+1}^{\infty} a_i \geqslant (8/9)a_{i+1} \tag{1.4.7}$$

and

$$|r_i(\theta)| \leqslant a_{i+1}\, c(i+1)^{-2} + \sum_{j=i+2}^{\infty} a_j \quad , \text{ if } |\cos(c(i+1)\theta)| \leqslant c(i+1)^{-2}.$$

$$\leqslant a_{i+1} ((1/9) + c(i+1)^{-2}) \tag{1.4.8}$$

(1.4.7), (1.4.8) together imply that any minima/maxima of $r_i(\theta)$ occuring where $|\cos(c(i+1)\theta)| \leqslant c(i+1)^{-2}$ cannot be global maxima or minima. (1.4.6) then implies $r_i(\theta)$ has global maxima and minima only at the extreme points θ^* of $\cos(c(i+1)\theta)$, where

$$\mathrm{sgn}(r_i(\theta^*)) = (-1)^{i+1} \mathrm{sgn}(\cos(c(i+1)\theta^*)), \tag{1.4.9}$$

and

$$|r_i(\theta^*)| = \|r_i\| = \left|\sum_{j=i+1}^{\infty} (-1)^j a_j\right| . \tag{1.4.10}$$

Let

$$h_i(\theta) = \sum_{j=1}^{i} (-1)^j a_j \cos(c(j)\theta). \tag{1.4.11}$$

The above results about the global maxima and minima of

$$r_i(\theta) = (f - h_i)(\theta)$$

show that $(f - h_i)(\theta)$ alternates between $\pm \|f - h_i\|$, $2c(i+1)$ times on T. Hence h_i is the best uniform approximation to f from $N_{c(i)}$, with error

$$E_{c(i)}(f) = \|r_i\| = \left|\sum_{j=i+1}^{\infty} (-1)^j a_j\right|. \tag{1.4.12}$$

Note that as i increases $h_i(0) = \sum_{j=1}^{i} (-1)^j a_j$ alternately underestimates and overestimates $f(0) = -\|f\|$, by $E_{c(i)}(f)$.

For the remainder of the proof choose i to be odd. Then

$$\|h_i\| \geqslant |h_i(0)| = \|f\| + E_{c(i)} > \|f\|. \tag{1.4.13}$$

Let t_i be any function in $N_{c(i)} \cap A$. i.e. $t_i \in N_{c(i)}$ and $\|t_i\| \leqslant \|f\|$. Then

$$\|f - t_i\| = \|(f - h_i) - (t_i - h_i)\| = \|r_i - p_i\| , \tag{1.4.14}$$

where p_i the perturbation of the best uniform approximation is given by

$$p_i = t_i - h_i .$$

The argument now proceeds using that

$$p_i(0) \geqslant E_{c(i)} = r_i(0) \quad \text{while} \quad r_i\left(\frac{\pi}{c(i+1)}\right) = -E_{c(i)} ;$$

and that the slope of $p_i(\theta)$ is related to its norm by Bernstein's inequality. Treat two cases:

Case 1. $\|p_i\| \geqslant 3E_{c(i)}$.

Then

$$\|r_i - p_i\| \geqslant \|p_i\| - \|r_i\| \geqslant 2E_{c(i)} . \tag{1.4.15}$$

Case 2. $\|p_i\| \leqslant 3E_{c(i)}$.

Then using Bernstein's inequality, and $c(i) = 3^{(2^i)}$,

$$\begin{aligned} p_i\left(\frac{\pi}{c(i+1)}\right) &\geqslant E_{c(i)} + 0\left(3E_{c(i)}\, c(i)\, \frac{\pi}{c(i+1)}\right) \quad \text{as } i \to \infty \\ &= E_{c(i)}\left(1 + \mathbf{0}\left(\frac{1}{3^{(2^i)}}\right)\right) \quad \text{as } i \to \infty. \end{aligned} \tag{1.4.16}$$

Now since $r_i\left(\frac{\pi}{c(i+1)}\right) = -E_{c(i)}$ we find

$$\|r_i - p_i\| \geq \left|(r_i - p_i)\left(\frac{\pi}{c(i+1)}\right)\right| \geq 2E_{c(i)}(1+o(1)). \qquad (1.4.17)$$

By (1.4.14), (1.4.15) and (1.4.17)

$$\limsup_{\nu \to \infty} \; E_\nu(f,A)/E_\nu(f) \geq 2.$$

Remarks. The function f constructed above also has $\limsup_{\nu \to \infty} E_\nu(f,B)/E_\nu(f) = 2$ where B is the set, $B = \{g \in C(T) : \|g\| \geq \|f\|\}$. Transformation of the example into an example concerning approximation by algebraic polynomials on $[-1,1]$ is accomplished in the usual fashion.

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CHAPTER 2

JACKSON TYPE THEOREMS FOR APPROXIMATION WITH HERMITE-BIRKHOFF INTERPOLATORY SIDE CONDITIONS

§2.1 SUMMARY

Jackson type theorems are obtained for approximation with Hermite-Birkhoff interpolatory side conditions. Let $E_\nu(f,A_\kappa)$ denote the degree of uniform approximation of a function, f, on an interval, by polynomials of degree not exceeding ν satisfying a fixed set of Hermite-Birkhoff interpolatory side conditions of order κ. Then

(i) If the polynomials are the trigonometric polynomials then there exists D_κ, not depending on $f \in C^{*\kappa}[-\pi,\pi]$, such that

$$E_\nu(f,A_\kappa) \leqslant D_\kappa \nu^{-\kappa} e_\nu(f^{(\kappa)}) \qquad \forall \text{ sufficiently large } \nu; \qquad (2.1.1)$$

where $e_\nu(f^{(\kappa)})$ is the degree of approximation of $f^{(\kappa)}$, by trigonometric polynomials of degree not exceeding ν, with constant part zero.

(ii) If the polynomials are the algebraic polynomials then there exists $L_k (k \geqslant \kappa)$, not depending on $f \in C^k[-1,1]$, such that

$$E_\nu(f,A_\kappa) \leqslant L_k \nu^{-k} \omega(f^{(k)}, \nu^{-1}) \qquad \forall \text{ sufficiently large } \nu. \qquad (2.1.2)$$

Also for certain restricted classes of function f and Hermite-Birkhoff interpolatory side conditions A_κ,

$$E_\nu(f,A_\kappa) \leqslant D_k \nu^{-k} e_\nu(g^{(k)}) \qquad \forall \text{ sufficiently large } \nu; \qquad (2.1.3)$$

where $g \in C^{*k}[-\pi,\pi]$ is defined by $g(\theta) = f(\cos\theta)$.

The direct comparison

$$E_\nu(f,A_\kappa) = O(E_\nu(f)^{1-\varepsilon}) \quad , \quad \forall \varepsilon > 0,$$

holds when (2.1.1) or (2.1.3) apply. In the other direction, given a set of interpolatory side conditions including at least one derivative constraint, *there does not exist* a sequence $\{G(\nu)\}$ such that

$$E_\nu(f,A_\kappa) = O(G(\nu)E_\nu(f)), \qquad \forall f \in C^{*\kappa}[-\pi,\pi].$$

§2.2 INTRODUCTION

Given a positive integer κ; a finite set $t_1, \ldots, t_\gamma$ of distinct points in $-\pi \leqslant t < \pi$; for each $i = 1, \ldots, \gamma$ a nonempty subset K_i of the set $\{0, 1, \ldots, \kappa\}$; and $f \in C^{*\kappa}[-\pi, \pi]$, the set of κ times continuously differentiable 2π periodic functions; define the set

$A_\kappa = \{g \in C^{*\kappa}[-\pi, \pi] : g^{(j)}(t_i) = f^{(j)}(t_i) ; j \in K_i; i = 1, \ldots, \gamma\}$.

Let N_ν be the space of trigonometric polynomials of degree not exceeding ν. For each $\nu = 0, 1, 2, 3, \ldots$ define

$$E_\nu(f) = \inf_{g \in N_\nu} \|f - g\| \tag{2.2.1}$$

where

$$\|f - g\| = \sup_{-\pi \leqslant t \leqslant \pi} |f(t) - g(t)|. \tag{2.2.2}$$

Similarly define $e_\nu(f)$ as the infimum of (2.2.2) over those g in N_ν with constant part zero; and if $N_\nu \cap A_\kappa$ is non empty, $E_\nu(f, A_\kappa)$ as the infimum of (2.2.2) over g in $N_\nu \cap A_\kappa$.

It is natural to ask whether Jackson type estimates hold for $E_\nu(f, A_\kappa)$; and whether they hold for the analogous algebraic polynomial problem. Also can $E_\nu(f, A_\kappa)$ and $E_\nu(f)$ be directly compared? These questions will be considered in the rest of the chapter.

§2.3 DEGREE OF TRIGONOMETRIC POLYNOMIAL APPROXIMATION WITH HERMITE-BIRKHOFF INTERPOLATORY SIDE CONDITIONS

<u>THEOREM 2.1.</u> *For each* $\kappa = 1, 2, 3, \ldots$ *there exists an* $M_\kappa > 0$, *and for each set of side conditions* A_κ, *a* $\nu_1 = \nu_1(\kappa, t_1, \ldots, t_\gamma)$, *not depending on the function* $f \in C^{*\kappa}[-\pi, \pi]$, *such that* $E_\nu(f, A_\kappa)$ *exists and satisfies*

$$E_\nu(f, A_\kappa) \leqslant M_\kappa \nu^{-\kappa} e_\nu(f^{(\kappa)}), \qquad \forall \nu \geqslant \nu_1.$$

Proof: We need the following version of one of the standard Jackson theorems (For the standard theorem see for example Cheney [3,pp.145-146]).

LEMMA 2.2. *For all positive integers* κ, *there exists a positive constant* c_κ, *and for each* $f \in C^{*\kappa}[-\pi,\pi]$ *a sequence of trigonometric polynomials* $\{T_\nu : T_\nu \in N_\nu\}$ *such that*

$$\|(f - T_\nu)^{(j)}\| \leqslant c_\kappa \frac{1}{\nu^{\kappa-j}} e_\nu(f^{(\kappa)}); \quad j = 0,1,\ldots,\kappa; \quad \nu = 1,2,3,\ldots .$$

Proof: Let j_ν be the Jackson kernel normalized so that

$$\int_{-\pi}^{\pi} j_\nu(t)\,dt = 1. \tag{2.3.1}$$

The corresponding trigonometric polynomial operator mapping $C^*[-\pi,\pi]$ into N_ν is

$$J_\nu(f,x) = \int_{-\pi}^{\pi} f(x+t)\,j_\nu(t)\,dt .$$

It is well known that there exists an $M > 0$ such that

$$\|f - J_\nu(f)\| \leqslant \frac{M}{\nu}\| f^{(1)}\| , \quad \nu = 1,2,3,\ldots \tag{2.3.2}$$

for all $f \in C^{*1}[-\pi,\pi]$. The proof now proceeds by induction.

Induction basis: Let t_ν be the best approximation to $f^{(\kappa)}$ from N_ν, with constant part zero. Let $P(g)$, $g \in C^*[-\pi,\pi]$, be the indefinite integral of g such that $\int_{-\pi}^{\pi} P(g) = 0$. Let P^κ be the κ-wise composition of operators P, and $S_\nu = P^\kappa(t_\nu)$. Then $S_\nu \in N_\nu$ and

$$\|f^{(\kappa)} - S_\nu^{(\kappa)}\| = e_\nu(f^{(\kappa)}), \quad \nu = 1,2,3,\ldots .$$

Induction step. If for some $m = 0,1,\ldots,\kappa-1$ and some $c_m > 0$, there exists a sequence of trig polynomials $\{S_\nu : S_\nu \in N_\nu\}$ such that

$$\|(f - S_\nu)^{(\kappa-j)}\| \leqslant c_m \nu^{-j} e_\nu(f^{(\kappa)}); \quad j = 0,\ldots,m; \quad \nu = 1,2,3,\ldots ;$$

then $(S_\nu + J_\nu(f - S_\nu)) \in N_\nu$ and with a constant $c_{m+1} \leqslant c_m(M+2)$

$$\|(f - (S_\nu + J_\nu(f - S_\nu)))^{(\kappa-j)}\| \leqslant c_{m+1}\, \nu^{-j} e_\nu(f^{(\kappa)})$$

for $j = 0,\ldots,m+1$; $\nu = 1,2,3,\ldots$. To prove this use the identity

$$[J_\nu(f-S_\nu)]^{(\kappa-j)} = J_\nu((f-S_\nu)^{(\kappa-j)}).$$

Now the induction step for $j = m+1$ follows from the Jackson theorem (2.3.2); and that for $j = 0,\ldots, m$ is a consequence of $\|J_\nu\| = 1$. //

Proof of Theorem 2.1 continued. Let T be the unit circle. Let f; $t_i, i = 1,\ldots, \gamma$; $K_i, i = 1,\ldots, \gamma$ satisfy the conditions of Theorem 2.1; and $\{T_\nu\}$ be a sequence of trigonometric polynomials providing the estimate of Lemma 2.2.

