

The University of Canterbury
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**The Constructive Theory
of Operator Algebras**

A thesis submitted
in fulfilment of the requirements
of the degree for

Doctorate of Philosophy

at

The University of Canterbury

by

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April 20, 2000

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Abstract

The present work is a first step towards a systematic constructive development of the theory of operator algebras over a Hilbert space H . Among the topics investigated in the thesis are locally convex topologies, the extension and characterisation of ultraweakly continuous linear functionals on $\mathcal{B}(H)$, and conditions that ensure the (constructive) existence of the adjoint of a bounded linear operator on H . We also study the relationship between a linear subset of $\mathcal{B}(H)$ and the dual of its predual, and the comparison of projections in a von Neumann algebra. The two appendices to the thesis deal, respectively, with weak continuity properties and the locatedness of the range of an operator.

Acknowledgements

I would like to express my deep gratitude to Professor Douglas Bridges, my supervisor, for his continuous advice, support and encouragement.

Special thanks to Professor Cristian Calude for his scientific cooperation and guidance over the last 10 years.

A warm thank you to Associate Professor Hajime Ishihara, Dr. Peter Schuster, Associate Professor Doru Ștefănescu, and Professor Ioan Tomescu for their help.

I would also like to thank my colleagues Robin Havea and Mihaela Baroni for their kind and efficient support, Bas Spitters for his suggestions and Julian Visch for his assistance. Thanks to Viv Bridges, Jane Strachan and Glenys Williams for their friendship and their being there for me.

The financial support offered by the Royal Marsden Foundation of New Zealand has made possible this project and is acknowledged with thanks. Many thanks to the University of Waikato and the University of Canterbury.

Last, but not least, a word of thanks to my parents, Petru and Cornelia Vâță, for their love and faith in me.

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Chapter 1

Preface

1.1 A little bit of history

The beginning of the twentieth century was clearly marked for the mathematical community by the optimism expressed in projects such as Hilbert's proof theory (which tried to set the notion of a proof on a sound basis, thereby avoiding the paradoxes) and Russell and Whitehead's *Principia Mathematica* (in which logic took precedence over mathematics). When, in 1907, the Dutch mathematician L.E.J. Brouwer published his doctoral thesis [23] initiating the first fully developed alternative to what nowadays we call classical mathematics (**CLASS**), not many mathematicians greeted it enthusiastically. Brouwer's **intuitionistic mathematics** (**INT**) was a radically different approach to mathematics, born from the need to create a bridge across the ever-widening gap between what exists formally and what can be obtained effectively. In the years that followed, unexpected results like Gödel's incompleteness theorem, the independence of the axiom of choice from Zermelo–Fraenkel set theory, and the undecidability of important formal systems proved that Brouwer's initiative was truly motivated.

In Brouwer's intuitionism, logic is secondary and derives from the mental constructions that form the basis of mathematics. A mathematical object comes into

existence precisely when it is constructed. Thus the distinction in meaning that forms the basis of Brouwer's revolution is that between

- **idealistic existence**, where we are allowed to conclude that an object exists by proving that its non-existence is impossible, and
- **constructive existence**, where, in order to prove that an object exists, we must provide a method for finding it.

In his 1908 essay "The Unreliability of the Logical Principles" [24] Brouwer criticised the unrestricted use of the **law of excluded middle (LEM)**

$$P \vee \neg P$$

in logic. Subsequently, he introduced into INT some principles that led to results apparently contradicting aspects of classical mathematics. However, to regard Brouwer's mathematics as inconsistent with its classical counterpart is a serious oversimplification of the situation: it would be better to regard the two types of mathematics as incomparable. Nevertheless, in light of the intractability of Brouwer and some other leading exponents of constructivism, it is not surprising that most mathematicians remained sceptical and that many reacted to constructive mathematics with vigorous opposition. A common view was (and remains) that too much mathematics had to be given up in order to accommodate Brouwer's ideas. For example, Hilbert expressed his disagreement with Brouwer by saying that

No-one, though he speak with the tongues of angels, will keep people from using the law of excluded middle.

In 1952 Kleene wrote

Intuitionistic mathematics employs concepts and makes distinctions not found in classical mathematics; and it is very attractive on its own account. As a substitute for classical mathematics it has turned out to be less powerful and in many ways more complicated to develop. ([44], page 50.)

Despite continuing opposition, constructive approaches to mathematics survived. In 1930, Brouwer’s most famous pupil, Arend Heyting, published the first formalisation of intuitionistic logic, abstracted from the practice of intuitionistic mathematics. A completely different approach to constructive mathematics—essentially recursive mathematics with intuitionistic logic (**RUSS**)—was initiated by A.A. Markov in the Soviet Union in 1948–49, and has achieved a number of technical successes [47, 45].

By the mid–1960s constructive mathematics was, when compared with its classical counterpart, virtually stagnant. The situation changed in 1967 with the publication of Errett Bishop’s monograph *Foundations of Constructive Mathematics* [2]. This book represents the most far-reaching and systematic presentation of constructive mathematics to date. In it, Bishop revealed, by thorough-going constructive means but without resorting to either Brouwer’s principles or the formalism of recursive function theory, a vast panorama of constructive mathematics, covering elementary analysis, metric and normed spaces, abstract measure and integration, the spectral theory of selfadjoint operators on a Hilbert space, Haar measure and duality on locally compact groups, and Banach algebras. Bishop’s constructive mathematics (**BISH**) was founded on a primitive, unspecified notion of *algorithm* and the Peano properties of natural numbers, and kept strictly to the interpretation of “existence” as “computability”. His refusal to pin down the notion of algorithm led to criticism, particularly from philosophers of mathematics and from those committed to Church’s thesis. But this very imprecision enabled Bishop’s work to have a variety of interpretations: his results are valid in classical mathematics, intuitionism, and all reasonable models of computable mathematics—such as recursive function theory [45] or Weihrauch’s Type II Effectivity theory [58, 59, 60].

An interesting formal system for Bishop’s mathematics, an intuitionistic Zermelo–Fraenkel set theory, was produced by Myhill [49, 50]. Other foundational systems for BISH are found in [31, 30, 48, 9].

The modern view of constructive mathematics, as propounded by Fred Richman [52], is that in practice

constructive mathematics is none other than mathematics carried out with intuitionistic logic.

From this point of view, each of CLASS, RUSS, and INT can be regarded as BISH plus some additional principles.

1.2 Why use intuitionistic logic?

The study of computability in mathematics can be carried out in two ways:

- ▷ Use classical logic, but pin down the notion of algorithm in order to avoid making decisions that a computer cannot make.
- ▷ Use intuitionistic logic, which automatically takes care of the types of decisions that are permitted, and then argue with whatever mathematical objects one pleases.

But why would anyone choose the second way? Because not only, as we have mentioned before, does mathematics developed with intuitionistic logic have more models than its classical counterpart, but also a constructive proof usually provides more information than a classical one.

Unlike intuitionistic logic, classical logic permits “decisions” that no computer (real or idealistic) can make in general. For example, if x is a nonnegative real number, a computer may be unable to decide between the alternatives $x = 0$ and $x > 0$: the input x may be positive but so close to zero that the computer sets its floating-point representation equal to 0. (This is the problem of *underflow*). Thus we cannot expect the statement

$$\forall x \in \mathbf{R} (x = 0 \vee x \neq 0),$$

to be provable with a logic that truly incorporates the principles used in computation.

The desire for algorithmic interpretability forces us to reconsider the meaning of all logical connectives and quantifiers. The standard constructive/computational interpretations are as follows.

- $P \vee Q$: we have either a proof of P or a proof of Q .
- $P \wedge Q$: we have a proof of P and a proof of Q .
- $\neg P$: assuming P , we can derive a contradiction (such as $0 = 1$).
- $P \implies Q$: we can convert any proof of P into a proof of Q .
- $\exists x P(x)$: there is an algorithm that computes an object x and demonstrates that $P(x)$ holds.
- $\forall x \in A P(x)$: there is an algorithm that, applied to an object x and a proof that $x \in A$, demonstrates that $P(x)$ holds.

Note that in order to recover axioms for classical logic, one need only add the law of excluded middle to Heyting's intuitionistic axioms.

From the time of Brouwer, constructive mathematicians have excluded from their practice a number of intuitionistically undecidable principles. Among these are

- **LPO**, the **limited principle of omniscience**: for each binary sequence (a_n) either $a_n = 0$ for all n , or else there exists n such that $a_n = 1$.
- **LLPO**, the **lesser limited principle of omniscience**: for each binary sequence (a_n) with at most one term equal to 1, either $a_{2n} = 0$ for all n or else $a_{2n+1} = 0$ for all n ,
- **WLPO**, the **weak limited principles of omniscience**: for each binary sequence (a_n) , either $\neg \forall n (a_n = 0)$ or $\neg \neg \forall n (a_n = 0)$.

- **MP, Markov's principle**: for each binary sequence (a_n) such that $\neg\forall n(a_n = 0)$, there exists n such that $a_n = 1$.

Each of the first three is a special case of the law of excluded middle. It is easy to prove that LPO implies WLPO, and that WLPO implies LLPO. These three principles are false in INT and RUSS. On the other hand, Markov's principle is used with caution by the practitioners of RUSS, is in contradiction with some principles in INT, and, since it represents an unbounded search, is rejected outright by most constructive mathematicians. There are models which show that each of LPO, LLPO, LEM, and MP is independent of Heyting arithmetic (that is, Peano arithmetic with intuitionistic logic); see under *Kripke and Beth models* in [29].

By a **Brouwerian counterexample** to a classical proposition P , we mean a (constructive) proof that P implies some essentially nonconstructive principle like LPO, LLPO, There are Brouwerian counterexamples to many widely used classical propositions. Here are some examples, with the implied principle in parentheses:¹

- $\forall x \in \mathbf{R} (x = 0 \vee x \neq 0)$ (LPO)
- **The least-upper-bound principle**: Each nonempty subset S of \mathbf{R} that is bounded above has a least upper bound. (LPO)
- Every real number is either rational or irrational. (LPO)
- $\forall x \in \mathbf{R} (x \geq 0 \vee x \leq 0)$. (LLPO)
- If $x, y \in \mathbf{R}$ and $xy = 0$, then $x = 0$ or $y = 0$. (LLPO)
- A uniformly continuous function from $[0, 1]$ to \mathbf{R} attains its bounds. (LLPO)

¹Chapter 1 of [20] contains more information about Brouwerian counterexamples.

- The **Intermediate Value Theorem**: If $f : [0, 1] \rightarrow \mathbf{R}$ is a continuous function with $f(0) < 0 < f(1)$, then there exists $x \in (0, 1)$ such that $f(x) = 0$.

(LLPO)

Since in constructive mathematics we are more interested in positive results than in the rejection of classical theorems, we desire constructive substitutes for such inadmissible propositions. Fortunately, such substitutes have been found:

- ▷ Although the comparison of two real numbers is a problem, it is shown ([4], page 26, (2.17)) that if $a < b$, then for all $x \in \mathbf{R}$ either $a < x$ or $x < b$ (this result is very often used to split a proof in two cases).
- ▷ If x, y are real numbers such that the assumption $x > y$ implies $0 = 1$, then $x \leq y$ ([4], page 26, (2.18)).
- ▷ The conclusion of the least-upper-bound principle holds if we add the hypothesis that for all real numbers a, b with $a < b$, either b is an upper bound of S or else there exists $x \in S$ with $x > a$ ([4], page 37, (4.3)).
- ▷ The conclusion of the Intermediate Value Theorem holds if we add the hypothesis that f is **locally nonzero**, in the sense that each subinterval of $[0, 1]$ contains points at which the value of f is different from 0 ([4], page 63, Problem 15).

1.3 An example: the infimum of two projections

The following definition introduces one notion that plays an important role in the constructive theory of metric spaces.

Definition 1.3.1 *Let (X, ρ) be a metric space. A subset S of X is said to be **located** (in X) if we can compute the distance*

$$\rho(x, S) = \inf_{s \in S} \rho(x, s)$$

from S to any point x in X .

It follows from the constructive least–upper–bound principle that S is located if and only if for each $x \in X$, and all real numbers α, β with $0 \leq \alpha < \beta$,

either $\rho(x, s) > \alpha$ for all $s \in S$

or else there exists $s \in S$ such that $\rho(x, s) < \beta$.

Now consider a complex Hilbert space H . We define the **infimum of the projections** E and F of H to be the unique projection G (if it exists) that satisfies the following two conditions:

(G1) $G \leq E$ and $G \leq F$.

(G2) If H is a projection such that $H \leq E$ and $H \leq F$, then $H \leq G$,

where \leq is the usual ordering of projections on H . We then denote the infimum by $E \wedge F$. Classically, $E \wedge F$ always exists, and is the projection on the intersection of $\text{ran } E$ (the range of E) and $\text{ran } F$ ([40], page 111). Constructively, the projection on a closed linear subset M of a Hilbert space H exists if and only if M is located (see [4], page 366, Theorem (8.7)). Since there is no guarantee that the intersection of two located sets is also located, the intersection of two projections may not exist.

It can be shown classically that the decreasing sequence $((EFE)^n)_{n=1}^\infty$ of projections converges strongly to a projection G on H , in the sense that

$$Gx = \lim_{n \rightarrow \infty} (EFE)^n x$$

for all $x \in H$; it then follows that G satisfies conditions (G1) and (G2), and is therefore the infimum of the projections E and F ([33], page 257). Thus in classical mathematics there is an analytic characterisation of the infimum of two projections. Can we use this to produce interesting conditions under which $E \wedge F$ exists constructively?

A famous theorem of Specker [54] shows that the monotone convergence theorem in \mathbf{R} is false in RUSS. It follows that the monotone convergence theorem for projections of a Hilbert space is also false in RUSS, and hence that Halmos's classical proof of the statement

$$(E \wedge F) x = \lim_{n \rightarrow \infty} (EFE)^n x \quad (x \in H) \quad (1.1)$$

fails constructively.

Now, it is easy to adapt Halmos's argument to show constructively that if either

$$\lim_{n \rightarrow \infty} (EFE)^n x$$

exists for all $x \in H$ or (equivalently)

$$\lim_{n \rightarrow \infty} \langle (EFE)^n x, y \rangle$$

exists for all $x, y \in H$, then $E \wedge F$ exists and satisfies (1.1). Can we prove, conversely, that if $E \wedge F$ exists, then the sequence $((EFE)^n x)_{n=1}^{\infty}$ converges for each $x \in H$? To see that the answer is “no”, consider the case where $H = \mathbf{R}^2$, E is the projection of H on the subspace $\mathbf{R}(1, 0)$, and F is the projection on $\mathbf{R}(\cos \theta, \sin \theta)$, where $\neg(\theta = 0)$. In this situation we have $E \wedge F = 0$. Taking $e = (1, 0)$, we also have

$$(EFE)^n e = (\cos^{4n} \theta, 0)$$

for each n ; but this converges to $(0, 0)$ if and only if $\cos \theta \neq 1$ (that is, $|\cos \theta| > 0$) and therefore $\theta \neq 0$. Thus if the answer to our last question were “yes”, we could prove that

$$\forall \theta \in \mathbf{R} \quad (\neg(\theta = 0) \Rightarrow \theta \neq 0),$$

which is easily shown to be equivalent to Markov's Principle.

As a sample of positive constructive mathematics, we have the following necessary and sufficient condition for the existence of the infimum of two projections.

Proposition 1.3.2 *Let E, F be projections of the Hilbert space H , and let $S = \text{ran } E \cap \text{ran } F$. Then the following two conditions are equivalent.*

- (i) *The infimum $E \wedge F$ exists.*
- (ii) *For each $z \in H$ there exist $y \in S$ and $N \geq 1$ such that $(EFE)^N z - y$ is orthogonal to S .*

Proof. Suppose that $G = E \wedge F$ exists; then S is closed, G is the projection of H onto S , and for each $z \in H$, $z - Gz$ is orthogonal to S . Since $G \leq (EFE)^n$, it follows that for each $x \in S$ and each n we have

$$\begin{aligned}
 \langle (EFE)^n z - Gz, x \rangle &= \langle (EFE)^n z - (EFE)^n Gz, x \rangle \\
 &= \langle z - Gz, (EFE)^n x \rangle \\
 &= \langle z - Gz, x \rangle \\
 &= 0.
 \end{aligned}$$

Thus we may take $y = Gz$ to complete the proof that (i) \Rightarrow (ii).

Conversely, assume (ii). Then, for each $x \in S$ we have

$$\begin{aligned}
 \langle (EFE)^{N+1} z - y, x \rangle &= \langle (EFE)^{N+1} z - (EFE)y, x \rangle \\
 &= \langle (EFE)((EFE)^N z - y), x \rangle \\
 &= \langle (EFE)^N z - y, EFE x \rangle \\
 &= \langle (EFE)^N z - y, x \rangle \\
 &= 0.
 \end{aligned}$$

Replacing N successively by $N + 1, N + 2, \dots, 2N - 1$ in this argument, we obtain

$$\langle (EFE)^{2N} z - y, x \rangle = 0.$$

Thus

$$\langle (EFE)^N z - z, x \rangle = \langle (EFE)^N z - z, (EFE)^N x \rangle$$

$$\begin{aligned}
&= \langle (EFE)^{2N}z - (EFE)^Nz, x \rangle \\
&= \langle (EFE)^{2N}z - y, x \rangle - \langle (EFE)^Nz - y, x \rangle \\
&= 0
\end{aligned}$$

and therefore

$$\begin{aligned}
\langle z - y, x \rangle &= \langle (EFE)^Nz - z, x \rangle + \langle z - y, x \rangle \\
&= \langle (EFE)^Nz - y, x \rangle \\
&= 0.
\end{aligned}$$

Hence $z - y$ is orthogonal to S . Elementary Hilbert space theory now tells us that $\rho(z, S)$ exists and equals $\|z - y\|$. Since $z \in H$ is arbitrary, we conclude that S is located and hence that $E \wedge F$ exists. Q.E.D

1.4 How the Thesis is organised

In Chapter 2 we introduce some elementary notions in the constructive theory of uniform and locally convex spaces. In particular, we show that if the unit ball of a locally convex space X is totally bounded, then so is the intersection of that ball with the kernel of any uniformly continuous linear functional on X . This result is used later, in Chapter 4.

Chapter 3 deals with the standard operator topologies on $\mathcal{B}(H)$. We show that the unit ball of $\mathcal{B}(H)$ is weak-operator totally bounded, and then use this to prove that the weak-operator continuity of the left multiplication mapping $T \mapsto AT$ on $\mathcal{B}_1(H)$ is equivalent to the existence of the adjoint of A . We also examine the sequential continuity of left multiplication.

The main result of the Thesis, a Hahn–Banach type theorem for linear functionals on a linear subset of $\mathcal{B}(H)$, is presented in Chapter 4. There we also explore the embedding of a linear set of bounded operators on a Hilbert space as a dense subset

of the dual of its predual.

Chapter 5 is the first step towards a constructive development of the theory of projections on a Hilbert space. We investigate finite and infinite projections, the countable additivity of equivalence; we introduce two types of equivalence for projections and study the relationship between them.

There are two appendices to the thesis. The first of these introduces conditions that ensure weak continuity properties of mappings between metric and normed spaces. The second deals with the relation between the locatedness of the range of a positive operator T on H and the strong convergence of the sequence $(T^{1/n})_{n=1}^{\infty}$.

1.5 Notations

Throughout the Thesis we will be using the following notations:

H : a complex (real) Hilbert space.

$\mathcal{B}(H)$: the linear space of all bounded operators on H .

H_1 : the unit ball of H .

$\mathcal{B}_1(H)$: the unit ball of $\mathcal{B}(H)$.

H_{∞} : the direct sum $\bigoplus_{n=1}^{\infty} H_n$ of infinitely many copies of H .

M^{\perp} : the orthogonal complement of M .

Chapter 2

Uniform and Locally Convex Spaces

2.1 Introduction

Locally convex spaces are regarded by many authors as the most important class of topological vector spaces. Although Errett Bishop considered that “in most cases of interest it seems to be unnecessary to make use of any deep facts from the general theory of locally convex spaces”, recent developments in constructive analysis (in particular, operator algebra theory) increasingly depend on such a theory. In turn, that theory draws on the general theory of uniform spaces, the beginnings of which were outlined in Problems 17–21 on pages 110–111 of [4]. Some basic definitions in the theory of locally convex spaces also appear in Chapter 8 of [34].

We first introduce the basic terminology and establish some fundamental facts about uniform spaces; in general, we do not define notions, or prove facts, that carry over unchanged from the classical to the constructive setting.

Definition 2.1.1 *A **uniform space** is a set X together with a family $(\rho_i)_{i \in I}$ of pseudometrics on X . The **equality** and **inequality** on X are defined, respectively,*

as follows:

$$x = y \quad \text{if and only if} \quad \forall i \in I \ (\rho_i(x, y) = 0),$$

$$x \neq y \quad \text{if and only if} \quad \exists i \in I \ (\rho_i(x, y) > 0).$$

The corresponding **uniform topology** on X is the topology in which, for each $x_0 \in X$, the sets

$$V(x_0, F, \varepsilon) = \left\{ x \in X : \sum_{i \in F} \rho_i(x, x_0) < \varepsilon \right\},$$

with $\varepsilon > 0$ and F a finitely enumerable subset of I , form a basis of neighbourhoods of x_0 ; the pseudometrics ρ_i are called the **defining pseudometrics** for this topology.

An inequality relation \neq on a set is said to be **tight** if $\neg(x \neq y)$ implies that $x = y$. Note that the inequality on a uniform space is tight. Indeed, we have $\neg(x \neq y) \iff \neg(\exists i \in I (\rho_i(x, y) > 0)) \iff \forall i \in I \neg(\rho_i(x, y) > 0) \iff \forall i \in I (\rho_i(x, y) = 0)$.

Metric and uniform spaces are viewed as uniform spaces in the obvious way. For our purposes, a more important type of uniform space is a **locally convex space**, which consists of a linear space X over \mathbf{F} , together with a family $(p_i)_{i \in I}$ of seminorms for which the corresponding family \mathcal{F} of pseudometrics

$$(x, y) \mapsto p_i(x - y)$$

defines the topology (and, incidentally, the inequality) on X . In this case we refer to the seminorms p_i as **defining seminorms** for the **locally convex topology**—that is, the uniform topology defined by \mathcal{F} ; and we call the set

$$X_1 = \{x \in X : \forall i \in I (p_i(x) \leq 1)\}$$

the **unit ball** of the locally convex space X .

