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BOUNDARY ALGEBRA: A SIMPLE NOTATION FOR BOOLEAN ALGEBRA AND THE TRUTH FUNCTORS

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WORKING PAPER

No. 02/2007

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May 2007

Abstract

Boundary algebra [BA] is a simpler notation for Spencer-Brown's (1969) *primary algebra* [pa], the Boolean algebra **2**, and the truth functors. The primary arithmetic [PA] is built up from the atoms, '()' and the blank page, by enclosure between '(' and ')', denoting the primitive notion of *distinction*, and concatenation. Inserting letters denoting the presence or absence of () into a PA formula yields a BA formula. The BA axioms are "()()=()" (A1), and "(()) [=⊥] may be written or erased at will" (A2). Repeated application of these axioms to a PA formula yields a member of B= {(),⊥} called its simplification. If (a)b [dually (a(b))] ⇔ $a \le b$, then $\perp \le ()$ [$() \le \perp$] follows trivially, so that B is a poset. (a) has two intended interpretations: (a) ⇔ a' (Boolean algebra **2**), and (a) ⇔ ~a (sentential logic). BA is a self-dual notation for **2**: () ⇔ 1 [0] so that B is the carrier for **2**, and ab ⇔ $a \cup b$ [$a \cap b$]. The BA basis abc=bca (Dilworth 1938), a(ab)= a(b), and a()=() (Bricken 2002) facilitates clausal reasoning and proof by calculation. BA also simplifies the usual normal forms and Quine's (1982) truth value analysis. () ⇔ *true* [*false*] yields *boundary logic*.

Keywords: G. Spencer Brown, boundary algebra, boundary logic, primary algebra, primary arithmetic, Boolean algebra, calculation proof, C.S. Peirce, existential graphs.

Acknowledgement.

This paper revises and extends Meguire (2003). In addition to revisions of detail everywhere, §§3.4, 4.3, 5.5, 6.3, and the first Appendix of this paper are new, and §§2.3, 3.3, 5.0-6.2, and the second Appendix are substantially revised. I have renumbered §3.4 as §4.1, §6.0 as §6.1, and moved material from §§3.1, 5.0, 5.2, and 6.1 to §§4.4, 5.4, and 6.2-3. This version, unlike the earlier one, does not discuss the monadic predicate calculus.

I owe my awareness of Peirce's importance for boundary methods to Kauffman (2001), who kindly hosted me for a week at the University of Waterloo. I thank William Bricken, Howard DeLong, Michael Pittarelli, and two anonymous referees for finding errors, Rolf Eberle for a careful reading, and Art Collings, David Glynn, Ivor Grattan-Guiness, and Joao Leao for encouragement. To Bricken I owe my appreciation of boundary algebra's planar nature, such that the commutativity and associativity of juxtaposition are metalinguistic devices. Simona Vita translated one of the references. Finally, I thank the inventors of Google, Sergey Brin and Larry Page, for keeping access to Google free of charge, making possible many of the intellectual connections discussed herein.

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Biography.

Philip Meguire (Ph.D., University of Chicago) is an academic economist whose formal study of mathematics ceased with a 1971 introduction to linear algebra. At age 13, he chanced on a reprint of Shannon (1938) and the *Encyclopedia Brittanica*'s article on Frege, Peano, and *Principia Mathematica*, thus discovering the enchanted realm of mathematical logic. He still has the paperback copy of *Laws of Form* he acquired in 1974, and was for a time the only New Zealand subscriber to the *Transactions of the C. S. Peirce Society*. He wishes that mereology and Peirce's beta graphs were better known.

1. Introduction.

"No one should fear that the contemplation of characters will lead us away from the things themselves; on the contrary, it will lead us into the interior of things. For nowadays our notions are often confused because the characters we use are badly arranged, but with the aid of characters we will easily have the most distinct notions, for we will have at hand a mechanical thread of meditation, as it were, with whose aid we can easily resolve any idea whatever into its components."

Leibniz (1969: 193).[1]

"…to unfold all truths of mathematics down to their ultimate grounds, and thereby provide all concepts of this science with the greatest possible clarity, correctness, and order, is an endeavour which will not only promote the thoroughness *of education but also make it* easier."

Bolzano, *Considerations on Some Objects of Elementary Geometry*, written in 1804 and republished in Ewald (1996: 172). Emphasis in original.

"Symbols have the same importance for thought that discovering how to use the wind to sail against the wind had for navigation. Thus, let no one despise symbols! A great deal depends on choosing them properly… without symbols we would scarcely lift ourselves to conceptual thinking." Frege (1972: 84), writing in 1882.

"By relieving the brain of… unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of [humanity]."

Whitehead (1948: 39), quoted in Roberts (1973: 118).

"*...a proper notation is like a live teacher, gently guiding us into the clear and keeping us from error and wooliness. A real effort should be made to express* [logical] *principles in as perspicuous a notation as possible*." Martin (1978: 41).

In his *Laws of Form*[2] (hereinafter *LoF*), in print since 1969, George Spencer-Brown[3] proposed a minimalist formal system, called the primary arithmetic, arising from the primitive mental act of making a distinction. He reached the next rung on the ladder of abstraction by letting letters denote, indifferently, a distinction or its absence, resulting in the primary algebra. The primary arithmetic and algebra featured a single primitive symbol —'⏋' in *LoF* and '()' here— indicating the *boundary* between the two states generated by a distinction. *Boundary algebra* (BA), a phrase coined in Meguire (2003), unites the primary arithmetic and algebra and modifies the notation somewhat. BA has two intended interpretations: the Boolean algebra **2** (i.e., one whose carrier has cardinality two; cf. Halmos and Givant 1998: 55) and classical bivalent sentential logic (hereinafter the

1. Letter to Tchirnhaus, dated May 1678, quoted in translation by Ishiguro (1990: 44).

2. Page numbers and the like refer to the 1972 American paperback edition.

3. George Spencer-Brown entered Cambridge in 1947, the year Wittgenstein resigned his Chair; hence he cannot have studied under Wittgenstein (who only taught advanced classes), as some claim. Graduated with Honours in philosophy and psychology, then taught philosophy at Oxford, 1952–58, from which (as well as from Cambridge) he obtained an M.A. Published *Probability and Scientific Inference* in 1957. Taught mathematics in the University of London's extramural program, 1963–68, and has held visiting appointments at Maryland, Stanford, and Western Australia. Source: http://www.lawsofform.org/gsb/vita.html .

calculus of truth values[4], CTV). The latter interpretation I call *boundary logic*, a phrase Bricken apparently coined in the 1980s. This paper stems from my admiration of the simplicity and elegance of boundary logic.

The primary algebra, as set out in *LoF*, consists of 2 axioms, 11 consequences in the form of equations, the usual rules governing uniform replacement and the substitution of equals for equals, and 9 "canons." There are 18 (meta)theorems, including but not limited to the standard metatheory of the CTV: soundness, completeness, and postulate independence. Definitions are informal. All this fills less than 55 small pages of large type.[5] The balance of the book consists of 20pp of front matter, a chapter claiming that certain recursive Boolean equations have "imaginary" solutions, and 60pp of notes and appendices relating the primary algebra to the CTV as it was understood circa 1940, and to elementary Boolean algebra and syllogistic logic. Much of this peripheral material is frankly speculative and digressive. I do not evaluate here *LoF*'s claim that one can usefully think of certain recursive Boolean equations as having "imaginary" solutions.

I take it as given that notational innovation can facilitate the teaching of extant mathematics, and the invention of new mathematics. Since the dawning of modern logic in 1847, three notations for the truth functors have acquired a significant following:

- The notation begun by Boole, revised by C. S. Peirce, and systematised by Schröder in the 1890s. This became the Boolean algebra **2**, with the following arithmetic/set theory/logic interpretations: $a+b-ab$ / $a \cup b$ / $a \vee b$, $a \times b$ / $a \cap b$ / $a \wedge b$, and $1-a$ / $\overline{a}$ / $\sim a$. Boolean algebra can be cast in terms of equations and inequalities ('≤' interprets the conditional) with unknowns. A classic treatment is Lewis (1918: chpts. II, III); for a thorough modern exposition, see Rudeanu (1974);[6]
- The binary prefix (Polish) notation introduced by Lukasiewicz in the 1920s. Fully exploiting the fact that formulae are ordered trees, this notation is free of the ambiguities that plague its infix rivals and so enjoys the advantage of requiring no brackets;
- The standard notation for first order logic, originated by Peano and modified by Whitehead and Russell in their *Principia Mathematica* (*PM*). While the notation of *PM*

4. Synonyms for the CTV include Bostock's (1997) *logic of truth functors*, *sentential calculus* (Kalish et al 1980), *logic of connectives* (LeBlanc and Wisdom 1976), *propositional calculus* (Church 1956; Mendelson 1997; Halmos & Givant 1998; Cori and Lascar 2000), *propositional logic* (Smullyan 1968; Epstein 1995; Wolf 1998; Hodges 2001), *statement calculus*, (Stoll 1974), *Propcal* (Machover 1996), *truth functional logic* (Hunter 1971; Quine 1982), the *theory of deduction* (Whitehead and Russell 1925; hereinafter *PM*).

5. Smullyan (1968: chpts. I, II) covers the same ground as *LoF* in 27 pages. Nidditch (1962), Goodstein (1963: chpt. 4), and Mendelson (1997: chpt. 1) require 81, 17, and 35 pages, respectively, and include the Deduction Theorem. Hunter (1971: §§15-36), Cori & Lascar (2000: chpt. 1, §§4.1-2), Stoll (1974: §§2.1-4, 3.5), Machover (1996: chpt. 7), Epstein (1995: §§II.J-L), and Schütte (1977: chpt. I) require 79, 62, 43, 40, 31, and 12 pages, respectively. Some treatments include compactness as well as consistency and completeness.