By the Hausdorff property of T there exist disjoint open sets $B_1,\ldots, B_\gamma$ in T containing $t_1,\ldots, t_\gamma$ respectively. Urysohn's theorem now guarantees the existence of functions $f_j \in C(T)$, $j = 1,\ldots, \gamma$, such that

$$f_j(t_j) = 1 ,$$
$$0 \leqslant f_j(t) \leqslant 1, \quad t \in B_j ,$$
$$f_j(t) = 0 , \quad t \in T\backslash B_j .$$

By the SAIN property of trigonometric approximation in conjunction with point evaluations, Deutsch and Morris [4; Theorem 4.1], there exists a ν_2 such that $\nu \geqslant \nu_2$ there exist approximations $q_{\nu j}$ from N_ν to the f_j satisfying

$$\|q_{\nu j}\| = 1;$$

$$q_{\nu j}(t_i) = f_j(t_i), \quad i = 1,\ldots, \gamma; \quad j = 1,\ldots, \gamma;$$

and if $\delta_\nu = \max\limits_{j=1,\ldots,\gamma} \|q_{\nu j} - f_j\|$ then

$$\lim_{\nu\to\infty} \delta_\nu = 0 . \tag{2.3.3}$$

Let $\lambda = [\nu/(\kappa+1)]$, $\lambda_1 = [\lambda/\kappa+1]$, where $[\,.]$ is the integral part function and ν_3 be so large that $\lambda_1 \geqslant \max(\nu_2, 1)$. Suppose throughout the following that $\nu \geqslant \nu_3$. Note

$$\lambda^j \leqslant \nu^j \leqslant (2(\kappa+1)\lambda)^j, \quad j = 1,\ldots, \kappa. \tag{2.3.4}$$

Define trigonometric polynomials of degree not exceeding ν

$$h_{ij} = (q_{\lambda_1,i})^{\kappa+1}(\sin \lambda(t - t_i))^j, \quad j = 0,\dots,\kappa; \quad i = 1,\dots,\gamma.$$

h_{ij} plays a role similar to that of the correcting "bump functions" of Theorems 1.4, 1.6. Loosely speaking it is a correcting "bump" for the jth derivative at t_i. From the definition

$$\|h_{ij}\| \leqslant 1 \quad ; \tag{2.3.5}$$

$$h_{ij}^{(r)}(t_e) = 0 \; ; \quad r = 0,\dots,\kappa; \quad e \neq i \quad ; \tag{2.3.6}$$

$$h_{ij}^{(r)}(t_i) = 0 \; , \quad r < j \quad ; \tag{2.3.7}$$

and

$$h_{i,j}^{(j)}(t_i) = j!\,\lambda^j \; . \tag{2.3.8}$$

Also by the Bernstein inequality, (2.3.5) and (2.3.4)

$$\|h_{ij}^{(k)}\| \leqslant \nu^k \leqslant (2(\kappa+1)\lambda)^k \; , \quad k = 0,1,2,\dots \; . \tag{2.3.9}$$

Now fix i. *Let* $j_1,\dots,j_p$ *be the members of* K_i *in ascending order. We seek a linear combination of* $h_{io},\dots,h_{i\kappa}$ *which will correct the values of* $T_\nu^{(j)}(t_i)$, $j \in K_i$ *to the* $f^{(j)}(t_i)$. From (2.3.7) we seek a solution **b** to the equation

$$\begin{bmatrix} h_{ij_1}^{(j_1)}(t_i) & 0\dots 0 \\ \vdots & \vdots \\ h_{ij_1}^{(j_p)}(t_i) \;\dots & h_{ij_p}^{(j_p)}(t_i) \end{bmatrix} \begin{bmatrix} b_{j_1} \\ \\ \\ b_{j_p} \end{bmatrix} = \begin{bmatrix} (f - T_\nu)^{(j_1)}(t_i) \\ \\ \\ (f - T_\nu)^{(j_p)}(t_i) \end{bmatrix} \tag{2.3.10}$$

Dividing the k-th row of the matrix above, and the k-th element of the product vector by $j_k!\,\lambda^{j_k}$; and using (2.3.8) the equation may be written

$$\begin{bmatrix} 1 & & \\ a_{21} & 1 & \\ & & \\ a_{p1} & a_{p,p-1} & 1 \end{bmatrix} \begin{bmatrix} b_{j_1} \\ \\ \\ b_{j_p} \end{bmatrix} = \begin{bmatrix} c_{j_1} \\ \\ \\ c_{j_p} \end{bmatrix} \tag{2.3.11}$$

Since the matrix $A = (a_{ke})$ above is lower triangular and has determinant one a solution exists. By (2.3.9) there exists an M, depending only on κ such that

$$|a_{ke}| \leq M, \quad k = 1,\ldots, p, \quad e = 1,\ldots, p.$$

By Lemma 2.2 there exists an L depending only on κ such that

$$|c_{j_k}| \leq Le_\nu(f^{(\kappa)})/\nu^\kappa \qquad , k = 1,\ldots, p.$$

Employing Cramer's rule;

$$|b_{j_k}| \leq (\kappa+1)!M^\kappa Le_\nu(f^{(\kappa)})/\nu^\kappa \qquad ; k = 1,\ldots, p. \tag{2.3.12}$$

Writing $H_i = \sum_{k=1}^{p} b_{j_k} h_{i,j_k}$, and using (2.3.12)

$$|H_i(t)| \leq \begin{cases} D_\kappa e_\nu(f^{(\kappa)})/\nu^\kappa , & t \in B_i , \\ D_\kappa \delta_{\lambda_1} e_\nu(f^{(\kappa)})/\nu^\kappa , & t \in T\backslash B_i , \end{cases} \tag{2.3.13}$$

where

$$D_\kappa = (\kappa+1)!(\kappa+1)M^\kappa L.$$

The analysis above holds for $i = 1,\ldots, \gamma$. Also since by (2.3.6)

$$H_i^{(r)}(x_e) = 0, \quad e \neq i, \quad r = 0,\ldots, \kappa,$$

we can find $H_1,\ldots, H_\gamma$ separately, by the above, and

$$H = T_\nu + \sum_{i=1}^{\gamma} H_i$$

will belong to A_κ, the set of functions satisfying the interpolatory side conditions. It remains to estimate $\|f - H\|$; using (2.3.13) we find

$$|(f-H)|(t) \leq |f - T_\nu|(t) + |\sum_{i=1}^{\gamma} H_i|(t)$$

$$\leq \begin{cases} C_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa} + D_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa}(1+(\gamma-1)\delta_{\lambda_1}) \; ; & t \in \bigcup_{i=1}^{\gamma} B_i \\ C_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa} + \gamma\delta_{\lambda_1} D_\kappa \dfrac{e_\nu(f^{(\kappa)})}{\nu^\kappa} \; ; & t \in T\backslash \bigcup_{i=1}^{\gamma} B_i \end{cases}$$

Thus

$$\|f - H\| \leqslant (C_\kappa + 2D_\kappa) \frac{e_\nu(f^{(\kappa)})}{\nu^\kappa} \quad , \quad \nu \geqslant \nu_1$$

where $\nu_1 \geqslant \nu_3$ is chosen so that $\delta_{\lambda_1} \leqslant \frac{1}{\gamma}$, $\nu \geqslant \nu_1$.

This concludes the proof. //

§2.4 DEGREE OF ALGEBRAIC POLYNOMIAL APPROXIMATION WITH HERMITE-BIRKHOFF INTERPOLATORY SIDE CONDITIONS.

Consider now approximation of $f \in C^\kappa[-1,1]$ by algebraic polynomials satisfying Hermite-Birkhoff interpolatory side conditions. Redefining A_κ, $E_\nu(f)$, and $E_\nu(f,A_\kappa)$ appropriately; Platte [7, Theorem 2.3.1] has shown

<u>THEOREM 2.3</u>. *If* $f \in C^\kappa[-1,1]$ *then there exists a constant* C, *independent of* ν, *such that*

$$E_\nu(f,A_\kappa) \leqslant C\, E_{\nu-\kappa}(f^{(\kappa)}) \quad , \qquad \forall \nu \geqslant ((\kappa+1)\gamma - 1).$$

He also shows that if f can be extended to an analytic function in some domain of the complex plane which contains $[-1,1]$, then

$$E_\nu(f,A_\kappa) = O(E_\nu(f)^{1-\varepsilon}) \quad , \qquad \forall \varepsilon > 0.$$

Platte's results can be greatly improved. In the following the estimate $E_\nu(f,A_\kappa) = O(\nu^{-k}\omega(f^{(k)}, \nu^{-1}))$ will be shown for $f \in C^k[-1,1]$, $(k \geqslant \kappa)$. Also estimates $E_\nu(f,A_\kappa) = O(\nu^{-k} e_\nu(g^{(k)}, \nu^{-1}))$ with $g(\theta) = f(\cos\theta)$, will be shown for restricted sets of interpolation conditions, $(k = \kappa)$, and for functions in $C^{2\kappa}[-1,1]$, $(k = 2\kappa)$. These last estimates imply $E_\nu(f,A_\kappa) = O(E_\nu(f)^{1-\varepsilon})$, $\forall \varepsilon > 0$; whereas the first estimate in terms of $\nu^{-k}\omega(f^{(k)}, \nu^{-1})$ does not. (This point is discussed in Section 2.5).

THEOREM 2.4. *For each* $k = 1,2,3,\ldots$ *, there exists an* L_k*, and for each set of side conditions* A_κ *with* $\kappa \leqslant k$*, a* ν_1*, not depending on the function* $f \in C^k[-1,1]$*, such that* $E_\nu(f,A_\kappa)$ *exists and satisfies*

$$E_\nu(f,A_\kappa) \leqslant L_k \nu^{-k}\omega(f^{(k)},\nu^{-1}), \qquad \forall\nu \geqslant \nu_1.$$

Proof: (Sketch only). The proof is analogous to that of Theorem 2.1. Lemma 2.2 is replaced by the following theorem of Trigub [9].

THEOREM 2.5. *There exists for* $\nu > k$ *a polynomial* p_ν *of degree* ν *satisfying the estimate*

$$\|f^{(j)} - p_\nu^{(j)}\|_{[-1,1]} \leqslant (R/\nu^{k-j})\omega(f^{(k)},\nu^{-1}), \qquad j = 0,1,\ldots, k,$$

where R *depends only on* k.

For the remainder of the proof let κ and $k \geqslant \kappa$ be fixed. Let p_ν be the polynomial approximation to f whose existence is guaranteed by Trigub's theorem.

It remains to construct algebraic polynomial "bump functions" h^*_{ij} linear combinations of which will be used to correct the values of $p_\nu^{(j)}(t_i)$ to the $f^{(j)}(t_i)$. Clearly there exist disjoint intervals $B^*_1,\ldots, B^*_\gamma$ in $[-1,1]$ containing $t_1,\ldots, t_\gamma$ respectively. Arguing as in Theorem 2.1 we can find a ν_2 and for $\nu \geqslant \nu_2$ algebraic polynomials $q^*_{\nu j}$ of degree not exceeding ν satisfying

$$\|q^*_{\nu j}\|_{[-1,1]} = 1 \ ; \qquad q^*_{\nu j}(t_j) = 1 \ ; \qquad q^*_{\nu j}(t_i) = 0 \ , \qquad i \neq j \ ;$$

and $\delta_\nu^* = \max\limits_{j=1,\ldots,\gamma} \|q^*_{\nu j}\|_{[-1,1]\setminus B_j}$ goes to zero as $\nu \to \infty$.

By renumbering the $q^*_{\nu j}$ and redefining ν_2 (if necessary) it is possible to impose the extra condition $\|q^{*(\ell)}_{\nu j}\|_{[-1,1]} \leqslant \nu^\ell$, $\ell = 1,\ldots, \kappa$. In the construction of the h^*_{ij} the function $\sin(\lambda x)$ used to construct the h_{ij} is replaced by the Maclaurin polynomial of degree λ corresponding to $\sin([\frac{\lambda}{8}]x)$, where $[.]$ is the integral part function, that is by

$$p_\lambda(x) = \sum_{i=0}^{\lambda} \frac{x^i}{i!} \left\{ \sin^{(i)} \left(\left[\frac{\lambda}{8}\right] x \right) \Big|_{x=0} \right\} .$$

The jth derivative of $p_\lambda(x)$ is the Maclaurin polynomial of degree $\lambda-j$ corresponding to $\sin^{(j)}([\frac{\lambda}{8}]\, x)$. Hence the error expression for Maclaurin expansions shows

$$\|\sin^{(j)}([\tfrac{\lambda}{8}]\, x) - p_\lambda^{(j)}(x)\|_{[-2,2]} \leqslant \frac{2^{\lambda-j+1}}{(\lambda-j+1)!} \| \sin^{(\lambda-j+1)}([\tfrac{\lambda}{8}]\, x)\|_{[-2,2]}$$

Stirling's formula implies that the right hand side is

$$\leqslant \frac{2^{\lambda-j+1}(\lambda/8)^{\lambda-j+1}}{\sqrt{2\pi}\,(\lambda-j+1)^{\lambda-j+3/2}\, e^{-(\lambda-j+1)}\,(1+o(1))}$$

$$= \frac{(1+o(1))}{\sqrt{2\pi}} (\lambda-j+1)^{-\frac{1}{2}} \left(\frac{\lambda}{\lambda-j+1}\right)^{\lambda-j+1} \left(\frac{e}{4}\right)^{\lambda-j+1}$$

$$\leqslant C_1 \left(\frac{\lambda}{\lambda-\kappa+1}\right)^{\lambda+1} \left(\frac{e}{4}\right)^{\lambda+1}, \quad j = 0,1,\ldots,\kappa, \quad \lambda > \kappa ,$$

where C_1 depends only on κ. Since $\lim_{\lambda\to\infty} \left(\frac{\lambda}{\lambda-\kappa+1}\right)^{\lambda+1} = \exp(\kappa-1)$, there exists C_2 depending only on $\kappa(\leqslant k)$ such that

$$\|\sin^{(j)}([\tfrac{\lambda}{8}]\, x) - p_\lambda^{(j)}(x)\|_{[-2,2]} \leqslant C_2(e/4)^\lambda$$

for $j = 0,1,\ldots,\kappa$ and $\lambda > \kappa$.

Now define algebraic polynomials of degree not exceeding ν

$$h_{ij}^* = (q_{\lambda_1,i}(x))^{\kappa+1}(p_\lambda(x-t_i))^j; \quad j = 0,1,\ldots,\kappa; \quad i = 1,\ldots,\gamma;$$

where $\lambda = [\nu/(\kappa+1)]$, $\lambda_1 = [\lambda/\kappa+1]$. Then if $\lambda_1 \geqslant \max(\nu_2,1)$

$$\|h_{ij}^*\|_{[-1,1]} \leqslant 1 + C_2 ; \tag{2.3.5'}$$

$$h_{ij}^{*(r)}(t_e) = 0 , \ r = 0,1,\ldots,\kappa; \ e \neq i ; \tag{2.3.6'}$$

$$h_{ij}^{*(r)}(t_i) = 0 , \quad r < j ; \tag{2.3.7'}$$

and

$$h_{ij}^{*(j)}(t_i) = j![\tfrac{\lambda}{8}]^j . \tag{2.3.8'}$$

Also from the degree of approximation of the functions $p_\lambda^{(j)}(x)$ to the functions $\sin^{(j)}([\frac{\lambda}{8}]x)$; and the condition $\|q^{*(\ell)}_{\nu j}\|_{[-1,1]} \leq \nu^{\ell}$, $\ell = 0,\ldots,\kappa$; there exists an C_3 depending on κ alone such that

$$\|h^{*(\ell)}_{ij}\|_{[-1,1]} \leq C_3\nu^{\ell}, \qquad \ell = 0,1,\ldots,\kappa. \tag{2.3.9'}$$

The remainder of the proof is analogous to that of Theorem 2.1 from equation (2.3.9) on. In brief the matrix equation to be satisfied if

$$H^*_i(x) = \sum_{\ell=1}^{p} b_{j_\ell} h^*_{i,j_\ell}(x)$$

is to provide the correction/perturbation

$$H^{*(j)}_i(t_i) = (f-p_\nu)^{(j)}(t_i), \qquad j = j_1,\ldots,j_p,$$

is non-singular and has a unique solution $b_{j_1},\ldots,b_{j_p}$. Also $|b_{j_\ell}| < C_4\nu^{-k}\omega(f^{(k)},\nu^{-1})$, $\ell = 1,\ldots,p$; with C_4 depending only on k.

Hence

$$|H^*_i(x)| \leq \begin{cases} C_5\nu^{-k}\omega(f^{(k)},\nu^{-1}) & \text{if } x \in B_i, \\ o(1)\nu^{-k}\omega(f^{(k)},\nu^{-1}) & \text{if } x \in [-1,1]\backslash B_i. \end{cases}$$

By choice of the h^*_{ij} the correction equations separate and

$$p^*_\nu = p_\nu(x) + \sum_{i=1}^{\gamma} H^*_i(x)$$

is an algebraic polynomial of degree not exceeding ν which satisfies the Hermite-Birkhoff interpolation conditions and provides the order of approximation required. //

The following corollaries to Theorem 2.1 provide the estimate $E_\nu(f,A_\kappa) = 0(E_\nu(f)^{1-\varepsilon})$ for all $\varepsilon > 0$ whenever they apply (see Section 2.5).