In the rest of this section, unless we specify otherwise, $(X, (\rho_i)_{i \in I})$ and $(Y, (\sigma_j)_{j \in J})$ are uniform spaces.

Definition 2.1.2 A mapping $f : X \rightarrow Y$ is **uniformly continuous on X** if for every $\epsilon > 0$ and every finitely enumerable subset G of J there exist $\delta > 0$ and a finitely enumerable subset F of I such that if $x, y \in X$ and $\sum_{i \in F} \rho_i(x, y) < \delta$, then $\sum_{j \in G} \sigma_j(f(x), f(y)) < \epsilon$.

Notice that each defining seminorm p_i on a locally convex space is uniformly continuous.

Definition 2.1.3 Let (J, \geq) be a partially ordered set. We say that J is **directed by the binary relation \geq** if for each pair a, b of elements from J , there exists an element c in J such that $c \geq a$ and $c \geq b$.

A **net** in a set S is a mapping $j \mapsto x_j$ from a directed set J to S ; we denote this net by $(x_j)_{j \in J}$.

Definition 2.1.4 A net (x_j) in the uniform space X

- **converges to an element x in X** if for each neighbourhood V of x there exists an index j_0 in J such that $x_j \in V$ whenever $j \geq j_0$.
- is a **Cauchy net** if given any $\epsilon > 0$ and any finitely enumerable subset F of I , there exists an index j_0 in J such that $\sum_{i \in F} \rho_i(x_j, x_k) < \epsilon$ whenever $j, k \geq j_0$.

A subset S of an uniform space X is said to be **dense** in X if given any point $x \in X$ and a neighbourhood V of x there exists an element $s \in S$ such that $s \in V$.

The following result is useful in constructing uniformly continuous functions with values in a complete locally convex space.

Proposition 2.1.5 Let Y be a dense subset of a locally convex space X , and $f : Y \rightarrow Z$ a uniformly continuous function from Y to a complete locally convex space Z . Then there exists a uniformly continuous function $g : X \rightarrow Z$ such that $f(y) = g(y)$ for all y in Y .

Proof. Let $(\rho_i)_{i \in I}$ and $(\sigma_j)_{j \in J}$ be the defining families of seminorms on X and Z , respectively. Since Y is dense, for each x in X there exists a net $(y_m)_{m \in M}$ in Y converging to x . This also means that $(y_m)_{m \in M}$ is a Cauchy net and since f is uniformly continuous, $(f(y_m))_{m \in M}$ is a Cauchy net in Z ; whence it converges to a limit z . We will show that the operation assigning to x the element z so defined is a function. Consider another net $(y'_l)_{l \in L}$ in Y converging to x , and denote by z' the limit of $(f(y'_l))$. Suppose that $z \neq z'$. Then there exists $j_0 \in J$ such that $\sigma_{j_0}(z - z') > \varepsilon$ for some $\varepsilon > 0$. As all the seminorms σ_j are continuous, we get

$$\sigma_{j_0}(z - z_0) = \sigma_{j_0}(\lim_m f(y_m) - \lim_l f(y'_l)) = \lim_m \lim_l \sigma_{j_0}(f(y_m) - f(y'_l)) > \varepsilon. \quad (2.1)$$

Let $\delta > 0$ be as in Definition 2.1.2 corresponding to ε and j_0 . Since both (y_m) and (y'_l) converge to x , for each finitely enumerable subset F of I there exist $m_0 \in M$ and $l_0 \in L$ such that

$$\sum_{i \in F} \rho_i(y_m - y'_l) < \delta \quad \text{for all } m \geq m_0 \text{ and all } l \geq l_0. \quad (2.2)$$

Using again the uniform continuity of f , we can find a finitely enumerable subset F of I such that whenever (2.2) is satisfied, we have

$$\sigma_{j_0}(f(y_m) - f(y'_l)) < \varepsilon \quad \text{for all } m \geq m_0 \text{ and all } l \geq l_0,$$

which contradicts (2.1). Since the inequality on a locally convex space is tight and the assumption $z \neq z'$ is contradictory, it follows that $z = z'$ and hence that $x \mapsto g(x) = z$ is a function. Clearly, $g(y) = f(y)$ for all $y \in Y$. By an argument similar to the one above, it easily follows that g is uniformly continuous on X .
Q.E.D.

Definition 2.1.6 *A subset S of X is **totally bounded with respect to the finitely enumerable subset F of I** if for each $\varepsilon > 0$ there exists a finitely enumerable subset S_ε of S such that for each $x \in S$ there exists $s \in S_\varepsilon$ with $\sum_{i \in F} \rho_i(x, s) < \varepsilon$.*

The set S_ε is called a **finitely enumerable ε -approximation** to S relative to F . If S is totally bounded with respect to each finitely enumerable subset of I , then we say that S is **totally bounded**.

Many of the most important results in classical analysis depend on the existence of the supremum or infimum of certain sets. Since constructively the least upper bound principle does not hold, the following proposition and its corollary will show the important role played by total boundedness in this respect.

Proposition 2.1.7 *If X is totally bounded and $f : X \rightarrow Y$ is uniformly continuous, then $f(X)$ is totally bounded.*

Proof. Let x be any point of X , let $\varepsilon > 0$, and consider a finitely enumerable subset G of J . Since f is uniformly continuous, there exist $\delta > 0$ and a finitely enumerable subset F of I such that

$$\sum_{j \in G} \sigma_j(f(y), f(x)) < \varepsilon$$

whenever $x, y \in X$ are such that

$$\sum_{i \in F} \rho_i(x, y) < \delta.$$

Since X is totally bounded, there exists a δ -approximation $\{x_1, \dots, x_N\}$ to X corresponding to F . Pick k such that

$$\sum_{i \in F} \rho_i(x, x_k) < \delta.$$

Then

$$\sum_{j \in G} \sigma_j(f(x), f(x_k)) < \varepsilon.$$

As x was chosen arbitrarily, we conclude that $\{f(x_1), \dots, f(x_N)\}$ is an ε -approximation to the image of f with respect to G . Q.E.D.

Corollary 2.1.8 *If f is a uniformly continuous function from a totally bounded uniform space X to \mathbf{R} , then the supremum and infimum of f exist.*

Proof. Since $f(X)$ is totally bounded, the result follows from (4.4) page 38 of [4].
Q.E.D.

The following definition generalises the notion of a located set to the present context.

Definition 2.1.9 *A subset S of X is **located** if*

$$\inf \left\{ \sum_{i \in F} \rho_i(x, y) : y \in S \right\}$$

exists for each $x \in X$ and each finitely enumerable subset F of I .

It follows from the constructive least-upper-bound principle that S is located if and only if for each $x \in X$, each finitely enumerable subset F of I , and all real numbers α, β with $0 \leq \alpha < \beta$,

either $\sum_{i \in F} \rho_i(x, y) > \alpha$ for all $y \in S$

or else there exists $y \in S$ such that $\sum_{i \in F} \rho_i(x, y) < \beta$.

Proposition 2.1.10 *A totally bounded subset of a uniform space is located.*

Proof. Let S be a totally bounded subset of X . Let $x \in X$, let F be a finitely enumerable subset of I , and let $0 \leq \alpha < \beta$. Writing $\varepsilon = \frac{1}{2}(\alpha + \beta)$, construct a finitely enumerable ε -approximation $\{s_1, \dots, s_n\}$ to S relative to F . Let

$$d = \inf \left\{ \sum_{i \in F} \rho_i(x, s_k) : 1 \leq k \leq n \right\},$$

which exists as the infimum of a finitely enumerable subset of \mathbf{R} . Either $d > \alpha + \varepsilon$ or $d < \beta$. In the first case, given $y \in S$ and choosing k ($1 \leq k \leq n$) such that $\sum_{i \in F} \rho_i(y, s_k) < \varepsilon$, we have

$$\sum_{i \in F} \rho_i(x, y) \geq \sum_{i \in F} \rho_i(x, s_k) - \sum_{i \in F} \rho_i(y, s_k) > d - \varepsilon > \alpha.$$

In the second case, there exists k ($1 \leq k \leq n$) such that $\sum_{i \in F} \rho_i(x, s_k) < \beta$. Q.E.D.

Proposition 2.1.11 *A located subset of a totally bounded uniform space is totally bounded.*

Proof. Assume that X is totally bounded, and let S be a located subset of X . Given $\varepsilon > 0$ and a finitely enumerable subset F of I , choose a finitely enumerable $\frac{\varepsilon}{3}$ -approximation $\{x_1, \dots, x_n\}$ to X . Since S is located, we can write $\{1, \dots, n\}$ as a union of subsets P, Q such that

if $k \in P$, then $\sum_{i \in F} \rho_i(s, x_k) > \varepsilon/3$ for all $s \in S$, and

if $k \in Q$, then there exists $s \in S$ such that $\sum_{i \in F} \rho_i(s, x_k) < 2\varepsilon/3$.

For each $j \in Q$ choose $s_j \in S$ such that $\sum_{i \in F} \rho_i(x_j, s_j) < 2\varepsilon/3$. Given $s \in S$, choose k ($1 \leq k \leq n$) such that $\sum_{i \in F} \rho_i(s, x_k) < \varepsilon/3$. Then $k \in Q$ and so

$$\sum_{i \in F} \rho_i(s, s_k) \leq \sum_{i \in F} \rho_i(s, x_k) + \sum_{i \in F} \rho_i(x_k, s_k) < \varepsilon.$$

Thus $\{s_k : k \in Q\}$ is a finitely enumerable ε -approximation to S . Q.E.D.

We omit the proofs of the next three results since they are simple adaptations of (4.7) page 30 and (4.8) page 31 of [20], and (4.9) page 98 of [4], respectively.

Theorem 2.1.12 *Let (E, ρ) be a totally bounded pseudometric space, x_0 a point of E , and r a positive number. Then there exists a closed, totally bounded subset K of E such that $B(x_0, r) \subset K \subset B(x_0, 8r)$.*

Corollary 2.1.13 *If E is a totally bounded pseudometric space, then for each $\varepsilon > 0$ there exist totally bounded sets K_1, \dots, K_n , each of diameter less than ε , such that $E = \bigcup_{i=1}^n K_i$.*

Proposition 2.1.14 *Let f be an uniformly continuous mapping on a totally bounded subset S of a pseudometric space E . Then for all but countably many real numbers $t > m = \inf\{f(x) : x \in S\}$ the set*

$$S_t = \{x \in E : |f(x)| \leq t\}$$

is totally bounded; in other words, there exists a sequence $(t_n)_{n=1}^{\infty}$ in the interval (m, ∞) such that S_t is totally bounded whenever $t > m$ and $t \neq t_n$ for each n .

2.2 Continuous linear functionals on locally convex spaces

One important branch of the theory of locally convex spaces is the theory of linear operators. Of special interest therein is the role played by linear functionals on those spaces. The duality theory of locally convex spaces provides a powerful tool to translate a problem on the space (or on linear operators between locally convex spaces) into one concerning linear forms.

In the rest of this chapter, unless otherwise specified, X denotes a locally convex space with its topology defined by the family $(p_i)_{i \in I}$ of seminorms. Our main result, Theorem 2.2.4, shows that, under suitable hypotheses, the intersection of the unit ball with the kernel of a linear functional on X is totally bounded. This result has at least one substantial application, in the theory of operator algebras acting on a Hilbert space which is not assumed to be separable.

The proof of our theorem requires some preliminaries; for the first of these, we recall that a mapping f between linear spaces is **homogeneous** if $f(\lambda x) = \lambda f(x)$ for all scalars λ and vectors x .

Proposition 2.2.1 *Let (E, p) be a seminormed space and let S be a balanced, totally bounded subset of E . If $f : E \rightarrow \mathbf{F}$ is a homogeneous mapping, uniformly continuous on S , then for all $t > 0$ the set*

$$S_t = \{x \in S : |f(x)| \leq t\}$$

is totally bounded.

Proof. Since S is balanced, it contains 0, and therefore $\inf\{f(x) : x \in S\} = 0$. Being totally bounded, S is bounded: there exists $M > 0$ such that $p(x) \leq M$ for all $x \in S$. Let $t > 0$ and let $0 < \varepsilon < 1$. By Proposition 2.1.14 there exists $t' < t$ such that

$$\frac{t}{t'} < \min \left\{ 2, 1 + \frac{\varepsilon}{2M} \right\}$$

and $S_{t'}$ is totally bounded. Let $\{x_1, \dots, x_n\}$ be an $\frac{\varepsilon}{4}$ -approximation to $S_{t'}$. If $x \in S_t$, then $\frac{t'}{t}x \in S_{t'}$, so there exists j ($1 \leq j \leq n$) such that

$$p \left(\frac{t'}{t}x - x_j \right) < \frac{\varepsilon}{4}.$$

Then

$$p \left(x - \frac{t}{t'}x_j \right) < \frac{t}{t'} \cdot \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$$

and so

$$\begin{aligned} p(x - x_j) &\leq p \left(x - \frac{t}{t'}x_j \right) + p \left(\left(\frac{t}{t'} - 1 \right) x_j \right) \\ &< \frac{\varepsilon}{2} + \left(\frac{t}{t'} - 1 \right) p(x_j) \\ &< \frac{\varepsilon}{2} + \left(\frac{t}{t'} - 1 \right) M \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus the set $\{x_1, \dots, x_n\}$ is a finitely enumerable ε -approximation to S_t . Q.E.D.

The following criterion for the continuity of linear functionals in terms of families of defining seminorms enables us to show that in a locally convex space, the sets S_t of Proposition 2.2.1 are totally bounded with respect to any finitely enumerable family of defining seminorms.

Proposition 2.2.2 *A linear functional f on X is uniformly continuous if and only if there exist a positive real number C and a finitely enumerable subset F of I such*

that

$$|f(x)| \leq C \sup_{i \in F} p_i(x) \quad (2.3)$$

for each $x \in X$.

Proof. We include the (slightly adapted) well-known argument for the sake of completeness. Since f is continuous and $f(0) = 0$, the set $\{x \in X : |f(x)| < 1\}$ is open in X ; so there exist $\delta > 0$ and a finitely enumerable subset F of I such that if $\sum_{i \in F} p_i(x) < \delta$, then $|f(x)| < 1$. It follows that for each $x \in X$ and each $\varepsilon > 0$,

$$\left| f \left(\frac{\delta x}{\sum_{i \in F} p_i(x) + \varepsilon} \right) \right| < 1$$

and therefore

$$|f(x)| < \delta^{-1} \left(\sum_{i \in F} p_i(x) + \varepsilon \right).$$

Since $\varepsilon > 0$ is arbitrary, we see that (2.3) holds. Q.E.D.

In the presence of linearity we can improve Proposition 2.1.14 substantially.

Proposition 2.2.3 *Let f be a uniformly continuous linear functional on X , and S a balanced, totally bounded subset of X . Then for all $t > 0$ the sets*

$$S_t = \{x \in S : |f(x)| \leq t\}$$

are totally bounded.

Proof. Choose a finitely enumerable subset F of I such that (2.3) holds for some $C > 0$ and all $x \in X$, and let G be an arbitrary finitely enumerable subset of I . Since

$$|f(x)| \leq C \sum_{i \in F \cup G} p_i(x) \quad (x \in X),$$

f is uniformly continuous with respect to the seminorm $\sum_{i \in F \cup G} p_i$ on X . It follows from Proposition 2.2.1 that for each $t > 0$ the set S_t is totally bounded with respect

to $F \cup G$. Given $\varepsilon > 0$, choose a finitely enumerable ε -approximation $\{x_1, \dots, x_n\}$ to S_t relative to $F \cup G$. Then for each $x \in S_t$ we have

$$\sum_{i \in G} p_i(x - x_j) \leq \sum_{i \in F \cup G} p_i(x - x_j) < \varepsilon$$

for some j ($1 \leq j \leq n$). Since $\varepsilon > 0$ is arbitrary, we conclude that S_t is totally bounded relative to G . Q.E.D.

Theorem 2.2.4 *Let X be a locally convex space, Y a linear subset of X such that $Y_1 \equiv Y \cap X_1$ is totally bounded. If f is a nonzero linear functional on X , uniformly continuous on Y_1 , then $Y_1 \cap \ker f$ is totally bounded.*

Proof. Since f is uniformly continuous on Y_1 ,

$$C = \sup\{|f(x)| : x \in Y_1\}$$

exists. By definition of "supremum", we can find $x \in Y_1$ with $|f(x)| > \frac{C}{2}$. Then

$$x_0 = \frac{C}{2|f(x)|}x$$

belongs to Y_1 , and $|f(x_0)| = \frac{C}{2}$. Let ε be a positive number and $\{p_1, \dots, p_m\}$ a finitely enumerable set of defining seminorms on X . Let t be a positive number such that

$$0 < t < \frac{\varepsilon}{1 + 4C^{-1}}.$$

Since

$$S_t = \{x \in Y_1 : |f(x)| \leq t\}$$

is totally bounded, there exists a t -approximation $\{s_1, \dots, s_n\}$ corresponding to the above finitely enumerable set of seminorms. Set

$$x_k = (1 + 2C^{-1}t)^{-1}(s_k - 2C^{-1}f(s_k)x_0) \quad (1 \leq k \leq n).$$

Then $x_k \in \ker f$. If p_λ is any defining seminorm on X , then we have

$$\begin{aligned} p_\lambda(x_k) &\leq (1 + 2C^{-1}t)^{-1}(p_\lambda(s_k) + 2C^{-1}|f(s_k)|p_\lambda(x_0)) \\ &\leq (1 + 2C^{-1}t)^{-1}(1 + 2C^{-1}t) \\ &= 1. \end{aligned}$$

So x_k belongs to $Y_1 \cap \ker f$.

Now consider any element x of $Y_1 \cap \ker f$. Since $x \in S_t$, for some k ($1 \leq k \leq n$) and each i ($1 \leq i \leq m$) we have $p_i(x - s_k) < t$ and therefore

$$\begin{aligned} p_i(x - x_k) &\leq p_i(x - s_k) + p_i(s_k - x_k) \\ &\leq t + 2(C + 2t)^{-1}p_i(ts_k + f(s_k)x_0) \\ &\leq t + 2C^{-1}(tp_i(s_k) + tp_i(x_0)) \\ &\leq t(1 + 4C^{-1}) \\ &< \varepsilon. \end{aligned}$$

Thus, $\{x_1, \dots, x_n\}$ is a finitely enumerable ε -approximation to $Y_1 \cap \ker f$. Q.E.D.

The following Brouwerian example shows that we cannot expect to prove that $Y_1 \cap \ker f$ is totally bounded unless we know that $f = 0$ or $f \neq 0$.

Let $a \in \mathbf{R}$, and define a linear functional $f : \mathbf{R} \rightarrow \mathbf{R}$ by $f(x) = ax$. Then f is bounded—it has norm equal to a —and the unit ball $[-1, 1]$ of \mathbf{R} is balanced, convex, and totally bounded. Suppose that

$$K = [-1, 1] \cap \ker f$$

is totally bounded, so that $s = \sup K$ exists. Either $s > 0$ or $s < 1$. In the first case there exists $x \neq 0$ in \mathbf{R} with $f(x) = 0$, so $a = 0$; in the second we have $\neg(f(1) = 0)$, so $\neg(a = 0)$. Thus Theorem 2.2.4 without the hypothesis $f = 0 \vee f \neq 0$ implies that

$$\forall x \in \mathbf{R} (x = 0 \vee \neg(x = 0)),$$

a statement that is known to be essentially nonconstructive.

2.3 The Minkowski functional on a locally convex space

In Bishop's constructive mathematics, the existence of certain functionals is not always a trivial consequence of the logic used, as it is in the classical setting. One such functional is the Minkowski functional of a convex absorbing set. In the first part of this section we study the existence of the Minkowski functional on a locally convex space; in the second part we establish some basic results about these functionals.

Definition 2.3.1 *A convex subset C of X is said to be **absorbing** if for each $x \in X$ there exists a positive number r such that $x \in rC$. If, for such C ,*

$$\mu_C(x) = \inf\{r > 0 : x \in rC\} \tag{2.4}$$

*exists for all $x \in X$, then equation (2.4) defines the **Minkowski functional** of C , and we say that C has a **Minkowski functional**.*

It is shown in [35] that the existence of the Minkowski functional μ_C entails LPO. So in order to ensure constructively the existence of the Minkowski functional, we need additional hypotheses on the convex absorbing set.

The following lemma of Ishihara [35] plays a crucial role in the proofs of subsequent results.

Lemma 2.3.2 *Let C be a convex absorbing subset of X . Then C has a Minkowski functional if and only if for each $x \in X$ and all positive real numbers s, t with $s < t$, either $x \notin sC$ or else $x \in tC$.*

Proposition 2.3.3 *A located convex absorbing subset of X with nonempty interior has a Minkowski functional.*

Proof. Let C be such a subset of X . Without loss of generality, we may assume that 0 is an interior point of C . Hence there exist $i \in I$ and a positive number δ such that

$$V_0 = \{x \in X : p_i(x) < \delta\} \subset C.$$

Let s, t be positive real numbers such that $s < t$, and let $x \in X$. Since C is located, either $p_i(x - sy) > 0$ for all $y \in C$, or else there exists $y \in C$ such that $p_i(x - sy) < \delta(t - s)$. In the first case, $x \notin sC$. In the second we have $p_i((t - s)^{-1}(x - sy)) < \delta$, so

$$z = (t - s)^{-1}(x - sy)$$

belongs to V_0 and therefore to C . Hence

$$t^{-1}x = t^{-1}sy + t^{-1}(x - sy) = t^{-1}sy + (1 - t^{-1}s)z$$

and therefore $x \in tC$. Lemma 2.3.2 now shows that the Minkowski functional of C exists. Q.E.D.

We now prove a very elementary but useful lemma which, to our surprise, we cannot find anywhere in the literature.

Lemma 2.3.4 *Let f, g be two functions from \mathbf{R}^{0+} into a set S with a tight inequality, such that $f(t) = g(t)$ for all $t \in \mathbf{R}^+ \cup \{0\}$. Then $f = g$.*

Proof. Let $t \in \mathbf{R}^{0+}$, and suppose that $f(t) \neq g(t)$. Then $\neg(t > 0)$, so $t \leq 0$ and therefore $t = 0$; but this is impossible, since $f(0) = g(0)$. Hence $\neg(f(t) \neq g(t))$, and so, as the inequality on S is tight, $f(t) = g(t)$. Q.E.D.