6. Boolean algebra is distinct from Boole's "algebra of logic," on which see Kneebone (1963: 184-88) and Lewis (1918: §I.V), in good part because Boole's alternation was exclusive, not inclusive. Hailperin (1986: 140) argues that Boole's algebra is a "commutative ring with unit, having neither additive nor multiplicative nonzero nilpotents." On Boolean algebra, see the references in the Bibliographic Postscript.

had canonical status during much of the 20th century, a variant due to Hilbert, Tarski, and their students, one employing '∧', '→', and '↔' in place of the '.', '⊃', and '≡', and brackets instead of the dots of Peano and *PM*, has largely displaced it. I make free metalinguistic use of this notation herein.[7]

Boundary algebra is, arguably, a notational innovation of the same order as the above. The notation of *LoF*, however, cannot be reproduced using standard word processing software. Hence I employ an alternative notation employed by Bricken and others to discuss *LoF*, a notation in which Latin letters, '()', and the blank page are atomic. I believe that Croskin (1978: 187) was the first to employ this notation, a mild variant of one in Peirce (4.378-383, 1902).[8] The syntax of the primary arithmetic consists of balanced parentheses strings. Inserting Latin letters into a primary arithmetic formula yields a primary algebra formula.

The domain of analysis is the numbers, real and complex; that of geometry, space, Euclidian and otherwise. Algebra studies mathematical structure without commiting to a specific domain. Logic studies the mathematical structure of statements and deductive systems so that, as Peirce and Halmos and Givant (1998: §§35-39) have maintained, logic can be viewed as an application of algebra. Boundary logic is very much a case in point.

My discussion comes under four headings: syntax, semantics, proof, and the history of ideas. My purpose is mainly expository, in that much of what I say has already appeared somewhere in the literature. *LoF* was written as if the mathematics it advocates were wholly new, when in fact, Spencer-Brown was reinventing the syntactic wheel. Kauffman (2001) shows how the notation of *LoF* was anticipated by C. S. Peirce—in papers written in 1886 but not published in full until 1993—and Nicod (1917). Kauffman also touches on Peirce's *alpha* existential graphs, discussed in §6, whose semantics and proof theory are very much in the spirit of BA. In §7, I give possible reasons why the mathematics and logic of *LoF* have made no headway since *LoF* was first published nearly 40 years ago.

This paper includes a "Precis of Mathematical Logic" (hereinafter "Precis"), for the benefit of readers lacking prior exposure to formal logic. I assume the reader has an intuitive grasp of elementary set theory, including the notion of function. *LoF*, however, is utterly innocent of set theory, other than a brief mention, near the end and free of all rigor, of the Boolean algebra of sets.

2. The Primary Arithmetic (PA).

"*The theme of* [LoF] *is that a universe comes into being when a space is severed or taken apart… By tracing the way we represent such a severance, we can begin to reconstruct… the basic forms*

7. On the importance of Peano's notational innovations, see Quine (1995: §28). Kneebone (1963: 49-51, §6.4, 87) discusses Polish notation and the notation of Frege (never emulated). The notations '∀*x*' and '∃*x*' descend from the '∏*x*' and '∑*x*' of Peirce (*W*5: 162-90) and his student Mitchell, a notation the Poles adopted. While Prior (1962) and Zeman (1973) adopted Polish notation, and texts still mention it, it appears to have died out. The syntactic ('⊢') and semantic ('⊨') turnstiles should be seen as part of the standard metalanguage.

8. To Bricken I owe my awareness of Croskin's work. Here and elsewhere, I cite Peirce's *Collected Papers* (Peirce 1931-35) in the following standard manner: *x.y*, *z* refers to a passage, first published or written in year *z*, reprinted in section *y* of volume *x* of the *Collected Papers*.

underlying linguistic, mathematical, physical, and biological science, and can begin to see how the familiar laws of our experience follow inexorably from the original act of severance."

LoF, p. v.

"*A common* image schema *of great importance in mathematics is the* Container schema [having] *three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides. The structure is topological in… that the Boundary can be made larger, smaller, or distorted and still remain the Boundary of a Container schema*." Lakoff & Núñez (2001: 33).[9]

In the beginning there is a *space*, normally a plane surface, that is featureless but upon which *symbols* (a primitive notion) may be inscribed. We are notably interested in the symbols '(' and ')'. Acting in tandem, they divide the space into two parts, one inside or between '(' and ')' and the balance being outside '(' and ')'. A *sign* is one or more symbols representing some human intention. The sign '()' marks the *boundary* between these two parts of the space. Letting *x* be a token or marker, *x* can be inside a boundary, as in '(*x*)', or outside, as in '()*x*'. Each side of a boundary forms one of a pair of mutually exclusive and exhaustive entities. '()' can also indicate either member of this pair of entities, in which case '()' signifies a boundary's *content* as well as its *fact*. A boundary '()' has meaning to the extent one wishes to distinguish that which is inside '()' (which may be nothing) from the remainder of the space. 2.0.1 attempts to codify these admittedly enigmatic ruminations:

2.0.1. Definition. The sign '()' inscribed in some *space* both signifies a *boundary* and denotes the *marked state*. Any space on which '()' does not appear represents the *unmarked state*.

The unmarked state can also be called "not ()", "nothing", "the void"; I will grant it a symbol shortly. The formal system whose sole atoms are '()' and "the void" is the *primary arithmetic* (*LoF* also employs the term "calculus of indications"), which I abbreviate to PA. *Boundary logic* begins by interpreting '()' as one of *true* or *false*. (I will often refer to the usual notation for first order logic as *conventional logic*.)

Now consider a plane surface, of indefinite extent, on which the following four symbols (not necessarily interpreted or related to each other) are inscribed, in the pattern shown:

```
#          ∴
     ϶
∴          ♦
```

Figure 1.

Note that '∴' appears twice. If one wished to represent Figure 1 more concisely, one could assert that the symbols in that Figure form a list and write '#∴϶∴♦', without separators such as commas. Note that multiple instances of the same object, namely '∴', are allowed. One could also depict Figure 1 as a set and write {#,϶,∴,♦}, keeping in mind

9. *Container schema* and *image schema* are terms of art in cognitive science.

that a set cannot have multiple instances of any of its members; hence the set corresponding to Fig. 1 has four members but the list has five elements.

Recall that for both lists and sets, the order in which elements are listed is immaterial. Hence when using list or set notation, the elements can be permuted at will without affecting meaning, consistent with the symbols having no necessary relation to each other. BA should be thought of as consisting of spatial arrangements such as Figure 1. In order to save space, and deferring to typographical custom in mathematics and conventional logic, I represent such arrangements as lists.

A boundary can be depicted by any closed curve that does not intersect itself, and with a distinguishable 'inside' and 'outside'. Boundaries can also be nested at will. The objects of BA should be seen as inscribed on a surface of dimension at least 2. All objects on a given side of a boundary, other boundaries excepted, have equal status. Hence boundary purists deem jejune the algebraic notions of commutativity and associativity when these are applied to BA. I do invoke these notions, but only as metalinguistic manners of speaking, doing so mainly out of deference to readers accustomed to conventional notations for logic and Boolean algebra. I revisit this syntactic curiosity in §3.2.[10]

2.1. PA: Syntax.

"*Although some material may be very familiar, it should be remembered that one of our main themes is the development of new perspectives for familiar concepts. Hence… these concepts* [should] *be re-appraised, and explicit discussion be provided of things that to many will have become second nature*." Goldblatt (1984: 4).

2.1.1. Definition. A PA *symbol* is an instance of '(', ')', '⊥', and '='. 'Symbol' is otherwise undefined. The symbol '⊥' is *improper*; the remaining symbols are *proper*.

2.1.2. Definition. A PA *string* consists of a single symbol, or of two or more symbols juxtaposed.[11] A PA *formula* is a string constructed recursively as follows:

Base case. Any *atomic formula*. These are the string '()', the blank page, and the space between symbols.

Recursive rule. If 'α' and 'β' are formulae, '(α)' and 'αβ', i.e., 'α' and 'β' juxtaposed, are formulae. This rule may be repeated any finite number of times.[12]

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10. Kauffman (2001) uses the term *boundary mathematics* to refer to formal systems with one or more syntactical boundaries, having mathematical as well as logical interpretations. Bricken (2001), building on James (1993), uses boundary notation to explore analysis (e.g., by representing e^X as (x)) and the integers, rationals, reals, etc. 'Boundary,' as employed herein, is unrelated to the use of that term in topology. At the same time, I do not wish to deny possible links among boundary mathematics, topology, and the *mereotopology* of Simons (1987: §2.10) and Casati and Varzi (1999: chpts. 4,5); see fn 60. By grounding BA in the "mental act" of drawing a distinction I distance boundary logic, for good or ill, from Kneebone's (1963: §12.2.1) reading of *PM*, in which "…logic deals with propositions, not with mental acts; and it follows that… mathematics likewise is essentially propositional."

11. 'Juxtaposition' (a term I appropriate from Hehner 2004) and 'concatenation', terms which I deem synonymous even while not defining them, are not mathematically trivial; see Halmos and Givant (1998: §12).

I revisit the blank page as formula in §2.2. 2.1.2 introduces an important notational convention: *Greek letters are metalogical symbols standing for arbitrary formulae or strings*. (N.B. *LoF* also invokes this convention, albeit silently.)

2.1.3. Definition. If the formula ‘α’ can be obtained by applying the recursive rule in 2.1.2 to the formula ‘β’ one or more times, then ‘β’ is a *proper sub-formula* of ‘α’. A *subformula* of ‘α’ is either ‘α’ itself or a proper subformula of ‘α’.

LoF has no synonym for subformula, a lacuna giving rise to occasional awkward periphrases. ‘String’ and ‘formula’ are not intrinsic to BA. I employ these words mainly to facilitate expositing BA to those who have learned standard formal systems. ‘String’ and ‘formula’ are synonyms for ‘arrangement’ and ‘expression,’ which *LoF* employs without defining. I will write ‘arrangement’ only when quoting *LoF*.

In everyday language, a formula must satisfy the rule: reading from left to right, *any left parenthesis must eventually be paired with a right parenthesis*. A string satisfying this rule is known as a *balanced parenthesis string*. The following algorithm determines whether a given string is balanced and hence a formula.