<u>COROLLARY 2.6</u>. *For each* $\kappa = 1,2,3,\ldots$ *there exists an* $M_\kappa > 0$; *and for each set of side conditions* A_κ, *provided that* $-1 < t_i < 1$, $i = 1,\ldots,\gamma$, *a* ν_1; *not depending on* $f \in C^\kappa[-1,1]$;

such that $E_\nu(f,A_\kappa)$ *exists and satisfies*

$$E_\nu(f,A_\kappa) \leq M_\kappa \nu^{-\kappa} e_\nu(g^{(\kappa)}) ,$$

for all ν *greater than* ν_1*, where* $g \in C^{*\kappa}[-\pi,\pi]$ *is defined by* $g(\theta) = f(\cos\theta)$.

<u>*Proof:*</u> Write $g(\theta)$, $g^{(1)}(\theta) = \frac{dg(\theta)}{d\theta}$, ..., $g^{(\kappa)}(\theta) = \frac{d^\kappa g(\theta)}{d\theta^\kappa}$,

in terms of $f(x)$, $\frac{df}{dx}(x)$, ..., $\frac{d^\kappa f}{dx^\kappa}(x)$; as

$$\begin{bmatrix} g(\theta) \\ g^{(1)}(\theta) \\ g^{(2)}(\theta) \\ \\ \\ g^{(\kappa)}(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & . & . & 0 & 0 \\ 0 & -\sin\theta & & & & 0 \\ 0 & a_{21}(\theta)\sin^2\theta & & & & . \\ & & & & & . \\ & & & & & 0 \\ 0 & a_{\kappa_1}(\theta)a_{\kappa_2}(\theta) & & & & (-\sin\theta)^\kappa \end{bmatrix} \begin{bmatrix} f(x) \\ \frac{df}{dx}(x) \\ \frac{d^2f}{dx^2}(x) \\ \\ \\ \frac{d^\kappa f}{dx^\kappa}(x) \end{bmatrix}$$

Note that the lower triangular matrix involved is invertible, $x \neq \pm 1$, and therefore $g(\theta)$, $g^{(1)}(\theta)$, ..., $g^{(\kappa)}(\theta)$ are uniquely determined by $f(x)$, ..., $\frac{d^\kappa f}{dx^k}(x)$, $x \neq \pm 1$, and vice versa. Thus the algebraic interpolation conditions are equivalent to trigonometric interpolation conditions of the same order, κ. To each node t_i, of the algebraic problem, there correspond two nodes θ_{2i-1}, θ_{2i} of the trigonometric where

$$0 < \theta_{2i-1} = \cos^{-1} t_i < \pi \quad \text{and} \quad \theta_{2i} = -\theta_{2i-1} \tag{2.4.1}$$

We apply Theorem 2.1 and find a ν_1 and a sequence $\{T_\nu\}_{\nu=\nu_1}^\infty$ satisfying the trigonometric interpolation conditions, such that

$$\|g - T_\nu\| \leq M_\kappa \nu^{-\kappa} e_\nu(g^{(\kappa)}) . \tag{2.4.2}$$

Since the interpolation conditions occur in pairs (see (2.4.1)) and $g(\theta)$ is even, the even functions $\tilde{T}_\nu$ given by $\tilde{T}_\nu(\theta) = (T_\nu(\theta) + T_\nu(-\theta))/2$ satisfy them also. Let $p_\nu(x) = \tilde{T}_\nu(\cos^{-1} x)$. As discussed previously p_ν satisfies the interpolation conditions of the algebraic problem.

Since $\tilde{T}_\nu$ is an even trigonometric polynomial of degree not exceeding ν, $p_\nu(x)$ is an algebraic polynomial of degree not exceeding ν. Now the eveness of g implies

$$\|g - \tilde{T}_\nu\|_{[-\pi,\pi]} \leq \|g - T_\nu\|_{[-\pi,\pi]} \ ;$$

hence by (2.4.2)

$$\|f - p_\nu\| = \| g(\cos^{-1} x) - \tilde{T}_\nu(\cos^{-1} x)\| \leq M_\kappa \nu^{-\kappa} e_\nu(g^{(\kappa)}).$$

Hence $p_\nu(x)$ provides the estimate of the Corollary. //

COROLLARY 2.7. *For each* $\kappa = 1,2,3,\ldots$ *there exists an* $M_{2\kappa} > 0$; *and for each set of side conditions* A_κ, *a* ν_1, *not depending on* $f \in C^{2\kappa}[-1,1]$; *such that* $E_\nu(f,A_\kappa)$ *exists and satisfies*

$$E_\nu(f,A_\kappa) \leq M_{2\kappa}\nu^{-2\kappa}e_\nu(g^{2\kappa}) \quad ,$$

for all ν *greater than* ν_1, *where* $g \in C^{*2\kappa}[-\pi,\pi]$ *is defined by* $g(\theta) = f(\cos\theta)$.

Proof: As in the last Corollary the argument is that the given algebraic interpolation conditions are equivalent, under the transformation $g(\theta) = f(\cos\theta)$, to certain trigonometric interpolation conditions. However the relationship between the algebraic and the trigonometric interpolation conditions, at t_i $(i \in \{1,\ldots,\gamma\})$, varies with the position of t_i in $[-1,1]$.

If t_i *with* $|t_i| < 1$ *is a node of the algebraic problem,* then algebraic interpolation conditions of order κ at t_i are equivalent to trigonometric interpolation conditions of the same order at two points $-\pi < \theta_{2i} < 0 < \theta_{2i-1} < \pi$. This has been discussed already in the previous corollary.

If $t_i = 1$ *is a node of the algebraic problem* then the previous argument fails. That the algebraic interpolation conditions at $t_i = 1$ cannot always be equivalent to trigonometric interpolation conditions of the same order at $\theta_i = 0$ may be deduced from the eveness of g. This since the function mapping $(f(1), \frac{df}{dx}(1), \ldots, \frac{d^K f}{dx^K}(1))$ into $(g(0), \frac{dg}{d\theta}(0), \ldots, \frac{d^K g}{d\theta^K}(0))$ cannot have an inverse when $\frac{d^j g}{d\theta^j}(0) = 0$ for all odd j between 1 and K. Indeed algebraic interpolation conditions of order K at $t_i = 1$ are equivalent to trigonometric interpolation conditions of order 2K at $\theta = 0$. *To see this* note

$$\frac{d^k g(0)}{d\theta^k} = \sum_{i=0}^{k} a_{k,i} \frac{d^i f(1)}{dx^i}, \qquad k > 0, \tag{2.4.3}$$

where the $a_{k,i}$ are constants not depending on $f \in C^k[-1,1]$. Consider the function $f(x) = (x-1)^k/k!$ with

$$\frac{d^j f}{dx^j}(1) = 0, \quad j \neq k; \quad \text{and} \quad \frac{d^k f}{dx^k}(1) = 1.$$

The corresponding periodic function g is given by the everywhere convergent power series

$$g(\theta) = \frac{(\cos\theta - 1)^k}{k!} = \left(-\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)^k \Big/ k!$$

$$= \frac{(-1)^k \theta^{2k}}{k!(2!)^k} + O(\theta^{2k+2}) .$$

This series is differentiable term by term with

$$\frac{d^j g}{d\theta^j}(0) = 0, \qquad j = 0, \ldots, 2k-1 \qquad (k > 0),$$

and

$$\frac{d^{2k} g}{d\theta^{2k}}(0) = \frac{(-1)^k (2k)!}{k!\, 2^k} .$$

Since the only non-zero derivative $\frac{d^j f}{dx^j}$ at x = 1 is the kth derivative which is 1, the equations above and (2.4.3) imply

$$a_{j,k} = 0 \quad , \quad j = k, \ldots, 2k-1 \quad , \quad (k > 0) \; ,$$

while

$$a_{2k,k} = \frac{(-1)^k (2k)!}{k!\, 2^k} .$$

Hence the matrix equation

$$\begin{bmatrix} g(0) \\ \frac{d^2g}{d\theta^2}(0) \\ \frac{d^4g}{d\theta^4}(0) \\ \\ \frac{d^{2K}g}{d\theta^{2K}}(0) \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & 0 \\ 0 & a_{21} & & & . \\ 0 & a_{41} & a_{42} & & . \\ & & & & 0 \\ 0 & a_{2K,1} & a_{2K,2} & & a_{2K,K} \end{bmatrix} \begin{bmatrix} f(1) \\ \frac{df}{dx}(1) \\ \frac{d^2f}{dx^2}(1) \\ \\ \frac{d^Kf}{dx^K}(1) \end{bmatrix}$$

, in which only even order derivatives of $g(\theta)$ occur, is nonsingular since all the diagonal elements of the lower triangular matrix involved are non-zero. Thus algebraic interpolation conditions of order K at $x = 1$ are equivalent to trigonometric interpolation conditions of order $2K$ at $\theta = 0$, under the transformation $g(\theta) = f(\cos\theta)$.

If $x = -1$ *is a node of the algebraic problem* then arguments, similar to those above, show that algebraic interpolation conditions of order K at -1 are equivalent to trigonometric interpolation conditions of order $2K$ at $\theta = \pi$ and $\theta = -\pi$.

All choices of nodes in $[-1,1]$ have now been considered. Hence algebraic interpolation conditions of order K on $[-1,1]$ are equivalent , under the transformation $g(\theta) = f(\cos\theta)$, to trigonometric interpolation conditions of order not exceeding $2K$ on $[-\pi,\pi]$. Moreover the nodes of the trigonometric problem occur in pairs symmetrical about $\theta = 0$.

Apply Theorem 2.1 to find an approximation T_ν to $g(\theta)$ satisfying the trigonometric interpolation conditions. Let $\tilde{T}_\nu$ be the even part of this trigonometric polynomial. The argument of the last part of Corollary 2.6 shows the algebraic polynomial, $p_\nu(x) = \tilde{T}_\nu(\cos^{-1} x)$, satisfies the algebraic interpolation conditions and provides the order of approximation of the present Corollary. //

§2.5 COMPARISON OF $E_\nu(f)$ AND $E_\nu(f,A_\kappa)$

The question of a direct comparison of $E_\nu(f)$ and $E_\nu(f,A_\kappa)$, as opposed to a comparison of $e_\nu(f^{(\kappa)})/\nu^\kappa$ and $E_\nu(f,A_\kappa)$ remains. Below are results in two opposing directions.

NOTATION: The following convention will be adopted. If it is specified that $f \in C^{*\kappa}[-\pi,\pi]$ then $E_\nu(f)$, $E_\nu(f,A_\kappa)$, etc. refer to approximation by trigonometric polynomials. If, on the other hand, it is specified the $f \in C^k[-1,1]$ then $E_\nu(f)$, $E_\nu(f,A_\kappa)$, etc. refer to approximation by algebraic polynomials.

<u>LEMMA 2.8</u>. *If* $f \in C^{*\kappa}[-\pi,\pi]$, $\kappa \geq 1$, *then for all* $\varepsilon > 0$

$$e_\nu(f^{(\kappa)})/\nu^\kappa = O(E_\nu(f)^{1-\varepsilon}).$$

<u>*Proof:*</u> Either f has only a finite number, k, of continuous derivatives or f has an infinite number of continuous derivatives.

In the first case, using the well known Jackson and Bernstein Theorems (see e.g. Butzer and Nessel [2, Corollary 2.2.4 and Theorem 2.3.6]) characterizing the rate at which $E_\nu(f)$ goes to zero in terms of the order of magnitude of the second modulus $\omega_2(f^{(k)},\delta)$, defined by

$$\omega_2(f^{(k)},\delta) = \sup_{|h|\leq\delta} \| f^{(k)}(t+h) + f^{(k)}(t-h) - 2f^{(k)}(t)\|$$

we find <u>either</u>

(i) $e_\nu(f^{(k)}) = o(1)$ but $E_\nu(f)\nu^{k+\varepsilon}$ is unbounded for all $\varepsilon > 0$;

<u>or</u>

(ii) there exists α, $0 < \alpha \leq 1$, such that

$$e_\nu(f^{(k)}) = O(\nu^{-\alpha}) \text{ but } E_\nu(f)\nu^{k+\alpha+\varepsilon} \text{ is unbounded for all } \varepsilon > 0.$$

In either case

$$\frac{1}{\nu^k} e_\nu(f^{(k)})/E_\nu(f) = O(\nu^\varepsilon) \text{ for all } \varepsilon > 0 ,$$

and since $\nu = O(E_\nu(f)^{-\frac{1}{k}})$ this implies

$$\frac{1}{\nu^k} e_\nu(f^{(k)}) = O(E_\nu(f)^{1-\varepsilon}), \text{ for all } \varepsilon > 0 .$$

The desired result follows as $e_\nu(f^{(\kappa)}) = 0\left(\dfrac{e_\nu(f^{(k)})}{\nu^{k-\kappa}}\right)$.

If f *has an infinite number of continuous derivatives,* let T_ν be the best approximation to f from N_ν; then for p = 1,2,... ,

$$\|f^{(p)} - T_\nu^{(p)}\| = 0(E_\nu(f)^{1-\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

This follows from a modification of the argument of Platte [7, Theorem 2.3.3]. Briefly fixing ε, 1 > ε > 0, and p, write

$$\|f^{(p)} - T_\nu^{(p)}\| \leqslant \sum_{n=\nu}^{\infty} \|T_{n+1}^{(p)} - T_n^{(p)}\|$$

$$\leqslant 2 \sum_{n=\nu}^{\infty} (n+1)^p E_n$$

$$\leqslant \left\langle 2 \sum_{n=\nu}^{\infty} (n+1)^p E_n^{\varepsilon} \right\rangle E_\nu^{1-\varepsilon},$$

where the term in angular brackets is bounded since

$$E_n(f) = 0(\frac{1}{n^k}) \quad , k = 1,2,3,\ldots \quad .$$

Hence $e_\nu(f^{(p)}) = 0(E_\nu(f)^{1-\varepsilon})$, for p = 1;2,... . In particular $e_\nu(f^{(\kappa)}) = 0(E_\nu(f)^{1-\varepsilon})$. //

Remarks. Analogous to Lemma 2.8 is the following: If to $f \in C^{*\kappa}[-\pi,\pi]$ corresponds a quantity $B_\nu = 0(\nu^{-k}\,\omega(f^{(k)},\nu^{-1}))$ whenever $f \in C^{*k}[-\pi,\pi]$ and $B_\nu = 0(e_\nu(f^{(\kappa)}))$; then $B_\nu = 0(E_\nu(f)^{1-\varepsilon})$, $\forall\varepsilon > 0$.

A similar result does not hold when $E_\nu(f)$ denotes the degree of approximation by algebraic polynomials of degree not exceeding ν of $f \in C^\kappa[-1,1]$. i.e. In this case $B_\nu = 0(\nu^{-k}\,\omega(f^{(k)},\nu^{-1}))$ and $f \in C^k[-1,1]\backslash C^{k+1}[-1,1]$; <u>does not</u> imply $B_\nu = 0(E_\nu(f)^{1-\varepsilon})$ for all ε > 0. The reason for this is the lack of an inverse theorem of the Bernstein type. For example $f^{(k)} \in C[-1,1]$, and $f^{(k)} \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, is sufficient but not necessary for $E_\nu(f) = 0(\nu^{-k-\alpha})$. The following example occurs in Timan [8,pp.342-343]. Let $f(x) = \sqrt[4]{1-x^2}$. Then $E_\nu(f,[-1,1]) = E_\nu^*(\sqrt{|\sin t|}) = 0(\nu^{-1/2})$; here E_ν^* is the degree of trigonometric polynomial approximation. At the same time as $\nu \to \infty$, the modulus of continuity $\omega(f,\nu^{-1})$ on the whole segment $[-1,1]$ is of the exact order $\nu^{-1/4}$. Similarly if $f'(x) = \sqrt[4]{1-x^2}$ then $E_\nu(f) = O(\nu^{-1}E_{\nu-1}(f')) = O(\nu^{-3/2})$ while $\nu^{-1}\omega(f',\nu^{-1})$ is of the exact order $\nu^{-5/4}$.