Proposition 2.3.5 *Let C be a convex absorbing subset of X for which the Minkowski functional exists. Then μ_C is a sublinear functional on X . If also C is balanced, then μ_C is a seminorm.*

Proof. Let $x, y \in X$ and $a > 0$. For each $r > \mu_C(ax)$ —such r exists, since C is absorbing—we have $x \in a^{-1}rC$ and so $a^{-1}r \geq \mu_C(x)$. Thus $a\mu_C(x) \leq r$ and therefore $a\mu_C(x) \leq \mu_C(ax)$. Conversely, if $t > \mu_C(x)$, then $x \in tC$, so $ax \in atC$ and therefore $at \geq \mu_C(ax)$. Hence $a\mu_C(x) \geq \mu_C(ax)$, and therefore $a\mu_C(x) = \mu_C(ax)$. Note that this equality also holds for $a = 0$, since $\mu_C(0) = 0$; so, by the preceding lemma, it holds for all nonnegative a .

Now choose positive numbers r_1, r_2 such that

$$r_1 < \mu_C(x) + \frac{\varepsilon}{2}, \quad r_2 < \mu_C(y) + \frac{\varepsilon}{2} \quad \text{and} \quad x \in r_1C, \quad y \in r_2C.$$

Then $x + y = r_1c_1 + r_2c_2$ for some $c_1, c_2 \in C$. Since C is convex, we have

$$x + y = (r_1 + r_2) \left(\frac{r_1}{r_1 + r_2}c_1 + \frac{r_2}{r_1 + r_2}c_2 \right) \in (r_1 + r_2)C.$$

Therefore

$$\mu_C(x + y) \leq r_1 + r_2 < \mu_C(x) + \mu_C(y) + \varepsilon.$$

Since ε is arbitrary, it follows that $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

Suppose now that C is also balanced, and let a be a nonzero real number. Since $\left| \frac{|a|}{a} \right| = 1$, we see that $ax \in rC$ is equivalent to $|a|x \in rC$ —that is, $\mu_C(ax) = \mu_C(|a|x) = |a|\mu_C(x)$. Q.E.D.

Proposition 2.3.6 *Let p be a seminorm on X . Then the following hold.*

- (i) *The set $S = \{x \in X : p(x) < 1\}$ is convex, balanced, absorbing, and has Minkowski functional p .*
- (ii) *If A is a convex absorbing subset of X that has a Minkowski functional, then the sets*

$$\begin{aligned} B &= \{x \in X : \mu_A(x) < 1\}, \\ C &= \{x \in X : \mu_A(x) \leq 1\} \end{aligned}$$

have Minkowski functionals, $B \subset A \subset C$, and $\mu_A = \mu_B = \mu_C$.

Proof. Let $x, y \in S$ and let α, β be positive real numbers with sum 1. Since p is a seminorm,

$$p(\alpha x + \beta y) \leq \alpha p(x) + \beta p(y) < \alpha + \beta = 1,$$

so S is convex. Let $|\alpha| \leq 1$ and $x \in S$. Then

$$p(\alpha x) = |\alpha|p(x) < |\alpha| \leq 1,$$

so S is balanced. In order to prove that S is absorbing, choose r such that $p(x) < r$; then $p(r^{-1}x) < 1$, which means that $x \in rS$. We have proved so far that S is convex, balanced and absorbing.

The Minkowski functional for S will exist if and only if for each $x \in X$, and arbitrary positive numbers s, t with $s < t$, either $x \in tS$ or else $x \notin sS$. There are two possibilities: either $p(x) < t$, or $p(x) > s$. In the first case $t^{-1}x \in S$, hence $x \in tS$. In the second case, suppose that $x \in sS$. Then $p(x) < s$, a contradiction, so $x \notin sS$.

We have already seen that if $r > p(x)$ then $x \in rS$. This shows that $\mu_S(x) \leq r$ and $\mu_S \leq p$. Given x , suppose that $\mu_S(x) < p(x)$ and choose t such that $\mu_S(x) < t < p(x)$. Then $p(t^{-1}x) > 1$, and so $t^{-1}x \notin S$. But $\mu_S(x) < t$, so $x \in tS$ and therefore $t^{-1}x \in S$, a contradiction. Thus, in fact, $\mu_S(x) \geq p(x)$ and therefore $\mu_S(x) = p(x)$.

The inclusions $B \subset A \subset C$ are obvious, as is the fact that both B and C are convex and absorbing. If x is an arbitrary vector and $0 < s < t$, then either $\mu_A(x) < t$ or $\mu_A(x) > s$. In the first case, $x \in tB$; in the second, if we assume that $x \in sB$, then, as $B \subset A$, we get $\mu_A(x) < s$, a contradiction. Thus μ_B , and similarly μ_C , exists. It is immediate that $\mu_C \leq \mu_A \leq \mu_B$. To complete the proof, it suffices to show that $\mu_B \leq \mu_C$. If $\mu_C(x) < s < t$, then $s^{-1}x \in C$, $\mu_A(s^{-1}x) \leq 1$, and $\mu_A(t^{-1}x) \leq st^{-1} < 1$; whence $t^{-1}x \in B$, and therefore $\mu_B(x) \leq t$. Since this inequality holds for every $t > \mu_C(x)$, it follows that $\mu_B(x) \leq \mu_C(x)$. Finally, since x is arbitrary, we conclude that $\mu_A = \mu_B = \mu_C$. Q.E.D.

Thus every seminorm on a locally convex space is the Minkowski functional of some convex, balanced, absorbing subset. The last result of this section shows that a locally convex space has a plentiful supply of subsets with the Minkowski functional and that such a subset can be described analytically in terms of this functional.

Theorem 2.3.7 *Let X be a locally convex space with the topology given by the defining family of seminorms $(p_i)_{i \in I}$. Let \mathcal{B} be the local base consisting of all neighbourhoods of 0 with the form*

$$V_{F,\varepsilon} = \{x \in X : p_i(x) < \varepsilon \text{ for all } i \in F\},$$

where $\varepsilon > 0$, $F \subset I$, and F is finitely enumerable. Then

- (i) Each set $V_{F,\varepsilon}$ has a Minkowski functional, and $V_{F,\varepsilon} = \{x \in X : \mu_{F,\varepsilon}(x) < 1\}$.
- (ii) The set $\{\mu_{F,\varepsilon} : V_{F,\varepsilon} \in \mathcal{B}\}$ is a family of continuous seminorms on X .

Proof. Let ε be a positive number, and F a finitely enumerable subset of I . It is immediate that $V_{F,\varepsilon}$ is convex and balanced. Let x be an element of X , write

$$M = \sup \{p_i(x) : i \in F\},$$

and choose $t > M\varepsilon^{-1}$. Then $p_i(x) < t\varepsilon$ for all $i \in F$, and therefore $V_{F,\varepsilon}$ is absorbing.

Now let $0 < s < t$. Then either $M < t\varepsilon$ or else $M > s\varepsilon$. In the first case it follows that $x \in tV_{F,\varepsilon}$. In the second case there exists $i_0 \in F$ such that $p_{i_0}(x) > s\varepsilon$, so $x \notin sV_{F,\varepsilon}$; hence $V_{F,\varepsilon}$ has a Minkowski functional.

If $x \in V_{F,\varepsilon}$, then since $V_{F,\varepsilon}$ is a balanced set, $t^{-1}x \in V_{F,\varepsilon}$ for some $t \in (0, 1)$; whence $\mu_{F,\varepsilon}(x) < 1$. Conversely, if x is an element of X such that $\mu_{F,\varepsilon}(x) < 1$, then for some $t < 1$ we have $p_i(x) < t\varepsilon < \varepsilon$ for all $i \in F$. This completes the proof of (i).

Since $V_{F,\varepsilon}$ is balanced, the Minkowski functional $\mu_{F,\varepsilon}$ is a seminorm on X . Given $\alpha > 0$, let F be a finitely enumerable subset of I , and let x, y be points of X such that $p_i(x - y) < \alpha\varepsilon$ for all $i \in F$. Then

$$\mu_{F,\varepsilon}(x - y) = \inf \left\{ r > 0 : \forall i \in F \left(p_i \left(\varepsilon^{-1}(x - y) \right) < r \right) \right\} \leq \alpha$$

and therefore

$$|\mu_{F,\varepsilon}(x) - \mu_{F,\varepsilon}(y)| \leq \mu_{F,\varepsilon}(x - y) \leq \alpha.$$

Since $\alpha > 0$ is arbitrary, $\mu_{F,\varepsilon}$ is continuous.

Finally, if x is a nonzero element of X , then there exist $i_0 \in I$ and $\varepsilon > 0$ such that $p_{i_0}(x) > \varepsilon$; so $x \notin V_{\{i_0\},\varepsilon}$, and therefore $\mu_{F,\varepsilon}(x) \geq 1$. Q.E.D.

Proposition (5.1) on page 35 of [20] shows that if the closed subset B of X is convex, balanced, and absorbing, and has a Minkowski functional, then μ_B defines the unique norm on X with respect to which B is the closed unit ball.

Chapter 3

Operator Topologies on $\mathcal{B}(H)$

3.1 Introduction

Let H be a Hilbert space, $\mathcal{B}(H)$ the space of bounded linear operators on H , and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. Given Hilbert spaces H_n ($n \geq 1$), the direct sum $\bigoplus_{n=1}^{\infty} H_n$ consists of all sequences $(x_n)_{n \geq 1}$ such that $x_n \in H_n$ ($n \geq 1$) and the series $\sum_{n=1}^{\infty} \|x_n\|^2$ converges. It is a Hilbert space with respect to the inner product defined by

$$\langle (x_n), (y_n) \rangle = \sum_{n \geq 1} \langle x_n, y_n \rangle.$$

There are four important topologies on $\mathcal{B}(H)$ that will concern us in this and subsequent chapters:

- The **uniform topology**, in which two operators S, T are close if $S - T$ has a small positive bound. (Classically, this is just the norm topology, but we may not be able to prove constructively that a given element of $\mathcal{B}(H)$ has a norm. Nevertheless, we shall adopt the convention that an inequality of the form $\|S - T\| < \varepsilon$, where $S, T \in \mathcal{B}(H)$, means that for some α with $0 < \alpha < \varepsilon$ and for all x in the unit ball of H , we have $\|Sx - Tx\| \leq \alpha$, whether or not the operator norm $\|S - T\|$ exists as a supremum.)

- The **strong-operator topology** τ_s : the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \mapsto Tx$ is continuous for each $x \in H$.
- the **weak-operator topology** τ_w : the weakest topology with respect to which the mapping $T \mapsto \langle Tx, y \rangle$ is continuous for all $x, y \in H$. We denote by $p_{x,y}$ the seminorm $T \mapsto |\langle Tx, y \rangle|$. The weak-operator topology is generated on $\mathcal{B}(H)$ by the seminorms $p_{x,y}$ as x and y range over H .
- the **ultraweak-operator topology** τ_{ow} : the weakest topology such that the mapping $T \mapsto \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle$ is continuous on $\mathcal{B}(H)$ for all elements $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ of the direct sum $H_{\infty} = \bigoplus_{n=1}^{\infty} H$ of a sequence of copies of H .

Definition 3.1.1 An *orthonormal basis* for a Hilbert space H is a set of pairwise orthogonal unit vectors that generate a dense subspace of H .

This definition is different from, and more tractable than, the sequential one given by Bishop [2] in the separable case.

We assume that H has an orthonormal basis E . Classically, an application of Zorn's Lemma shows that this assumption is redundant; but Zorn's Lemma has every appearance of being essentially nonconstructive. (Note that the axiom of choice, to which Zorn's Lemma is classically equivalent, entails the law of excluded middle [32].)

In order to make sense of Parseval's formula and related matters associated with orthonormal bases, we need to clarify what we mean by a sum over an arbitrary index set (see also Chapter 4). If $(r_i)_{i \in I}$ is a family of nonnegative real numbers, then we define $\sum_{i \in I} r_i$ to be $\sup_F \sum_{i \in F} r_i$, where F ranges over the finite subsets of I . This agrees with the standard definition when I is the set \mathbf{N}^+ of positive integers. With this definition at hand, we can establish identities, such as

$$\|x\|^2 = \sum_{e \in E} |\langle x, e \rangle|^2, \quad (3.1)$$

familiar from the classical theory of Hilbert spaces.

The following result improves upon Lemma (2.5) on page 308 of [4].

Proposition 3.1.2 *Let K be a nonempty finitely enumerable subset of H , and $\varepsilon > 0$. Then there exists a finite subset F of E , generating a finite-dimensional subspace V of H , such that $\rho(x, V) < \varepsilon$ for all $x \in K$.*

Proof. Let $K = \{x_1, \dots, x_n\}$, and use (3.1) to construct a finite subset F_1 of E , generating a finite-dimensional subspace V_1 of H , such that $\rho(x_1, V_1) < \varepsilon$. Suppose that, for some $k < n$, we have constructed finite subsets $F_1 \subset F_2 \subset \dots \subset F_k$ of E such that $\rho(x_i, V_i) < \varepsilon$ ($1 \leq i \leq k$), where V_i is the finite-dimensional subspace of H generated by F_i . Either $\rho(x_{k+1}, V_k) < \varepsilon$ or $\rho(x_{k+1}, V_{k+1}) > 0$. In the first case we take $F_{k+1} = F_k$. In the second, we choose a finite subset S of E such that the distance from x_{k+1} to the subspace of H generated by S is less than ε , and we set $F_{k+1} = F_k \cup S$. This completes the inductive construction of F_{k+1} . To complete the proof, we take $F = F_n$. Q.E.D.

The following lemma shows that when dealing with the weak-operator topology on $\mathcal{B}_1(H)$, we need only concern ourselves with the seminorms $p_{e,e'}$ where $e, e' \in E$.

Lemma 3.1.3 *The sets of the form*

$$U(F, \varepsilon) = \{T \in \mathcal{B}_1(H) : \forall e, e' \in F (p_{e,e'}(T) < \varepsilon)\},$$

with F a finite subset of E and $\varepsilon > 0$, form a base of neighbourhoods of 0 in the weak-operator topology on $\mathcal{B}_1(H)$.

Proof. A typical basic τ_w -neighbourhood of 0 in $\mathcal{B}_1(H)$ has the form

$$V(S, \varepsilon) = \{T \in \mathcal{B}_1(H) : \forall x, y \in S (p_{x,y}(T) < \varepsilon)\},$$

where S is a finitely enumerable subset of H and $\varepsilon > 0$. Let

$$M = \max_{x \in S} \|x\|.$$

It will suffice to prove that there exists a finite subset F of E such that

$$U(F, \varepsilon) \subset V(S, \varepsilon(M^2 + 2M)).$$

To this end, choose a finite subset F of E such that $\|x - Px\| < \varepsilon$ for each $x \in S$, where P is the projection on the finite-dimensional subspace of H generated by F .

If $T \in U(F, \varepsilon)$, then for all $x, y \in S$ we have

$$\begin{aligned} p_{x,y}(T) &= |\langle Tx, y \rangle| \\ &\leq |\langle TPx, Py \rangle| + |\langle T(I - P)x, y \rangle| + |\langle TPx, (I - P)y \rangle| \\ &\leq \left| \sum_{e, e' \in E} \langle x, e \rangle \langle e', y \rangle p_{e, e'}(T) \right| + \|x - Px\| \|y\| + \|x\| \|y - Py\| \\ &\leq \varepsilon \sum_{e, e' \in E} |\langle x, e \rangle \langle e', y \rangle| + 2\varepsilon M \\ &\leq \varepsilon \left(\sum_{e \in E} |\langle x, e \rangle|^2 \right)^{1/2} \left(\sum_{e' \in E} |\langle e', y \rangle|^2 \right)^{1/2} + 2\varepsilon M \\ &\leq \varepsilon \|x\| \|y\| + 2\varepsilon M \\ &= \varepsilon(M^2 + 2M), \end{aligned}$$

as we require. Q.E.D.

For the proof of the next Lemma the reader is referred to page 34 of [28].

Lemma 3.1.4 *The weak-operator topology τ_w and the ultraweak-operator topology $\tau_{\sigma w}$ are equivalent on $\mathcal{B}_1(H)$.*

Define a linear mapping $T \mapsto \tilde{T}$ of $\mathcal{B}(H)$ into $\mathcal{B}(H_\infty)$ as follows:

$$\tilde{T}(\mathbf{x}) = (Tx_n)_{n=1}^\infty$$

for each T in $\mathcal{B}(H)$ and each $\mathbf{x} = (x_n)_{n=1}^\infty$ in H_∞ . This mapping is **isometric**:

$$\|T\| = \|\tilde{T}\| \text{ for all } T \in \mathcal{B}(H).$$

Lemma 3.1.5 *Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$, and let*

$$\widetilde{\mathcal{R}} = \{\widetilde{T} : T \in \mathcal{R}\}.$$

If the unit ball $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{B}_1(H)$ is $\tau_{\sigma w}$ -totally bounded, then the unit ball $\widetilde{\mathcal{R}}_1 = \widetilde{\mathcal{R}} \cap \mathcal{B}_1(H_\infty)$ is τ_w -totally bounded.

Proof. Let F be a finitely enumerable subset of H_∞ and let ε be a positive number. For $\mathbf{x} = (x_n)_{n=1}^\infty, \mathbf{y} = (y_n)_{n=1}^\infty$ in F , since the series $\sum_{n=1}^\infty \|x_n\| \|y_n\|$ converges, we can choose N such that

$$\sum_{\mathbf{x}, \mathbf{y} \in F} \sum_{n=N+1}^\infty \|x_n\| \|y_n\| < \varepsilon.$$

Let $\{T_1, \dots, T_m\}$ be a finitely enumerable ε -approximation to \mathcal{R}_1 relative to the seminorm $T \mapsto \sum_{\mathbf{x}, \mathbf{y} \in F} \sum_{n=1}^N |\langle T x_n, y_n \rangle|$. Given $T \in \mathcal{R}_1$, choose k such that

$$\sum_{\mathbf{x}, \mathbf{y} \in F} \sum_{n=1}^N |\langle (T - T_k) x_n, y_n \rangle| < \varepsilon.$$

Then

$$\begin{aligned} & \sum_{\mathbf{x}, \mathbf{y} \in F} \left| \langle (\widetilde{T} - \widetilde{T}_k) \mathbf{x}, \mathbf{y} \rangle \right| \\ & \leq \sum_{\mathbf{x}, \mathbf{y} \in F} \left| \sum_{n=1}^N \langle (T - T_k) x_n, y_n \rangle \right| + \sum_{\mathbf{x}, \mathbf{y} \in F} \left| \sum_{n=N+1}^\infty \langle (T - T_k) x_n, y_n \rangle \right| \\ & < \varepsilon + 2 \sum_{\mathbf{x}, \mathbf{y} \in F} \sum_{n=N+1}^\infty \|x_n\| \|y_n\| \\ & < 3\varepsilon. \end{aligned}$$

Hence $\{\widetilde{T}_1, \dots, \widetilde{T}_m\}$ is a finitely enumerable 3ε -approximation to $\widetilde{\mathcal{R}}_1$ relative to the seminorm $\widetilde{T} \mapsto \sum_{\mathbf{x}, \mathbf{y} \in F} |\langle \widetilde{T} \mathbf{x}, \mathbf{y} \rangle|$. Since F and ε are arbitrary, this concludes the proof. Q.E.D.

Recall that an element T of $\mathcal{B}(H)$ is **positive** if $\langle T x, x \rangle \geq 0$ for all $x \in X$; in that case, T is selfadjoint. We denote by $\mathcal{B}(H)^+$ (respectively, $\mathcal{B}_1(H)^+$) the set of positive elements of $\mathcal{B}(H)$ (respectively, $\mathcal{B}_1(H)$).

Lemma 3.1.6 *The identity mapping of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}_1(H), \tau_s)$ is uniformly continuous on the set of positive elements of $\mathcal{B}_1(H)$.*

Proof. For each $T \in \mathcal{B}_1(H)^+$ and each $x \in H$ we have $\|T^{1/2}x\|^2 = \langle Tx, x \rangle$, where the square root of T is obtained by the functional calculus (see Chapter 7 of [4]). It follows that the mapping $T \mapsto T^{1/2}$ of $(\mathcal{B}_1(H)^+, \tau_w)$ into $(\mathcal{B}_1(H)^+, \tau_s)$ is uniformly continuous. Since multiplication, and hence squaring, is strong-operator continuous on bounded sets of operators ([40], (2.5.10)), we obtain the desired conclusion. Q.E.D.

3.2 A monotone convergence theorem for operator nets

We define the partial ordering \leq on $\mathcal{B}(H)$ in the usual way: $S \leq T$ if and only if $T - S \geq 0$. A well-known classical theorem states that, relative to this partial ordering, an increasing net of selfadjoint operators in $\mathcal{B}_1(H)$ converges in the strong-operator topology to a selfadjoint operator which is the least-upper-bound of the net [40]. It is an almost trivial consequence of the failure of the classical least upper bound principle in constructive analysis that we cannot prove that theorem about nets of operators constructively. However, we can prove the following results.

Proposition 3.2.1 *Let $(T_j)_{j \in J}$ be a net of operators of $\mathcal{B}_1(H)$ where the index set J is directed by the partial order \geq . Then the following statements are equivalent.*

- (i) (T_j) is τ_s -convergent
- (ii) (T_j) is τ_w -convergent

Proof. The proof is a direct consequence of Lemma 3.1.6. Q.E.D.