2.1.4. Algorithm.
a) Let *d* be a counter variable, and initialise it to 0.
b) Starting from one end of the string and working towards the other end, increase *d* by 1 for each ‘(’ and reduce *d* by 1 for each ‘)’.
c) IF *d* is ever positive [negative], it must always be nonnegative [nonpositive].
d) ELSE the string is not a formula. STOP
e) ELSE IF *d* is nonzero when the end of the string is reached, the string is not a formula. STOP
f) ELSE the string is a formula.
End of Algorithm

2.1.4 is required only because not all possible strings involving only ‘(’ and ‘)’ are well-formed. This is a (minor, I trust) drawback of the notation I propose. This problem does not arise with the notation of *LoF*, its signal virtue. That notation is based on the symbol ‘⏋’, called the *mark*, and placed to the *right* and *over* that which ‘()’ encloses. For instance, ‘⏋⏋⏋’ in *LoF* corresponds to ‘(((()())()))’ here. All possible concatenations and nestings of ‘⏋’ are well-formed, as long as the upper part of any ‘⏋’ extends over the left extremity of all ‘⏋’ under it.

12. I crafted 2.1.2 so as to resemble the recursive formula definitions in conventional treatments of logic and other formal systems; e.g., Bostock (1997: 21) and Machover (1996: §7.1.2). In linguistics and computer science, 2.1.2 defines a *Dyck language of order 1* with a *null alphabet*, the simplest instance of a Chomskian *context-free language* (Davis et al 1994: §10.7).

2.1.5. Definition. When applying 2.1.4 to 'α', the absolute value of the counter d at any point inside 'α' is the *depth* of 'α' at that point. Henceforth, d refers to this absolute value. The place in 'α' where d attains its largest value is the *greatest depth,* d^*_α, of 'α'.[13]

2.1.6. Definition. Corresponding to every value of d is a *subspace* s_d. The subspace s_d *pervades* any sub-formula situated at depth d. Given some formula 'α', the subspace of depth 0, s_0, is the *pervasive space* of 'α'. Let 'β' be a sub-formula occurring at depth d of 'α'. Then the subspace s_d *pervades* 'β' (is the *pervasive space* of 'β').

'()' marks the boundary between the subspace "inside" and the pervasive space "outside." Each subspace s_j contains all subspaces s_k, for which $i<k\leq d^*$. Hence a formula creates a system of d^* nested subspaces. (The terms *space* and *subspace* are also fundamental to analysis and linear algebra, but this is coincidental.) A subspace can pervade more than one sub-formula, a fact giving rise to the following definition:

2.1.7. Definition. Given a formula of the form '…((β)(γ)…)…', the sub-formulae '(β)', '(γ)', etc. constitute *divisions* of the subspace containing '(β)', '(γ)', etc, and these sub-formulae are said to *divide* the subspace.

I now explicate '⊥':

2.1.8. Definition. The symbol '⊥' represents the *null formula* and designates "the unmarked state", "nothing", "the void".

'⊥' is a sub-formula of every formula, so that the null formula is a formula in the same sense that the empty set is a set.[14]

2.1.9. Definition. () and ⊥ are the *primitive values* of BA. B={(),⊥}.

B is the defining set of BA. In Boolean algebra, B is known as the *carrier*.

LoF invokes the Principle of Relevance (cf. §3.1) to argue that there is no need for a symbol to denote the unmarked state or the null formula: "...a recessive value is common to every [PA formula] and... by this Principle, has no necessary indicator there" (*LoF*, p. 43). Although *LoF* is silent about the null formula, it (pp. 15-18, 37, 47-48, 56-57) repeatedly employs '*n*' to refer to the unmarked state; the semantics of '*n*' and '⊥' are identical. The counterpart to '*n*' is '*m*', denoting the marked state. *n* and *m* are presumably metalinguistic.

By including the symbol '⊥' in BA, I defer to established usage in logic and Boolean algebra, which all feature a symbol akin to '⊥'. Moreover, '=' with nothing to one side (and strings of this form do occur in *LoF*) leaves the mind guessing at a possible typographic-

13. A BA formula can be seen as a *finite ordered tree* (Smullyan 1968: 4-5), whose *level* corresponds to depth in the sense of 2.1.5. *LoF* operationalizes formula depth in a different manner.

14. 'T' ('top') and '⊥' ('bottom') are standard notation for the lattice bounds (3.3.5), and are the primitives of Hehner's (2004) binary algebra. '⊥' is also analogous to Bostock's (1997: 12-13) *empty sequent*, false by definition.

al problem. '⊥' is a placeholder like the number 0; I created it in good part simply out of respect for the dyadic character of '='. The controversy surrounding the role of '⊥' in BA is analogous to that surrounding a possible role for the null individual in mereology, the formal theory of the relation of part to whole. For a review of this controversy, see the Appendix titled "The Controverted Ontology of the Null Individual." It would seem that both controversies stem from assuming that a name necessarily refers to at least one thing, when in fact the null individual and '⊥' are names that do not denote. They refer not to things, but to "nothing" and "the unmarked state," respectively.

2.2. PA: Axiomatics, Simplification, Semantics.

"I have for a long time been urging… the importance of demonstrating all the secondary axioms… by bringing them back to axioms which are primary, i.e., immediate and indemonstrable…" Leibniz (1996: 408).

The *LoF* axioms are:

A1. ()() = () A2. (()) = ⊥.

In Spencer-Brown's inimitable Zen-like words (*LoF*, pp. 1-2):
A1. The value of a call made again is the value of the call. *Calling*
A2. The value of a crossing made again is not the value of the crossing. *Crossing*

A2 is arguably self-evident, A1 perhaps less so. I ignore the distinction *LoF* draws between "axiom," meaning Calling and Crossing (also referred to as *Number* and *Order*) stated in natural language, and the corresponding "arithmetic initials" ()()=() and (())=⊥.

A1 and A2 reveal that '()' has an "inside" distinguishable from its "outside" by virtue of what each does to another instance of '()' with which it is in contact. A1 lays down that the exterior is idempotent; A2, that the interior is nilpotent. A1 and A2 can also be seen as defining '(' and ')' as the two halves of a single operator, with A1 [A2] being the defining property of the convex [concave] side of a parenthesis. When a pair of parentheses enclose a sub-formula, the pair functions as an (unary) operator; the subformula—which may be no more than an instance of '()'—is the corresponding operand. Note that '(())' has an interior while the constant '⊥' does not. This is the sense in which '⊥' is not a mere redundant synonym for '(())'.

Pp. 104-06 of *LoF*, perhaps the most sweeping and poetic pages of the book, were written to lead the reader to a deeper understanding of the plausibility of A1 and A2. These pages include the following enigmatic passage:

> "It seems hard to find an acceptable answer to the question of how or why the world conceives a desire, and discovers an ability, to see itself, and appears to suffer the process. That it does so is sometimes called the *original mystery*. Perhaps, in view of the form in which we presently take ourselves to exist, the mystery arises from our insistence on framing a question where there is, in reality, nothing to question. However it may appear, if such desire, ability, and sufferance be granted, the state or condition that arises as an outcome is, according to the laws here formulated, absolutely unavoidable." *LoF*, pp. 105-06; emphasis in original.

Spencer-Brown apparently saw A1 and A2 as ineluctable features of the universe, material as well as abstract, and of how humans interact with it.

Croskin (1978) concludes that A1 and A2 assert a one-to-one mapping between *B* and '()()' and '(())', the two simplest nonatomic formulae. I prefer to see A1 and A2 as instances of the sort of arbitrary defining choices that ground all formal systems. '()()' and '(())' could just as well be mapped onto any member of {⊥, (), ()(), (())}, giving rise to 4^2=16 possible pairs of axioms, one pair being (A1, A2) and four pairs consisting of the same axiom iterated. Exploring the mathematical significance of the 11 remaining axiom pairs I leave to future research.

A1 and A2 do not make explicit what value to assign to a formula containing both '⊥' and parentheses. *LoF* is not to blame for this lacuna, because it has no null formula and so blithely states A2 with nothing to the right of '='. I propose to remedy this omission in either of two ways:

1. Invoke a third axiom making explicit that when '⊥' is combined with '()' in any way, nothing is altered:

A3. ⊥() = (⊥) = ()⊥ = (); ⊥⊥ = ⊥.

Or, in the Zen-like manner of *LoF*:

A3. The void can only be a sign of itself.

2. Restate A2 as follows: *An instance of* '(())' *or* '⊥' *may be written anywhere or erased at will*. I submit that a generous reading of *LoF* points to this definition. The four cases covered by A3 above then all follow.

Under either approach, '⊥' can appear anywhere in a formula without affecting its meaning or value. '⊥' is a synonym for '(())' and as such is, in all essentials, optional. Hence parentheses alone suffice to build any PA formula. A2 as restated above has a curious and deeper consequence. Since '(())' aliases with the blank page, and since by 2.1.9 '(())' is a primitive value, the following three things stand for the same atomic formula and can denote the same primitive value: the "space" between any two juxtaposed symbols, an entire blank page, and any blank part thereof. We shall see in §6.1 that Peirce reached a related conclusion at the end of the 19th century while devising a related formal system.

Table 2-1 summarises the discussion thus far, with each cell in that Table giving one of the six possible ways of forming pairs from '()' and '⊥', keeping in mind that '()' has both an "interior" and "exterior". Table 2-1 and definition 2.1.2 essentially define the PA. A1 and A2 each yield the value of one cell. The remaining four cells contain the string "A3", which stands for either of the two paths proposed above: either invoke a new axiom, A3, or alter A2 to allow '⊥' to be erased at will. Henceforth, I will take the latter course, so that A2 includes the four equalities in the cells labelled A3. The cell '⊥⊥ =⊥' implies that strings consisting of iterated instances of '⊥' all refer to the null formula. Once I define Boolean algebra (§3.3), it will be clear that Table 2-1 defines the corresponding Boolean arithmetic. A numerical interpretation of that arithmetic is that 1, 1−α, and α×β interpret (), (α) and αβ, respectively.[15]

Table 2-1. Axiomatic PA operation table, With a numerical interpretation.				
1a. Juxtaposition			1b. Enclosure	
()() = () A1 0×0 = 0	⊥() = () A3 1×0 = 0		(()) = ⊥ A2 1−0 = 1	
()⊥ = () A3 0×1 = 0	⊥⊥ = ⊥ A3 1×1 = 1		(⊥) = () A3 1−1 = 0	

A1, A2, and Table 2-1 may appear trivial. However, in 3.3.2 we shall see that A1 and A2 imply that *B* is a partially ordered set. Ordered sets are rich in mathematical (cf., e.g., Davey and Priestley: 2002) and logical (Curry 1963: chpt. 4) content.