The above remark shows that in a sense Corollaries 2.6, 2.7 are stronger than Theorem 2.4.

We also have the following, showing that no inequality of the form

$$E_\nu(f,A_\kappa) = O(G(\nu)E_\nu(f)) \quad ,$$

can exist where $G(\nu)$ does not depend on f. The proof is an adaptation to the trigonometric case of the argument of Lorentz and Zeller[6].

<u>LEMMA 2.9.</u> *Given any sequence* $\{h_\nu\}_{\nu=1}^{\infty}$ *of positive numbers, and a set of interpolatory side conditions* A_κ $(\kappa \geq 1)$ *including at least one constraint on* $f^{(\kappa)}$*, there exists* $f \in C^{*\kappa}[-\pi,\pi]$ *such that*

$$\limsup_{\nu \to \infty} E_\nu(f,A_\kappa)/h_\nu E_\nu(f) \geq 1 \quad .$$

<u>Proof:</u> We assume, without loss of generality, that the constraint on $f^{(\kappa)}$ is at $\theta = 0$. If $(\kappa \geq 1)$ is odd we take $g_i = \sin(i\theta)$, $i = 1,2,3,\ldots$; if κ (≥ 1) is even take $g_i = \cos(i\theta)$, $i = 1,2,3,\ldots$. Given any $b > 0$ we can clearly choose an N such that

$$\sum_{i=1}^{N} i^\kappa/N \geq b \quad . \tag{2.5.1}$$

Now with $H = \sum_{i=1}^{N} g_i/N$

$$|H^{(\kappa)}(0)| \geq b \quad , \quad \|H\| = 1 \quad . \tag{2.5.2}$$

Take

$$b_\nu = 2\nu^\kappa(h_\nu+1) \quad , \quad \nu = 1,2,\ldots \quad , \tag{2.5.3}$$

and $N_0 = 1$. Given N_{j-1} $(j \geq 1)$, there exists, according to (2.5.2), a polynomial, f_j, such that

$$|f_j^{(\kappa)}(0)| \geq b_{N_{j-1}} \quad , \quad \|f_j\| = 1 \quad . \tag{2.5.4}$$

We denote the degree of this polynomial by N_j.

The function f of the Lemma will be given by the series

$$f = \sum_{j=1}^{\infty} c_j f_j$$

where the $c_j > 0$ satisfy

$$c_j \leqslant j^{-2} M_j^{-1}, \quad M_j = \max (\| f_j \|, \ldots, \| f_j^{(\kappa)} \|) \quad , \tag{2.5.5}$$

and

$$\sum_{j=\nu+1}^{\infty} c_j \leqslant c_\nu \| f_\nu \| \tag{2.5.6}$$

For instance, we can define the numbers c_i inductively by means of the relation

$$c_i = \min \{ \tfrac{1}{2} c_{i-1} \| f_{i-1} \| , \ldots, \frac{1}{2^{i-1}} c_1 \| f_1 \| , i^{-2} M_i^{-1} \}$$

$$, i = 2,3,\ldots \quad .$$

Note that (2.5.5) implies f is κ times continuously differentiable.

Let $F_\nu = \sum_{i=1}^{\nu} c_i f_i$. Clearly

$$E_{N_{\nu-1}} (f) \leqslant \| f - F_{\nu-1} \| = \| \sum_{i=\nu}^{\infty} c_i f_i \| \quad , \nu = 2,3,\ldots \quad ;$$

and using (2.5.6)

$$E_{N_{\nu-1}} (f) \leqslant 2c_\nu \| f_\nu \| \quad . \tag{2.5.7}$$

Let Q be any trigonometric polynomial of degree not exceeding $N_{\nu-1}$ such that $Q^{(\kappa)}(0) = f^{(\kappa)}(0)$.

Writing

$$\| Q - f \| \leqslant \| Q - F_{\nu-1} \| - \| F_{\nu-1} - f \|$$

it follows using Bernstein's inequality that

$$\| Q - f \| \geqslant c_\nu | f_\nu^{(\kappa)} (0) | / N_{\nu-1}^{\kappa} - 2c_\nu \| f_\nu \| \quad ,$$

and by (2.5.3), (2.5.4) that

$$\| Q - f \| \geqslant 2c_\nu h_{N_{\nu-1}} \quad .$$

Since Q was an arbitrary polynomial subject to $Q^{(\kappa)}(0) = f^{(\kappa)}(0)$ it follows that

$$E_{N_{\nu-1}} (f, A_\kappa) \geqslant 2c_\nu h_{N_{\nu-1}} \quad . \tag{2.5.8}$$

(2.5.7) and (2.5.8) together imply

$$E_{N_{\nu-1}}(f,A_K)/E_{N_{\nu-1}}(f) \geq h_{N_{\nu-1}}, \quad \nu = 2,3,\ldots \quad ;$$

the desired result.

Remark. Lemma 2.9 may be used to show that if $E_\nu^*(f)$ denotes the error in SAIN approximation, with interpolation at a node t_i where $|f(t_i)| = \|f\|$, then no relation of the form $E_\nu^*(f) = O(G(\nu)E_\nu(f))$ exists with $G(\nu)$ not depending on f. This is interesting in view of Theorem 1.3 and Corollary 1.9; which imply that if $E_\nu^*(f)$ denotes the error in SAIN approximation, by trigonometric polynomials of degree ν or less, with interpolation only at nodes t_i where $|f(t_i)| < \|f\|$, then $E_\nu^*(f) = O(E_\nu(f))$.

To prove this; given $\{h_\nu\}$ and with $A_1 = \{g : g'(0) = f'(0)\}$ construct as in Lemma 2.9 a function $f \in C^{*2}[-\pi,\pi]$ for which

$$\limsup_{\nu\to\infty} (E_\nu(f,A_1)/h_\nu E_\nu(f)) \geq 1.$$

Take $\quad f_1(\theta) = f(\theta) - f'(0)\sin\theta + C(1+\cos\theta)^2 \quad$ where C will be chosen later. Note $f_1'(0) = 0$. Also since $f''(\theta)$ is bounded and $\frac{d^2}{d\theta^2}(1+\cos\theta)^2$ is negative on $[-\frac{\pi}{4},\frac{\pi}{4}]$, $f''(\theta)$ will be negative on $[-\pi/4,\pi/4]$ for all C greater than some C_0.

Also $\quad \|(1+\cos\theta)^2\|_{[-\pi,\pi]\setminus[-\pi/4,\pi/4]} < \|(1+\cos\theta)^2\|_{[-\pi,\pi]}$

implies the maximum modulus of $f_1(\theta)$ on $[-\pi,\pi]$ occurs within $[-\pi/4,\pi/4]$ for all C greater than some C_1. Hence with $C = \max(C_0,C_1)$, $\|f_1\| = |f(0)|$, and each SAIN approximation to f_1 has $f_1'(0) = 0$. Then for $\nu \geq 3$

$$E_\nu(f,A_1) = E_\nu(f_1,A_1) \leq E_\nu^*(f_1) \quad \text{and} \quad E_\nu(f) = E_\nu(f_1)$$

implying $$\limsup_{\nu\to\infty} (E_\nu^*(f_1)/h_\nu E_\nu(f_1)) \geq 1.$$

Final Remark. After the author had completed this investigation the results of Hill, Passow, and Raymon [5] came to his attention. These include a Jackson type theorem for algebraic polynomial approximation with Hermite-Birkhoff interpolatory side conditions. This theorem [5, Theorem 2] is similar in intent to Theorem 2.4 of this thesis. Note however that both its statement and proof are incorrect. An initial mis-statement of a lemma attributed to Teljakovskii has propagated through the Theorem. [The lemma should read as Theorem 2.5 of this thesis does; instead $\omega(f^{(k)}, n^{-1})$ has been replaced by $\omega(f^{(i)}, n^{-1})$. The stated lemma is untrue. For example with $k = 1$, $i = 0$ and $Z = \{f \in C^1[-1,1] : f'(0) = 0$ and $n^{-\frac{1}{2}} < \omega(f', n^{-1}) < 2n^{-\frac{1}{2}}$ for $n = 1,2,3,\ldots\}$ it guarantees

$E_n(Z) = \max_{f \in Z} E_n(f) = O(n^{-2})$ which is untrue; $E_n(Z)$ is well-known to be of the order $n^{-3/2}$ exactly.]

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CHAPTER 3

THE DEGREE OF APPROXIMATION BY RECIPROCALS OF POLYNOMIALS ON $[0,\infty)$.

§3.1 SUMMARY

Asymptotic estimates are obtained for the error in uniform approximation by reciprocals of algebraic polynomials of degree n on $[0,\infty)$. The theorems concern the approximation of 1/f where; either f is an entire function of finite order (logarithmic order) and type (logarithmic type); or f has $k(\geqslant 1)$ continuous derivatives on $[0,\infty)$.

§3.2 INTRODUCTION

Many results are known concerning the degree of approximation of differentiable functions by reciprocals of polynomials on $[0,\infty)$. However most of these results concern approximation of 1/f where

$$f^{(j)}(x) \geqslant 0, \quad \forall x \geqslant 0, \quad j = 0,1,2,\ldots \quad . \tag{3.2.1}$$

(see for example [6]). This chapter extends these results in two directions: weakening the positivity condition (3.2.1) for entire functions, and showing some results for functions with only a finite number of derivatives. In each of the proofs that follow rational approximations to 1/f on $[0,\infty)$ are derived from polynomial approximations, p_n, to f on $[0, r(n)]$ satisfying the side condition that $p_n'(x) \geqslant 0, \quad \forall x \geqslant r(n)$.

Notation: As usual the degree of approximation by reciprocals of polynomials is denoted by

$$\lambda_{0,n}(f^{-1}) \equiv \inf_{p \in \Pi_n} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{[0,\infty)} \quad ,$$

where throughout $\|\cdot\|_I$ indicates the uniform norm on the interval I, and Π_n is the class of polynomials of degree $\leq n$.

§3.3 ORDER OF APPROXIMATION RESULTS FOR CERTAIN ENTIRE FUNCTIONS

Some of the known results on the order of approximation of reciprocals of entire functions, f, follow from Taylor series expansion of the function f about zero. The basis of these arguments is: take p_n^*, the Maclaurin polynomial of degree n corresponding to f; choose upper end points r(n), and discuss

$$\|f - p_n^*\|_{[0,r(n)]} \quad , \quad \inf_{x \geq r(n)} f(x), \quad \inf_{x \geq r(n)} p_n^*(x)$$

using the positivity conditions to deduce that $p_n^{*\prime}(x) \geq 0, \quad \forall x \geq 0$. It is easily seen that it is not necessary for these arguments that the approximation, p_n to f, increase for all $x \geq 0$, only that it increase for all $x \geq r(n)$. Thus in Theorems 3.1, 3.3 a Taylor expansion about r(n), as opposed to a Maclaurin expansion about zero, is used. This allows the positivity conditions to be weakened dramatically. The results obtained are best possible in the sense that the power of $\lambda_{o,n}$; n^{-1}, $n^{-(1+1/\Lambda)}$; in Theorems 3.1, 3.3 respectively; is known to be best possible ([4],[5]).

If f(x) has n + 1 continuous derivatives on $[0,r]$ then with $p_n(x) = \sum_{j=0}^{n} (f^{(j)}(r)\, x^j/j!)$,

$$\|f - p_n\|_{[0,r]} \leq \|f^{(n+1)}\|_{[0,r]}\, r^{n+1}/(n+1)! \quad , \tag{3.3.1}$$

a classical formula for the error in truncated Taylor series expansion. If also f(z) is entire then by Cauchy's inequalities (see e.g. [3, p.202]).

$$\|f^{(n+1)}\|_{[0,r]} \leq (n+1)!M(s)/(s-r)^{n+1} \quad , \quad s > r > 0, \tag{3.3.2}$$

where M(.) is the maximum modulus function. Combining (3.3.1) and (3.3.2)

$$\|f - p_n\|_{[0,r]} \leq M(s)\ (r/(s-r))^{n+1} \quad , \quad \forall s > r. \tag{3.3.3}$$

If $f(z)$ is a non-constant entire function then the order ρ of f is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log\log M(r)}{\log r} \quad , \quad 0 \leqslant \rho \leqslant \infty \ .$$

If $0 < \rho < \infty$, then the type τ of f is defined by

$$\tau = \limsup_{r \to \infty} \frac{\log M(r)}{r^{\rho}} \quad , \quad 0 \leqslant \tau \leqslant \infty \, .$$

THEOREM 3.1. *Let $f(z)$ be an entire function of order ρ, type τ, positive on $[0,\infty)$ and satisfying*

$$\liminf_{r \to \infty} r^{-\rho} \log f(r) = \omega \quad (0 < \rho < \infty,\ 0 < \omega < \tau < \infty).$$

Choose $r = r(n) = \alpha n^{1/\rho}$, and assume that for all sufficiently large n

$$f^{(j)}(r(n)) \geqslant 0, \qquad j = 1, \ldots, n;$$

where $\alpha > 0$ is the unique α, minimizing over α, $\beta > 0$ the maximum of

$$-\,\omega\alpha^{\rho} \tag{3.3.4}$$

and

$$\log\alpha - \log\beta + \tau(\alpha+\beta)^{\rho} \ . \tag{3.3.5}$$

Then

$$\limsup_{n \to \infty} \left(\lambda_{o,n}\left(\frac{1}{f}\right)\right)^{n^{-1}} \leqslant \exp(-\omega\alpha^{\rho}). \tag{3.3.6}$$

Proof. We first discuss the nonlinear program contained in the statement of the theorem. Let

$$\theta(\alpha,\beta) = \max(-\omega\alpha^{\rho}, \log\alpha - \log\beta + \tau(\alpha+\beta)^{\rho}).$$

Since for each $\beta > 0$; (3.3.4) decreases from 0 to $-\infty$, and (3.3.5) increases from $-\infty$ to $+\infty$, as α increases from 0; there is a unique $\alpha(\beta)$ minimizing $\theta(\alpha,\beta)$ for each fixed β, and

$$\theta(\alpha(\beta),\beta) = -\omega\alpha(\beta)^{\rho} = \log\alpha(\beta) - \log\beta + \tau(\alpha(\beta)+\beta)^{\rho} \ . \tag{3.3.7}$$

Choosing $\beta = 1$ and letting $\alpha \to 0^{+}$ it is clear

$\theta(\alpha(1),1) < 0$ and $\alpha(1) > 0$. Now from (3.3.4)

$$\theta(\alpha,\beta) > -\,\omega\alpha(1)^{\rho} \quad , \quad \forall\alpha < \alpha(1) \quad ,$$

and also

$$\theta(\alpha,\beta) > 0, \qquad \forall\beta \leqslant \alpha \quad .$$

Therefore in seeking to minimize $\theta(\alpha,\beta)$ we may assume $\alpha,\beta \geqslant \alpha(1)$. Given this, it follows from (3.3.5) that α,β may also be restricted from above. The existence of some minimizing α,β follows from the uniform continuity of (3.3.4), (3.3.5) and therefore their maximum, on the restricted range. The unicity of the minimizing α follows from (3.3.7).