Theorem 3.2.2 *Let $(T_j)_{j \in J}$ be an increasing net of positive selfadjoint operators of $\mathcal{B}_1(H)$. If it is τ_w -totally bounded, then it is τ_s -convergent in $\mathcal{B}_1(H)$.*

Proof. If $(T_j)_{j \in J}$ is τ_w -totally bounded, then for each vector x of H the set $\{\langle T_j x, x \rangle : j \in J\}$ is a totally bounded subset of \mathbf{R} . Fix $x \in H$; for simplicity, for each $j \in J$ write $t_j = \langle T_j x, x \rangle$. We show that (t_j) is a Cauchy net in \mathbf{R} . By total boundedness, for each $\varepsilon > 0$ there exists a finitely enumerable subset I of J such that $(t_i)_{i \in I}$ is an ε -approximation to $\{t_j : j \in J\}$. Since J is directed, we can pick $j_0 \in J$ such that $j_0 \geq i$ for all $i \in I$. Let j be an element of J such that $j \geq j_0$, and choose $i \in I$ such that $|t_j - t_i| < \varepsilon$. Then as our net is increasing,

$$0 \leq t_j - t_{j_0} \leq t_j - t_i < \varepsilon.$$

It follows that if $j \geq k \geq j_0$, then

$$0 \leq t_j - t_k \leq t_j - t_{j_0} < \varepsilon.$$

Hence $(\langle T_j x, x \rangle)_{j \in J}$ is a Cauchy net for each $x \in H$.

We now show that (T_j) is a τ_w -Cauchy net. To this end, let $\varepsilon > 0$ and let F be a finite subset of H . For each $x \in F$ compute $j_x \in J$ such that

$$\langle (T_j - T_k)x, x \rangle < \varepsilon \quad (j, k \geq j_x).$$

Since $\{j_x : x \in F\}$ is finitely enumerable, there exists $j_F \in J$ such that $j_F \geq j_x$ for all $x \in F$. It then follows that

$$\langle (T_j - T_k)x, x \rangle < \varepsilon \quad (j, k \geq j_F; x \in F).$$

Hence (T_j) is a τ_w -Cauchy net in $\mathcal{B}_1(H)^+$. By Lemma 3.1.6, it is a strong-operator Cauchy net in $\mathcal{B}_1(H)^+$. Since $\mathcal{B}_1(H)$, and therefore $\mathcal{B}_1(H)^+$, is strong-operator complete ([40], (2.5.11)), we conclude that the net (T_j) converges strongly to an element T of $\mathcal{B}_1(H)^+$. Q.E.D.

The last result is the best that we can hope for, in the sense that its converse does not hold constructively. For, although classically it is trivial that any subset of a totally bounded subset of a uniform space is totally bounded, if every increasing Cauchy net in $[0, 1]$ is totally bounded, then LPO holds. To see this, consider any nonnegative real number a , and let

$$J \equiv \left(\mathbf{R}a \cap \left[0, \frac{1}{2}\right] \right) \cup \left[\frac{3}{4}, 1 \right].$$

with the ordering induced by the standard ordering on \mathbf{R} . Suppose that the net $(j)_{j \in J}$ is totally bounded and therefore located. Compute the distance

$$d \equiv \inf \left\{ \left| \frac{1}{2} - j \right| : j \in J \right\}.$$

Either $d < 1/4$ or else $d > 0$. In the first case there exists $j \in J$ such that $\left| \frac{1}{2} - j \right| < \frac{1}{4}$; whence $j \in \mathbf{R}a \cap \left[\frac{1}{4}, \frac{1}{2} \right]$ and therefore $a > 0$. In the second case the equation $ax = \frac{1}{2}$ has no solution in \mathbf{R} , which implies that $a = 0$.

Note that for sequences of operators we can prove the following result.

Theorem 3.2.3 *Let (T_n) be a sequence of operators on H such that $0 \leq T_n \leq T_{n+1} \leq I$ for each n . Then the following conditions are equivalent.*

- (i) $\sup_{n \geq 1} \langle T_n x, x \rangle$ exists for each $x \in H$.
- (ii) (T_n) is weak-operator totally bounded.
- (iii) (T_n) is weak-operator convergent to an element of $\mathcal{B}(H)$.

Proof. Suppose that (i) holds. Let ε be a positive number, F a finite subset of H , and x any element of F . Since $(\langle T_n x, x \rangle)_{n=1}^{\infty}$ is an increasing sequence of real numbers whose supremum exists, it converges to that supremum and is therefore a Cauchy sequence. Thus there exists N_x such that

$$|\langle (T_m - T_n)x, x \rangle| < \varepsilon \quad (m \geq n \geq N_x).$$

Setting $N = \max_{x \in F} N_x$, we see that for each $x \in F$,

$$|\langle (T_m - T_n)x, x \rangle| < \varepsilon \quad (m \geq n \geq N).$$

Hence $(T_n)_{n=1}^{\infty}$ is a weak-operator Cauchy, and therefore τ_w -totally bounded, sequence.

The implication (ii) \Rightarrow (iii) is a direct application of Theorem 3.2.2.

Now assume that the sequence $(T_n)_{n=1}^{\infty}$ is weak-operator convergent to an element T of $\mathcal{B}(H)$. Then for each $x \in H$, the sequence $(\langle T_n x, x \rangle)_{n=1}^{\infty}$ converges to a limit in \mathbf{R} . This limit is its supremum. Q.E.D.

3.3 The total boundedness of $\mathcal{B}_1(H)$

Classically, $\mathcal{B}_1(H)$ is τ_w -compact. Constructively, Bridges [6] showed that when H is separable, $\mathcal{B}_1(H)$ is τ_w -totally bounded, but the τ_w -completeness of $\mathcal{B}_1(l^2)$ implies LPO. We now establish the τ_w -total boundedness of $\mathcal{B}_1(H)$ in the general case.

Theorem 3.3.1 *Let H be a Hilbert space. Then $\mathcal{B}_1(H)$ is weak-operator totally bounded.*

Proof. Let x_1, \dots, x_n be unit vectors in H , and let $\varepsilon > 0$. Then, by Lemma 2.5 on page 308 of [4], there exists a finite dimensional subspace H_0 of H such that

$$\rho(x_i, H_0) < \frac{\varepsilon}{5} \quad (1 \leq i \leq n).$$

Let P be the projection of H onto H_0 . Then

$$\|x_i - Px_i\| < \frac{\varepsilon}{5} \quad (1 \leq i \leq n).$$

Since H_0 is finite dimensional, the operator norm of each element of $\mathcal{B}(H_0)$ is computable, and $\mathcal{B}(H_0)$ is a finite-dimensional Banach space under this norm; therefore its unit ball $\mathcal{B}_1(H_0)$ is totally bounded in the operator norm.

Let $\{T_1^0, \dots, T_r^0\}$ be an ε -approximation to $\mathcal{B}_1(H_0)$, and let $T \in \mathcal{B}_1(H)$. Then $(PT)_0$, the restriction of PT to H_0 , belongs to $\mathcal{B}_1(H_0)$, and hence there exists m ($1 \leq m \leq r$) such that

$$\|(PT)_0 - T_m^0\| < \frac{\varepsilon}{5}.$$

It follows that

$$\sup \left\{ \|PTx - T_m^0x\| : x \in H_0, \|x\| \leq 1 \right\} < \frac{\varepsilon}{5}.$$

On the other hand, since P is the projection on H_0 , and T_j^0 maps H_0 to H_0 , we have $PT_j^0 = T_j^0$ for each j , and therefore

$$\begin{aligned} \left| \langle (T - T_m^0P)x_i, x_j \rangle \right| &\leq \left| \langle (T - T_m^0P)x_i, Px_j \rangle \right| + \left| \langle (T - T_m^0P)x_i, x_j - Px_j \rangle \right| \\ &= \left| \langle (PT - T_m^0P)x_i, x_j \rangle \right| + 2\|x_i\| \|x_j - Px_j\| \\ &< \left| \langle (PT - T_m^0P)Px_i, x_j \rangle \right| + \left| \langle PT - T_m^0P \rangle (x_i - Px_i), x_j \right| + \frac{2\varepsilon}{5} \\ &\leq \|(PT)_0 - T_m^0\| + 2\|x_i - Px_i\| \|x_j\| + \frac{2\varepsilon}{5} \\ &< \varepsilon. \end{aligned}$$

It follows now that $\{T_1^0P, \dots, T_r^0P\}$ is a weak-operator ε -approximation to the unit ball $\mathcal{B}_1(H)$. Hence $\mathcal{B}_1(H)$ is weak-operator totally bounded. Q.E.D.

3.4 Weak-operator continuity and the existence of adjoints

Consider the following classical proposition ([40], pages 304–306).

Proposition 3.4.1 *For each $A \in \mathcal{B}(H)$ the mappings $T \mapsto TA$ and $T \mapsto AT$ of $(\mathcal{B}(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$ are continuous.*

The proof of the continuity of the mapping $T \mapsto TA$ on $\mathcal{B}_1(H)$ is relatively trivial, both classically and constructively. Classically, it is an immediate consequence of

the compactness of $\mathcal{B}_1(H)$ that $T \mapsto TA$ is uniformly continuous as a mapping of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$. Constructively, we must do a little more work, since we cannot prove the uniform continuity theorem (see [20], Chapter 6), and even if we could, we know only that $\mathcal{B}_1(H)$ is totally bounded. Using an argument similar to that in the proof of Proposition 2.2.2, we first show that the mapping $T \mapsto TA$ is uniformly continuous on $\mathcal{B}_1(H)$ if and only if for each pair of vectors $x, y \in H$ there exist $M > 0$ and a finitely enumerable subset F of H such that

$$|\langle TAx, y \rangle| \leq M \sup_{e, f \in F} |\langle Te, f \rangle|.$$

Since the mapping $T \mapsto |\langle T(Ax), y \rangle|$ is τ_w -uniformly continuous on $\mathcal{B}_1(H)$, we can now apply Proposition 2.2.2 to complete the required proof.

The classical proof of the continuity of left multiplication with respect to the weak-operator topology is a trivial consequence of the identity

$$\langle ATx, y \rangle = \langle Tx, A^*y \rangle \quad (x, y \in H)$$

and the uniform continuity of the mapping $T \mapsto \langle Tx, z \rangle$ on $\mathcal{B}_1(H)$ for all x, z in H . Constructively, this proof is fine when A^* exists, but will not work in general since, as will be shown overleaf, the statement “Every element of $\mathcal{B}(H)$ has an adjoint” implies LPO.

At this point one might ask, “What is the problem, constructively, with the classical method of obtaining A^*y for any $A \in \mathcal{B}(H)$ and $y \in H$: namely, apply the Riesz Representation Theorem to the bounded linear functional $x \mapsto \langle Ax, y \rangle$?”. In order to apply the Riesz Representation Theorem constructively to a linear functional f on H , we need to know that f is not just bounded but **has a norm**, in the sense that

$$\sup \{ |f(x)| : x \in H, \|x\| \leq 1 \}$$

exists ([4], page 419, (2.3)); since the classical least upper bound principle does not hold constructively, we may not be able to find the supremum in question.

However, the classical proof of the existence of A^*y will work for us if we know that

$$\sup \{ |\langle Ax, y \rangle| : x \in H, \|x\| \leq 1 \}$$

exists for each $y \in H$.

Definition 3.4.2 A mapping $f : X \rightarrow Y$ between uniform spaces $(X, (\rho_i)_{i \in I})$ and $(Y, (\sigma_j)_{j \in J})$ is **sequentially continuous at** $x \in X$ if for each sequence (x_n) converging to x in X , the sequence $(f(x_n))$ converges to $f(x)$ in Y .

Some general results on sequential continuity in constructive analysis are proved in Appendix A. We now show, by a Brouwerian example, that we cannot hope to prove constructively even the sequential continuity of the mapping $T \mapsto AT$ at 0 with respect to the weak-operator topology. Let (e_n) be the usual orthonormal basis of unit vectors in the Hilbert space l^2 , and for each positive integer n define $T_n \in \mathcal{B}_1(H)$ such that

$$\begin{aligned} T_n e_1 &= e_n, \\ T_n e_k &= 0 \quad (k \geq 2). \end{aligned}$$

Then T_n has an adjoint, and the sequence (T_n) is weak-operator convergent to 0. Now let (a_n) be a binary sequence with at most one term equal to 1, and define $A \in \mathcal{B}_1(H)$ by

$$Ax = \left(\sum_{n=1}^{\infty} a_n \langle x, e_n \rangle \right) e_1.$$

Suppose that the mapping $T \mapsto AT$ of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$ is sequentially continuous at 0—that is, maps sequences converging to 0 to sequences converging to 0. Then there exists N such that

$$a_n = |\langle AT_n e_1, e_1 \rangle| < 1,$$

and therefore $a_n = 0$, for all $n \geq N$. By testing a_1, \dots, a_{N-1} , we can therefore prove that

$$\forall n (a_n = 0) \vee \exists n (a_n = 1).$$

Hence the proposition

For each $A \in \mathcal{B}(H)$ the mapping $T \mapsto AT$ of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$ is sequentially continuous at 0.

implies LPO. It follows that if A^* exists for each $A \in \mathcal{B}(H)$, then LPO holds.¹

This example raises the (classically vacuous) question:

If, for a given element A of $\mathcal{B}(H)$, the mapping $T \mapsto AT$ of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$ is continuous—or, in this case equivalently, uniformly continuous—does A^ exist?*

We show in the next theorem that for A^* to exist it suffices that the mapping $T \mapsto AT$ preserve total boundedness.

Theorem 3.4.3 *Let H be a Hilbert space, let $A \in \mathcal{B}(H)$, and let f_A be the linear mapping $T \mapsto AT$ of $(\mathcal{B}_1(H), \tau_w)$ into $(\mathcal{B}(H), \tau_w)$. Then the following are equivalent conditions.*

- (i) f_A is continuous at 0.
- (ii) f_A is τ_w -uniformly continuous on $\mathcal{B}_1(H)$.
- (iii) f_A maps totally bounded subsets of $(\mathcal{B}_1(H), \tau_w)$ to totally bounded subsets of $(\mathcal{B}(H), \tau_w)$.
- (iv) A has an adjoint.

Proof. It is routine to show that (i) \Rightarrow (ii); the implication (ii) \Rightarrow (iii) is a special case of the general result that uniform continuity preserves total boundedness. ([4], (4.2) page 94).

¹This is shown directly in [21].

Assuming (iii), fix y and a unit vector e in H . For each x in the unit ball H_1 of H define $T_x \in \mathcal{B}_1(H)$ such that $T_x e = x$, and $T_x z = 0$ for all $z \perp e$. Since $T e \in H_1$ for each $T \in \mathcal{B}_1(H)$, we see that

$$H_1 = \{T e : T \in \mathcal{B}_1(H)\}$$

and therefore

$$\{\langle Ax, y \rangle : x \in H_1\} = \{\langle AT e, y \rangle : T \in \mathcal{B}_1(H)\}.$$

So, in order to apply the Riesz Representation Theorem to construct A^*y , it suffices to show that the set

$$C = \{|\langle AT e, y \rangle| : T \in \mathcal{B}_1(H)\}$$

has a supremum in \mathbf{R} . As $\mathcal{B}_1(H)$ is weak-operator totally bounded, we see from (c) that $\{AT : T \in \mathcal{B}_1(H)\}$ is also weak-operator totally bounded. The uniform continuity of the mapping $S \mapsto \langle S e, y \rangle$ on norm-bounded subsets of $\mathcal{B}(H)$ now ensures that the set C is totally bounded, and therefore has a supremum, in \mathbf{R} .

Since we have already noted that (iv) \Rightarrow (i), our proof is complete. Q.E.D.

We now investigate the weak-operator sequential continuity of the mapping $T \mapsto AT$ for a fixed $A \in \mathcal{B}(H)$. Our final aim in the chapter is to prove the following result.

Proposition 3.4.4 *Let H be a separable Hilbert space, A an element of $\mathcal{B}(H)$, and f_A the restriction to $\mathcal{B}_1(H)$ of the mapping $T \mapsto AT$. Suppose that f_A maps τ_w -Cauchy sequences to τ_w -Cauchy sequences. Then f_A is (τ_w, τ_w) -sequentially continuous at 0.*

The proof of requires a couple of preliminaries. For the first of these we note that, as Richman [51] has recently shown, the adjoint of $A \in \mathcal{B}(H)$ exists if and only if $A(H_1)$ is located.

Lemma 3.4.5 *LPO implies that if H is a separable Hilbert space, then every element of $\mathcal{B}(H)$ has an adjoint.*

Proof. Let H be a separable Hilbert space, H_1 its (separable) unit ball, and A an element of $\mathcal{B}(H)$. Then $A(H_1)$ is separable. If LPO holds, then every separable subset of a metric space is located: this is easily shown using the constructive least-upper-bound principle ([4], page 37, Proposition (4.3)). In that case, $A(H_1)$ is located, and therefore A has an adjoint. Q.E.D.

Definition 3.4.6 *A mapping $f : X \rightarrow Y$ between uniform spaces $(X, (\rho_i)_{i \in I})$ and $(Y, (\sigma_j)_{j \in J})$ is **sequentially nondiscontinuous** at $x \in X$ if, whenever $\varepsilon \in \mathbf{R}$ and (x_n) is a sequence converging to x in X such that $|f(x_n) - f(x)| \geq \varepsilon$ for all n , we have $\varepsilon < 0$.*

Lemma 3.4.7 *Under the hypotheses of Proposition 2.5.1, f_A is sequentially nondiscontinuous at 0.*

Proof. Let $(T_n)_{n=1}^\infty$ be a sequence in $\mathcal{B}_1(H)$ that is τ_w -convergent to 0, let $\xi \in H$, and let ε be a real number such that $|\langle AT_n \xi, \xi \rangle| > \varepsilon$ for each n . Suppose that $\varepsilon > 0$. Given a binary sequence $(a_n)_{n=1}^\infty$, construct an increasing binary sequence $(\lambda_n)_{n=1}^\infty$ such that

$$\lambda_n = 0 \Rightarrow \forall k \leq n (a_k = 0),$$

$$\lambda_n = 1 \Rightarrow \exists k \leq n (a_k = 1).$$

Without loss of generality, we may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $S_n = 0$; if $\lambda_n = 1 - \lambda_{n-1}$, set $S_k = T_n$ for all $k \geq n$. Then $(S_n)_{n=1}^\infty$ is a τ_w -Cauchy sequence in $\mathcal{B}_1(H)$; so, by our hypotheses on f_A , $(\langle AS_n \xi, \xi \rangle)_{n=1}^\infty$ is a Cauchy sequence in \mathbf{C} . Choose N such that

$$|\langle AS_m \xi, \xi \rangle - \langle AS_n \xi, \xi \rangle| < \varepsilon \quad (m, n \geq N).$$

Either $\lambda_N = 1$ and there exists $n \leq N$ such that $a_n = 1$, or else $\lambda_N = 0$. In the latter case, if $n > N$ and $\lambda_n = 1 - \lambda_{n-1}$, then

$$\varepsilon > |\langle AS_n \xi, \xi \rangle - \langle AS_{n-1} \xi, \xi \rangle| = |\langle AS_n \xi, \xi \rangle| > \varepsilon,$$

a contradiction; so $\lambda_n = 0$ for all $n > N$ and therefore for all n ; whence $a_k = 0$ for all k . Thus LPO holds, and therefore, by Lemma 3.4.5, A has an adjoint. It follows that f_A is (τ_w, τ_w) -continuous, which is impossible in view of our choice of ε . We conclude that $\varepsilon \leq 0$. Q.E.D.

Ishihara [36] has proved a nondiscontinuity result related to the preceding one. However, his theorem requires the completeness of the domain of the function, which we cannot guarantee for f_A . We trade completeness for the Cauchy-sequence-preserving property.

We now give the

Proof of Proposition 3.4.4. To establish the sequential continuity of f_A at 0, it suffices to prove that for each $\xi \in H$ the mapping $T \mapsto \langle AT\xi, \xi \rangle$ on $\mathcal{B}_1(H)$ is τ_w -sequentially continuous at 0. Accordingly, let $(T_n)_{n=1}^\infty$ be a sequence in $\mathcal{B}_1(H)$ converging to 0 in the topology τ_w ; then for each k , $(\langle AT_n \xi, \xi \rangle)_{n=k}^\infty$ is a Cauchy, and therefore totally bounded, sequence in \mathbf{C} ; so

$$s_k = \sup_{n \geq k} |\langle AT_n \xi, \xi \rangle|$$

exists. Given $\varepsilon > 0$, we need only find k such that $s_k < 2\varepsilon$; clearly, we may assume that $s_1 > \varepsilon$. Taking $n_0 = 0$, construct an increasing binary sequence (λ_n) , and an increasing sequence $(n_i)_{i=1}^\infty$ of positive integers, such that

- if $\lambda_i = 0$, then $n_i > n_{i-1}$ and $|\langle AT_{n_i} \xi, \xi \rangle| > \varepsilon$;
- if $\lambda_i = 1$, then $n_i = n_{i-1}$ and $s_{n_i} < 2\varepsilon$.

If $\lambda_i = 0$, set $S_i = 0$; if $\lambda_i = 1 - \lambda_{i-1}$, set $S_k = T_{n_{i-1}}$ for all $k > n$. Then $(S_k)_{k=1}^\infty$ is a τ_w -Cauchy sequence in $\mathcal{B}_1(H)$; so $(\langle AS_k\xi, \xi \rangle)_{k=1}^\infty$ is a Cauchy sequence in \mathbb{C} , and therefore

$$s = \sup_{k \geq 1} |\langle AS_k\xi, \xi \rangle|$$

exists. Either $s > 0$ or $s < \varepsilon$. In the latter case, if there exists i such that $\lambda_i = 1 - \lambda_{i-1}$, then

$$s \geq |\langle AS_i\xi, \xi \rangle| = |\langle AT_{n_{i-1}}\xi, \xi \rangle| > \varepsilon,$$

a contradiction; whence $\lambda_i = 0$, and therefore $|\langle AT_{n_i}\xi, \xi \rangle| > \varepsilon$, for all i . This is absurd, in view of Lemma 3.4.7. Thus the case $s < \varepsilon$ is ruled out, and so $s > 0$. Hence there exists i such that $|\langle AS_i\xi, \xi \rangle| > 0$; then $\lambda_i = 1$ and therefore $s_{n_i} < 2\varepsilon$. This completes the proof. Q.E.D.

Chapter 4

Ultraweakly Continuous Linear Functionals

4.1 An extension theorem

Let \mathcal{R} be a linear subset of the space $\mathcal{B}(H)$ of all bounded linear operators on H . In this section we consider the extension and characterisation of those linear functionals on \mathcal{R} that are continuous with respect to the weak-operator topology τ_w and the ultraweak-operator topology $\tau_{\sigma w}$.