Table 2-1 implies that a pair of parentheses can serve as either an operator or an operand. In a subformula of the form '·(·)·', the parentheses can be seen as denoting a three-place operator (functor), such that one or more of '·' can be left blank. Leaving all three places blank takes us back to the boundary sign '()', a primitive value and hence an operand. Sections 3 and 4 will say more about '()' as operand and operator. In the PA, the distinction between operator and operand is purely contextual and has effectively degenerated, a situation to which *LoF* (p. 88) refers as the "partial identity of operator and operand."[16] Any notation proposed for the PA must do justice to this degeneration. I chose parentheses with this degeneracy uppermost in mind.

Definitions 2.2.1, 2.2.2, and 2.2.4 lay out the principle use of Table 2-1, and the meaning of the symbol '='.

2.2.1. Definition. A *step* is any alteration of a formula justified by invoking the contents of a cell in Table 2-1. Steps are of two kinds: slimming and expansion. Let α be some PA formula. To replace '()()' with '()', or to erase an instance of '(())' occurring in α is to *slim* α. To replace an instance of '()' with '()()', or to insert '(())' anywhere in α, is to *expand* α.

---

15. Since the two cells in Table 2-1b form a *dual pair* (4.1.3), duality suggests that only one of these cells is strictly necessary. Another numerical interpretation of Table 2-1 is: ⊥⇔−1 < 1⇔(), αβ ⇔ max(α,β) [min(α,β)], and (α) ⇔ −α. Obtain the dual (cf. §4.1) of this interprêtation by interchanging the numerical interpretations of ⊥ and (), and applying the operation in square brackets. The dual of the interpretation in Table 2-1 is obtained by setting αβ ⇔ (α+β)−(α×β). A Boolean interpretation of the PA is: ()⇔1, ⊥⇔0, (α)⇔−α, and αβ⇔α⋃β [α⋂β]. In this case, A1 is 1⋃1=1; A2, −1=0, 0⋃1=1⋃0=1, 0⋃0=0, −0=1. Given this interprêtation, A1 and A2 are (3b) and (7b) in Shannon (1938). For other arithmetical axiomatizations of Ba and sentential logic, see fn 82.

16. *LoF* (p. 88) asserts that this partial identity characterizes Ba as well, albeit in disguised form.

A1 (A2) justifies the first (second) action mentioned in the last two sentences of 2.2.1. Slimming and expansion can be thought of as an inverse pair of operations, whose arguments are formulae.

2.2.2. Definition. To *simplify* a formula 'α' is to slim 'α' one or more times until the result is a member of *B*. That member is the *value* of 'α', denoted '$|\alpha|$'.

2.2.3. Algorithm. The following algorithm operationalises what is meant by simplifying the formula 'α' with greatest depth d_α^*:

1. Go to the subspace of 'α' whose depth is $d_\alpha^* - 1$.
2. IF this subspace pervades one or more sub-formulae of the form '()()', THEN invoke A1 and replace every '()()' with '()'. Repeat this step until only one '()' is left.
3. ELSE go to the subspace at depth $d_\alpha^* - 2$, which must pervade one or more sub-formulae of the form '(())'. Invoke A2 and eliminate all instances of '(())'.
4. IF what remains of α is '()' or '⊥', THEN STOP.
5. REPEAT 2 through 4, decrementing the depth of the subspace by 1 each time.[17]

End of Algorithm

Remark. The *simplification* of 'α' is the value of α when the algorithm 2.2.3 terminates. For more about on simplification and expansion see §2.3, specifically T3, T4, and the Hypothesis of Simplification.

I now define the symbol '=' (the 'equal sign') as follows:

2.2.4. Definition. The string 'α=β', called an *equation*, signifies that the formulae 'α' and 'β' have the same simplification and hence are *equivalent*.[18]

Corresponding to the adjective 'equivalent' is the noun 'equivalence.' I revisit equivalence after 2.3.8. Following *LoF*, I denote equivalence by the mathematician's common-garden '='. The equation 'α=β' implies nothing whatsoever about the literal appearance of 'α' and 'β'.

The semantics of the PA are an elusive aspect of *LoF*, and perhaps the greatest obstacle to a wider appreciation of BA. *LoF* leaves the PA uninterpreted, saying little more about the possible semantics of the PA than what can be found in the first few paragraphs of its Appendix II, barely hinting at what I say in §2.0 above. Could the degeneracy of the PA extend even to the distinction between syntax and semantics? Setting this possibly deep question aside, I intend by "semantics of the PA" no more than some asserted *interpretation* of the PA, defined as:

2.2.5. Definition. An *interpretation* of the PA is a one-to-one correspondence between *B* and another two-member set.

17. Compare this algorithm to that on p. 13 of *LoF*. After devising 2.2.3, I discovered the following related definitions in Machover (1996): *degree of complexity* of a formula (§§7.1.7, 8.1.8), *weight* of a string (§§7.1.9, 8.2.1), *parity* of a formula (§8.8.3).

18. Synonyms for 'α=β' include 'α↔β' (common in conventional logic), and 'α ⫤⊨ β' (Bostock 1997: 36).

One such two-member set is {1,0}, containing the Boolean primitives in Table 2-1. Other possibilities include {**V**,**Λ**} (set theory), {top 'T', bottom '⊥'} (lattice theory), {True,False} (logic), and the everyday meaning of {On,Off}. Hehner (2004) even proposes the numerical reading {∞,-∞}. I cannot claim that these minimalist semantics necessarily do justice to the philosophical intent of *LoF*.

A Technical Digression on PA *Semantics.*
Given some standard notions from mathematical logic, the following argument renders plausible interpreting '()' and '(())' as the classical bivalent truth values. Let the *extension* of an *n*-place atomic formula be the set of ordered *n*-tuples of individuals that satisfy it (i.e., for which it comes out true) (Carnap 1958: §10b). Let a sentential variable be a 0-place atomic formula; its extension is a classical truth value by definition. An ordered 2-tuple is known as an *ordered pair*, whose standard set theoretic definition is $\langle a,b \rangle =_{\mathbf{df}} \{\{a\},\{a,b\}\}$, where *a*,*b* are individuals. Ordered *n*-tuples for any *n*>2 may be constructed from ordered pairs in a well-known recursive way (cf. Stoll 1963: §1.6). Dana Scott has argued (Bostock 1997: 83, fn 11, 12) that the extension of a sentential variable can also be seen as the empty ordered pair (ordered 0-tuple), {{},{}}, equal to {{}} by elementary set theory. Hence **T** interprets {{}}. Reading {} as **F** follows naturally, if the curly braces of set notation are given a boundary reading.

One can go much further. Angell (1960) showed how to notate denial, conjunction, quantified variables, and set membership using only parentheses. Angell's notation requires setting A1 and A2 aside. Angell codes the first quantified variable as '()', the second as '()()', and so on. Letting ϕ and φ be metalogical notation for formulae, the truth functional part of Angell's notation is $(\phi) \Leftrightarrow \sim\phi$ and $(\varphi\phi) \Leftrightarrow \phi\wedge\varphi$. The outer parentheses of $(\varphi\phi)$ are needed only because of peculiarities of Angell's notation for quantification and membership, not described here. Because his notation allows for set membership, Angell unwittingly showed that PA syntax suffices for set theory. Angell supplied no axioms or proof theory, as he only wished to show his notation capable of expressing the system of Quine (1951), whose primitives were the Sheffer stroke, universal quantification, and set membership.

2.3. PA: Canons and (Meta)theorems.

"*The more important structures of command are sometimes called canons. They are the ways in which the guiding injunctions appear to group themselves in constellations, and are thus by no means independent of each other. A canon bears the distinction of being outside (i.e., describing) the system under construction, but a command to construct (e.g., 'draw a distinction'), even though it may be of central importance, is not a canon. A canon is an order, or set of orders, to permit or allow, but not to construct or create.*" *LoF*, p. 80.

"*…the primary form of mathematical communication is not description but injunction… Music is a similar art form, the composer does not even attempt to describe the set of sounds he has in mind, much less the set of feelings occasioned through them, but writes down a set of commands which, if they are obeyed by the performer, can result in a reproduction, to the listener, of the composer's original experience.*" *LoF*, p. 77.

The Six Canons.
LoF includes nine canons, which Spencer-Brown intended to serve mainly as *injunctions*, i.e., directives (see quote above). The PA *canons* and *theorems* establish protocols for al-

tering, and reasoning about, PA formulae. They are *about* boundary algebra, hence metamathematical. The PA is too elementary for logical/mathematical proof as conventionally understood to apply to its formulae. In a sense, the canons and theorems stand in for the absent PA proof theory.

As best as I can determine, the mathematical and philosophical literatures include no counterpart to *LoF*'s concept of canon. I list below the PA canons in the order in which they appear in *LoF*, referring to them by the names *LoF* gives them. In what follows, I have taken the liberty of replacing the *LoF* term "expression" by the term "formula." Letting *X* be some word or phrase, any sentence below of the form "*X* is undefined" (or words to that effect) is shorthand for "*LoF* does not define *X* with the precision that generally characterizes mathematics, mathematical logic in particular."

Convention of Intention
What is not allowed is forbidden.

Remark: I trust no reader has so misunderstood my purpose as to take BA as a basis for political philosophy!

Contraction of Reference
Let injunctions be contracted to any degree in which they can still be followed.

Remark: "Injunction", "contract", and "degree" are not defined. *LoF* (p. 8) states that this canon is shorthand for the following list of instructions:
1. Write '()' in some space.
2. Mark '()' with a name, eg, *a*.
3. Let *a* be the name of '()'.
4. Let the name *a* indicate '()'.

Expansion of Reference
Let any form of reference be divisible without limit.

Remark: I take "form of reference" to mean "space or subspace" in the sense of 2.1.6, and "divisible without limit" to mean that divisions of a subspace in the sense of 2.1.7 can be created at will, using A1. More generally, this canon permits expanding a formula, with each step justified by A1 or A2.