Let $s = s(n) = (\alpha+\beta)n^{1/\rho}$, where α,β are some minimizing pair. Now using the estimate (3.3.3)

$$\|f - p_n\|_{[0,r(n)]} \leqslant M(s)\ \ (r/(s-r))^{n+1}$$

$$\leqslant \exp((\tau+\varepsilon)n(\alpha+\beta)^{\rho}).(\alpha/\beta)^{n+1}$$

where $\varepsilon \to 0$ as $n \to \infty$; implying

$$\log(\|f - p_n\|_{[0,r(n)]}) \leqslant O(1) + n[(\tau+\varepsilon)(\alpha+\beta)^{\rho} + \log\alpha - \log\beta]$$

and by (3.3.7)

$$\lim\sup \|f - p_n\|^{n^{-1}}_{[0,r(n)]} \leqslant \exp(-\omega\alpha^{\rho}).$$

Writing

$$\left\|\frac{1}{f} - \frac{1}{p_n}\right\|_{[0,r(n)]} \leqslant \frac{\|f - p_n\|_{[0,r(n)]}}{\inf\limits_{x\in[0,r(n)]} f(x)\ p_n(x)} \quad ,$$

we conclude

$$\limsup_{n\to\infty} \left\|\frac{1}{f} - \frac{1}{p_n}\right\|^{n^{-1}}_{[0,r(n)]} \leqslant \exp(-\omega\alpha^{\rho}). \tag{3.3.8}$$

Since p_n is the truncated Taylor expansion of f about r(n), the positivity conditions imply that p_n increases to the right of r(n) for all sufficiently large n. Since also p(r(n)) = f(r(n)), and

$$f(x) \geqslant \exp((\omega-\varepsilon)\ r(n)^{\rho}) \quad , \qquad x \geqslant r(n) \quad ,$$

where $\varepsilon \to 0$ as $n \to \infty$;

$$\left\|\frac{1}{f} - \frac{1}{p_n}\right\|_{[r(n),\infty)} \leqslant 2\ \exp(-(\omega-\varepsilon)\alpha^{\rho}n), \tag{3.3.9}$$

where $\varepsilon \to 0$ as $n \to \infty$. Combining estimates (3.3.8) and (3.3.9) gives the theorem. //

Remarks 3.2. Certain limits on the degree of relaxation of the positivity conditions are inherent in this argument. In the preceding theorem the positivity conditions (i.e. α) have been chosen so as to give the best result, in terms of order of approximation, using the method of Taylor expansion about the upper end point. Alternatively α_1 (and thus $r(n)$ where $r(n) = \alpha_1 n^{1/\rho}$ and $f^{(j)}(r(n)) \geqslant 0$, $j = 1,\ldots, n$, for all sufficiently large n) could be fixed and the result optimized for this value of α_1. If $\alpha_1 \leqslant \alpha$ and $f^{(j)}(x) \geqslant 0$; $j = 1,\ldots, n$; $x \geqslant r(n)$; for all sufficiently large n, this "optimal" result will be the previous theorem. If $\alpha_1 > \alpha$ then geometric convergence can still be proved provided

$$\inf_{\beta > 0} \log \alpha_1 - \log \beta + \tau(\alpha_1 + \beta)^{\rho} \tag{3.3.10}$$

is negative. It is clear that there exists an $\alpha_2 > \alpha$ such that

$$\inf_{\beta>0} \log \alpha_2 - \log \beta + \tau(\alpha_2 + \beta)^{\rho} = 0,$$

and geometric convergence can be shown with this argument only if $0 \leqslant \alpha_1 < \alpha_2$.

Note that very slight modifications of the proof of Theorem 3.1 give a result for Taylor expansion about $r(n)/2$. //

If $f(z)$ is a nonconstant entire function of growth $\rho = 0$; then the logarithmic order $\Lambda + 1$ of f is defined by

$$\Lambda + 1 = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log \log r}, \quad 0 \leqslant \Lambda \leqslant \infty .$$

If $0 < \Lambda < \infty$, then the logarithmic type τ_ℓ of f is defined by

$$\tau_\ell = \limsup_{r \to \infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}}, \quad 0 \leqslant \tau_\ell \leqslant \infty .$$

<u>THEOREM 3.3.</u> *Let* $f(z)$ *be an entire function of logarithmic order* $\Lambda + 1$, *type* τ, *positive on* $[0,\infty)$ *and satisfying*

$$\liminf_{r \to \infty} \frac{\log f(r)}{(\log r)^{\Lambda+1}} = \omega, \quad 0 < \Lambda < \infty, \; 0 < \omega \leqslant \tau < \infty .$$

Choose $r = r(n) = \exp(\alpha n^{1/\Lambda})$ *and assume that for all sufficiently large* n

$$f^{(j)}(r(n)) \geq 0 \quad , \quad j = 1,\ldots, n,$$

where $\alpha > 0$ *is the unique positive solution of the system*

$$(\alpha+\beta)^{\Lambda} = 1/(\tau(1+\Lambda)) \tag{3.3.11}$$

$$f_1(\alpha) = f_2(\alpha,\beta) \tag{3.3.12}$$

where

$$f_1(\alpha) = -\omega\alpha^{1+\Lambda} \quad , \tag{3.3.13}$$

$$f_2(\alpha,\beta) = \tau(\alpha+\beta)^{1+\Lambda} + \alpha - \beta \ . \tag{3.3.14}$$

Then

$$\limsup_{n\to\infty} \left(\lambda_{0,n}\left(\frac{1}{f}\right)\right)^{n^{-(1+1/\Lambda)}} \leq \exp(-\omega\alpha^{1+\Lambda}) .$$

<u>*Proof*</u>. First we discuss the nonlinear program:

$$\underset{\alpha,\beta\geq 0}{\text{minimize}} \quad \max(f_1(\alpha),\ f_2(\alpha,\beta)).$$

Let

$$\theta(\alpha,\beta) = \max(-\omega\alpha^{1+\Lambda},\ \tau(\alpha+\beta)^{1+\Lambda} + \alpha - \beta).$$

Choosing $0 < \alpha_1{}^{\Lambda} < 1/(3^{\Lambda+1}\tau)$ we find

$$\theta(\alpha_1,\ 2\alpha_1) < 0 \quad . \tag{3.3.15}$$

Taking $\alpha^{\Lambda} \geq 1/(\tau(1+\Lambda))$ observe (3.3.14) has a positive minimum, as a function of $\beta \geq 0$ where $\beta = 0$. Taking $\beta^{\Lambda} \geq 1/\tau$ observe (3.3.14) is non-negative for all nonnegative α. Thus ((3.3.15)) in seeking to minimize $\theta(\alpha,\beta)$ we may assume

$$0 \leq \alpha^{\Lambda} \leq 1/(\tau(1+\Lambda)), \qquad 0 \leq \beta^{\Lambda} \leq 1/\tau.$$

The existence of some minimizing α,β now follows from the uniform continuity of (3.3.13), (3.3.14) on the restricted range. Write the program equivalently as

$$\underset{0\leq\alpha^{\Lambda}\leq 1/(\tau(1+\Lambda))}{\text{minimize}} \quad \max\left[f_1(\alpha),\ \underset{0\leq\beta^{\Lambda}\leq 1/\tau}{\min} f_2(\alpha,\beta)\right] ;$$

or

$$\underset{0\leq\alpha^{\Lambda}\leq 1/(\tau(1+\Lambda))}{\text{minimize}} \quad \max[f_1(\alpha),\ f_2(\alpha,\beta)] \text{ with } (\alpha+\beta)^{\Lambda} = 1/(\tau(1+\Lambda)).$$

(3.3.15) and elementary arguments about increasing and decreasing functions now show there is a unique $\alpha,\beta > 0$ solving the program; given by the system in the statement of the theorem.

Using these unique values $\alpha,\beta > 0$, let $s = s(n) = \exp((\alpha+\beta)n^{1/\Lambda})$. Using the estimate (3.3.3)

$$\|f - p_n\|_{[0,r(n)]} \leqslant M(s)(r/(s-r))^{n+1} ,$$

implying

$$\log\|f - p_n\|_{[0,r(n)]} \leqslant O(n^{1/\Lambda}) + n^{1+1/\Lambda}((\tau+\varepsilon)(\alpha+\beta)^{1+\Lambda} + \alpha - \beta), \quad (3.3.16)$$

where $\varepsilon \to 0$ as $n \to \infty$.

Since $p_n(x) \geqslant f(r(n))$, $x \geqslant r(n)$, and

$$f(x) \geqslant \exp((\omega-\varepsilon)\alpha^{1+\Lambda}n^{1+1/\Lambda}), \qquad x \geqslant r(n) ;$$

where $\varepsilon \to 0$ as $n \to \infty$;

$$\left\|\frac{1}{f(x)} - \frac{1}{p_n(x)}\right\|_{[r(n),\infty)]} \leqslant 2\exp(-(\omega-\varepsilon)\alpha^{1+\Lambda}n^{1+1/\Lambda}) , \quad (3.3.17)$$

where $\varepsilon \to 0$ as $n \to \infty$.

Combining (3.3.16) and (3.3.17) with an argument analogous to the latter part of the proof of Theorem 3.1 will give the estimate of this theorem. //

Remark 3.4. Remarks analogous to Remarks 3.2 apply to Theorem 3.3.

§3.4 THE DEGREE OF APPROXIMATION OF THE RECIPROCALS OF FUNCTIONS POSSESSING A FINITE NUMBER OF DERIVATIVES.

In contrast to the case when f is entire relatively few estimates are known for $\lambda_{o,n}(f^{-1})$ when f has only a finite number of derivatives. (There are some results in Blatt [1] and in Freud and Szabados [2]).

In this section we prove a lemma on the degree of approximation, by polynomials, satisfying a side condition, of functions possessing k continuous derivatives and satisfying another side condition. Corollaries to this Lemma are theorems on the degree of approximation by reciprocals of polynomials.

LEMMA 3.5. *For each* $k = 1,2,\ldots$ *there is a constant* B $(=B(k))$ *with the property that if*

$$f \in C^k[0,r], \qquad \exists\, r > 0 \quad ;$$

$$f(r),\ f^{(1)}(r),\ldots,\ f^{(k-1)}(r) \geq 0 \quad ;$$

there exists, for every $n \geq k$, a polynomial q_n of degree not exceeding n such that

$$q_n(x) \geq f(x), \qquad 0 \leq x \leq r \quad ;$$

$$q_n'(x) \geq 0 \quad , \qquad x \geq r \quad ;$$

and $$\|f - q_n\|_{[0,r]} \leq Br^k \|f^{(k)}\|_{[0,r]}/n^k .$$

Proof. If f is a polynomial of degree not exceeding k we take $q_n = f$ and obtain the result.

Otherwise we may assume $\|f^{(k)}\|_{[0,r]} \neq 0$ and proceed to construct the q_n. Since the transformation

$$g(x) \to g^*(x^*)$$

where

$$x^* = (r/2)x - 1, \quad g^*(x^*) = g(x)/(\|f^{(k)}\|_{[0,r]}(r/2)^k)$$

is invertible, preserves the signs of derivatives, and takes polynomials into polynomials; we have only to prove the result for all $f \in C^k[-1,1]$ with $\|f^{(k)}\|_{[-1,1]} = 1$. More precisely the result is equivalent to the existence of an R such that for each $f \in C^k[-1,1]$ with

$$\|f^{(k)}\|_{[-1,1]} = 1 \quad ;$$

$$f^{(1)}(1),\ f^{(2)}(1),\ldots,\ f^{(k-1)}(1) \geq 0 \quad ;$$

there exists a polynomial s_n of degree not exceeding n satisfying

$$s_n^{(1)}(x) \geqslant 0 \quad , \qquad x \geqslant 1 \quad , \tag{3.4.1}$$

$$\| f - s_n \|_{[-1,1]} \leqslant R/n^k \quad ; \tag{3.4.2}$$

since then $s_n^* = s_n + R/n^k$ satisfies

$$s_n^*(x) \geqslant f(x), \qquad -1 \leqslant x \leqslant 1 \quad , \tag{3.4.3}$$

and (3.4.1), (3.4.2) with $R^* = 2R$.

We proceed to prove this equivalent result. Using a theorem of Trigub [8] there exists C depending only on k, and p_n a polynomial of degree not exceeding n with

$$\| f^{(j)} - p_n^{(j)} \|_{[-1,1]} \leqslant C \| f^{(k)} \|_{[-1,1]} / n^{k-j} \leqslant C/n^{k-j}, \qquad j = 0,\ldots, k. \tag{3.4.4}$$

In particular

$$\| p_n^{(k)} \|_{[-1,1]} \leqslant C + 1.$$

We perturb p_n in order to obtain an approximation increasing to the right of $x = 1$. By a classical result (see Rogosinski [7]) if h_n is any kth indefinite integral of $(C+1)T_{n-k}$, T_{n-k} being the Tchebycheff polynomial of the first kind of degree $n-k$, then

$$(h_n + p_n)^{(k)}(x) \geqslant 0 \quad , \qquad x \geqslant 1. \tag{3.4.5}$$

We use the formula

$$I_1(T_i) = \begin{cases} T_1 \quad , & i = 0, \\ T_2/4 \quad , & i = 1, \\ \dfrac{T_{i+1}}{2(i+1)} - \dfrac{T_{i-1}}{2(i-1)} \quad , & i \geqslant 2; \end{cases}$$

to specify a particular indefinite integral of the T_i with certain desirable properties. Let I_j represent the j-fold composition of operators I. It is easy to see that if $n - 2k \geqslant 1$ and $1 \leqslant j \leqslant k$, then

$$I_j(T_{n-k}) = \sum_{i=-j}^{j} a_i T_{n-k+i} \quad ,$$

with

$$\|I_j(T_{n-k})\|_{[-1,1]} \leqslant (2j+1)/(2\prod_{i=1}^{j}(n-k-i)).$$

Also

$$I_k^{(j)}(T_{n-k}) = I_{k-j}(T_{n-k}) \quad , \quad 0 \leqslant j \leqslant k ;$$

and the number of n such that $k \leqslant n \leqslant 2k$ is finite. Therefore there exists E, depending on k alone, such that

$$\|I_k^{(j)}(T_{n-k})\|_{[-1,1]} \leqslant E/n^{k-j} \quad , \quad 0 \leqslant j \leqslant k, \quad \forall n \geqslant k. \tag{3.4.6}$$

Taking

$$h_n = (C+1)I_k(T_{n-k}) \ ,$$

(3.4.4) and (3.4.6) imply the existence of R_1, depending only on k, such that

$$\|(f-h_n-p_n)^{(j)}\|_{[-1,1]} \leqslant R_1/n^{k-j} \tag{3.4.7}$$

$$, \ j = 0,\ldots, k; \quad n \geqslant k.$$

Define A as the least positive real such that

$$An!/((n-j)!n^k) \geqslant R_1/n^{k-j} \quad , \quad j = 0,1,\ldots, k-1, \quad n \geqslant k.$$

Take

$$s_n = h_n + p_n + Ax^n/n^k \quad ,$$

then

$$s_n^{(k)}(x) \geqslant 0 \quad , \quad x \geqslant 1 \quad ; \tag{3.4.8}$$

$$s_n^{(j)}(1) \geqslant 0 \quad , \quad j = 0,\ldots, k-1, \tag{3.4.9}$$

$$\|f - s_n\|_{[-1,1]} \leqslant R/n^k \tag{3.4.10}$$

where

$$R = k!A + R_1 \ .$$

(3.4.8) and (3.4.9) imply

$$s_n'(x) \geqslant 0 \quad , \quad x \geqslant 1. \tag{3.4.11}$$

(3.4.10) and (3.4.11) together imply the Lemma. //

Clearly this Lemma has many corollaries concerning rational approximation on $[0,\infty)$. We cite two of the simplest.