Definition 4.1.1 *A von Neumann algebra over H is a subalgebra \mathcal{R} of $\mathcal{B}(H)$ with the following properties:*

- (i) *If $T \in \mathcal{R}$ and T^* exists, then $T^* \in \mathcal{R}$.*
- (ii) *The unit ball $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{B}_1(H)$ of \mathcal{R} is closed and totally bounded with respect to the weak-operator topology.*

Classically, if f is a $\tau_{\sigma w}$ -continuous linear functional on \mathcal{R} , then we can extend it by continuity to the von Neumann algebra \mathcal{A} generated by \mathcal{R} . It follows classically from (7.4.5) on page 483 of [41] that f is $\tau_{\sigma w}$ -continuous on \mathcal{A} . The Hahn-Banach

Theorem then enables us to extend f to a $\tau_{\sigma w}$ -continuous linear functional on $\mathcal{B}(H)$. In turn, a beautiful argument ([28],) using the Hahn–Banach Theorem and the Riesz Representation Theorem shows that the extended functional has the form

$$T \mapsto \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle, \quad (4.1)$$

where $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are elements of H_{∞} . (It is easy to show, conversely, that any linear functional on $\mathcal{B}(H)$ of this form is $\tau_{\sigma w}$ -continuous.) This conclusion can also be established, not just for $\tau_{\sigma w}$ -continuous linear functionals on $\mathcal{B}(H)$ but also for those on a general von Neumann algebra, by deeper results in von Neumann algebra theory; see pages 481–483 of [41].

What can we say constructively about this situation? It is shown in [22] that f has the form (4.1) in the case $\mathcal{R} = \mathcal{B}(H)$. This is not enough to deal with the general case, in which the extension of f to $\mathcal{B}(H)$ cannot be accomplished by a simple application of the Hahn–Banach Theorem, since the constructive form of that theorem requires stronger hypotheses than those of its classical counterpart; nor is the constructive theory of von Neumann algebras sufficiently developed—indeed, the work presented below is one of the first steps towards such a development—to accommodate the more advanced classical method of characterising $\tau_{\sigma w}$ -continuous linear functionals. Nevertheless, as we show in this section, we can extend f to $\mathcal{B}(H)$ constructively under the assumption (which holds in classical mathematics) that \mathcal{R}_1 is τ_w -totally bounded; the extension is accomplished by an iterative use of our main lemma, and ultimately incorporates a new demonstration that f has the form (4.1).

Note that

- we cannot prove constructively that every bounded linear mapping u between

normed spaces has a norm (in Bishop' phrase, is **normable**)¹

$$\|u\| = \sup \{ \|u(x)\| : \|x\| \leq 1 \};$$

- a nonzero bounded linear functional f is normable if and only if its kernel is located ([4], page 303, Proposition (1.10)).

As before, we denote by $p_{x,y}$ the seminorm $T \mapsto |\langle Tx, y \rangle|$ on $\mathcal{B}(H)$. We already know that when dealing with the weak-operator topology, which is the topology generated on $\mathcal{B}(H)$ by $p_{x,y}$ as x and y range over H , we need only concern ourselves with the seminorms $p_{e,e'}$ where $e, e' \in E$ (3.1.3).

To each element T of $\mathcal{B}(H)$ there corresponds an element \tilde{T} of $\mathcal{B}(H_\infty)$ defined for each $\mathbf{x} = (x_n)_{n=1}^\infty$ in H_∞ by

$$\tilde{T}\mathbf{x} = (Tx_n)_{n=1}^\infty.$$

Lemma 4.1.2 *Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$, and let*

$$\tilde{\mathcal{R}} = \{ \tilde{T} : T \in \mathcal{R} \}.$$

If f is a $\tau_{\sigma w}$ -continuous linear functional on \mathcal{R} , then $\tilde{f}(\tilde{T}) = f(T)$ defines a τ_w -continuous linear functional on $\tilde{\mathcal{R}}$. If also \mathcal{R}_1 is τ_w -totally bounded, then both \tilde{f} and f have norms, and these norms are equal.

Proof. By Proposition 2.2.2 of Chapter 2, there exist a positive constant C and a finitely enumerable set $F \subset H_\infty$ such that for each $T \in \mathcal{R}$,

$$|\tilde{f}(\tilde{T})| = |f(T)| \leq C \sup_{\mathbf{x}, \mathbf{y} \in F} \left| \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle \right| = C \sup_{\mathbf{x}, \mathbf{y} \in F} |\langle \tilde{T}\mathbf{x}, \mathbf{y} \rangle|.$$

¹Nevertheless, we write $\|u\| \leq c$ when $\|u(x)\| \leq c\|x\|$ for all x , even if we do not know that u has a norm. Other such inequalities involving classical norms that may not exist constructively will be interpreted in the obvious, analogous manner.

Hence \tilde{f} is τ_w -(uniformly) continuous on $\widetilde{\mathcal{R}}$. If also \mathcal{R}_1 is τ_w -totally bounded, then it is $\tau_{\sigma w}$ -totally bounded; so, by 3.1.5, the unit ball $\widetilde{\mathcal{R}}_1$ of $\widetilde{\mathcal{R}}$ is τ_w -totally bounded. In view of Corollary 2.1.8, both \tilde{f} and f have norms. Finally,

$$\|\tilde{f}\| = \sup \{ |\tilde{f}(\tilde{T})| : \tilde{T} \in \widetilde{\mathcal{R}}_1 \} = \sup \{ |f(T)| : T \in \mathcal{R}_1 \} = \|f\|. \quad \text{Q.E.D.}$$

Our aim in the remainder of this section is to prove the following extension-characterisation theorem for $\tau_{\sigma w}$ -continuous linear functionals.

Theorem 4.1.3 *Let H be a Hilbert space with an orthonormal basis E , and let \mathcal{R} be a linear subset of $\mathcal{B}(H)$ whose unit ball is τ_w -totally bounded. Then each $\tau_{\sigma w}$ -continuous linear functional f on \mathcal{R} extends to a $\tau_{\sigma w}$ -continuous linear functional on $\mathcal{B}(H)$ and has the form*

$$f(T) = \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle \quad (4.2)$$

with $(x_n), (y_n)$ elements of the direct sum $H_{\infty} = \bigoplus_{n=1}^{\infty} H$.

To this end, we now establish a number of technical results. The proofs of the first two of these, which are fundamental results in the duality theory of normed spaces, are found on page 341 of [4].

Proposition 4.1.4 *Let x_0 be an element of a separable normed space X , and S a bounded, balanced, convex subset of X . Suppose that $\{x_0 - x : x \in S\}$ is located and that*

$$0 < d = \inf \{ \|x - x_0\| : x \in S \}.$$

Then for each $\varepsilon > 0$ there exists a linear functional u on X such that $\|u\| = 1$ and

$$u(x_0) > |u(x)| + d - \varepsilon \quad (x \in S).$$

Proposition 4.1.5 *Let x_0 be an element of a nontrivial separable normed space X , and ε a positive number. Then there exists a linear functional u on X such that $\|u\| = 1$ and $u(x_0) > \|x_0\| - \varepsilon$.*

Proposition 4.1.6 *Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$, and let f be a τ_w -continuous linear functional on \mathcal{R} . For each $\varepsilon > 0$ there exist $\delta > 0$ and a finite subset F of E with the following property: if $T \in \mathcal{R}_1$, if $\{e_1, \dots, e_n\}$ is a finite subset of E containing F , and if*

$$\left| \sum_{k=1}^n \langle Tx_k, e_k \rangle \right| < \delta$$

for all x_1, \dots, x_n in the unit ball of $\text{span}\{e_1, \dots, e_n\}$, then $|f(T)| < \varepsilon$.

Proof. Given $\varepsilon > 0$, use the uniform continuity of f on \mathcal{R}_1 to find $\delta > 0$ and a finite subset F of E such that $|f(T)| < \varepsilon$ whenever $T \in \mathcal{R}_1$ and $p_{e,e'}(T) < \delta$ for all $e, e' \in F$. Let $\{e_1, \dots, e_n\}$ be a finite subset of E containing F , and let T be an element of \mathcal{R}_1 such that $|\sum_{k=1}^n \langle Tx_k, e_k \rangle| < \delta$ for all vectors x_1, \dots, x_n in the unit ball of $\text{span}\{e_1, \dots, e_n\}$. Given i, j with $1 \leq i, j \leq n$, take

$$x_k = \begin{cases} e_j & \text{if } k = i \\ 0 & \text{if } k \neq i, \end{cases}$$

to obtain $p_{e_j, e_i}(T) = |\langle Te_j, e_i \rangle| < \delta$. In particular, $p_{e, e'}(T) < \delta$ for all $e, e' \in F$, so $|f(T)| < \varepsilon$. Q.E.D.

The proofs of the next lemma and our main theorem are modelled on Bishop's proof of the characterisation of linear functionals on the dual of a normed space ([4], pages 354–357).

Lemma 4.1.7 *Let \mathcal{R} and f be as in Theorem 4.1.3, and let $\varepsilon > 0$. There exists a finite subset F of E with the following property: if $\{e_1, \dots, e_n\}$ is a finite subset of E containing F , then there exist x_1, \dots, x_n and y_1, \dots, y_n in the linear span of $\{e_1, \dots, e_n\}$ such that*

$$\begin{aligned} \left(\sum_{k=1}^n \|x_k\|^2 \right)^{1/2} &\leq 5 \|f\|, \\ \left(\sum_{k=1}^n \|y_k\|^2 \right)^{1/2} &= n^{-1}, \end{aligned}$$

and

$$\left| f(T) - \sum_{k=1}^n \langle Tx_k, y_k \rangle \right| < \varepsilon \quad (T \in \mathcal{R}_1).$$

Proof. We may assume that there exists $T_0 \in \mathcal{R}_1$ such that $f(T_0) = 1$. Given ε such that $0 < \varepsilon < 1/4$, set

$$\alpha = \frac{\varepsilon}{5\|f\|(1+\|f\|)}.$$

Using Lemma 4.1.6, find $\delta > 0$ and a finite subset F of E such that if $T \in \mathcal{R}_1$, if $S = \{e_1, \dots, e_n\}$ is a finite subset of E containing F , and if

$$\left| \sum_{k=1}^n \langle Tx_k, e_k \rangle \right| < \delta$$

for all x_1, \dots, x_n in the unit ball of H_0 , the linear span of $\{e_1, \dots, e_n\}$, then $|f(T)| < \alpha/2$. For such S let

$$y_k = n^{-3/2}e_k \quad (1 \leq k \leq n);$$

then $(\sum_{k=1}^n \|y_k\|^2)^{1/2} = n^{-1}$. For all $\mathbf{x} = (x_1, \dots, x_n)$ in the direct sum $H_0^n = \bigoplus_{k=1}^n H_0$, since the unit kernel

$$\mathcal{N}_1 = \mathcal{R}_1 \cap f^{-1}(0)$$

of f is τ_w -totally bounded, by Lemma 2.2.4, and since the mappings $T \mapsto |\langle Tx_k, y_k \rangle|$ ($1 \leq k \leq n$) are uniformly continuous on \mathcal{N}_1 , the real number

$$\|\mathbf{x}\|_0 = \sup \left\{ \left| \sum_{k=1}^n \langle Tx_k, y_k \rangle \right| : T \in \mathcal{N}_1 \right\}$$

exists. Note that $\|\cdot\|_0$ is a seminorm on H_0^n . Since, for each $T \in \mathcal{B}(H)$,

$$\left| \sum_{k=1}^n \langle Tx_k, y_k \rangle \right| \leq \left(\sum_{k=1}^n \|Tx_k\|^2 \right)^{1/2} \left(\sum_{k=1}^n \|y_k\|^2 \right)^{1/2} \leq \|T\| \left(\sum_{k=1}^n \|x_k\|^2 \right)^{1/2} n^{-1},$$

we see that

$$\|\mathbf{x}\|_0 \leq \|\mathbf{x}\| = \left(\sum_{k=1}^n \|x_k\|^2 \right)^{1/2};$$

whence the mapping $\mathbf{x} \mapsto \|\mathbf{x}\|_0$ is uniformly continuous. As H_0^n is finite-dimensional, it follows that

$$\beta = \inf \{ \|\mathbf{x}\|_0 : \mathbf{x} \in H_0^n, \|\mathbf{x}\| = 1 \}$$

exists. We show that $\beta < \alpha$.

Since either $\beta > 0$ or $\beta < \alpha$, we may assume that $\beta > 0$. It follows from the definition of β that $\beta\|\mathbf{x}\| \leq \|\mathbf{x}\|_0$ for each $\mathbf{x} \in H_0^n$; whence $\|\cdot\|_0$ is a norm on H_0^n , and $(H_0^n, \|\cdot\|_0)$ is a finite-dimensional Banach space. Define the norm on the dual space $(H_0^n)^*$ in the usual way:

$$\|u\|^0 = \sup \{ \|u(\mathbf{x})\| : \mathbf{x} \in H_0^n, \|\mathbf{x}\| \leq 1 \};$$

and define a mapping $F : \mathcal{N}_1 \rightarrow (H_0^n)^*$ by

$$F(T)(\mathbf{x}) = \sum_{k=1}^n \langle Tx_k, y_k \rangle \quad (T \in \mathcal{N}_1, \mathbf{x} \in H_0^n).$$

For each $\mathbf{x} \in H_0^n$ the mapping $T \mapsto F(T)(\mathbf{x})$ is τ_w -uniformly continuous on \mathcal{N}_1 ; so, as H_0 is finite-dimensional, F is τ_w -uniformly continuous on \mathcal{N}_1 . Therefore, \mathcal{N}_1 being τ_w -totally bounded, the range $\text{ran } F$ of F is totally bounded and therefore located in $(H_0^n)^*$. We show that $\text{ran } F$ is dense in the unit ball $(S_0^*, \|\cdot\|^0)$ of $(H_0^n)^*$. To this end, fix u in S_0^* and suppose that

$$0 < \gamma = \rho_0(u, \text{ran } F) = \inf \{ \|u - F(T)\| : T \in \mathcal{N}_1 \}.$$

Since $\text{ran } F$ is bounded, balanced, and convex, we see from Proposition 4.1.4 that there exists a normable linear functional Φ on $((H_0^n)^*, \|\cdot\|^0)$ such that

$$\Phi(u) > |\Phi(F(T))| + \gamma/2 \quad (T \in \mathcal{N}_1).$$

Because $(H_0^n)^*$ is a finite-dimensional Banach space, the topology induced on it by the norm $\|\cdot\|^0$ is equivalent to the weak* topology; so, by Corollary (6.9) on page 357 of [4], there exists $\xi \in H_0$ such that $\Phi(v) = v(\xi)$ for each $v \in (H_0^n)^*$. Therefore

$$\begin{aligned} u(\xi) &> \sup \{ |F(T)(\xi)| : T \in \mathcal{N}_1 \} \\ &= \sup \left\{ \left| \sum_{k=1}^n \langle T\xi_k, y_k \rangle \right| : T \in \mathcal{N}_1 \right\} = \|\xi\|_0, \end{aligned}$$

which is absurd as u belongs to S_0^* . We conclude that $\gamma = 0$, and hence that $\text{ran } F$ is dense in S_0^* .

Now let \mathbf{x} be any element of the unit ball of $(H_0^n, \|\cdot\|_0)$. Since

$$\left| \sum_{k=1}^n \langle \beta T_0 x_k, y_k \rangle \right| \leq \beta \|\mathbf{x}\| \leq \|\mathbf{x}\|_0,$$

the linear functional $\mathbf{x} \mapsto \sum_{k=1}^n \langle \beta T_0 x_k, y_k \rangle$ belongs to S_0^* . As $\text{ran } F$ is dense in S_0^* , there exists an element T of \mathcal{N}_1 such that

$$\left| \sum_{k=1}^n \langle (\beta T_0 - T) x_k, y_k \rangle \right| < 2n^{-1}\delta$$

and therefore

$$\left| \sum_{k=1}^n \left\langle \frac{1}{2} (\beta T_0 - T) x_k, y_k \right\rangle \right| < \delta$$

for each \mathbf{x} in the unit ball of H_0^n . Now, $\beta \leq 1$ and both T and T_0 belong to \mathcal{R}_1 , so $\frac{1}{2}(\beta T_0 - T) \in \mathcal{R}_1$; hence

$$\left| f\left(\frac{1}{2}(\beta T_0 - T)\right) \right| < \alpha/2$$

and therefore

$$\beta = \beta f(T_0) - f(T) = 2f\left(\frac{1}{2}(\beta T_0 - T)\right) < \alpha,$$

which embodies the inequality that we wanted to establish.

Now choose \mathbf{z} in the unit ball of H_0^n such that

$$\left| \sum_{k=1}^n \langle T z_k, y_k \rangle \right| < \alpha \quad (T \in \mathcal{N}_1).$$

For each $T \in \mathcal{R}_1$, since

$$(1 + \|f\|)^{-1} (T - f(T)T_0) \in \mathcal{N}_1,$$

we have

$$(1 + \|f\|)^{-1} \left| \sum_{k=1}^n \langle (T - f(T)T_0) z_k, y_k \rangle \right| < \alpha.$$

On the other hand, by Proposition 4.1.5, there exists $v \in S_0^*$ such that $v(\mathbf{z}) = 1/2$.

Since $\text{ran } F$ is dense in S_0^* , there exists $T_1 \in \mathcal{N}_1$ such that

$$\left| v(\mathbf{x}) - \sum_{k=1}^n \langle T_1 x_k, y_k \rangle \right| < \frac{1}{4} \quad (\mathbf{x} \in H_0, \|\mathbf{x}\|_0 \leq 1).$$

In particular,

$$\left| \frac{1}{2} - \sum_{k=1}^n \langle T_1 z_k, y_k \rangle \right| < \frac{1}{4}.$$

Since

$$\left| \frac{1}{2} - \left| \sum_{k=1}^n \langle T_1 z_k, y_k \rangle \right| \right| \leq \left| \frac{1}{2} - \sum_{k=1}^n \langle T_1 z_k, y_k \rangle \right|,$$

it follows that

$$\begin{aligned} \frac{1}{4} &\leq \sum_{k=1}^n \langle T_1 z_k, y_k \rangle \\ &\leq \left| \sum_{k=1}^n \langle (T_1 - f(T_1)T_0) z_k, y_k \rangle \right| + \left| \sum_{k=1}^n \langle f(T_1)T_0 z_k, y_k \rangle \right| \\ &\leq (1 + \|f\|) \alpha + \|f\| \sum_{k=1}^n \langle T_0 z_k, y_k \rangle. \end{aligned}$$

Hence

$$\sum_{k=1}^n \langle T_0 z_k, y_k \rangle \geq \|f\|^{-1} \left(\frac{1}{4} - (1 + \|f\|) \alpha \right) > \frac{1}{5} \|f\|^{-1}.$$

Writing

$$\mathbf{x} = \left(\sum_{k=1}^n \langle T_0 z_k, y_k \rangle \right)^{-1} \mathbf{z},$$

we have $\|\mathbf{x}\| \leq 5 \|f\|$. Also, for each $T \in \mathcal{R}_1$,

$$\begin{aligned} \left| \sum_{k=1}^n \langle T x_k, y_k \rangle - f(T) \right| &= \left| \sum_{k=1}^n \langle T_0 z_k, y_k \rangle \right|^{-1} \left| \sum_{k=1}^n \langle (T - f(T)T_0) z_k, y_k \rangle \right| \\ &< 5 \|f\| (1 + \|f\|) \alpha \\ &= \varepsilon. \end{aligned} \tag{4.3}$$

This completes the proof. Q.E.D.

We are now in a position to prove the main result of this chapter.

Proof of Theorem 4.1.3 We first consider the case where f is a τ_w -continuous linear functional on \mathcal{R} . We may assume that $\|f\| < 1$. Setting $n_1 = 1$ and $x_{1,1} = y_{1,1} = 0$, we construct a strictly increasing sequence $(n_j)_{j=1}^{\infty}$ of positive integers, and

for each j elements $x_{j,1}, \dots, x_{j,n_j}$ and $y_{j,1}, \dots, y_{j,n_j}$ of H , such that

$$\begin{aligned} \left(\sum_{k=1}^{n_j} \|x_{j,k}\|^2 \right)^{1/2} &\leq 5 \cdot 2^{-j}, \\ \left(\sum_{k=1}^{n_j} \|y_{j,k}\|^2 \right)^{1/2} &= n_j^{-1}, \end{aligned}$$

and

$$\left| f(T) - \sum_{i=1}^j \sum_{k=1}^{n_i} \langle T x_{i,k}, y_{i,k} \rangle \right| < 2^{-j} \quad (T \in \mathcal{R}_1). \quad (4.4)$$

To do so, we use induction on j . Supposing that n_j and the corresponding elements $x_{j,k}$ and $y_{j,k}$ ($1 \leq k \leq n_j$) of H have been constructed, we have either $\|f\| < 2^{-j-1}$ or $\|f\| > 0$. In the first case we set $n_{j+1} = n_j + 1$ and $x_{j+1,n_{j+1}} = y_{j+1,n_{j+1}} = 0$. In the second, applying Lemma 4.1.7 to the τ_{σ_w} -continuous linear functional

$$T \mapsto f(T) - \sum_{i=1}^j \sum_{k=1}^{n_i} \langle T x_{i,k}, y_{i,k} \rangle$$

on \mathcal{R} , we obtain a positive integer $n_{j+1} > n_j$, and elements $x_{j+1,k}, y_{j+1,k}$ ($1 \leq k \leq n_{j+1}$) of H , such that

$$\begin{aligned} \left(\sum_{k=1}^{n_{j+1}} \|x_{j+1,k}\|^2 \right)^{1/2} &\leq 5 \cdot 2^{-j-1}, \\ \left(\sum_{k=1}^{n_{j+1}} \|y_{j+1,k}\|^2 \right)^{1/2} &= n_{j+1}^{-1}, \end{aligned}$$

and

$$\left| f(T) - \sum_{i=1}^{j+1} \sum_{k=1}^{n_i} \langle T x_{i,k}, y_{i,k} \rangle \right| < 2^{-j-1} \quad (T \in \mathcal{R}_1).$$

This completes the induction.