Convention of Substitution
In any formula, any subformula can be replaced by an equivalent subformula.

Remark: *LoF* wrote "arrangement" and "changed" where I write "subformula" and "replaced." This canon is:
- An important example of what is meant by a "step";
- The first *LoF* canon or theorem to mention "equivalent," a term *LoF* does not discuss until 13 pages later. I shall revisit "equivalent" below when discussing T5-T7 and 2.3.8.

LoF distils the sense of 2.2.1–3 into the following canon:

Hypothesis of Simplification
Suppose the value of a formula, $|\alpha|$, to be its simplification.

Remark. Thus *LoF* defines "simplification."

Rule of Dominance
If a formula α shows a *dominant value*, then |α|=(). Otherwise, |α|=⊥.

Remark: This canon introduces "dominant value" without defining it.

The canons would seem to be assertions of the sort requiring proof; in fact, they are informally motivated at best.[19] The canons sometimes serve as definitions; e.g., the Hypothesis of Simplification and the Rule of Dominance effectively define the *value* of a formula. Curiously, for a work of logic/mathematics, *LoF* contains only one sentence preceded by the word "definition". That sentence, the third one in the body of *LoF*, simply reads: *Distinction is perfect continence.* What this sentence purports to define is less than obvious. §3.1 presents three more canons, bearing on the primary algebra.

The Seven Meta-Theorems of the PA.

2.3.1. Definition. A *theorem* is metalinguistic statement asserted true because it is the last of an ordered finite sequence of metalinguistic statements known as an *informal proof*.

All theorems about the BA are proved informally in the metalanguage, the academic dialect of contemporary written English, using devices that tacitly draw on the reader's previous mathematical experience. (Those lacking such experience will find *LoF* and BA challenging.) An informal proof may draw on concepts that, strictly speaking, are not defined or proved within BA as of the point at which they are invoked. In particular, an informal proof relies strongly on natural language, and may invoke informal reasoning and mathematical concepts that are not part of BA. "Informal proof" is in contrast to "formal proof," defined in the Precis.

Following *LoF*, I number BA theorems consecutively, with the *m*th theorem denoted T*m*. The proofs are freely adapted from *LoF*.

Establishing Consistency
2.3.2 (T1). Any string composed of finite instances of '(' and ')', and satisfying the formation rule 2.1.2, is a formula.

Remarks.
1. A formula must be finite in order for the algorithm 2.1.4 to terminate.
2. I state T1 only out of loyalty to *LoF*; the formula formation rule 2.1.2 renders it unnecessary. T1 in *LoF* (p. 12) says that, starting from '()', "any conceivable arrangement" can be contructed by repeated application of A1 and A2. This version of T1 sounds trivial because it is predicated on *LoF*'s '⏋' notation, in which all possible strings are formulae. *LoF* does not articulate the operational meaning of "conceivable." *LoF* (p. 22-24) unaccountably invokes T1 in the proofs of J1 and J2, to justify asserting that the only values a pa variable can take on are '()' and '⊥'.

19. *LoF* (pp. 40-41) calls T14 and T15 'canons,' thereby sowing terminological confusion.

2.3.3 (T2). If any space pervades the formula '()', the value indicated in the space is the marked state. Notation: ()α=().

Proof. If |α|=(), then ()α is ()(), which simplifies to () by A1. If |α|=⊥, then ()α simplifies to () by A2. □

Remark. T2 is the PA version of the primary algebra consequence C3 (§3.1). *LoF* makes frequent use of T2, which arguably defines the "marked state."

2.3.4 (T3). The simplification of a formula is unique.

Proof. Review the algorithm 2.2.3. This algorithm systematically reduces a formula, starting from its greatest depth. Each step has only two possible outcomes: ()α or (()). By T2, ()α reduces to (); by A2, (()) can be erased. Given each outcome, the next step is unambiguous. Hence there is only one possible simplification. □

Remarks. Let *A* be the set of all possible PA formulae. Simplification can be represented by the mapping *f*: *A*→*B*⊂*A*. T3 implies that *f* is a *homomorphism* (Halmos and Givant 1998: §27) and an *isotone* function (Rudeanu 1974: §11.3), whose *fixed points* are '()' and '⊥'.

2.3.5 (T4). The value of a formula constructed by taking steps starting from a primitive value is that same primitive value.

Proof. Let α be a formula constructed by taking steps starting with the primitive value *x*. The steps can be retraced back to *x*, so that *x* is *a* possible simplification of α. By T3, all possible simplifications of α must yield *x*, hence *x* is also *the* simplification of α. Hence we can write |α|=*x*. □

Remark. T3 [T4] says that the value of a PA formula is invariant under simplification [complication]. T3 and T4 together imply that every PA formula has a unique value. Hence the PA is *consistent*, and *LoF* refers to T1-T4 as the "theorems of consistency."

Procedural Theorems

2.3.6 (T5). Identical formulae express the same value. Notation: α=α.

Proof. Use 2.2.3 to simplify the formula α to some member of *B*; call that member *x*. By T3, *x* exists and is unique. Hence α is equivalent to *x*, so that we write α=*x*. Beginning with *x*, we reverse each step in the simplification of α, recreating α. By T4, the value of this recreated α will also be *x*, so that we can write α=α. □

Remark. The verb "express" in T5 is undefined.

2.3.7 (T6). Formulae having the same value can be equated. Notation: Let *x*∈*B*. If α=*x* and β=*x*, then α=β.

Proof. Identical to the proof of T5, except that we proceed by steps from *x* to β rather than α, by reversing the simplification of β. □

Remark. T6 in effect means "if |α|=|β|, then α=β."

T7 requires some preliminary definitions.

2.3.8. Definition (Wolf 1998: §§6.1-2; Stoll 1974: §1.7). Let A be a set. A binary *relation* R whose *field* is A is a subset of $A \times A$. Hence R is a set whose members are all ordered pairs. The notation xRy denotes that the ordered pair (x,y) is a member of R. R is *Euclidian* iff $(aRc \wedge bRc) \rightarrow (aRb)$. R is an *equivalence relation* iff $\forall a,b,c \in A$, $(aRa) \in R$ (R is *reflexive*), $aRb \leftrightarrow bRa$ (*symmetric*), and $(aRb \wedge bRc) \rightarrow aRc$ (*transitive*). If R is an equivalence relation whose field is A, an *equivalence class* is a set $A^* \subset A$, such that $\forall x,y \in A^*$, xRy comes out true.[20]

2.3.9 (**T7**). Formulae equivalent to the same formula are equivalent to one another. Notation: if $\alpha = \nu$ and $\beta = \nu$, then $\alpha = \beta$.

Proof. Let $|\nu| = e$. Then $|\alpha| = e$ and $|\beta| = e$, by hypothesis. Now simplify α to e, then retrace the simplification of β, starting from e and ending with β. Since by T3 and T4, no allowed step alters value, $|\alpha| = |\beta|$, so that $\alpha = \beta$. □

Remark. T7 is T6 with ν replacing x. T7 can be recast as "the relation of logical equivalence is Euclidian." *LoF* invokes T7 repeatedly, but invokes T5 and T6 only to prove the pa initials J1 and J2.

I now invoke a result from the logic of relations.

2.3.10. **Theorem**. R is an equivalence relation iff R is reflexive and Euclidian.

Proof. Even though the proof is neither long nor difficult, I relegate it to A.2 as it employs features of BA not yet explained.

Let A in 2.3.8 be the set of all possible PA formulae, and let R be logical equivalence, '='. Then '=' is reflexive (by T5) and Euclidian (by T7). Hence by 2.3.10, '=' is an equivalence relation. T3 says that logical equivalence partitions A into two equivalence classes, each corresponding to an element of B. Hence T3 is but an instance of the very well-known result that *an equivalence relation partitions its field into equivalence classes* (Wolf 1998: Th. 6.6). Let $[\alpha]$ denote the equivalence class of which α is a member. T4 can then be restated more formally as: $\forall \alpha \in A$, there exists a one-to-many relation g: $B \rightarrow A$, corresponding to expansion, such that $[f(g(f(\alpha)))] = [f(\alpha)] = [\alpha]$. When one of True or False interprets '()', '=' denotes *logical equivalence*.

I denote equivalence by '⇔' when one or both formulae linked by '⇔' are not BA formula. The sign '⇔' is part of the metalanguage, and two formulae linked by '⇔' form a *sequent*, a metalinguistic term. More generally, '⇔' can be thought of as indicating a translation from one syntax to another.

2.3.11. Recapitulation. The *primary arithmetic* (abbreviated PA) is a very elementary formal system whose *primitive basis* (cf. Précis) consists of:

- The symbols '(', ')', '⊥', and '=';

---

20. A slightly more general definition of relation goes as follows. If A,B are sets, a *binary relation* is a subset of $A \times B$ and its *field* is $A \cup B$. The term *Euclidian* honors the first of Euclid's "common notions" (Eves 1990: 35). Introductory logic texts usually do not mention relations; exceptions include Carnap (1958: §§29-38) and Suppes (1957: chpt. 10-11; 1960: chpt. 3).

- The operator-operand '()', which can have itself as argument, resulting in the formula (()). The defined constant ⊥ is a synonym for (());
- '()' and the blank page as primitive values;
- The definitions of a formula (2.1.2) and the null formula '⊥' (2.1.8), and the algorithms for verifying (2.1.4) and simplifying (2.2.3) formulae;
- Table 2-1, taken as axiomatic;
- Six procedural "canons";
- Equivalence of formulae, an equivalence relation by virtue of T5 and T7, denoted by infix '='. Two formulae linked by '=' form an *equation*.

The PA is sound by virtue of T1-T4; its intended interpretation is Boolean arithmetic.

3. The Primary Algebra (pa): Syntax.

"It is… valuable to meditate on algebraic notation; the whole of the formal and symbolic part, having gradually broken away and developed immensely, is of great interest."