COROLLARY 3.6. *Suppose the function* f *satisfies:*

$$f^{(k)} \text{ is continuous on } [0,\infty) \tag{3.4.12}$$

$$\|f^{(k)}\|_{[0,r]} \leqslant p(r)g(r); \quad f(x) \geqslant g(r), \quad x \geqslant r \geqslant 0; \tag{3.4.13}$$

where g(r) *is a positive, increasing, continuous function,* p(r) *any positive function and*

$$\lim_{r\to\infty} \frac{\max(\log p(r), \log (r))}{\log g(r)} = 0, \tag{3.4.14}$$

$$(1 - \operatorname{sign}(f^{(j)}(r)))f^{(j)}(r) = 0\left[p(r)(g(r))^{\frac{2j-k}{k}}\right], \quad j = 1,\dots, k-1. \tag{3.4.15}$$

Then

$$\lambda_{0,n}\left(\frac{1}{f}\right) = 0\left(n^{-\frac{k}{2}+\varepsilon}\right) \text{ for every } \varepsilon > 0.$$

Proof. Take $N_1 > k$ so large that for $n > N_1$ there exists $r(n) > 1$, with

$$g(r(n)) = n^{k/2}. \tag{3.4.16}$$

Assume in what follows that $n > N_1$. Defining the function $h(n) = r(n)^k$, (3.4.14) implies

$$h(n) = 0(n^{\varepsilon}), \quad p(r(n)) = 0(n^{\varepsilon}) \ ; \quad \forall\varepsilon > 0. \tag{3.4.17}$$

Let $q_n(x) = x^n/r(n)^n$ then

$$q_n^{(j)}(r(n)) = n!/((n-j)!r(n)^{n-j}), \quad j = 0,1,\dots, n;$$

and therefore there exist $0 < C < D < \infty$ such that

$$C(h(n))^{-1}n^j \leqslant q_n^{(j)}(r(n)) \leqslant Dn^j, \quad j = 0,1,\dots, k. \tag{3.4.18}$$

(3.4.15) and (3.4.16) together imply the existence of a constant E such that

$$(1-\operatorname{sign}(f^{(j)}(r(n))))f^{(j)}(r(n)) \leqslant En^{j-\frac{k}{2}}p(r(n)), \quad j = 1,\dots, k-1. \tag{3.4.19}$$

By (3.4.18), (3.4.19) there exists a constant F such that

$$[1 - \operatorname{sign}(f^{(j)}(r(n)))]\, f^{(j)}(r(n))/2 \leqslant Fn^{-k/2} p(r(n))h(r(n))q_n^{(j)}(r(n)),$$
$$j = 1,\dots, k-1.$$

Let

$$f_n = Fn^{-k/2}\, p(r(n))h(r(n))q_n(x) + f(x). \tag{3.4.20}$$

Then using (3.4.17)

$$\| f - f_n \|_{[0,r(n)]} = 0\left(n^{\frac{-k}{2}+\varepsilon}\right), \qquad \forall\varepsilon > 0 ; \tag{3.4.21}$$

also

$$f_n(x) \geqslant f(x), \qquad 0 \leqslant x < \infty \quad ; \tag{3.4.22}$$

$$f_n^{(j)}(r(n)) \geqslant 0, \qquad j = 1,\dots, k-1 ;$$

and by (3.4.13), (3.4.17), (3.4.18)

$$\| f_n^{(k)} \|_{[0,r(n)]} = 0\left(n^{\frac{k}{2}+\varepsilon}\right), \qquad \varepsilon > 0. \tag{3.4.23}$$

We apply the Lemma to the sequence of functions $\{f_n\}$ on the sequence of intervals $\{[0,r(n)]\}$ to obtain the corollary. It follows from the Lemma, (3.4.22) and (3.4.23) that there exists a sequence of polynomials $\{p_n\}$, p_n of degree not exceeding n, such that

$$p_n(x) \geqslant f_n(x) \quad , \qquad 0 \leqslant x \leqslant r(n) \quad ;$$

$$p_n'(x) \geqslant 0 \quad , \qquad x \geqslant r(n) \quad ; \tag{3.4.24}$$

$$\| f_n - p_n \|_{[0,r(n)]} = 0\left(n^{\frac{-k}{2}+\varepsilon}\right), \qquad \varepsilon > 0.$$

By (3.4.21), (3.4.22) also

$$p_n(x) \geqslant f(x), \qquad 0 \leqslant x \leqslant r(n), \tag{3.4.25}$$

$$\| f - p_n \|_{[0,r(n)]} = 0\left(n^{\frac{-k}{2}+\varepsilon}\right), \qquad \varepsilon > 0. \tag{3.4.26}$$

Since

$$f(x) \geqslant g(r(n)) = n^{k/2} , \qquad x \geqslant r(n),$$

(3.4.24), (3.4.25) imply

$$f(x),\ p_n(x) \geqslant n^{k/2}, \qquad x \geqslant r(n). \tag{3.4.27}$$

Combining the bound ((3.4.26))

$$\left\|\frac{1}{f} - \frac{1}{p_n}\right\|_{[0,r(n)]} \leqslant \frac{\|p_n - f\|_{[0,r(n)]}}{g^2(0)} = 0\left(n^{\frac{-k}{2}+\varepsilon}\right), \qquad \forall\varepsilon > 0;$$

with the bound ((3.4.27))

$$\left\|\frac{1}{f} - \frac{1}{p_n}\right\|_{[r(n),\infty)} \leqslant \left\|\frac{1}{f}\right\|_{[r(n),\infty)} + \left\|\frac{1}{p_n}\right\|_{[r(n),\infty)} = 0\left(n^{\frac{-k}{2}}\right);$$

gives the required result. //

Remark 3.7. In constructing Corollary 3.6, the conditions (3.4.15) on the derivatives $f^{(1)},\ldots, f^{(k-1)}$ were chosen in order to preserve the order of approximation obtained, by the same argument, under the stricter condition

$$f^{(1)}(r(n)),\ldots, f^{(k-1)}(r(n)) \geqslant 0 . \tag{3.4.15*}$$

Results concerning other trade offs between conditions on f ; $f^{(k)}$; $f^{(1)},\ldots, f^{(k-1)}$; can clearly be shown by arguments resembling the corollary. //

COROLLARY 3.8. *Suppose the positive function* f *satisfies*

$$f^{(k)} \textit{ is continuous on } [0,\infty) \ ; \tag{3.4.28}$$

$$\|f^{(k)}\|_{[0,x]} = 0(x^{y_1}), \text{ as } x \to \infty \ ; \tag{3.4.29}$$

$$f(x) \geqslant Cx^{y_2}, \qquad x > 0 \ ; \tag{3.4.30}$$

where $\infty > y_1 \geqslant 0,\quad \infty > y_2 > 0,\quad C > 0$;

$$f^{(j)}(x) \geqslant 0, \qquad x \geqslant M, \qquad j = 1,\ldots, k-1, \tag{3.4.31}$$

with M *some positive constant. Then*

$$\lambda_{o,n}\left(\frac{1}{f}\right) = 0\left(n^{-k\left(\frac{y_2}{k+y_1+y_2}\right)}\right), \text{ as } n \to \infty .$$

Proof. Write $\beta = y_2/(k+y_1)$ and let $(r(n))^{k+y_1} = n^{k\alpha}$, where $\alpha > 0$ is to be chosen later. By an argument analogous to the second part of the previous corollary there exists a sequence of polynomials $\{p_n\}$ such that

$$\|\frac{1}{f} - \frac{1}{p_n}\|_{[0,r(n)]} = 0(n^{-k(1-\alpha)}),$$

$$\|\frac{1}{f} - \frac{1}{p_n}\|_{[r(n),\ )} = 0(n^{-k\beta\alpha}).$$

Maximizing the minimum of $(1-\alpha)$, $\alpha\beta$ by taking $\alpha = 1/(1+\beta)$ gives the corollary. //

Remark 3.9. The condition (3.4.31) may be relaxed as in the last corollary, the relaxing perturbation again satisfying conditions on its norm and the norm of its k-th derivative. The writer does not know of a general form for the relaxed conditions.

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CHAPTER 4

THE DEGREE OF MONOTONE APPROXIMATION

§4.1 SUMMARY

Jackson type theorems are obtained for generalized monotone approximation. Let $E_{n,k}(f)$ be the degree of approximation of f by nth degree polynomials with kth derivative non-negative on $[-\frac{1}{4}, \frac{1}{4}]$. Then for each $k \geq 2$ there exists an absolute constant D_k, such that for all $f \in C[-\frac{1}{4}, \frac{1}{4}]$ with kth forward difference non-negative on $[-\frac{1}{4}, \frac{1}{4}]$; $E_{n,k}(f) \leq D_k\, \omega(f,n^{-1})$. If in addition $f' \in C[-\frac{1}{4}, \frac{1}{4}]$ then $E_{n,k}(f) \leq D_k n^{-1} \omega(f',n^{-1})$. Let $E^*_{n,2}(f)$ be the degree of approximation on $[-1,1]$, of f, by nth degree polynomials convex on the whole real line. Then there exists a constant M such that for each f convex on $[-1,1]$; $E^*_{n,2}(f) \leq M\omega(f,n^{-1})$. The results concerning $E_{n,k}$ are to appear in Beatson [1].

§4.2 INTRODUCTION

Let f be a function with non-negative kth forward difference over each set of k equally spaced points in $[-\frac{1}{4}, \frac{1}{4}]$ (equivalently any finite real interval). It is natural to ask whether Jackson type estimates hold for

$$E_{n,k}(f) = \inf_{\{p \in \Pi_n : p^{(k)}(x) \geq 0, x \in [-\frac{1}{4}, \frac{1}{4}]\}} \|f-p\|_{[\frac{1}{4}, \frac{1}{4}]} \quad ;$$

where the norm is the uniform norm, and Π_n is the space of algebraic polynomials of degree not exceeding n. In the case $k = 1$, Lorentz and Zeller [5] and Lorentz [6] have shown that there exists a constant D_1 such that if f is increasing on $[-\frac{1}{4}, \frac{1}{4}]$

$$E_{n,1}(f) \leqslant D_1 \omega(f, n^{-1}), \qquad n = 1,2,\ldots \quad , \tag{4.2.1}$$

where $\omega(f,.)$ denotes the modulus of continuity of f. If, in addition $f' \in C[\frac{1}{4}, \frac{1}{4}]$ then

$$E_{n,1}(f) \leqslant D_1 \, n^{-1} \, \omega(f', n^{-1}), \qquad n = 1,2,\ldots \quad . \tag{4.2.2}$$

Let f be a function convex on $[-1,1]$, and

$$E^*_{n,2}(f) = \inf_{\{p \in \Pi_n \,:\, p''(x) \geqslant 0,\ \forall x \in \mathbb{R}\}} \|f-p\|_{[-1,1]} \ .$$

The lowest order Jackson type estimate will be shown for $E^*_{n,2}$. Higher order Jackson type estimates for $E^*_{n,2}$, if they exist, would have immediate practical application. Combined with standard arguments they would yield results concerning uniform approximation by reciprocals of polynomials on semi-infinite or infinite intervals.

§4.3 TWO JACKSON TYPE ESTIMATES OF $E_{n,k}$.

DeVore [3,4] has given a much simpler proof of the k = 1 results. Partly similar arguments, are used in this section, to show Jackson estimates analogous to (4.2.1), (4.2.2) for $E_{n,k}$.

Notation. Throughout C_1, C_2,... denote positive constants depending on k, but not depending on f, x or $n \geqslant k$. Whenever it causes no confusion, $\|.\|_\beta$ denotes $\|.\|_{[-\beta,\beta]}$ and $\omega(e,.)$ denotes $\omega_{[-\frac{1}{4},\frac{1}{4}]}(e,.)$.

A function with non-negative kth difference on $[a,b]$ cannot, in general, be extended to a function with non-negative kth difference on a larger interval. For example the piecewise linear and convex function, $f \in C[0, \sum_{n=1}^{\infty} n^{-3}]$, with slope n on the interval $\left(\sum_{i=1}^{n-1} i^{-3}, \sum_{i=1}^{n} i^{-3}\right)$, cannot be extended to the right and remain convex. This motivates the construction of a pre-approximation (see Lemma 4.1) to f, to which we will apply appropriate polynomial convolution operators (see Lemma 4.2).

LEMMA 4.1. *Suppose* $k \geqslant 2$. *Let*

$$L_n(h,x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} h(x+t_1+\ldots+t_k)\,dt_1 \ldots dt_k \qquad (4.3.1)$$

where $h \in C[-\frac{1}{4}, \frac{1}{4}]$ *and*

$$\lambda = 1/8n, \qquad n = k, k+1, \ldots \quad . \qquad (4.3.2)$$

Extend the definition of $L_n(h)$ *from*

$$[-\alpha,\alpha] = [-\tfrac{1}{4} + \frac{k}{8n}, \tfrac{1}{4} - \frac{k}{8n}] \qquad (4.3.3)$$

to $[-\frac{1}{2}, \frac{1}{2}]$ *by adjoining, to the right and left, the Taylor polynomials of degree* k*, corresponding to* $L_n(h)$ *at the points* α, $-\alpha$. *Then there exists constants* $E_k, F_k, G_k; \bar{E}_k, \bar{F}_k, \bar{G}_k;$ *such that; for all* $f \in C[-\frac{1}{4}, \frac{1}{4}]$ *with* $f(-\frac{1}{4}) = f(\frac{1}{4}) = 0$ *and non-negative* kth *difference on* $[-\frac{1}{4}, \frac{1}{4}]$; *for* $n = k, k+1, \ldots$;

$$L_n(f,x)^{(k)} \geqslant 0, \qquad x \in \mathbb{R} \quad , \qquad (4.3.4)$$

$$\|L_n(f)^{(j)}\|_{\frac{1}{4}} \leqslant E_k\, n^j\, \omega(f,n^{-1}) \qquad (j = 1,\ldots, k-1), \qquad (4.3.5)$$

$$\|L_n(f)^{(k)}\|_{\frac{1}{2}} \leqslant E_k\, n^k\, \omega(f,n^{-1}) \quad , \qquad (4.3.6)$$

$$\|f - L_n(f)\|_{\frac{1}{4}} \leqslant F_k\, \omega(f,n^{-1}) \quad , \qquad (4.3.7)$$

and

$$\|L_n(f)\|_{\frac{1}{4}} \leqslant G_k\, n\, \omega(f,n^{-1}) \quad . \qquad (4.3.8)$$

If in addition $f' \in C[-\frac{1}{4}, \frac{1}{4}]$ *then*

$$\|L_n(f)^{(j)}\|_{\frac{1}{4}} \leqslant \bar{E}_k\, n^{j-1}\, \omega(f',n^{-1}) \qquad (j = 2,\ldots, k-1), \qquad (4.3.5')$$

$$\|L_n(f)^{(k)}\|_{\frac{1}{2}} \leqslant \bar{E}_k\, n^{k-1}\, \omega(f',n^{-1}) \quad , \qquad (4.3.6')$$

$$\|f - L_n(f)\|_{\frac{1}{4}} \leqslant \bar{F}_k\, n^{-1}\, \omega(f',n^{-1}) \quad , \qquad (4.3.7')$$

and

$$\|L_n(f)^{(2-j)}\|_{\frac{1}{4}} \leqslant \bar{G}_k\, n^j \omega(f',n^{-1}) \qquad (j = 1,2). \qquad (4.3.8')$$

Proof. For $x \in [-\alpha,\alpha]$

$$L_n(f,x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \int_{x+t_2+\ldots+t_k-\lambda}^{x+t_2+\ldots+t_k+\lambda} f(\gamma)\,d\gamma\, dt_2 \ldots dt_k$$

implying

$$L_n(f,x)' = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \Delta_{2\lambda}\, f(x+t_2+\ldots+t_k-\lambda)\,dt_2 \ldots dt_k \ ;$$

repeating the argument, j times, j = 1,... k,

$$L_n(f,x)^{(j)} = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} .. \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j} f(x+t_{j+1}+\ldots+t_k - j\lambda)dt_{j+1} \ldots dt_k. \tag{4.3.9}$$

(4.3.4) follows immediately. (4.3.9) and the definition of λ imply

$$\|L_n(f)^{(j)}\|_\alpha \leqslant C_1 n^j \omega(f,n^{-1}) \qquad (j = 1,\ldots, k). \tag{4.3.10}$$

(4.3.5), (4.3.6) follow from (4.3.10) on estimating the derivatives of the Taylor polynomials extending $L_n(f)$ to the larger interval.