Now define sequences \mathbf{x}, \mathbf{y} of elements of H as follows:

$$\mathbf{x} = (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}, x_{3,1}, \dots, x_{3,n_3}, \dots)$$

and

$$\mathbf{y} = (y_{1,1}, \dots, y_{1,n_1}, y_{2,1}, \dots, y_{2,n_2}, y_{3,1}, \dots, y_{3,n_3}, \dots).$$

The series $\sum_{k=1}^{\infty} \|x_k\|^2$ and $\sum_{k=1}^{\infty} \|y_k\|^2$ converge, by comparison with $5 \sum_{j=1}^{\infty} 2^{-2j+2}$ and $\sum_{j=1}^{\infty} n_j^{-2}$, respectively; so $\mathbf{x}, \mathbf{y} \in H_{\infty}$, and the series $\sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle$ converges absolutely for each $T \in \mathcal{B}(H)$. It follows from (4.4) that (4.3) holds, and hence that f extends to a $\tau_{\sigma w}$ -continuous linear functional on $\mathcal{B}(H)$.

It remains to consider the case where f is $\tau_{\sigma w}$ -continuous. Define $\widetilde{\mathcal{R}}$ and \widetilde{f} as in Lemma 4.1.2. Since $\widetilde{\mathcal{R}}_1$ is weak-operator totally bounded and \widetilde{f} is weak-operator continuous on $\widetilde{\mathcal{R}}$, using Lemma 3.1.5 we obtain sequences $(\xi_k)_{k=1}^{\infty}, (\eta_k)_{k=1}^{\infty}$ of elements of H_{∞} such that

$$\widetilde{f}(\widetilde{T}) = \sum_{k=1}^{\infty} \langle \widetilde{T} \xi_k, \eta_k \rangle = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \langle T \xi_{k,i}, \eta_{k,i} \rangle \quad (T \in \mathcal{R}),$$

where $\xi_k = (\xi_{k,i})_{i=1}^{\infty}$ and $\eta_k = (\eta_{k,i})_{i=1}^{\infty}$. Let ϕ be a one-one mapping of \mathbf{N}^+ onto $\mathbf{N}^+ \times \mathbf{N}^+$, and set $x_n = \xi_{\phi(n)}$, $y_n = \eta_{\phi(n)}$. By Fubini's Theorem, the series $\sum_{k,i=1}^{\infty} \|\xi_{k,i}\|^2$, $\sum_{k,i=1}^{\infty} \|\eta_{k,i}\|^2$, and $\sum_{k,i} |\langle T \xi_{k,i}, \eta_{k,i} \rangle|$ converge. Hence the series $\sum_{k,i=1}^{\infty} \|\xi_{k,i}\|^2$, $\sum_{k,i=1}^{\infty} \|\eta_{k,i}\|^2$, and $\sum_{k,i=1}^{\infty} \langle T \xi_{k,i}, \eta_{k,i} \rangle$ converge, in each case to a sum that does not depend on the ordering of the terms of the series. Writing $\mathbf{x} = (x_n)_{n=1}^{\infty}$ and $\mathbf{y} = (y_n)_{n=1}^{\infty}$, we now see that \mathbf{x} and \mathbf{y} belong to H_{∞} , and that

$$f(T) = \sum_{k,i=1}^{\infty} \langle T \xi_{k,i}, \eta_{k,i} \rangle = \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle$$

for all $T \in \mathcal{R}$. Q.E.D.

4.2 The predual

In classical mathematics, every von Neumann algebra has a weak-operator compact unit ball. In view of the Alaoglu–Bourbaki Theorem, we therefore see that a von Neumann algebra exhibits one of the key features of the dual of a Banach space. Constructively, we cannot expect a von Neumann algebra \mathcal{R} to have this property, since for the case $\mathcal{R} = \mathcal{B}(H)$ the most we can say, in general, is that its unit ball is τ_w -totally bounded.

Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$, let $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{B}_1(H)$ be its unit ball, and let $\mathcal{R}_\#$ denote the linear space of all linear functionals on \mathcal{R} that are ultraweakly continuous, and hence τ_w -continuous on \mathcal{R}_1 . If \mathcal{R}_1 is τ_w -totally bounded, then

$$\|f\| = \sup \{|f(T)| : T \in \mathcal{R}_1\}$$

defines a norm on $\mathcal{R}_\#$; taken with this norm, $\mathcal{R}_\#$ is called the **predual** of \mathcal{R} .

For convenience, we denote by $f_{x,y}$ the ultraweakly continuous functional $T \mapsto \langle Tx, y \rangle$ on $\mathcal{B}(H)$.

Theorem 4.2.1 *Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$ such that \mathcal{R}_1 is totally bounded in the weak-operator topology τ_w , and define a mapping ϕ of \mathcal{R} into the dual space $\mathcal{R}_\#^*$ of $\mathcal{R}_\#$ by*

$$\phi(T)(f) = f(T) \quad (T \in \mathcal{R}).$$

Then ϕ is one-one and linear, and is uniformly continuous on \mathcal{R}_1 . Moreover, $\phi(\mathcal{R}_1)$ is weak-dense in the unit ball of $\mathcal{R}_\#^*$, and the restriction of ϕ^{-1} to $\phi(\mathcal{R}_1)$ is uniformly continuous with respect to the weak*-topology on $\mathcal{R}_\#$ and the weak-operator topology on \mathcal{R}_1 .*

Proof. Since ϕ is clearly linear, to prove that it is one-one we need only show that its kernel is $\{0\}$. But if $\phi(T) = 0$, then we have

$$\langle Tx, y \rangle = \phi(T)(f_{x,y}) = 0$$

for all $x, y \in H$; whence $T = 0$.

For each $f \in \mathcal{R}_\#$ the mapping $T \mapsto \phi(T)(f)$ equals f and so is uniformly continuous on \mathcal{R}_1 . It follows immediately that ϕ is uniformly continuous as a mapping of (\mathcal{R}_1, τ_w) into $\mathcal{R}_\#^*$ (with the weak*-topology). Hence $K = \phi(\mathcal{R}_1)$ is weak*-totally bounded, and therefore located, in $\mathcal{R}_\#^*$ [13, 14]. Let u be an element of the unit ball of $\mathcal{R}_\#^*$, let $\{f_1, \dots, f_N\}$ be a finitely enumerable subset of $\mathcal{R}_\#$, and let ε be a

positive number. To prove that $\phi(\mathcal{R}_1)$ is weak*-dense in the unit ball $(\mathcal{R}_\#^*)_1$ of $\mathcal{R}_\#^*$, it is enough to show that K intersects the neighbourhood

$$V = \left\{ v \in (\mathcal{R}_\#^*)_1 : |(u - v)(f_k)| < 3\varepsilon \ (1 \leq k \leq N) \right\}$$

of u in $(\mathcal{R}_\#^*)_1$. To this end, choose a finite-dimensional subspace \mathcal{G} of $\mathcal{R}_\#$ such that

$$\inf \{ \|f_k - g\| : g \in \mathcal{G} \} < \varepsilon \quad (1 \leq k \leq N)$$

([4], page 308, Lemma (2.5)); for each k ($1 \leq k \leq N$), then choose $g_k \in \mathcal{G}$ such that $\|f_k - g_k\| < 1$. The dual space \mathcal{G}^* of \mathcal{G} is a finite-dimensional Banach space with respect to the usual norm defined by

$$\|u\|' = \sup \{ |u(g)| : g \in \mathcal{G}, \|g\| \leq 1 \}.$$

Since $K \subset (\mathcal{R}_\#^*)_1$, and $(\mathcal{R}_\#^*)_1$ is a subset of the unit ball of $(\mathcal{G}^*)_1$, we can regard u and, for each $T \in \mathcal{R}_1$, the functional $\phi(T)$ as elements of $(\mathcal{G}^*)_1$. Now suppose that

$$\inf \{ \|u - \phi(T)\|' : T \in \mathcal{R}_1 \} > 0. \quad (4.5)$$

By Proposition 3.1.3, there exists a linear functional ψ , with norm 1, on \mathcal{G}^* such that

$$|\psi(u)| > \sup \{ |\psi(v)| : v \in K \}.$$

Since \mathcal{G} is finite-dimensional, ψ is weak*-uniformly continuous on $(\mathcal{G}^*)_1$; so, by Corollary (6.9) on page 357 of [4], there exists $g \in \mathcal{G}$ such that $\psi(v) = v(g)$ for all $v \in \mathcal{G}^*$. In particular,

$$\begin{aligned} |u(g)| &> \sup \{ |v(g)| : v \in K \} \\ &= \sup \{ |\phi(T)(g)| : T \in \mathcal{R}_1 \} \\ &= \sup \{ g(T) : T \in \mathcal{R}_1 \} \\ &= \|g\|, \end{aligned}$$

which is absurd since $u \in (\mathcal{R}_\sharp^*)_1$. We conclude that (4.4) is false, and hence that

$$\|u - \phi(T_0)\|' < \frac{\varepsilon}{M+1}$$

for some $T_0 \in \mathcal{R}_1$, where

$$M = \max_{1 \leq k \leq N} \|g_k\|.$$

For each k ($1 \leq k \leq N$) we now have

$$\begin{aligned} |(u - \phi(T_0))(f_k)| &\leq |(u - \phi(T_0))(f_k - g_k)| + |(u - \phi(T_0))(g_k)| \\ &\leq 2\|f_k - g_k\| + \|u - \phi(T_0)\|' \|g_k\| \\ &< 2\varepsilon + \frac{\varepsilon}{M+1}M \\ &< 3\varepsilon; \end{aligned}$$

in other words, $\phi(T_0) \in V$. Since $\phi(T_0) \in K$, this completes the proof that $\phi(\mathcal{R}_1)$ is dense in $(\mathcal{R}_\sharp^*)_1$.

Finally, the uniform continuity of the inverse mapping on K follows from the identity

$$|\langle Tx, y \rangle| = |\phi(T)(f_{x,y})| \quad (x, y \in H; T \in \mathcal{R}_1),$$

with reference to the definitions of the weak*- and weak-operator topologies, and to Proposition 1.2.8 on page 19 of [40]. Q.E.D.

Classically, any von Neumann algebra can be identified, via the mapping ϕ , with the dual of its predual \mathcal{R}_\sharp ([41], page 482). If this were provable constructively, then we could use Theorem 4.2.1 to prove that $\mathcal{B}_1(H)$ is τ_w -complete, which we cannot do. Thus Theorem 4.2.1 appears to be the best general constructive result of its type.

4.3 An application to trace class operators

Let H be a complex Hilbert space that has an orthonormal basis. An element A of $\mathcal{B}(H)$

- **has an absolute value** if there exists a (necessarily unique, positive, self-adjoint, and bounded) operator $|A|$ on H such that

$$\langle |A|x, |A|y \rangle = \langle Ax, Ay \rangle$$

for all $x, y \in H$. If A has an adjoint, then $|A|^2 = A^*A$, and this equation may be used to define $|A|$. An operator need not have an absolute value, even if its range is 1-dimensional;

- is a **trace class operator** if it has an absolute value and

$$\|A\|_1 = \sum_{e \in E} \langle |A|e, e \rangle$$

exists for some orthonormal basis E of H .

In the second case, the **trace class norm** of A is

$$\|A\|_1 = \sum_{e \in E} \left\| |A|^{1/2}e \right\|^2.$$

For every trace class operator A , the family $(\langle Ae, e \rangle)_{e \in E}$ is summable. We define the **trace of A** to be the sum

$$\text{Tr}(A) = \sum_{e \in E} \langle Ae, e \rangle.$$

Note that $\|A\|_1 = \text{Tr}(|A|)$. We denote the set of trace class operators on H by $\mathcal{T}(H)$.

As we proved in Section 4.1 (see also [22]), under the hypotheses of Theorem 4.2.1, the ultraweakly continuous linear functionals on a linear subset \mathcal{R} of $\mathcal{B}(H)$ whose unit ball is weak-operator totally bounded extend to ultraweakly continuous linear functionals on $\mathcal{B}(H)$ and are precisely those functionals f_A mapping T to $\text{Tr}(TA)$, with A a trace-class operator on H . The norm of f_A on \mathcal{R} is

$$\|f_A\|_{\mathcal{R}} = \sup \{ |\text{Tr}(TA)| : T \in \mathcal{R}_1 \},$$

which in the case $\mathcal{R} = \mathcal{B}(H)$ equals the trace-class norm

$$\|A\|_1 = \text{Tr}(A)$$

of A (see [7]). Taken with Theorem 4.2.1, these observations lead to

Theorem 4.3.1 *Let \mathcal{R} be a linear subset of $\mathcal{B}(H)$ such that \mathcal{R}_1 is totally bounded in the weak-operator topology τ_w . Let $\mathcal{T}(H)$ denote the set of trace-class operators on H , taken with the norm*

$$\|A\|_{\mathcal{R}} = \sup \{ |\operatorname{Tr}(TA)| : T \in \mathcal{R}_1 \}.$$

Then

$$\Phi(T)(A) = \operatorname{Tr}(TA) \quad (T \in \mathcal{R}, A \in \mathcal{T}(H))$$

defines a one-one linear mapping Φ of \mathcal{R} into the dual space $\mathcal{T}(H)^$ with the following properties.*

- (i) $\Phi(\mathcal{R}_1)$ is dense in the unit ball $\mathcal{T}(H)_1^*$ of $\mathcal{T}(H)^*$.
- (ii) Φ is uniformly continuous on \mathcal{R}_1 .
- (iii) the restriction of Φ^{-1} to $\Phi(\mathcal{R}_1)$ is uniformly continuous relative to the weak*-topology on $\Phi(\mathcal{R}_1)$ and the weak-operator topology on \mathcal{R}_1 .

Corollary 4.3.2 *Under the hypotheses of Theorem 4.3.1, the following conditions are equivalent.*

- (i) \mathcal{R}_1 is weak-operator complete.
- (ii) ϕ maps \mathcal{R}_1 onto the unit ball of \mathcal{R}_1^* .
- (iii) Φ maps \mathcal{R}_1 onto the unit ball of $\mathcal{T}(H)^*$ relative to the norm $\|\cdot\|_{\mathcal{R}}$.

Proof. This is a special case of the following general lemma about metric spaces, whose straightforward proof we omit. Q.E.D.

Lemma 4.3.3 *Let X be a metric space, Y a complete metric space, and ϕ a one-one uniformly continuous mapping of X onto a dense subset of Y such that ϕ^{-1} is uniformly continuous on $\phi(X)$. Then X is complete if and only if $\phi(X) = Y$.*

Chapter 5

The Geometry of Projections

5.1 A first constructive look at the comparison of projections

Throughout this section, \mathcal{R} will be a von Neumann algebra over H (that is, a $*$ -subalgebra of $\mathcal{B}(H)$ whose unit ball $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{B}_1(H)$ is τ_w -closed and totally bounded). We investigate the constructive comparison theory for projections in \mathcal{R} of $\mathcal{B}(H)$, beginning with some definitions and facts from [21].

Two operators T, R are said to be **isometric** if $\langle Tx, Ty \rangle = \langle Rx, Ry \rangle$ for all x, y .

In that case

- if either R or T has an absolute value, then so does the other and the absolute values are equal;
- if either R or T is bounded, then so is the other; and
- if either R or T has a norm, then so does the other.

As a converse of the first of these observations, two operators with absolute values are isometric if their absolute values are equal.

If T and R are isometric, then there is an isometry U from $\text{ran}R$ to $\text{ran}T$ such that $T = UR$. This is a kind of polar decomposition of T . (Classically, the domain of U can be extended to the whole of H by first extending U (uniquely) to the closure of $\text{ran}R$ and then defining U to be 0 on the orthogonal complement of $\text{ran}R$. Constructively, this procedure will not work unless $\text{ran}R$ is located.)

Let P be a projection, and U a bounded operator, on H . Then the following conditions are equivalent:

- $|U|$ exists and equals P ;
- U is an isometry on the range of P and is 0 on the kernel of P .

If these hold for some projection P , we say that U is a **partial isometry** with **initial projection** P and **initial space** $P(H)$. The following facts about partial isometries are well known or easily established (see [21], and [41], Chapter 6).

- If U is a partial isometry, then $U|U| = U$.
- If U is a partial isometry with initial projection P , then $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in E(H)$, and U maps any orthonormal sequence in its initial space to an orthonormal sequence.
- The adjoint (if it exists) of a partial isometry is a partial isometry.
- An operator U with an adjoint is a partial isometry if and only if U^*U is a projection, in which case U^*U is the initial projection of U . UU^* is also a projection, called the **final projection** of U , and its range is called the **final space** of U .

Definition 5.1.1 *We say that two projections $E, F \in \mathcal{B}(H)$ are **strongly equivalent** relative to \mathcal{R} , and we write*

$$E \approx_{\mathcal{R}} F,$$

if there exists a partial isometry $U \in \mathcal{R}$ such that $U^*U = E$ and $UU^* = F$.

Definition 5.1.2 We say that E is **weaker** than F , and that F is **stronger** than E , relative to \mathcal{R} , and we write

$$E \preceq_{\mathcal{R}} F,$$

if $E \approx_{\mathcal{R}} F'$ for some subprojection F' of F .

We write \approx, \preceq instead of $\approx_{\mathcal{R}}, \preceq_{\mathcal{R}}$ when it is clear which von Neumann algebra \mathcal{R} is under consideration.

Note that, as classically, if $E \approx_{\mathcal{R}} F$, then both E and F belong to \mathcal{R} , since U and U^* are in the subalgebra \mathcal{R} , ([41], foot of page 402). Also, $\approx_{\mathcal{R}}$ is an equivalence relation ([41], 6.1.5), and $\preceq_{\mathcal{R}}$ is transitive ([41], 6.2.5).

Relative to the von Neumann algebra $\mathcal{B}(H)$ we have the following properties.

Lemma 5.1.3 *If E is a finite-dimensional projection and $E \approx F$, then F is finite-dimensional, with the same dimension as that of E .*

Proof. The partial isometry implementing the equivalence between E and F maps an orthonormal basis of $E(H)$ to an orthonormal basis of $F(H)$. Q.E.D.

Proposition 5.1.4 *If H is separable, then any two infinite-dimensional projections on H are strongly equivalent.*

Proof. Let E and F be infinite-dimensional projections on H . Let (e_n) be an orthonormal basis of $E(H)$, and (f_n) an orthonormal basis of $F(H)$. Since E is a projection, its range is located; so we can define a linear mapping U as follows:

$$\begin{aligned} Ue_n &= f_n \quad \text{for all } n, \\ Ux &= 0 \quad \text{for all } x \in E(H)^\perp. \end{aligned}$$

Then for all $x \in H$ we have

$$\begin{aligned}
 Ux &= UEx + U(I - E)x \\
 &= UEx \\
 &= U \left(\sum_{n=1}^{\infty} \langle Ex, e_n \rangle e_n \right) \\
 &= \sum_{n=1}^{\infty} \langle x, e_n \rangle Ue_n \\
 &= \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n,
 \end{aligned}$$

the fourth equality following because the convergence of

$$\sum_{n=1}^{\infty} \langle Ex, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

ensures that of $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$. Note that $\text{ran } U \subset F(H)$. We easily show that the adjoint of U exists and maps each f_n to the corresponding e_n . Also, for all $x \in H$ we have

$$U^*Ux = U^* \left(\sum_{n=1}^{\infty} \langle Ux, f_n \rangle f_n \right) = \sum_{n=1}^{\infty} \langle x, U^*f_n \rangle U^*f_n = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = Ex.$$

Similarly, $UU^*y = Fy$ for all $y \in H$. Hence U is a partial isometry with initial projection E and final projection F . Q.E.D.

Proposition 5.1.5 *Let E, F be projections in such that*

$$E \approx_{\mathcal{B}(H)} F < E.$$

Then E is infinite-dimensional.

Proof. Let U be a partial isometry such that $U^*U = E$ and $UU^* = F$. Let K be a finite-dimensional subspace of $E(H)$ with dimension n . Then $U(K)$ is an n -dimensional subspace of F . Since $F < E$, there exists a unit vector $e \in E(H) \cap F(H)^\perp$. The partial isometry U^* maps $U(K)$ onto an n -dimensional subspace of $E(H)$, and maps e to a unit vector orthogonal to $U(K)$. Hence $U(K)$ and e span

an $(n + 1)$ -dimensional subspace L of $E(H)$. Since K is n -dimensional, there exists $x \in K - L$. Hence $E(H)$ is infinite-dimensional. Q.E.D.

Classically, the ordering \preceq is total on $\mathcal{B}(H)$. To see that this is not the case constructively, let (a_n) be a binary sequence containing at most one term equal to 1. Let (e_n) be an orthonormal basis of unit vectors in an infinite-dimensional Hilbert space H , and define projections E, F on H as follows.

$$Ee_n = \begin{cases} 0 & \text{if either } a_n = 0, \text{ or } a_n = 1 \text{ and } n \text{ is odd} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is even,} \end{cases}$$

$$Fe_n = \begin{cases} 0 & \text{if either } a_n = 0, \text{ or } a_n = 1 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is odd.} \end{cases}$$

If $a_n = 1$ for an even value of n , then E is the projection on $\mathbf{C}e_n$ and $F = 0$, so $F \preceq E$ and $F < E$; if $a_n = 1$ for an odd value of n , then $E = 0$ and F is the projection on $\mathbf{C}e_n$, so $E \preceq F$ and $E < F$. It follows that if $E \preceq F$, then $a_n = 0$ for all even n ; and if $F \preceq E$, then $a_n = 0$ for all odd n .

5.2 Countable additivity of equivalence

Classically, equivalence is **countably additive**:

If $(E_n), (F_n)$ are pairwise orthogonal families of projections in R such that $E_n \approx_{\mathcal{R}} F_n$ for each n , then $\sum_{n=1}^{\infty} E_n \approx_{\mathcal{R}} \sum_{n=1}^{\infty} F_n$ ([41], 6.2.2).

There are two constructive problems with this statement as it stands. First, there is no guarantee that $\sum_{n=1}^{\infty} E_n$ and $\sum_{n=1}^{\infty} F_n$ converge (strongly); and secondly, even if the sum $U = \sum_{n=1}^{\infty} U_n$ of the partial isometries implementing the equivalences between the E_n and the F_n converges, we cannot be certain that U will have an adjoint.

We prove the following constructive version of the countable additivity of equivalence.