Paul Valéry, quoted in Le Lionnais (1948: 10).[21]

At any point in a PA formula, one can insert a marker that can take on either primitive value. Latin letters, termed (sentential) *variables*, will serve as such markers. The set of possible values a variable can assume is its *domain*; the domain of a pa variable is *B*. Thus the *primary algebra* (hereinafter abbreviated pa, by analogy with the abbreviation PA for the primary arithmetic) is born. Like the PA, the pa consists of formulae and equations, and includes canons, rules, and theorems. We begin by setting out the pa symbols:

3.0.1. Definition. The notation of the pa consists of *proper* and *improper symbols*. The proper symbols are:
- The PA proper symbols '(', ')';
- Lower case Latin letters, '*a*', '*b*', etc., often called *statement letters* or *sentential variables*. A letter may have a positive integer subscript, so that the number of possible variables is denumerable.

The improper symbols are '⊥', the prime ''', and the ellipsis '…' combined with the subscript *i* ranging over some range of the positive integers. Improper symbols are merely convenient notational shorthand. Symbols are concatenated into formulae:

3.0.2. Definition. The recursive definition of a pa formula is identical to 2.1.2, except that the atomic formulae include any single Latin letter.

Definitions similar to 3.0.2, e.g., Bostock (1997: 21), are standard in the literature. Synonyms for formula include *well-formed formula* (*wff*) and *schema* (Quine 1982: 33).

Because '()' is an atomic formula, 3.0.2 implies that a PA formula is also a pa formula. The pairing rule for parentheses, and the algorithm 2.1.4, both hold in the pa as well as in the PA. A non-obvious implication of 3.0.2 is that inserting a string of Latin letters into a PA formula results in a pa formula. Subformulae, proper and otherwise, are defined by obvious analogy with 2.1.3. Informally speaking, a subformula is any "part of"

21. From a letter to Pierre Honnorat, dated February 1932. The translation is mine.

a pa formula that is itself a formula. An atomic formula has no proper subformulae other than '⊥', which is a proper subformula of all formulae other than itself.

In this paper, '$=_{\mathbf{df}}$' is part of the metalanguage and serves to define a new notation or concept. Let the string x contain an instance of some new symbol, and let the string y contain only familiar symbols. The notation '$x =_{\mathbf{df}} y$' defines the new symbol by asserting that the strings x and y, however they differ in appearance, have the same meaning *by definition*. Let a, b, and r be pa formulae. I now define the improper symbols the prime, ''', the ellipsis '…', and the letter i subscript as follows:

$$a' =_{\mathbf{df}} (a) \qquad a_i\ldots =_{\mathbf{df}} a_1a_2\ldots \qquad a'_i\ldots =_{\mathbf{df}} a'_1a'_2\ldots \qquad (a_ir)\ldots =_{\mathbf{df}} (a_1r)(a_2r)\ldots$$

'a'' is nothing more than a synonym for '(a)', in which case 'a' is said to be *primed*. Using 'a'' in place of '(a)' is purely a matter of convenience and aesthetics. The letter i subscript and the ellipse always appear in tandem. The improper symbols are not mentioned in 3.0.2, and hence play no essential role in the syntax of the pa. Foremost among the virtues of the pa is its succinct syntax. The notation I propose for BA is more compact than that of *LoF* and requires only standard typographic symbols.

Observant readers will have noticed that thus far, I have enclosed symbols and formulae between single quotation marks. I have done so hoping to steer clear of Quine's *bête noire*: metadiscourse confusing *use* of a symbol with the *mention* thereof. Henceforth, I will rely on the following general rule, adapted from Conventions (I) and (II) in Suppes (1957: 125-26): *all symbols and formulae from* BA *and conventional logic are to be taken as names of themselves*. I deferred invoking this convenient rule until the syntax of BA was fully set out.

3.1. Consequences, Canons, Theorems.

"*Algebra is a science of the eye*." Peirce (1.34).[22]

3.1.1. Definition. T (*true*) and F (*false*) are the possible *truth values*. A *statement* is an object language formula, or piece of metalinguistic discourse, that can be assigned a truth value.

Remark. An individual sentential variable is a trivial statement. The last paragraph of §4.0 operationalises the assignment of truth values to pa statements.

3.1.2. Definition. An n-ary *truth functor* [or *functor* when 'truth' can be omitted without ambiguity] is a symbol combining n statements into a single statement (Bostock 1997: §2.2; Quine 1982: §§20, 45). A *connective* [*operator*] is a truth functor such that $n \geq 2$ [1]. () and ⊥ are 0-ary functors by convention.[23]

22. Sylvester wrote "mathematics" not "algebra" (Ewald 1996: 515), but Peirce's misquotation is nonetheless apt.

23. "A functor is a sign that attaches to one or more expressions of a given grammatical kind or kinds, to produce an expression of a given grammatical kind. [A functor] is grammatical in import but logical in habitat…" (Quine 1982: 129). 'Statement' is the only grammatical kind that concerns us here. 'Functor' in Carnap (1958: §18) denotes what other authors call a "first order logic operator." I do not employ 'functor' in this sense.

Truth functors, such as ~, →, ∧, ∨, and ↔, make up the core of the CTV. BA consists of one unary functor, enclosure by parentheses, and one binary functor, juxtaposition. Table 4-2 gives BA translations of the usual CTV truth functors.

3.1.3. Definition. An *atomic valuation* assigns a member of *B* to each of the *n* sentential variables appearing in a formula. Equivalently, an atomic valuation maps these *n* sentential variables into a member of B^n.

The PA definition of *equation*, 2.2.4, carries over to the pa, *mutatis mutandis*.

3.1.4. Definition. If α evaluates to () or ⊥ for a given atomic valuation, that valuation *satisfies* α, and α is *satisfiable*. If all 2^n possible atomic valuations satisfy α in the same way, then α and the equation α=() [or α=⊥] are both *tautologies*. If α↔β is a tautology, α and β are *tautologically equivalent* so that we may write α=β.

The definition of tautology in 3.1.4 differs from the standard one, which defines a tautology as a formula evaluating to T for all possible atomic valuations. α=β says nothing about the truth value of α or β taken in isolation. It does say that, given any atomic valuation, α and β have the same value. Translating '=' as '↔' assumes that the biconditional can be seen as an equivalence relation. This is indeed the case; see §A.5. On occasion, I will refer to an equation as a tautology; strictly speaking, this is an *abus de language*.

The preceding can be put a bit more formally. Let ℵ be the set of possible PA formulae, and let *f* be the simplification function defined in 2.2.3. T3 and T4 assure us that *f* maps every member of ℵ uniquely onto one of () or ⊥. We can then say the following about *f*:

- The image of ℵ under *f* is *B*;
- *f* is order preserving;
- *f* partitions ℵ into two equivalence classes.

The pa is a bit more complicated. Let *A* be the set of possible pa formulae. Letting the domain of *f* be *A*, *f* then partitions *A* into two non-empty subsets. The subset of *A* whose image under *f* is () or ⊥ consists of tautologous formulae; the complement of that subset consists of satisfiable formulae.

I now turn to proof and related notions, beginning with the following definitions.

3.1.5. Definition. A *consequence* is a tautological equivalence. An *initial* is a consequence proved via a decision procedure; hence an initial is a PA theorem. A *demonstration* formally verifies ("proves") a consequence that is not an initial.

A decision procedure can verify any consequence. The *LoF* decision procedure is exposited in §5.1 and named *truth value analysis* (TVA). An initial is *not* an axiom, but can be invoked just like an axiom or consequence. Again, A1 and A2 in §2.2 are the only pa axioms. In conformity with standard mathematical practice, a demonstration consists of a sequence of steps, each relying on one or more BA axioms and theorems (especially the initials), canons, the rules of substitution/replacement (3.1.7-8), and consequences already proved. I revisit the notion of demonstration in §5.0 and the *Precis*.

Each step in a demonstration is justified by an *annotation*, enclosed in square brackets and formatted as follows:

α [*annotation*] = β [*another annotation*].

If a step requires more than one consequence, these and the substitutions they may require are listed sequentially, separated by semicolons. If a step includes a rearrangement of subformulae, 'OI' annotates that fact. If symbols appearing in α are absent from β, I indicate that fact by underlining the relevant parts of α. If β contains subformulae absent from α, the additions to α are shown in bold. When a subformula in α is moved or copied to a greater or lesser depth in β, the part of β that is freshly moved or copied is also printed in bold.

LoF numbers the initials and consequences consecutively, as is the case for theorems, using a letter (initials begin with J, consequences with C) followed by an integer. I adopt this system to facilitate cross-references to *LoF*.[24] The *LoF* initials are:

3.1.6. J1. $(a'a) = \bot$ J2. $((ar)(br)) = (a'b')r$

Verifying J1 is trivial. J2 is an easy instance of truth value analysis (§5.1).

3.1.7. Theorem. J1 and J2 are tautological equivalences.

Proof:
J1: Let a=(). Then the lhs of J1 is $(\underline{(())}())$ [A2] = $\underline{(())}$ [A2] = ⊥. Now let a=⊥, so that the lhs of J1 becomes $((\underline{\bot})\underline{\bot})$ [A2] = $\underline{(())}$ [A2] = ⊥. By T1, () and ⊥ are the only possible values of a. Hence J1 always holds.
J2: By T2, α()=() for any formula α. Begin by setting r=(), in which case the lhs of J2 evaluates to $((a())(b()))$ [T2,2x] = $((())(()))$ [A2,2x] = (). The rhs evaluates to $(a'b')()$ [T2] = (). If r=⊥, then simply erase r from J2. Both sides of J2 then amount to the same thing, namely $(a'b')$, for all possible values of a and b. By T1, () and ⊥ are the only possible values of r. Hence J2 always holds. □

Remark. This proof is, in all essentials, a truth value analysis. An immediate consequence of J1 is that the formula (α'α), α being any subformula, can be inserted at will anywhere. By repeated application of J1, the maximum depth of a formula can be increased at will, without affecting its value under any atomic valuation.

In section 5, I argue that a trivial variant of J1, namely $a'a$=() which I call J0, is a bit more natural than J1. See A.1 for a demonstration of J0 from J1 and J2. Henceforth, I deem $a'ab$=() and $(a'ab) = \bot$ to be instances of J0 and J1, respectively. Doing so largely renders C3 redundant.