To prove (4.3.7). The definition of $L_n(f,x)$ clearly implies

$$\|f - L_n(f)\|_\alpha \leqslant C_2 \omega(f,n^{-1}). \tag{4.3.11}$$

Also

$$\|f-L_n(f)\|_{[\alpha,\frac14]} \leqslant \|f-f(\alpha)\|_{[\alpha,\frac14]} + |f(\alpha)-L_n(f,\alpha)| + \|L_n(f,\alpha) - L_n(f)\|_{[\alpha,\frac14]};$$

so by (4.3.2); (4.3.11); (4.3.5), (4.3.6); and the manner in which $L_n(f)$ was extended

$$\|f - L_n(f)\|_{[\alpha,\frac14]} \leqslant C_3 \omega(f,n^{-1}).$$

A similar result holds on $[-\frac14, -\alpha]$; (4.3.7) follows.

To prove (4.3.8). Note that (4.3.7) implies both

$$\omega(L_n(f),n^{-1}) \leqslant C_4 \omega(f,n^{-1})$$

and

$$L_n(f,-\tfrac14) \leqslant F_k \omega(f,n^{-1}) \quad ;$$

the second since $f(-\frac14) = 0$; (4.3.8) follows.

We proceed to prove the results for $f' \in C[-\frac14, \frac14]$. Arguments analogous to those leading from (4.3.9) to (4.3.5), (4.3.6); lead from

$$L_n(f,x)^{(j)} = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} .. \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j-1} f'(x+t_j+..+t_k - (j-1)\lambda)dt_j \ldots dt_k,$$

(j = 1,..., k) to (4.3.5'), (4.3.6').

To show (4.3.7') use the quantitative Korovkin type estimate (see e.g. DeVore [3, pp28-32])

$$|L_n(f,x) - f(x)| \leqslant |f(x)||1 - L_n(1,x)| + |f'(x)||L_n((t-x),x)|$$
$$+ (1+\sqrt{L_n(1,x)})\alpha_n(x)\omega(f';\alpha_n(x)) \tag{4.3.12}$$

where

$$\alpha_n^2(x) = L_n((t-x)^2, x). \tag{4.3.13}$$

Now $\qquad \|1 - L_n(1)\| = \|L_n((t-x), x)\| = 0,$

while

$$L_n((t-x)^2,x) = (2\lambda)^{-k}\int_{-\lambda}^{\lambda}..\int_{-\lambda}^{\lambda}(t_1+t_2+..+t_k)^2\,dt_1\,...\,dt_k$$
$$= k(2\lambda)^{-1}\int_{-\lambda}^{\lambda} t^2\,dt \leqslant C_5\,n^{-2}.$$

Substituting into (4.3.12), (4.3.13) we find

$$\|L_n(f) - f\|_\alpha \leqslant C_6\,n^{-1}\,\omega(f',n^{-1}). \tag{4.3.11'}$$

Since for this particular operator

$$L_n(f,x)' = L_n(f',x), \qquad x \in [-\alpha,\alpha]$$

and $L_n(f,x)'$ is continued outside $[-\alpha,\alpha]$ by adjoining the Taylor polynomials of degree $k-1$, corresponding to f', at either end point; reasoning, similar to that yielding (4.3.7), implies

$$\|f' - L_n(f)'\|_{\frac{1}{4}} \leqslant C_7\,\omega(f',n^{-1}). \tag{4.3.14}$$

Writing

$$\|f - L_n(f)\|_{[\alpha,\frac{1}{4}]} \leqslant |f(\alpha) - L_n(f,\alpha)| + \int_\alpha^{\frac{1}{4}} |f'(t) - L_n(f,t)'|dt\ ;$$

(4.3.11'); (4.3.2) and (4.3.14) imply

$$\|f - L_n(f)\|_{[\alpha,\frac{1}{4}]} \leqslant C_8\,n^{-1}\,\omega(f',n^{-1}).$$

Combining the above, the similar result on $[-\frac{1}{4}, -\alpha]$, and (4.3.11') proves (4.3.7').

To show (4.3.8'). Note (4.3.14) implies

$$\omega(L_n(f)', n^{-1}) \leqslant C_9\,\omega(f',n^{-1})$$

and also

$$|L_n(f,\xi)'| \leqslant C_7\,\omega(f',n^{-1}) \text{ where } f'(\xi) = 0,\ -\tfrac{1}{4} < \xi < \tfrac{1}{4};$$

the existence of such an ξ following from $f(-\frac{1}{4}) = f(\frac{1}{4}) = 0$.

Hence

$$\|L_n(f)'\|_{\frac{1}{4}} \leqslant C_{10}\; n\; \omega(f',n^{-1}).$$

(4.3.8') follows since (4.3.7') implies

$$|L_n(f,-\tfrac{1}{4})| \leqslant \bar{F}_k\; n^{-1}\; \omega(f',n^{-1}). \qquad //$$

We now know how well $L_n(f)$ approximates f, and concern ourselves with how well $L_n(f)$ may be approximated by convolutions with positive polynomials.

LEMMA 4.2. *Suppose* $k \geqslant 2$. *Then there exist constants* H_k, I_k *and a sequence of even positive algebraic polynomials* $\{\lambda_n\}_{n=k}^{\infty}$ *satisfying*

$$\int_{-1}^{1} \lambda_n(t)\,dt = 1 \quad , \qquad (4.3.15)$$

and

$$\|\lambda_n^{(j)}\|_{[-1,1]\setminus[-\frac{1}{4},\frac{1}{4}]} \leqslant H_k\; n^{2-4k+2j} (\leqslant H_k\; n^{-2k}), \qquad (j = 0,\ldots,k-1). \qquad (4.3.16)$$

Further if f *satisfies the conditions of Lemma* 4.1, $g = L_n(f)$ *and*

$$L_n^*(g) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\; \lambda_n(t-x)\,dt \quad ; \qquad (4.3.17)$$

then if $f \in C[-\frac{1}{4}, \frac{1}{4}]$

$$\|g - L_n^*(g)\|_{\frac{1}{4}} \leqslant I_k\; \omega(f,n^{-1}) \quad ; \qquad (4.3.18)$$

and if $f' \in C[-\frac{1}{4}, \frac{1}{4}]$

$$\|g - L_n^*(g)\|_{\frac{1}{4}} \leqslant I_k\; n^{-1}\; \omega(f',n^{-1}). \qquad (4.3.19)$$

Proof. Let $\lambda_k = \lambda_{k+1} = \ldots = \lambda_{4k-1} \equiv \frac{1}{2}$.

For $n \geqslant 2k$, let

$$\lambda_{4n-4k}(t) = c_n[P_{2n}(t)/((t^2 - x_{1,2n}^2)\ldots(t^2 - x_{k,2n}^2))]^2 \; , \qquad (4.3.20)$$

where P_{2n} is the Legendre polynomial of degree 2n and $x_{1,2n},\ldots,x_{n,2n}$ are its positive zeros in increasing order. c_n is a normalizing constant for (4.3.15). Define the remaining λ_n's with the relation

$$\lambda_{4n+1} = \lambda_{4n+2} = \lambda_{4n+3} = \lambda_{4n} \quad , \quad n \geqslant k.$$

Observe firstly that a theorem of Bruns (see e.g. DeVore [3, p.20]) implies

$$C_{11}\, n^{-1} \leqslant x_{1,2n} < \ldots < x_{k,2n} \leqslant C_{12}\, n^{-1}, \quad n > k \tag{4.3.21}$$

Using the normalization $\|P_n\|_{[-1,1]} = 1$ and the corresponding Taylor expansion of P_n (see e.g. Davis [2, p.365]),

$$|P_{2n}(0)| = 2^{-2n} \binom{2n}{n} = (1 + o(1))/\sqrt{\pi n} \quad , \tag{4.3.22}$$

the last equality being a consequence of Stirling's formula. (4.3.20), (4.3.21) and (4.3.22) together imply

$$\lambda_{4n-4k}(0) \geqslant C_{13}\, c_n\, n^{4k-1}, \quad n \geqslant 2k. \tag{4.3.23}$$

Let $n \geqslant 2k$. Write

$$1 = \int_{-1}^{1} \lambda_{4n-4k}(t)\,dt = \sum_{k=-n}^{n} A_k(2n+1)\lambda_{4n-4k}(x_{k,2n+1}) \;;$$

where the $A_k(2n+1)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree 4n+1, with nodes at the zeros of the Legendre polynomial of degree 2n+1. Therefore

$$1 \geqslant A_0(2n+1)\lambda_{4n-4k}(0)$$

and since (Szego [8, p.350]), $A_0\,(2n+1) = \frac{\pi}{2n+1}\,(1 + o(1))$

$$\lambda_{4n-4k}(0) \leqslant C_{14}\, n. \tag{4.3.24}$$

(4.3.23) and (4.3.24) imply

$$c_n \leqslant C_{15}\, n^{2-4k} \quad ;$$

which together with the normalization of the P_n, the definition of the λ_n, and (4.3.21) implies

$$\|\lambda_n\|_{[-1,1]\setminus[-\frac14,\frac14]} \leqslant C_{16}\, n^{2-4k} \quad .$$

(4.3.16) follows by means of Markov's inequality.

It remains to show the order of approximation results. We cannot use the standard quantitative Korovkin theorem as $\omega_{[-\frac{1}{2},\frac{1}{2}]}(g,n^{-1}) \neq O(\omega_{[-\frac{1}{4},\frac{1}{4}]}(f,n^{-1}))$; at least not in general. However a related method is applicable.

Again let $n \geqslant 2k$. The polynomial $t^{2k}\lambda_{4n-4k}(t)$ is of degree $4n - 2k$. Therefore for $j = 1,\ldots, k$

$$M_j = \int_{-1}^{1} t^{2j}\lambda_{4n-4k}(t)\,dt = 2\sum_{i=1}^{n} x_{i,2n}^{2j} A_i(2n)\lambda_{4n-4k}(x_{i,2n}) \quad ;$$

where the $A_i(2n)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n - 1$, with nodes at the zeros of the Legendre polynomial of degree $2n$. Since λ_{4n-4k} has zeros at $x_{k+1,2n},\ldots, x_{n,2n}$,

$$M_j = 2\sum_{i=1}^{k} x_{i,2n}^{2j} A_i(2n)\lambda_{4n-4k}(x_{i,2n}).$$

Since also λ_{4n-4k} has a local maximum on $[-x_{k+1,2n}, x_{k+1,2n}]$ at zero, and (Szego [8, p.350])

$$A_i(2n) \leqslant \frac{\pi}{2n}(1+o(1)) \qquad (i = 1,\ldots, k) \quad ,$$

(4.3.21), (4.3.24) imply

$$\int_{-1}^{1} t^{2j}\lambda_n(t)\,dt \leqslant C_{17}\, n^{-2j}, \quad j = 1, \ ,k, \ n \geqslant k. \tag{4.3.25}$$

(4.3.25), (4.3.16) and that $\lambda_n(t)$ is even and non-negative may be used to estimate certain quantities involving L_n^*. All the estimates are uniform in $|x| \leqslant \frac{1}{4}$.

$$o \leqslant 1-L_n^*(1,x) = \int_{-1}^{1}\lambda_n(t)\,dt - \int_{-\frac{1}{2}-x}^{\frac{1}{2}-x}\lambda_n(t)\,dt \leqslant 2\int_{\frac{1}{4}}^{1}\lambda_n(t)\,dt \leqslant C_{18}\, n^{2-4k}. \tag{4.3.26}$$

$$\begin{aligned} o \leqslant L_n^*((t-x)^{2j}, x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (t-x)^{2j}\lambda_n(t-x)\,dt \\ &= \int_{-\frac{1}{2}-x}^{\frac{1}{2}-x} t^{2j}\lambda_n(t)\,dt \\ &\leqslant \int_{-1}^{1} t^{2j}\lambda_n(t)\,dt \end{aligned}$$

and applying (4.3.25)

$$0 \leqslant L_n^*((t-x)^{2j},x) \leqslant C_{19}\, n^{-2j}, \qquad j = 1,\ldots, k. \tag{4.3.27}$$

$$\begin{aligned} 0 \leqslant L_n^*(|t-x|^k, x) &\leqslant \int_{-1}^{1} |t|^k \lambda_n(t)\,dt \\ &\leqslant \left(\int_{-1}^{1} t^{2k} \lambda_n(t)\,dt\right)^{\frac{1}{2}} \\ &\leqslant C_{20}\, n^{-k}, \end{aligned} \tag{4.3.28}$$

where we have used the Schwartz inequality, (4.3.15) and (4.3.27).