Proposition 5.2.1 *Let (E_n) and (F_n) be orthogonal families of projections in \mathcal{R} . Let (U_n) be a family of partial isometries in \mathcal{R} , each with an adjoint, such that $E_n = U_n^*U_n$ and $F_n = U_nU_n^*$ for each n . Then the following conditions are equivalent.*

- (i) $\sum_{n=1}^{\infty} E_n$ and $\sum_{n=1}^{\infty} F_n$ converge strongly in \mathcal{R} .
- (ii) $\sum_{n=1}^{\infty} U_n$ converges strongly to an operator in \mathcal{R} with an adjoint.
- (iii) $\sum_{n=1}^{\infty} U_n^*$ converges strongly to an operator in \mathcal{R} with an adjoint.
- (iv) $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} U_n^*$ converge in \mathcal{R} in the weak-operator topology.

In each—and therefore every—case, $\sum_{n=1}^{\infty} E_n \approx_{\mathcal{R}} \sum_{n=1}^{\infty} F_n$.

We need two simple lemmas for the proof.

Lemma 5.2.2 *Let (x_n) be an orthogonal sequence of elements of H . Then $\sum_{n=1}^{\infty} x_n$ converges in H if and only if $\sum_{n=1}^{\infty} \|x_n\|^2$ converges.*

Proof. For all $k > j$ we have

$$\left\| \sum_{n=j}^k x_n \right\|^2 = \sum_{n=j}^k \|x_n\|^2.$$

Thus the partial sums of $\sum_{n=1}^{\infty} x_n$ form a Cauchy sequence in H if and only if the partial sums of $\sum_{n=1}^{\infty} \|x_n\|^2$ form a Cauchy sequence in \mathbf{R} . Q.E.D.

Lemma 5.2.3 *Let (U_n) be a sequence of linear mappings of H into H such that*

- $\langle U_mx, U_nx \rangle = 0$ whenever $m \neq n$, and
- for each $x \in H$, $\sum_{n=1}^{\infty} U_nx$ converges weakly to an element Ux of H .

Then U is a linear mapping of H into H , and for each $x \in H$, $\sum_{n=1}^{\infty} U_n x$ converges strongly to Ux . Moreover, if each U_n belongs to \mathcal{R} , then so does U .

Proof. As the ranges of the operators U_n are pairwise orthogonal, whenever $M \geq N$ we have

$$\left\langle \sum_{m=1}^M U_m x, \sum_{n=1}^N U_n x \right\rangle = \sum_{n=1}^N \langle U_n x, U_n x \rangle = \sum_{n=1}^N \|U_n x\|^2.$$

Letting $M \rightarrow \infty$, we obtain

$$\left\langle Ux, \sum_{n=1}^N U_n x \right\rangle = \sum_{n=1}^N \|U_n x\|^2.$$

Now letting $N \rightarrow \infty$, we see that $\sum_{n=1}^{\infty} \|U_n x\|^2$ converges to $\langle Ux, Ux \rangle = \|Ux\|^2$. It follows from Lemma 5.2.2 that $\sum_{n=1}^{\infty} U_n x$ converges in H . Since $\sum_{n=1}^{\infty} U_n x$ converges weakly to Ux , we conclude that $Ux = \sum_{n=1}^{\infty} U_n x$. It is routine to show that U is a linear mapping. The final conclusion follows because a von Neumann algebra is strong-operator closed. Q.E.D.

We now have the **Proof of Proposition 5.2.1**. To begin with, we assume (i), letting

$$E = \sum_{n=1}^{\infty} E_n, \quad F = \sum_{n=1}^{\infty} F_n.$$

For each $x \in H$ we have

$$\sum_{n=1}^{\infty} \|U_n x\|^2 = \sum_{n=1}^{\infty} \langle U_n^* U_n x, x \rangle = \sum_{n=1}^{\infty} \langle E_n x, x \rangle = \sum_{n=1}^{\infty} \|E_n x\|^2,$$

so $\sum_{n=1}^{\infty} U_n x$ converges to a limit $Ux \in H$, by Lemma 5.2.2. Thus $\sum_{n=1}^{\infty} U_n$ converges in the strong-operator topology to a limit $U \in \mathcal{R}$. Similarly, $\sum_{n=1}^{\infty} U_n^*$ converges in \mathcal{R} in the strong-operator topology. For all $x, y \in H$ we have

$$\langle Ux, y \rangle = \lim_{N \rightarrow \infty} \left\langle \sum_{n=1}^N U_n x, y \right\rangle = \lim_{N \rightarrow \infty} \left\langle x, \sum_{n=1}^N U_n^* y \right\rangle = \left\langle x, \sum_{n=1}^{\infty} U_n^* y \right\rangle,$$

so $\sum_{n=1}^{\infty} U_n^*$ is strong-operator convergent to U^* .

Next, assume (ii), and let $U = \sum_{n=1}^{\infty} U_n^*$. For all $x, y \in H$ we have

$$\langle x, U^*y \rangle = \langle Ux, y \rangle = \left\langle \sum_{n=1}^{\infty} U_n x, y \right\rangle = \sum_{n=1}^{\infty} \langle x, U_n^* y \rangle,$$

so $\sum_{n=1}^{\infty} U_n^* y$ converges weakly to U^*y . It follows from Lemma 5.2.2 that (iii) holds. Interchanging the roles of U and U^* , we see that (ii) and (iii) are equivalent conditions.

Finally, assume (iv). Since the U_n are orthogonal partial isometries, so are their adjoints. Hence, by Lemma 5.2.3, both $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} U_n^*$ are strong-operator convergent in \mathcal{R} . Moreover,

$$\langle U^*x, U^*x \rangle = \sum_{k=1}^{\infty} \|U_k^*x\|^2 = \sum_{k=1}^{\infty} \langle U_k U_k^* x, x \rangle = \sum_{k=1}^{\infty} \langle F_k x, x \rangle = \sum_{k=1}^{\infty} \|F_k x\|^2.$$

Lemma 5.2.2 now shows that $\sum_{k=1}^{\infty} F_k$ converges strongly to an operator $F \in \mathcal{R}$, which must be a projection as the F_k are pairwise orthogonal projections. Using the orthogonality of the F_n , we find that

$$Fx = \sum_{n=1}^{\infty} U_n U_n^* x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_m U_n^* x = \sum_{m=1}^{\infty} U_m \left(\sum_{n=1}^{\infty} U_n^* x \right) = UU^*x.$$

Interchanging the roles of U and U^* , we see that $\sum_{n=1}^{\infty} E_n$ converges strongly to a projection $E \in \mathcal{R}$, that $E = U^*U$, and hence that E is equivalent to F relative to \mathcal{R} . In particular, (iv) implies (i). Q.E.D.

To see that the foregoing proposition is the best possible constructive result about countable additivity, let (a_n) be a binary sequence with at most one term equal to 1, and define

$$E_n x = \begin{cases} 0 & \text{if } a_n = 0 \\ \langle x, e_n \rangle e_n & \text{if } a_n = 1, \end{cases}$$

$$U_n x = \begin{cases} 0 & \text{if } a_n = 0 \\ \langle x, e_n \rangle e_1 & \text{if } a_n = 1. \end{cases}$$

Then (E_n) is an orthogonal sequence of projections, U_n is an isometry on the range of E_n , U_n has adjoint $y \mapsto a_n \langle y, e_1 \rangle e_n$, and U_n is 0 on the kernel of E_n . Let $F_n = U_n U_n^*$. Then F_n is a projection equivalent to E_n relative to $\mathcal{B}(H)$, and F_m is orthogonal to F_n when $m \neq n$. The series $\sum_{n=1}^{\infty} U_n x$ converges for each x : for, given $\varepsilon > 0$ and choosing N such that $|\langle x, e_n \rangle|^2 < \varepsilon$ for all $n \geq N$, we see that $\sum_{n=N}^{N+k} \|U_n x\|^2 < \varepsilon$ for each $k \geq 1$. Thus

$$Ux = \left(\sum_{n=1}^{\infty} a_n \langle x, e_n \rangle \right) e_1$$

is a well-defined operator on H with bound 1. But, as is shown in Chapter 3, if U has an adjoint, then either $a_n = 0$ for all n or else there exists n with $a_n = 1$.

Next, suppose that $\sum_{n=1}^{\infty} F_n$ converges strongly to a projection F . Then

$$Fx = \sum_{n=1}^{\infty} a_n \langle x, e_1 \rangle e_1,$$

so $F e_1 = \sum_{n=1}^{\infty} a_n e_1$. Either $F e_1 \neq 0$ or $\|F e_1\| < 1$. In the first case, there exists n such that $a_n = 1$; in the second, $a_n = 0$ for all n .

5.3 Finite and infinite projections

We call a projection E

- **finite** (relative to \mathcal{R}) if $E \approx_{\mathcal{R}} F \leq E$ implies that $F = E$;
- **infinite** (relative to \mathcal{R})¹ if, for any finite orthogonal family $(E_n)_{n=1}^N$ of finite subprojections of E , there exists a projection F such that

$$\left(E - \sum_{n=1}^N E_n \right) \approx_{\mathcal{R}} F < \left(E - \sum_{n=1}^N E_n \right).$$

¹Our definition of “infinite” is not the same as, but is classically equivalent to, the standard classical definition; see [41], 6.3.1, p. 411. We have chosen this definition to facilitate the proofs of Propositions 5.3.3 and 5.3.4.

While it is clear that if E is infinite, then it is not finite, we cannot prove constructively that if E is not finite, then it is infinite. Let (a_n) be an increasing binary sequence such that $a_1 = 0$ and $\neg\forall n (a_n = 0)$. Let (e_n) be an orthonormal basis of unit vectors in an infinite-dimensional separable Hilbert space, and define a projection E by

$$Ex = \sum_{n=1}^{\infty} a_n \langle x, e_n \rangle e_n.$$

Suppose that E is finite relative to the von Neumann algebra $\mathcal{B}(H)$. If there exists n such that $a_n = 1 - a_{n-1}$, then E is infinite, being the projection on the span of $\{e_{n+1}, e_{n+2}, \dots\}$; hence $a_n = 0$ for all n , a contradiction. Thus E is not finite. However, if E is infinite, then there exists a subprojection F such that $E \approx F < E$; in particular, $E \neq 0$, so we can find n such that $a_n = Ee_n \neq 0$. Hence the proposition “not-finite implies infinite” for projections on $\mathcal{B}(H)$ entails Markov’s Principle.

Note that if $E \approx F \leq E$ entails $F = E$, then we need not have E finite-dimensional. Let $a \in \mathbf{R}$, and take $H = \overline{\mathbf{R}a} \times \mathbf{R}$. Let E be the projection on $\overline{\mathbf{R}a} \times \{0\}$, and suppose that $E \approx F \leq E$. Given $x \in E(H)$, suppose that $Fx \neq x$. Then $a \neq 0$, so $E(H) = \mathbf{R} \times \{0\}$ is 1-dimensional, and therefore $F = E$, a contradiction. Hence $Fx = x$ for all $x \in E(H)$, so $F = E$. Thus E is finite; but if it has finite-dimensional range, then we can decide whether $a = 0$ or $a \neq 0$.

Note that this Brouwerian example has the apparently stronger property that $F < UU^*$ for each $F < E$ and each partial isometry U with initial projection E and final projection $\leq E$.

Here are some useful elementary results about finite and infinite projections.

Proposition 5.3.1 *Let E be a finite projection in the von Neumann algebra \mathcal{R} . Then each subprojection of E in \mathcal{R} is finite, and if $E \approx_{\mathcal{R}} F$, then F is finite.*

Proof. See [41], 6.3.2, page 411. Q.E.D.

Proposition 5.3.2 *If E is an infinite projection in the von Neumann algebra \mathcal{R} , and $E \approx_{\mathcal{R}} F$, then F is infinite. If $F_1 < F$ and F_1 is infinite, then F is infinite.*

Proof. With the help of the preceding proposition, this is a simple exercise. Q.E.D.

Proposition 5.3.3 *If E is an infinite projection in \mathcal{R} , and P_1, \dots, P_ν are orthogonal finite projections, then $E - \sum_{n=1}^{\nu} P_n$ is infinite.*

Proof. Let Q_1, \dots, Q_m be orthogonal finite projections in \mathcal{R} such that $\sum_{n=1}^m Q_n \leq E - \sum_{n=1}^{\nu} P_n$. Then $P_1, \dots, P_\nu, Q_1, \dots, Q_m$ are orthogonal finite projections in \mathcal{R} , so, as E is infinite, there exists a projection F in \mathcal{R} such that

$$\left(E - \sum_{n=1}^{\nu} P_n \right) - \sum_{n=1}^m Q_n \approx F \leq \left(E - \sum_{n=1}^{\nu} P_n \right) - \sum_{n=1}^m Q_n.$$

Hence $E - \sum_{n=1}^{\nu} P_n$ is infinite. Q.E.D.

Proposition 5.3.4 *Let E_0 be a finite subprojection of an infinite projection E in a von Neumann algebra \mathcal{R} . Then $E - E_0 \neq 0$.*

Proof. We easily reduce to the case where $E_0 \leq E$. Then, by our definition of “infinite”, there exists a projection F such that $E - E_0 \approx F < E - E_0$. Hence $E - E_0 \neq 0$. Q.E.D.

5.4 The weak and strong equivalence of two projections

Classically, the ordering \preceq is antisymmetric: if $E \preceq_{\mathcal{R}} F$ and $F \preceq_{\mathcal{R}} E$, then $E \approx_{\mathcal{R}} F$. The proofs in [41] (6.2.4, pages 406–407) and [56] (page 41–42) are both nonconstructive: the first proof fails constructively when the infimum of a descending sequence of projections is used, since there is no guarantee that such an infimum exists; the

second proof fails when a certain fixed-point theorem on a partially ordered set is applied.

An analysis of the classical proof of antisymmetry of $\preceq_{\mathcal{R}}$ from a constructive point of view led us to the idea of considering another type of equivalence.

Definition 5.4.1 *We say that projections E and F in $\mathcal{B}(H)$ are **weakly equivalent** relative to the von Neumann algebra \mathcal{R} if $E \preceq_{\mathcal{R}} F$ and $F \preceq_{\mathcal{R}} E$.*

It is straightforward that strong equivalence implies weak equivalence. We shall prove that for a certain class of von Neumann algebras, $\preceq_{\mathcal{R}}$ is (strongly) antisymmetric—that is, weak and strong equivalence coincide.

Theorem 5.4.2 *Let \mathcal{R} be a von Neumann algebra with the following properties:*

- *Each projection in \mathcal{R} can be written as the sum of a sequence of orthogonal finite (possibly 0) subprojections in \mathcal{R} .*
- *Any two infinite projections in \mathcal{R} are equivalent.*

If $E, F \in \mathcal{R}$ and $E \preceq_{\mathcal{R}} F \preceq_{\mathcal{R}} E$, then $E \approx_{\mathcal{R}} F$.

Proof. Let U, V be partial isometries in \mathcal{R} such that

$$\begin{aligned} U^*U &= E, & UU^* &= F_1 \leq F, \\ V^*V &= F, & VV^* &= E_1 \leq E. \end{aligned}$$

Then VF_1 is a partial isometry in \mathcal{R} implementing an equivalence between F_1 and a subprojection E_2 of E ; and $V(F - F_1)$ is a partial isometry in \mathcal{R} implementing an equivalence between $F - F_1$ and $E_1 - E_2$. Choose an orthogonal sequence $(P_n)_{n=1}^{\infty}$ of finite projections in \mathcal{R} such that $E = \sum_{n=1}^{\infty} P_n$. Let (g_n) be an orthonormal basis of $(F - F_1)(H)$, and construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \forall k \leq n (g_k = 0), \\ \lambda_n = 1 &\Rightarrow \exists k \leq n (g_k \neq 0). \end{aligned}$$

If $\lambda_n = 0$, define $WP_n x = Ux$ for all $x \in H$. If $\lambda_\nu = 1 - \lambda_{\nu-1}$, then

$$E_1 - E_2 \approx_{\mathcal{R}} F - F_1 \neq 0,$$

so $E \neq E_2$, $E_2 < E$, and therefore, as $E \approx_{\mathcal{R}} E_2$, E is infinite. It follows from Proposition 5.3.2 that F_1 , and therefore F , is infinite. Also, the orthogonal projections $WP_1, \dots, WP_{\nu-1}$ are finite subprojections of F in \mathcal{R} , by Proposition 5.3.1; so, as F is infinite, $F - \sum_{n=1}^{\nu-1} WP_n$ is infinite, by Proposition 5.3.3. Likewise, $E - \sum_{n=1}^{\nu-1} P_n$ is infinite; so

$$E - \sum_{n=1}^{\nu-1} P_n \approx_{\mathcal{R}} F - \sum_{n=1}^{\nu-1} WP_n.$$

By Proposition 5.2.1, we have

$$E = \sum_{n=1}^{\nu-1} P_n + \left(E - \sum_{n=1}^{\nu-1} P_n \right) \approx_{\mathcal{R}} \sum_{n=1}^{\nu-1} WP_n + \left(F - \sum_{n=1}^{\nu-1} WP_n \right) = F. \quad \text{Q.E.D.}$$

Classically, the hypotheses of Theorem 5.4.2 are satisfied in a separable Hilbert space by any Type I von Neumann algebra, and any factor in which each projection can be written as the sum of a pairwise orthogonal sequence of finite subprojections (see [41], 6.3.5). This has not been proved constructively, but it is clear that, when H is separable, Theorem 5.4.2 applies in the case $\mathcal{R} = \mathcal{B}(H)$, the sort of von Neumann algebra to which a Type I von Neumann algebra is classically *-isomorphic ([41], 6.6.1).

In the last part of this chapter we study conditions under which two weakly equivalent projections are strongly equivalent.

Let U, V be partial isometries in \mathcal{R} such that $UU^* \leq V^*V$ and $VV^* \leq U^*U$.

Write

$$U^*U = E_0, \quad V^*V = F_0,$$

$$UU^* = F_1, \quad VV^* = E_1.$$

Setting $U_0 = U$ and $V_0 = V$, we construct, inductively, sequences $(U_n), (V_n)$ of partial isometries, and sequences

$$E_0 \geq E_1 \geq E_2 \geq \dots, \quad F_0 \geq F_1 \geq F_2 \geq \dots$$

of projections, such that for each n ,

$$U_n = UE_n, U_n^*U_n = E_n, U_nU_n^* = F_{n+1}$$

and

$$V_n = VF_n, V_n^*V_n = F_n, V_nV_n^* = E_{n+1}.$$

Routine computations show that

$$\begin{aligned} E_{2n} &= (VU)^n ((VU)^*)^n, \\ E_{2n+1} &= (VU)^n E_1 ((VU)^*)^n, \\ F_{2n} &= (UV)^n ((UV)^*)^n, \quad \text{and} \\ F_{2n+1} &= (UV)^n E_1 ((UV)^*)^n. \end{aligned}$$

Also, for $n \geq 1$,

$$\begin{aligned} U_{2n-1} &= (UV)^n ((UV)^*)^{n-1} V^*, \\ U_{2n} &= (UV)^n F_1 ((UV)^*)^{n-1} V^* \end{aligned}$$

and

$$\begin{aligned} V_{2n-1} &= (VU)^n ((VU)^*)^{n-1} U^*, \\ V_{2n} &= (VU)^n E_1 ((VU)^*)^{n-1} U^*. \end{aligned}$$

We will refer to sequences (E_n) and (F_n) as the **associated descending chains** for E_0 and F_0 , respectively.

Proposition 5.4.3 *The following statements are equivalent.*

- (i) $(E_n)_{n=0}^\infty$ is weak-operator convergent.
- (ii) $\sum_{n=0}^\infty (E_n - E_{n+1})$ is strong-operator convergent.
- (iii) $\sum_{n=0}^\infty U(E_n - E_{n+1})$ is strong-operator convergent.

Proof. Indeed, if (i) holds, then (E_n) is strong-operator convergent in \mathcal{R} , and it is easy to see that its strong-operator limit is

$$E_\infty = \bigwedge_{n=0}^{\infty} E_n,$$

the infimum of the sequence (E_n) of projections. Moreover, since

$$\sum_{n=0}^N (E_n - E_{n+1}) = E - E_{N+1},$$

the series in (ii) converges strongly to $E - E_\infty$. It is clear that (ii) implies (iii). If (iii) holds, then

$$\sum_{n=1}^{\infty} (F_n - F_{n+1}) = \sum_{n=0}^{\infty} U(E_n - E_{n+1})$$

converges strongly, as therefore does

$$\sum_{n=0}^{\infty} (E_n - E_{n+1}) = E - E_2 + \sum_{n=1}^{\infty} V(F_n - F_{n+1}).$$

Hence $(E - E_n)$, and therefore (E_n) , converges strongly. Thus (iii) implies (i). Q.E.D.

Proposition 5.4.4 *If E and F are weakly equivalent projections relative to \mathcal{R} , and the associated descending chain (E_n) satisfies one of the equivalent conditions (i)–(iii) of Proposition 5.4.3, then E and F are strongly equivalent.*

Proof. If any, and therefore each, of the equivalent conditions (i)–(iii) holds, then we can prove that $E \sim F$ as follows. The identity

$$E = \sum_{n=0}^N (E_{2n} - E_{2n+1}) + \sum_{n=0}^N (E_{2n+1} - E_{2n+2}) + E_{2N+2}$$

and the strong-operator convergence of E_{2N+2} to E_∞ as $N \rightarrow \infty$ show that

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N (E_{2n} - E_{2n+1}) + \sum_{n=0}^N (E_{2n+1} - E_{2n+2}) \right) &= \lim_{N \rightarrow \infty} (E - E_{2N+2}) \\ &= E - E_\infty. \end{aligned}$$

For all $x \in H$, since $\langle (E_{2n+1} - E_{2n+2})x, x \rangle \geq 0$, we have

$$\langle (E_{2n} - E_{2n+1})x, x \rangle \leq \langle ((E_{2n} - E_{2n+1}) + (E_{2n+1} - E_{2n+2}))x, x \rangle.$$

Hence the series $\sum_{n=0}^{\infty} \langle (E_{2n} - E_{2n+1})x, x \rangle$ of nonnegative terms converges in \mathbf{R} . Thus $\sum_{n=0}^{\infty} (E_{2n} - E_{2n+1})$ converges weakly and therefore, being a series of orthogonal projections, strongly. Likewise, the series $\sum_{n=0}^{\infty} (E_{2n+1} - E_{2n+2})$, $\sum_{n=0}^{\infty} (F_{2n} - F_{2n+1})$, and $\sum_{n=0}^{\infty} (F_{2n+1} - F_{2n+2})$ all converge strongly. So we can now apply Proposition 5.2.1 to complete the proof of the equivalence of E and F . Q.E.D.