LoF demonstrates the nine consequences, C1-C9. I list them below, restated in the notation of this paper, with some indication of how they prove useful in *LoF*:

24. I number consequences as in *LoF*, even when I treat an *LoF* initial as a consequence or vice versa, as I am wont to do in §5.2 and later. *LoF*, Bricken, and Kauffman, following *PM* (p. xii), give Latinate names to the initials and consequences. I decline to follow their example, as the names they propose are not mnemonic.

Table 3-1. The Nine *LoF* Consequences.		
		How employed in LoF
C1	$((a)) =_{\mathbf{df}} (a') = a$	Invoked in many demonstrations.
C2	$(ba)a = b'a$	"
C3	$()a = ()$	Algebraic form of T2.
C4	$(a'b)a = a$	Helps demonstrate C5.
C5	$aa = a$	Algebraic form of A1. Helps prove T13.
C6	$(a'b')(a'b) = a$	Helps demonstrate C9.
C7	$((a'b)c) = (ac)(b'c)$	Helps prove T14; crucial for normal form.
C8	$(a'r')(b'r') = ((ab)r')$	Syntactic dual of J2. Invoked in the *LoF* demonstration of C9 and proof of T15.
C9	$((a'r')(b'r)) = (ar')(br)$	Helps prove T17.

The purpose of C7 will become clearer in §4.4. C8 is the dual of J2 (cf. §4.1) and by 4.1.4, requires no demonstration; it also finds no application in this paper. C9 above is simpler than its *LoF* equivalent, because the simpler form suffices to prove T17, the only use *LoF* and this paper have for C9. For a CTV interpretation of J1-C9, see Table 4-3.

pa: **Canons.**

The pa features three canons in addition to the PA canons described in §2.3. As before, I deviate from the *LoF* wording of the pa canons in an attempt to reduce their Zennish ambiguity and enhance their perspicuity.

Principle of relevance

If a property is common to every indication, it need not be indicated.

Remark. Could this be worded: "That which characterises everything distinguishes nothing"?

Principle of transmission

Let α be a sub-formula of the formula β, and let a be a variable appearing in α. Let the depth, relative to β, of α [a] be $d_\beta(\alpha)$ [$d_\beta(\alpha)+1$]. When the value of a changes, the value of α either changes or does not change. If it changes, the pervasive subspace of α is said to be *transparent* relative to a. Otherwise, the pervasive subspace is *opaque*.

Remark. The wording of this Principle differs from that in *LoF*, if only by employing "sub-formula" and "depth." This canon is also closely linked to T16.

Rule of demonstration

A demonstation rests in a finite number of steps.

Remark. Why did *LoF* say "rests in" rather than "consists of"? On "demonstration" and "steps," consult §5.0 and the Précis. Chapter 11 of *LoF* shows how formulae with infinite depth may violate T1-T4. I do not elaborate on this, nor do I explore formulae with infinitely many symbols but finite depth, as I wish to steer clear of all infinities and Cantorian paradoxes.

pa: **Substitution**

"*…there is* [no] *need for any other kind of proof than one which depends on the substitution of equivalents*." Leibniz.[25]

An *equational* formal system is one whose axioms and consequences consist of pairs of formulae linked by equality, denoted by '='. The inference rules for an equational system are the *substitutivity of equivalents*, R1 below, and the *uniform replacement of subformulae by subformulae*, R2 below. *LoF* (p. 26) states that "[R1 and R2] are commonly accepted as implicit in the use of the sign '='." Hence BA is equational, as is nearly all of mathematics, with conventional numerical algebra being paradigmatically so. The vast majority of extant formal logics, on the other hand, are *ponential*, so-called because their fundamental inference rule is *modus ponens* (cf. §5.3).[26] R1 and R2 are 3.1.7 and 3.18 below. In the interest of clarity, I restate these rules in a manner that deviates somewhat from *LoF*. As always, Greek letters are a metalogical device.

3.1.7. R1, **Substitution**. Let $\alpha\langle\varepsilon\rangle$ denote that the sub-formula ε appears at least once in the formula α. Let ϕ be a formula such that $\phi=\varepsilon$. Let $\alpha\langle\phi//\varepsilon\rangle$ be the formula formed by substituting ϕ for *any* (possibly 0) instance of ε in α. Then $\phi=\varepsilon \rightarrow \alpha\langle\varepsilon\rangle=\alpha\langle\phi//\varepsilon\rangle$.

Proof (adapted from Mendelson 1997: Prop. 1.4). The contribution of a subformula ε to the truth value of any formula α containing ε is fully determined by the truth value of ε. Hence the truth value of α is not affected by replacing ε by ϕ, whose truth value, by assumption, is identical to that of ε. □

Remark. α need not be a tautology but if it is, $\alpha\langle\phi//\varepsilon\rangle$ is also a tautology. *LoF* (p. 26) does not prove R1, instead justifying it as an algebraic version of the PA Convention of Substitution, and as an "inference from" T1-T4. Because R1 is essentially Leibniz's "identity of indiscernables," R1 and the reflexivity of '=' suffice to show that '=' is an equivalence relation (Kneebone 1963: §4.1). Some authors refer to R1 as the "substitutivity of the biconditional *or* of equivalents." R1 also answers to the familiar "substitution of equals for equals" of numerical algebra.[27]

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25. Letter to Placcius dated 16.11.1687, translated and quoted in Ishiguro (1990: 17).

26. Curry's (1963: §2.D.1) *relational-assertional* dichotomy inspired the equational-ponential dichotomy. A *relational* system consists of formulae linked by an unspecified dyadic relation; if that relation is an equivalence relation, the system is *equational* in the sense of Curry. Curry contrasted relational to *assertional*, by which he meant a system characterized by formulae prefixed by the syntactic turnstile, for him the defining aspect of conventional logic. On equational logic, see also Meredith and Prior (1968).

27. R1 also follows from '=' being a congruence relation (Stoll 1963: 260). For other proofs of R1, see Quine (1982: 64), Mendelson (1997: Prop. 1.4), and Cori and Lascar (2000: Th. 1.24). After devising the '//' notation, I encountered it in Simons (1987: 49). Quine (1982: 63f) refers to Substitution as "interchange," for which he proposes three laws, which I condense to two: (1) R1 holds if '↔' replaces '=', and (2) R1 preserves (in)equivalence, (un)satisfiability, (non)validity, and (non)implication. On '↔' and '=', see fn 18. A version of R1 is crucial to the system of Cole (1968).

3.1.8. R2, Replacement. Let $\alpha\langle v\rangle$ be a tautology. Let $\alpha\langle\omega/v\rangle$ be the formula formed by replacing *every* instance of v in $\alpha\langle v\rangle$ by the formula ω (v and ω are not necessarily equivalent). Then $\alpha\langle v\rangle=\alpha\langle\omega/v\rangle$.

Proof (adapted from Mendelson 1997: Prop. 1.3). By definition, the value of a tautology is not affected by the value of any (or all) of its statement letters. Hence any statement letter v can be replaced by some formula ω (with a value under any interpretation) without affecting the value of α, as long as this replacement applies to every instance of v. Hence R2 follows from $\alpha\langle v\rangle$ being a tautology. □

Remark. Stoll (1974: 124) refers to the result of applying R2 as an *instance*, making possible the following concise rewording of R2: "a tautology yields tautologous instances." Thanks to R2, all letters appearing in pa consequewnces can be taken as schematic variables; all consequences are *schemata*. Most formal systems dispense with Replacement by simply taking their axioms and theorems as schemata from the outset. Some authors refer to Replacement as (variable) 'Substitution'.[28]

R1 and R2 warrant close scrutiny in light of the careful treatment of Substitution by other authors. The reader should also ponder why R1 says "any" while R2 says "every". *LoF* explicitly invokes R1 and R2 only in the "pedantic" demonstration of C1 and the worked examples on pp. 44-47. In this paper, almost all use of R1 and R2 goes unremarked. But regardless of whether R1 and R2 are invoked tacitly or explicitly, the pa would be useless without them.

Some pa Theorems.

All PA theorems carry over to the pa; thus the pa inherits soundness and tautological equivalence from the PA. An additional 11 theorems, T8 through T18, bear on the pa only. I defer discussion of T14-T18 to §4.4. T8 and T9 merely restate J1 and J2. T10-T13 generalize certain consequences:

3.1.9. (T10). J2 generalizes to $((a_i r)\ldots) = (a_i' \ldots)r$.

T10 is needed only to prove T15. T11 and T12 are straightforward generalizations of C8 and C9 that have no application in this paper and so are omitted.

3.1.10 (T13). Let $\phi\langle\varepsilon\rangle$ be a formula containing one or more instances of the subformula ε, and let ϕ stand for $\phi\langle\varepsilon\rangle$ with all instances of ε erased. Then $\varepsilon(\phi\langle\varepsilon\rangle)=\varepsilon(\phi)$.

Informal Proof. Let the deepest instance of ε lie in depth k. Invoke C2 k-1x to create a copy of ε in each subspace of depth 1,…,k-1. One more instance of C2 then eliminates the instance of ε at depth k. Invoke to C5 to erase any multiple instances of ε at each of depths 1,…,k-1. Then invoke C2 k-1 times to undo the process described in the second sentence of this paragraph. Doing so erases all instances of ε in $\phi\langle\varepsilon\rangle$. □

28. *LoF* does not prove R2, instead asserting (p. 26): "R2 derives from the fact, proved with J1 and J2, that we can find formulae, [equivalent yet not identical,] which, considered arithmetically, are not wholly revealed." For other discussions of Replacement, see Prior (1962: 24-25), Quine (1982: 44), Bostock (1997: §2.5.D), Halmos and Givant (1998: §§13, 36), Wolf (1998: 86-7), and Cori and Lascar (2000: Cor. 1.23). Replacement is also a commonplace of ponential logic, as it preserves satisfiability and implication (Carnap 1958: T7-1).

Remarks.

1. T13 nicely generalizes C2 and is an example of a *theorem schema*. A formal proof of T13 would require induction on formula depth. Bricken (2002) was the first to propose T13 in this form, naming it Pervasion.
2. Viewing a formula as an ordered tree, any two instances of any subformula can be seen as connected by a *path*. A path is *monotone* if all parentheses crossed along the path are of one type, left or right. Repeated application of C2 allows ε to be copied into, or erased from, any subspace of ϕ whose depth exceeds that of ε, as long as a monotone path connects ε and its copy.