For j odd,

$$\begin{aligned} |L_n^*((t-x)^j,x)| &= \left|\int_{-\frac{1}{2}-x}^{\frac{1}{2}-x} t^j \lambda_n(t)\,dt\right| \\ &\leqslant 2\int_{\frac{1}{4}}^{1} t^j \lambda_n(t)\,dt \end{aligned}$$

since λ_n is even. Applying (4.3.16)

$$|L_n^*((t-x)^j,x)| \leqslant C_{21}\, n^{2-4k}, \qquad j = 1,3,5,\ldots \tag{4.3.29}$$

If $t \in [-\frac{1}{2},\frac{1}{2}]$ and $x \in [-\frac{1}{4},\frac{1}{4}]$, Taylor's theorem gives

$$g(t) = \left(\sum_{j=0}^{k-1} \frac{g^{(j)}(x)(t-x)^j}{j!}\right) + \frac{1}{(k-1)!}\int_x^t g^{(k)}(u)(t-u)^{k-1}du. \tag{4.3.30}$$

Since the last term on the right hand side is bounded in modulus by $\frac{1}{k!}|t-x|^k \|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]}$, the linearity and monotonicity of L_n^* imply

$$\left|L_n^*(g,x) - \sum_{j=0}^{k-1} \frac{g^{(j)}(x)}{j!} L_n^*((t-x)^j,x)\right| \leqslant \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} L_n^*(|t-x|^k,x);$$

or

$$\begin{aligned} |L_n^*(g,x) - g(x)| \leqslant |g(x)|\,|1 - L_n^*(1,x)| &+ \sum_{j=1}^{k-1} \frac{|g^{(j)}(x)|}{j!} |L_n^*((t-x)^j,x)| \\ &+ \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} L_n^*(|t-x|^k,x). \end{aligned}$$

Thus

$$\begin{aligned} \|L_n^*(g,x) - g(x)\|_{[-\frac{1}{4},\frac{1}{4}]} &\leqslant \|g\|_{[-\frac{1}{4},\frac{1}{4}]}\|1-L_n^*(1)\|_{[-\frac{1}{4},\frac{1}{4}]} \\ &+ \sum_{j=1}^{k-1} \frac{\|g^{(j)}\|_{[-\frac{1}{4},\frac{1}{4}]}}{j!}\|L_n^*((t-x)^j,x)\|_{[-\frac{1}{4},\frac{1}{4}]} \\ &+ \frac{1}{k!}\|g^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]}\|L_n^*(|t-x|^k,x)\|_{[-\frac{1}{4},\frac{1}{4}]}. \end{aligned}$$

Combining the above, the estimates of all the terms involving g from Lemma 4.1 ($g = L_n(f)$), and the estimates (4.3.26), (4.3.27), (4.3.28), (4.3.29) of all the $\|L_n^*(.,.)\|$'s yields (4.3.18), (4.3.19). //

Given Lemmas 4.1 and 4.2 it remains to discuss how close $L_n^*(g)$ is to a polynomial with non-negative kth derivative on $[-\frac{1}{4},\frac{1}{4}]$.

THEOREM 4.3. *For each* $k \geq 2$ *there exists a constant* D_k, *such that for all* $h \in C[-\frac{1}{4},\frac{1}{4}]$ *with kth forward difference non-negative on* $[-\frac{1}{4},\frac{1}{4}]$

$$E_{n,k}(h) \leq D_k\, \omega_{[-\frac{1}{4},\frac{1}{4}]}(h,n^{-1}), \qquad n = k,k+1,\ldots \quad .$$

If in addition $h' \in C[-\frac{1}{4},\frac{1}{4}]$ *then*

$$E_{n,k}(h) \leq D_k\, n^{-1}\omega_{[-\frac{1}{4},\frac{1}{4}]}(h',n^{-1}), \qquad n = k,k+1,\ldots \quad .$$

Proof. Fix $k \geq 2$. Let $f = h - \rho$ where

$$\rho(x) = h(-\tfrac{1}{4}) + 2(h(\tfrac{1}{4}) - h(-\tfrac{1}{4}))(x+\tfrac{1}{4}).$$

Clearly $\omega(f,n^{-1}) \leq 2\omega(h,n^{-1})$ and when h' exists $\omega(f',n^{-1}) = \omega(h',n^{-1})$. Lemmas 4.1, and 4.2 apply to f. Writing

$$\bar{L}_n(h) = \rho(x) + L_n^*(L_n(f))$$

Lemmas 4.1 and 4.2 imply

$$\begin{aligned} \|h - \bar{L}_n(h)\|_{\frac{1}{4}} &= \|f - L_n^*(L_n(f))\| \\ &\leq \|f - L_n(f)\|_{\frac{1}{4}} + \|L_n(f) - L_n^*(L_n(f))\|_{\frac{1}{4}} \\ &\leq \begin{cases} C_{22}\ \omega(h,n^{-1}), & h \in C[-\frac{1}{4},\frac{1}{4}], \\ C_{22}\ n^{-1}\ \omega(h',n^{-1}), & h' \in C[-\frac{1}{4},\frac{1}{4}]. \end{cases} \end{aligned} \tag{4.3.31}$$

Let $g = L_n(f)$. Then

$$\bar{L}_n(h) = \rho(x) + L_n^*(g) = \rho(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\lambda_n(t-x)\,dt,$$

$$\begin{aligned} \bar{L}_n(h,x)' &= \rho'(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\,.\,-\lambda_n'(t-x)\,dt \\ &= \rho'(x) + \left[-g(t)\lambda_n(t-x)\right]_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(t)\lambda_n(t-x)\,dt. \end{aligned}$$

$k \geq 2$ alternate differentiations and integrations by parts yield;

$$\bar{L}_n(h,x)^{(k)} = (-1)^k \left[\sum_{j=0}^{k-1} (-1)^j \left[g^{(j)}(t)\lambda_n^{(k-1-j)}(t-x)\right]_{t=-\frac{1}{2}}^{t=\frac{1}{2}}\right]$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{(k)}(t)\lambda_n(t-x)\,dt$$

$$= r(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{(k)}(t)\lambda_n(t-x)\,dt.$$

(4.3.4) and the positivity of the kernels imply the second term on the right hand side is non-negative. The estimates (4.3.5), (4.3.8); (4.3.16) imply

$$\|r\|_{\frac{1}{4}} \leqslant C_{23}\, n^{-2k+1}\omega(h,n^{-1}), \qquad h \in C[-\tfrac{1}{4}, \tfrac{1}{4}],$$

and the estimates (4.3.5'), (4.3.8'); (4.3.16) imply

$$\|r\|_{\frac{1}{4}} \leqslant C_{24}\, n^{-2k+2}\omega(h', n^{-1}), \qquad h' \in C[-\tfrac{1}{4}, \tfrac{1}{4}]$$

In the first case let

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{23}\, n^{-2k+1}\omega(h, n^{-1}),$$

and in the second let

$$p_n(x) = \bar{L}_n(h,x) + \frac{x^k}{k!} C_{24}\, n^{-2k+2}\omega(h', n^{-1}).$$

Then $p_n^{(k)}(x)$ is non-negative on $[-\frac{1}{4},\frac{1}{4}]$; and by (4.3.31) $p_n(x)$ provides the estimate of the theorem.

§4.4 A JACKSON TYPE ESTIMATE OF $E^*_{n,2}$.

The argument, used in this section, is derived from the delightful proof of Jackson's theorem given by Passow [7]. Define $E^*_{n,2}(f)$ as in section 4.2.

THEOREM 4.4. *There exists a constant* M, *such that for any function* f *convex on* [-1,1]

$$E^*_{n,2} \leqslant M\omega_{[-1,1]}(f,n^{-1}), \qquad n = 2,3,4,\ldots \qquad .$$

Proof. Following [7] construct the polygonal pre-approximation $L(x)$ with : $L(k/n) = f(k/n)$, $k = -n,\ldots, n$; and L linear in each of the intervals $[k/n\ ,\ (k+1)/n]$. Then $|L(x) - f(x)| \leqslant \omega_{[-1,1]}(f,n^{-1})$ for all

$x \in [-1,1]$ and L is convex with f. Let S_k be the slope of L(x) in $((k-1)/n,\ k/n)$ and let

$$a_k = (-S_k + S_{k+1})/2\ , \qquad -n+1 \leqslant k \leqslant n-1\ ;$$

$$a_n = -(S_n + S_{-n+1})/2 = -a_{-n}\ .$$

Then

$$L(x) = A_1 + \sum_{k=-n+1}^{n} a_k |x - k/n| = A_1 + \int_{-1}^{1} |x-t|\, dg_1(t)\ ; \qquad (4.4.1)$$

where $g_1(t)$ is the step function having jumps at $x = k/n$ $(k = -n+1,\dots,n)$ equal to a_k, $g(-1) = 0$, and A_1 a constant. Alternatively L(x) may be expanded as

$$L(x) = A_2 + \sum_{k=-n}^{n-1} a_k \left|x - \frac{k}{n}\right| = A_2 + \int_{-1}^{1} |x-t|\, dg_2(t)\ ; \qquad (4.4.2)$$

where $g_2(t)$ is the step function having jumps at $x = \frac{k}{n}$ $(k = -n+1,\dots,n-1)$ equal to a_k , $g(-1) = 0$, $g(x) = a_{-n}$ for $-1 < x < -1 + (1/n)$; and A_2 is a constant. These expansions are easily verified by calculating the slope of $\Sigma a_k |x - k/n|$ in each subinterval $((k-1)/n,\ k/n)$.

Since the slope of L is increasing, a_i will be non-negative for $i = -n+1,\dots,n-1$. Also a_n is either negative or non-negative; hence at least one of g_1, g_2 will be increasing. Let

$$L(x) = A + \int_{-1}^{1} |x - t|\, dg(t) \qquad (4.4.3)$$

be an expression (4.4.1), (4.4.2) with g_j increasing. Then

<u>LEMMA 4.5</u>. *Let* q(x) *be a polynomial of degree not exceeding* n, *convex on the whole real line, satisfying* q(0) = 0 and

$$\int_{-2}^{2} |d\{|x| - q(x)\}| \leqslant b/n.$$

Then

$$Q_n(x) = A + \int_{-1}^{1} q(x-t)\, dg(t)$$

is a polynomial of degree not exceeding n, *convex on the whole real line, satisfying*

$$\max_{-1\leq x\leq 1} |f(x) - Q_n(x)| \leq (2b+1)\ \omega_{[-1,1]}(f,n^{-1}).$$

Proof. The degree of the approximation follows exactly as in [7, Lemma 1]. The argument is repeated for completeness.

$$\left|f(x) - A - \int_{-1}^{1} q(x-t)\,dg(t)\right| \leq \left|f(x) - A - \int_{-1}^{1} |x-t|\,dg(t)\right| + \left|\int_{-1}^{1} \{|x-t| - q(x-t)\}dg(t)\right|$$

$$\leq \omega_{[-1,1]}(f,n^{-1}) + \left|\{|x-t| - q(x-t)\}g(t)\Big|_{-1}^{1} - \int_{-1}^{1} g(t)\ d\{|x-t| - q(x-t)\}\right|$$

$$\leq \omega_{[-1,1]}(f,n^{-1}) + |g(1)|\frac{b}{n} + \max_{-1\leq t\leq 1}|g(t)|b/n.$$

Now

$$\max_{-1\leq x\leq 1}|g(t)| = \begin{cases} \max\limits_{-n+1\leq j\leq n} \left|\sum\limits_{k=-n+1}^{j} a_k\right| , & \text{if } g = g_1 , \\ \max\limits_{-n\leq j\leq n-1} \left|\sum\limits_{k=-n}^{j} a_k\right| , & \text{if } g = g_2 , \end{cases}$$

$$\leq \max_j |s_j| \leq n\omega_{[-1,1]}(f,n^{-1}).$$

Thus

$$\left|f(x) - A - \int_{-1}^{1} q(x-t)\,dg(t)\right| \leq (2b+1)\omega_{[-1,1]}(f,n^{-1}).$$

The convexity of Q_n follows from the convexity of q and the monotonicity of g, since

$$Q_n''(x) = \int_{-1}^{1} q''(x-t)\,dg(t).$$ //

LEMMA 4.6. *There exists a constant* $c > 0$, *and for each* $n = 1,2,3,\ldots$ *a polynomial* q_{4n-2} *of degree* $4n-2$, *convex on the whole real line satisfying* $q_{4n-2}(0) = 0$ *and*

$$\int_{-2}^{2} |d\{|x| - q_{4n-2}(x)\}| \leq c/n. \qquad (4.4.4)$$

Proof. Let

$$\lambda_{4n-4}(t) = c_n(P_{2n}(t)/(t^2 - x_{1,2n}^2))^2$$

where P_{2n} is the Legendre polynomial of degree 2n ; $x_{1,2n}$ its smallest positive zero; and c_n is a normalizing constant chosen so that

$$\int_{-1}^{1} \lambda_{4n-4}(t)\,dt = 1. \tag{4.4.5}$$

Then [3, pp174-176] λ_{4n-4} is an even, non-negative, algebraic polynomial of degree $4n-4$ such that

$$0 < \int_{-1}^{1} \lambda_{4n-4}(t)\, t^2\, dt \leqslant C_1/n^2 \tag{4.4.6}$$

for some constant C_1, $n = 1,2,\ldots$. (4.4.5), (4.4.6) and the Schwartz inequality imply

$$0 < \int_{-1}^{1} |t|\lambda_{4n-4}(t)\,dt \leqslant C_2/n. \tag{4.4.7}$$

Take as the approximation to $|x|$

$$q_{4n-2}(x) = \int_0^x \left(\int_0^u \lambda_{4n-4}(t/2)\,dt\right) du\,. \tag{4.4.8}$$

The non-negativity of λ_{4n-4} implies the convexity of q_{4n-2}. Also ((4.4.8))

$$q_{4n-2}(0) = q'_{4n-2}(0) = 0\ ; \tag{4.4.9}$$

using in addition the properties of λ_{4n-4}

$$\begin{aligned} -1 = q'_{4n-2}(-2) \leqslant q'_{4n-2}(x) \leqslant 0\ , &\qquad -2 \leqslant x \leqslant 0, \\ 0 \leqslant q'_{4n-2}(x) \leqslant q'_{4n-2}(2) = 1\ , &\qquad 0 \leqslant x \leqslant 2\ . \end{aligned} \tag{4.4.10}$$

From (4.4.9), (4.4.10) and the eveness of q_{4n-2}, it follows that ; $|x| - q_{4n-2}(x)$ is monotone decreasing on $[-2,0]$, monotone increasing on $[0,2]$, with

$$\int_{-2}^{2} |d\{|x| - q_{4n-2}(x)\}| = 2(2 - q_{4n-2}(2))\ . \tag{4.4.11}$$

Taylor's theorem implies

$$q_{4n-2}(x) = q_{4n-2}(0) + xq'_{4n-2}(0) + \int_0^x q''_{4n-2}(u)(x-u)\,du$$
$$= \int_0^x \lambda_{4n-4}(u/2)(x-u)\,du\ .$$

Hence

$$2 - q_{4n-2}(2) = 2 - \int_0^2 \lambda_{4n-4}(u/2)(2-u)\,du$$
$$= 2 - 4\int_0^1 \lambda_{4n-4}(t)(1-|t|)\,dt$$
$$= 2\left(1 - \int_{-1}^1 \lambda_{4n-4}(t)(1-|t|)\,dt\right)$$
$$= 2\int_{-1}^1 \lambda_{4n-4}(t)|t|\,dt\ ,$$

by the eveness of λ_{4n-4} and (4.4.5). Now (4.4.7) implies
$0 < 2 - q_{4n-2}(2) \leqslant 2C_2/n$.

Substitute in (4.4.11) to obtain ;

$$\int_{-2}^2 |d\{|x| - q_{4n-2}(x)\}| \leqslant C/n\ ,$$

where $C = 4C_2$ does not depend on n; as required. //

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