Proposition 5.4.5 *Let U, V be partial isometries on H such that $UU^* \leq V^*V$ and $VV^* \leq U^*U$. If $((VU)^*)_{n=0}^{\infty}$ is strong-operator convergent in $\mathcal{B}_1(H)$ then the associated descending chain (E_n) is a strong-operator Cauchy sequence.*

Proof. For $m > n$ we have

$$\begin{aligned} \|E_{2n}x - E_{2m}x\|^2 &= \langle (E_{2n} - E_{2m})x, x \rangle \\ &= \langle E_{2n}x, x \rangle - \langle E_{2m}x, x \rangle \\ &= \|((VU)^*)^n x\|^2 - \|((VU)^*)^m x\|^2. \end{aligned}$$

So if $(\|((VU)^*)^n x\|)_{n=0}^{\infty}$ is a τ_s -Cauchy sequence, then so is $(E_{2n}x)_{n=0}^{\infty}$. Likewise, if $m > n$, then

$$\|E_{2n+1}x - E_{2m+1}x\|^2 = \|(E_1(VU)^*)^n x\|^2 - \|(E_1(VU)^*)^m x\|^2.$$

Since (E_n) is a Cauchy sequence if and only if both (E_{2n}) and (E_{2n+1}) are Cauchy sequences, the desired conclusion follows from the foregoing equalities. Q.E.D.

In conclusion, we have this result.

Proposition 5.4.6 *Let E and F be projections in a von Neumann algebra \mathcal{R} on a Hilbert space H . The following statements are equivalent.*

- (i) $E \approx_{\mathcal{R}} F$.
- (ii) *There exist partial isometries U and V such that $E = U^*U$, $F = V^*V$, $UU^* \leq F$, $VV^* \leq E$ and the sequence $((VU)^*)_{n=0}^{\infty}$ is strong-operator convergent in $\mathcal{B}_1(H)$.*

Proof. Suppose that $E \approx_{\mathcal{R}} F$, and let U be a partial isometry implementing the strong equivalence. Then (ii) holds true for $V = U^*$, and the sequence $((VU)^*)_{n=0}^{\infty}$ is constant (all terms equal E) and therefore convergent. Conversely, according to Proposition 5.4.4, the associated descending chain (E_n) is a strong-operator Cauchy, and hence convergent, sequence. A straightforward application of Proposition 5.4.3 completes the proof that (ii) implies (i). Q.E.D.

Appendix A

Weak Continuity Properties in Constructive Analysis

In this appendix we prove some results on continuity related to those in [36] and [18].

Let $f : (X, \rho) \rightarrow (Y, \rho)$ be a mapping between metric spaces. We say that f is **strongly extensional** if

$$f(x) \neq f(x') \Rightarrow x \neq x',$$

where¹, for example, $x \neq x'$ means that $\rho(x, x') > 0$.

A sequence (x_n) in a metric space X is said to be **weakly discriminating** if for all positive numbers α, β with $\alpha < \beta$, either $\rho(x_n, x_1) < \beta$ for all $n > 1$ or else $\rho(x_n, x_1) > \alpha$ for some $n > 1$. The constructive least-upper-bound principle shows that a bounded sequence (x_n) is weakly discriminating if and only if $\sup_{n \geq 2} \rho(x_n, x_1)$ exists. It follows from Corollary (4.4) on page 38 of [4] that every totally bounded sequence is weakly discriminating; in particular, every Cauchy sequence is weakly

¹Recall that the statement

$$\neg(x = y) \Rightarrow x \neq y$$

is equivalent to Markov's Principle

discriminating.

We begin with an improvement upon Ishihara's result that a sequentially continuous mapping on a complete metric space is strongly extensional ([36], Theorem 1).

Proposition A.0.1 *Let X, Y be metric spaces, and $f : X \rightarrow Y$ a function that maps Cauchy sequences to weakly discriminating sequences. Then f is strongly extensional.*

Proof. Let x, x' be points of X with $f(x) \neq f(x')$. Construct an increasing binary sequence (λ_n) such that

$$\begin{aligned}\lambda_n = 0 &\Rightarrow \rho(x, x') < 1/n, \\ \lambda_n = 1 &\Rightarrow x \neq x'.\end{aligned}$$

Note that if $\lambda_n = 0$ for all n , then $x = x'$ and so $f(x) = f(x')$, which is absurd. We show² that $\lambda_n = 1$ for some n ; we may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $\xi_n = x$; if $\lambda_n = 1 - \lambda_{n-1}$, set $\xi_k = x'$ for all $k \geq n$. Then (ξ_n) is a Cauchy sequence in X ; so $(f(\xi_n))$ is a weakly discriminating sequence in Y . Hence either

$$\rho(f(\xi_n), f(x)) = \rho(f(\xi_n), f(\xi_1)) < \rho(f(x'), f(x))$$

for all $n > 1$, or else there exists $n > 1$ such that $\rho(f(\xi_n), f(x)) > \frac{1}{2}\rho(f(x'), f(x))$.

In the first case we must have $\lambda_n = 0$ for all n , which is absurd. So the second case obtains, and there exists n such that $\rho(f(\xi_n), f(x)) > 0$. Then λ_n must equal 1.

Q.E.D.

Proposition A.0.2 *Let X, Y be metric spaces, and $f : X \rightarrow Y$ a function that maps convergent sequences to Cauchy sequences. Then f is sequentially continuous.*

²It is clear that $\neg\forall n(\lambda_n = 0)$; but without Markov's Principle we cannot immediately deduce from this that $\exists n(\lambda_n = 1)$.

The following result must be well-known classically, but we cannot find any reference to it in the literature.

Proposition A.0.3 *Let X, Y be metric spaces, and $f : X \rightarrow Y$ a function that maps convergent sequences to Cauchy sequences. Then f is sequentially continuous.*

Proof. Let (x_n) be a sequence converging to $x \in X$. Then the sequence

$$(x_1, x, x_2, x, x_3, x, \dots)$$

converges to x , so

$$(f(x_1), f(x), f(x_2), f(x), f(x_3), f(x), \dots)$$

is a Cauchy sequence. Since it contains a (constant) subsequence that converges to $f(x)$, the Cauchy sequence itself converges to $f(x)$, and hence $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$. Q.E.D.

A sequence (x_n) in a metric space X is called a³ **LEM–Cauchy sequence** if

$$\neg\neg((x_n) \text{ is a Cauchy sequence}).$$

Clearly, a Cauchy sequence is a LEM–Cauchy sequence.

If we assume that X is complete, then we can weaken the hypothesis of Proposition A.0.2.

Proposition A.0.4 *Let X be a complete metric space, and Y a metric space. Then the following conditions are equivalent on a mapping $f : X \rightarrow Y$.*

- (i) *f maps convergent sequences to weakly discriminating LEM–Cauchy sequences.*
- (ii) *f is sequentially continuous.*

³LEM stands for the law of excluded middle: $P \vee \neg P$.

Proof. Assuming (i), let (x_n) be a sequence in X converging to $x \in X$. In view of Proposition A.0.2, in order to prove that f is sequentially continuous at x , it is enough to prove that for each $\varepsilon > 0$ there exists ν such that $\rho(f(x_m), f(x_n)) < 4\varepsilon$ for all $m, n \geq \nu$. To this end, first note that for each positive integer N the sequence $(x_n)_{n>N}$ converges to x ; so, by hypothesis (i), the sequence $(f(x_n))_{n>N}$ is weakly discriminating. Using this observation and setting $n_1 = 1$, construct an increasing binary sequence (λ_k) , and an increasing sequence (n_k) of positive integers, such that

- if $\lambda_k = 0$, then $\rho(f(x_{n_{k+1}}), f(x_{n_k})) > \varepsilon$ and $n_{k+1} > n_k$;
- if $\lambda_k = 1$, then $\rho(f(x_n), f(x_{n_k})) < 2\varepsilon$ for all $n \geq n_k$, and $n_j = n_k$ for all $j > k$.

It suffices to find κ such that $\lambda_\kappa = 1$: for then there exists $k \leq \kappa$ such that $\rho(x_n, x_{n_k}) < 2\varepsilon$ for all $n \geq n_k$; whence $\rho(x_m, x_n) < 4\varepsilon$ for all $m, n \geq n_k$. Thus we may assume that $\lambda_1 = \lambda_2 = 0$. If $\lambda_k = 0$, set $\xi_k = \eta_k = x_{n_k}$. If $\lambda_k = 1 - \lambda_{k-1}$, set $\xi_j = x_{n_k}$ and $\eta_j = x_{n_{k-1}}$ for all $j \geq k$. Then $(\xi_k)_{k=1}^\infty$ and $(\eta_k)_{k=1}^\infty$ are Cauchy sequences in X , and therefore converge, respectively, to limits ξ and η in X . Either $f(\xi) \neq f(\eta)$ or else $\rho(f(\xi), f(\eta)) < \varepsilon$. In the latter case, if $\lambda_k = 1 - \lambda_{k-1}$ for some k , then

$$\rho(f(\xi), f(\eta)) = \rho(f(x_{n_k}), f(x_{n_{k-1}})) > \varepsilon,$$

a contradiction; so $\lambda_k = 0$, and therefore $\rho(f(x_{n_{k+1}}), f(x_{n_k})) > \varepsilon$, for all k . This is impossible, as $(f(x_{n_k}))_{k=1}^\infty$ is a LEM–Cauchy sequence. We conclude that $f(\xi) \neq f(\eta)$ and hence, by Proposition A.0.1, that $\xi \neq \eta$. Choosing κ such that $\xi_\kappa \neq \eta_\kappa$, we must have $\lambda_\kappa = 1$. This completes the proof that (i) implies (ii); the converse is trivial. Q.E.D.

For another characterisation of sequential continuity, see Theorem 1 of [36].

We next consider some criteria for the sequential continuity of functions on subsets of a normed space. For these, recall that a subset B of a normed space over \mathbf{F} (that is, \mathbf{R} or \mathbf{C}) is **balanced** if $\alpha x \in B$ whenever $x \in B$, $\alpha \in \mathbf{F}$, and $|\alpha| \leq 1$.

Proposition A.0.5 *Let X be a normed space over \mathbf{F} , B a balanced subset of X , and $f : B \rightarrow \mathbf{F}$ a function with the following properties.*

(i) *If (x_n) is a sequence converging to 0 in X , then*

$$\neg \exists \varepsilon > 0 \forall n (|f(x_{n+1}) - f(x_n)| \geq \varepsilon).$$

(ii) *$f(\alpha x) = \alpha f(x)$ whenever $x \in B$, $\alpha \in \mathbf{F}$, and $|\alpha| = 1$.*

Then f is sequentially nondiscontinuous at 0.

Proof. Note first that

$$f(0) = f(-1(0)) = -f(0),$$

so $f(0) = 0$. Let (ξ_n) be a sequence in B converging to 0, and ε a real number such that $|f(\xi_n)| \geq \varepsilon/2$ for each n . Suppose that $\varepsilon > 0$. Then we can find $\alpha_n \in \mathbf{F}$ such that $|\alpha_n| = 1$ and

$$f(\alpha_n \xi_n) = \alpha_n f(\xi_n) = |f(\xi_n)| \geq \frac{\varepsilon}{2}.$$

Setting $x_n = (-1)^n \alpha_n \xi_n$, we see that $x_n \rightarrow 0$ and

$$|f(x_{n+1}) - f(x_n)| = f(\alpha_{n+1} \xi_{n+1}) + f(\alpha_n \xi_n) \geq \varepsilon$$

for all n . Since this contradicts our hypotheses on f , we must have $\varepsilon \leq 0$. Q.E.D.

Corollary A.0.1 *Let X be a normed space over \mathbf{F} , B a balanced subset of X , and $f : B \rightarrow \mathbf{F}$ a function that maps convergent sequences to LEM–Cauchy sequences, such that $f(\alpha x) = \alpha f(x)$ whenever $x \in B$, $\alpha \in \mathbf{F}$, and $|\alpha| = 1$. The f is sequentially nondiscontinuous at 0.*

There are situations in constructive analysis—such as Banach’s Inverse Mapping Theorem [38]—in which a linear mapping between Banach spaces can be shown to

be sequentially continuous but we cannot be sure that it is bounded.⁴ Thus it is potentially useful to have conditions that enable us to pass from sequential continuity to boundedness for linear mappings. Such conditions were given in Proposition 2 of [10]. Our final result substantially improves that proposition by removing the hypothesis that the domain of the linear mapping be complete.

The **complement** of a set S in a metric space X is the set

$$\sim S = \{x \in X : x \neq s \text{ for all } s \in S\}.$$

Proposition A.0.6 *Let T be a sequentially continuous linear mapping of a normed space X into a normed space Y , and let B be the unit ball of the range of T . Then the following conditions are equivalent.*

- (i) T is bounded.
- (ii) $\sim T^{-1}(B)$ is bounded away from 0.
- (iii) For each $\varepsilon > 0$ either there exists $x \in \sim T^{-1}(B)$ such that $\|x\| < \varepsilon$, or else $\sim T^{-1}(B)$ is bounded away from 0.

Proof. It is straightforward to prove that (i) implies (ii); clearly (ii) implies (iii). Assuming (iii), construct an increasing binary sequence (λ_n) such that

- if $\lambda_n = 0$, then there exists $x_n \in \sim T^{-1}(B)$ such that $\|x_n\| < 1/n$;
- if $\lambda_n = 1$, then $\sim T^{-1}(B)$ is bounded away from 0.

In order to prove (ii), we may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $\xi_n = 0$; if $\lambda_n = 1 - \lambda_{n-1}$, set $\xi_k = x_{n-1}$ for all $k \geq n$. Then (ξ_n) is a Cauchy sequence in X . Since sequentially continuous linear maps preserve the Cauchy property [18], $(T\xi_n)$

⁴For strong indications that even for complete metric spaces there is a genuine gap, in constructive analysis, between sequential continuity and continuity, see Theorem 3 of [37] and Theorem 5 of [17].

is a Cauchy sequence in Y . So $(\|T\xi_n\|)$ is a Cauchy sequence in \mathbf{R} ; let s be its limit. Either $s > 0$ or $s < 1$. In the latter case, if $\lambda_n = 1 - \lambda_{n-1}$, then $\xi_k = x_{n-1} \in \sim T^{-1}(B)$ for all $k \geq n$, so $s = \|Tx_{n-1}\| \geq 1$, a contradiction. Hence $\lambda_n = 0$ for all n , and we have a sequence (x_n) such that $\|x_n\| < 1/n$ and $\|Tx_n\| \geq 1$ for all n . Since this contradicts the sequential continuity of T , we conclude that $s > 0$. Hence there exists N such that $T\xi_N \neq 0$. By Proposition A.0.1 above (recall again that, for linear mappings, sequential continuity preserves Cauchyness [18]), $\xi_N \neq 0$ and therefore $\lambda_N = 1$. Thus (iii) implies (ii).

It remains to prove that (ii) implies (i). Assuming (ii), choose $c > 0$ such that $\|x\| \geq c$ for all $x \in \sim T^{-1}(B)$. Consider $x \in X$ such that $\|x\| < c$, and suppose that $\|Tx\| > 1$. If $x' \in T^{-1}(B)$, then $Tx \neq Tx'$; since T preserves Cauchyness, it is strongly extensional, by Proposition A.0.1, and therefore $x \neq x'$. Hence $x \in \sim T^{-1}(B)$, and so $\|x\| \geq c$ —a contradiction. It follows that we must have $\|Tx\| \leq 1$ whenever $\|x\| < c$; whence T is bounded. Q.E.D.

Appendix B

Locating the range of an operator

A classical application of the monotone convergence theorem for operators shows that if $0 \leq T \leq I$, then the sequence $(T^{1/n})_{n=1}^{\infty}$ is strong-operator convergent to the projection P on the closure of $\text{ran } T$, the range of T . In constructive mathematics, not only is there no guarantee that the monotone sequence $(T^{1/n})$ is strong-operator convergent, but also we cannot be sure that the projection P exists; in fact, P exists if and only if $\text{ran } T$ is located. Not surprisingly, what we can say is this.

Theorem B.0.1 *Let T be an element of $\mathcal{B}(H)$ such that $0 \leq T \leq I$. Then $\text{ran } T$ is located if and only if the sequence $(T^{1/n})_{n=1}^{\infty}$ is strong-operator convergent. In that case, the strong-operator limit of $(T^{1/n})_{n=1}^{\infty}$ is the projection of H onto the closure of $\text{ran } T$.*

Note the obvious attempt at a Brouwerian counterexample to Theorem B.0.1: namely, the operator $T : \mathbf{C} \rightarrow \mathbf{C}$ defined by $Tx = ax$, where $0 \leq a \leq 1$. But if $(T^{1/n})$ is strong-operator convergent, then there exists α such that $a^{1/n}x \rightarrow \alpha x$ for all $x \in \mathbf{C}$. Taking $x = 1$, we get $a^{1/n} \rightarrow \alpha$. Either $\alpha > 0$, in which case $a > 0$, or else $\alpha < 1$ and therefore $a = 0$. This example shows that Theorem B.0.1 is the best we can hope for constructively.

For our proof of Theorem B.0.1 we need some information about the **functional calculus** for a selfadjoint operator T (see Chapter 7 of [4]). This is a pair consisting of a positive measure μ on \mathbf{R} , and a bound-preserving homomorphism $f \mapsto f(T)$ of $L_\infty(\mu)$ into a commutative algebra of operators that are “functions of T ”, such that

- if b is a bound for T , then μ is supported by the compact interval $[-b, b]$;
- if $f_n \rightarrow f$ in measure, then $f_n(T) \rightarrow f(T)$ strongly.

If T is a positive operator with a bound b , then μ is supported by $[0, b]$. Moreover, $\text{ran } T$ is located if and only if $\{0\}$ is μ -measurable, in which case the projection of H on the closure of $\text{ran } T$ is $\chi_{-\{0\}}(T)$, where $\chi_{-\{0\}}$ is the characteristic function of the complemented set¹ $(\{0\}, \{x \in \mathbf{R} : x \neq 0\})$; if also (r_n) is a sequence of positive numbers decreasing to 0 such that the complemented set

$$\llbracket |f| \leq r_n \rrbracket = (\{x : |f(x)| \leq r_n\}, \{x : |f(x)| > r_n\})$$

is μ -integrable for each n , then

$$\mu(\{0\}) = \lim_{n \rightarrow \infty} \mu(\llbracket |f| \leq r_n \rrbracket)$$

([8], Thm (4.6)).

Proof of Theorem B.0.1. Let $(\mu, f \mapsto f(T))$ be the functional calculus for T . Suppose that $\text{ran } T$ is located, and let P be the projection on its closure; then $\{0\}$ is μ -integrable, and $\chi_{-\{0\}}(T) = P$. For each positive integer n define $f_n : \mathbf{R} - \{0\} \rightarrow \mathbf{R}$ by

$$f_n(t) = \begin{cases} t^{1/n} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \text{ nor } t = 0. \end{cases}$$

¹For more on complemented sets and their role in the constructive theory of integration see Chapter 6 of [4].

Then $f_n(t) = t^{1/n}$ on a μ -full set, so $f_n(T) = T^{1/n}$. It is routine to show that (f_n) converges to $\chi_{-\{0\}}$ μ -almost everywhere. Hence $T^{1/n} = f_n(T) \rightarrow P$ strongly.

Suppose, conversely, that $(T^{1/n})$ converges strongly. The classical argument in [40] (5.1.5) shows that its strong limit P is a projection, and that $\ker T = \ker P$. Using an argument similar to that of [4] (Chapter 4, (5.9)), construct, for each n , a sequence $(p_{n,k}(T))$ of strict polynomials (ones without constant term) in T such that $p_{n,k}(T) \rightarrow T^{1/n}$ uniformly as $k \rightarrow \infty$. Then for each $x \in H$ we have

$$Px = \lim_{n \rightarrow \infty} T^{1/n}x = \lim_{n,k \rightarrow \infty} p_{n,k}(T)x \in \overline{\text{ran } T}.$$

Since $\text{ran } P + \ker P$ is dense in H , it follows that $\text{ran } T + \ker T$ is dense in H ; whence, by [8] (Lemma 1), $\text{ran } T$ is located. Q.E.D.

We end with an interesting, although longer, proof that if $(T^{1/n})$ converges strongly, then $\text{ran } T$ is located. For this proof we require the following special case of a result from [19]:

Let T be a selfadjoint operator on H such that $\ker T$ is located, and let P be the projection of H onto $(\ker T)^\perp$. Suppose that for any bounded sequence (x_n) in H , if (Tx_n) converges weakly to 0, then (Px_n) converges weakly to 0. Then $\text{ran } T$ is located.

First construct the strict polynomials $p_{n,k}(T)$ as before, noting that they can be chosen to have real coefficients and therefore be selfadjoint. Let (x_n) be a sequence in the unit ball of H such that (Tx_n) converges weakly to 0, and consider any $y \in H$. Given $\varepsilon > 0$, choose N such that $\|Py - T^{1/N}y\| < \varepsilon$. Then choose k such that $\|p_{N,k}(T) - T^{1/N}\| < \varepsilon$. Since $p_{N,k}(T)$ is a strict polynomial, we see that

$$\lim_{n \rightarrow \infty} |\langle p_{N,k}(T)x_n, y \rangle| = 0.$$

Hence

$$|\langle Px_n, y \rangle| = |\langle x_n, Py \rangle|$$

$$\begin{aligned}
&\leq |\langle x_n, p_{N,k}(T)y \rangle| + |\langle x_n, (p_{N,k}(T) - T^{1/N})y \rangle| \\
&\quad + |\langle x_n, (T^{1/N} - P)y \rangle| \\
&\leq |\langle x_n, p_{N,k}(T)y \rangle| + \varepsilon \|y\| + \|T^{1/N}y - Py\| \\
&< |\langle p_{N,k}(T)x_n, y \rangle| \varepsilon + \varepsilon \|y\| + \varepsilon \\
&< (2 + \|y\|) \varepsilon
\end{aligned}$$

for all sufficiently large n . Hence (Px_n) converges weakly to 0, and therefore $\text{ran } T$ is located in H .

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