3.2. Tacit Order Irrelevance.

"The spatial relations of written symbols on a two dimensional writing surface can be employed in far more diverse ways than the mere following and preceding in one-dimensional time, and this facilitates the apprehension of that to which we wish to direct our attention. In fact, simple sequential ordering in no way corresponds to the diversity of of logical relations through which thoughts are interconnected." Frege (1882: 87).

"…commutation…may be dispensed with by not recognizing any order of arrangement as significant. Associative transformations…will be dispensed with in the same way…"
Peirce (4.374, 1902).

We may think of *a* and *b* in the pa formula *ab* as linked by a tacit connective called juxtaposition, even though no explicit symbol separates *a* from *b*. Because this connective has been merely tacit thus far, I have said nothing about it. In particular, I have not assumed it commutative or associative, or restricted its scope to the binary. Nor have I assumed that the order or grouping of variables within a pa formula affects its value. I now show how the properties of the PA imply that variable order and binary scope are indeed irrelevant for juxtaposition. Let α, β, and γ be arbitrary pa formulae. Then:

- αβ *commutes*. This is a trivial implication of Table 2-1a. I became aware of the need for a broader axiomatic treatment of ⊥ when attempting to justify why juxtaposition commutes. More generally, all objects within a given subspace can be reordered at will. Hence a formula can be rearranged so that any multiple instances of a subformula in a given subspace can be juxtaposed. By virtue of C5, these multiple instances reduce to a single one. Hence a BA formula behaves not only like a list, as in section 2, but also like a set.[29]
- αβγ *associates*. () and ⊥ are the possible values that α, β, and γ can each take on. (By T3 and T4, α, β, and γ can each stand for any formulae whatsoever.) Then the value of αβγ is the value of a PA formula that is some concatenation of () and ⊥. According to Table 2-1a or T2, this concatenation simplifies to () if at least one of α, β, and γ has value (); otherwise, it simplifies to ⊥. Now T3 assures us that the simplification of any pa formula is unique. Hence the simplification of αβγ cannot depend in any way on the order in which α, β, and γ are paired. Associativity explains why the BA has no need for bracketing, thus freeing up brackets for another use.

29. Peirce's (4.372-584, 1902) logical graphs, discussed in section 6 below, clearly illustrate the irrelevance of order and grouping for the CTV. The first thorough treatment of modern logic, Frege (1879), also featured a two dimensional notation.

From the preceding, I conclude that the variables and sub-formulae that make up any formula may be reordered at will. *LoF* fully acknowledges this useful and important fact, but does not sufficiently highlight it. Juxtaposition indeed commutes and associates, but this fact should not be seen as a fundamental mathematical property. Rather, it merely offsets a metalinguistic typographic convention. All this merely restates in a bit of algebraic dress what I asserted about the BA at the beginning of section 2.

Using parentheses to denote an algebraic operation would appear to have two drawbacks. First, parentheses are already widely employed in mathematics, e.g., to denote ordered pairs and open intervals. Second, parentheses now cannot serve as brackets, i.e., as devices for resolving ambiguity in formulae or for overriding operator precedence. But the discussion in this section shows that the pa is free of all ambiguity arising from infix notation (a defect of the conventional notation for both logic and Boolean algebra). Hence the pa requires neither parentheses for grouping nor the notion of operator precedence.[30] Any ambiguity arising from expressions that mix the pa and conventional mathematical or logical notation (and I do not wish to rule out such expressions) can be resolved by square brackets, '[' and ']'. Since the comma plays no role in BA, I trust that the notations (x,y), denoting the ordered pair consisting of x and y, and $[x,y]=\{t|x\leq t\leq y\}$, denoting a closed interval, are free of ambiguity.

That '()' appears to have a two-sided "exterior" is a drawback of my preferred notation. The following geometric analogy may clarify this. Fold a circle upon itself along any diameter. The two resulting half circles will coincide, meaning that a circle is symmetric about any line going through its diameter. Now carry out this same exercise on '()'. Although '()' is not a regular polygon, it definitely has a geometric centre. But the only lines through that centre that preserve folding symmetry are the two lines parallel to the Cartesian axes.

Moral: A boundary can be thought of as a circle, because a circle appears the same from all angles. I intend "circle" as a metaphor, to suggest that a formula mixing statement variables and boundaries commutes and associates. The sign '()', not being symmetric from all perspectives, regrettably does not highlight this key property of boundaries.

3.3. From Anti-Symmetry to Boolean Algebra via Lattices.

(N.B. *The axiomatization of lattice theory proposed below is not fully satisfactory, but is not invoked elsewhere in this paper*.)

"*When logically analyzed, order turns out to be… inconceivable and incomprehensible to us unless we had the idea… expressed by the term 'negation'. Thus it is that negation, which is always also something intensely positive, not only aids us in giving order to life, and in finding order in the world, but logically determines the very essence of order*." Royce (1917: 540)

I now modify the definition of an equivalence relation (2.3.8) in a crucial way.

3.3.1. Definition. Let A be a set with typical members a and b. Let an infix '=' denote an equivalence relation whose field is a superset of A and whose intended reading is 'equality.' Let aRb denote a reflexive and transitive binary relation whose field is A. If

30. Polish notation dispenses with brackets by writing truth functors in prefix form. Is the persistence of infix notation a form of path dependence?

$(aRb \wedge bRa) \rightarrow a{=}b$, the relation R is *anti-symmetric* and a *partial ordering*. In this case, R is distinct from = and we write $a{\leq}b$. R is said to *partially order A*, and A is a *partially ordered set* (*poset*).

Equivalence relations and partial orderings are both reflexive and transitive, and recur in mathematical logic and foundational mathematics. (The only relation in common use that is neither reflexive nor transitive is set membership itself.) Equivalence relations are also symmetric, but a partial ordering is anti-symmetric, meaning that symmetry holds only for those members of A that are also members of some equivalence class under equality, '='. Anti-symmetry does not require that $a{\leq}b$ be defined for all ordered pairs (a,b). If one or both of $a{\leq}b$ or $b{\leq}a$ is the case, then a and b are *comparable*. If any two members of A are comparable, then A is *linear* or *totally ordered*. Th. 3.3.2 is an important reason why partial order is of interest:

3.3.2. **Theorem**. B is partially ordered, in two ways.

Proof. *Case 1*. Let $(a)b \Leftrightarrow a{\leq}b$ and *true* $\Leftrightarrow$ (). Then $\bot{\leq}() \Leftrightarrow (\bot)()$ [A2] = ()() [A1] = ().
Case 2. Let $(a(b)) \Leftrightarrow a{\leq}b$ and *true* $\Leftrightarrow$ (()). Then $(){\leq}\bot \Leftrightarrow (()(\bot))$ [A2] = (()()) [A1] = (()). □

3.3.2., whose proof requires merely A1 and A2, grounds much of the mathematical substance of BA/Ba. I will not elaborate on this substance much beyond what I say in this section and in §4.1. Posets have a rich mathematical structure, beginning with the next three definitions.

3.3.3. Definition (Machover 1996: 4.2.23). Let ≤ partially order the set A, and let $B{\subseteq}A$. If $\forall x[x{\in}B \wedge x{\leq}a]$ holds for some $a{\in}A$, then a is an *upper bound* of B. If b is an upper bound of B such that $b{\leq}a$, where a is any upper bound of B, then b is the *least upper bound* of B. Replacing '≤' in the preceding three sentences by '≥' requires changing *upper* to *lower* and *least* to *greatest*.

The syntax of lattices is very simple. Let Latin letters range over some poset L. Let α and β be formulae. A lattice features two binary operations, *concatenation*, denoted 'αβ', and *enclosure*, denoted '[αβ]'.

3.3.4a. Definition (Donnellan 1968: 49). L is a *lattice* if ab denotes one of the pair (least upper bound, greatest lower bound) of a and b, and $[ab]$ denotes the other member of the pair.

Note that a matched pair of '[' and ']' simply toggles from one lattice operation to the other. Corresponding the set theoretic definition 3.3.4a is the algebraic definition:

3.3.4b. Definition. A *lattice* is an algebra $\langle L,\cdot\cdot,[\cdot\cdot]\rangle$ of type $\langle 2,2\rangle$ such that $\forall\alpha,\beta{\in}L$, the axioms in Table 3-2 hold.

If ab is the *meet* of a and b, then $[ab]$ is their *join*, and vice versa. 3.3.4a can be derived from 3.3.4b, and conversely (Donnellan 1968: §8). While the underlying algebraic concepts are standard, the notation and terminology in 3.3.4b, and the axioms L1 and L2 are new. The conventional axiomatization of lattices is OI and L3a-b.

Table 3-2. Lattice Axioms.		
L0	Closure	$\forall a,b \in L$, $ab \in L$ and $[ab] \in L$.
OI	Order Irrelevance	Concatenation commutes and associates.
L1	Involution	$[[ab]] = ab$.
L2	Duality	$a = b$ iff $[a] = [b]$.
L3a	Absorption	$a[ab] = a$.
L3b	"	$[a[ab]] = a$.

The idempotence of concatenation, L4, follows immediately from L3a-b:

L4a: $aa=a$. *Dem*. aa [L3b] = $a[a[aa]]$ [L3a] = a.
L4b: $[aa]=a$. *Dem:* $[aa]$ [L3b] = $[a[a[aa]]]$ [L3b] = a. □

A trivial corollary of L4 is $[a]$ [L4a] = $[aa]$ [L4b] = a.

$[a[ab]] = a$ [L2] ↔ $[[a[ab]]] = [a]$ [L1;] ↔ $a[ab] = a$. □

A lattice L is *bounded* iff L has two distinct members called *bounds*, denoted $\bot$ and $\top$ and governed by the axiom:

L5: $\top a = \top \leftrightarrow [\bot a] = \bot$. $[\top a] = \top \leftrightarrow \bot a = \bot$.

L5 is equivalent to $\forall a \in L, \bot \leq a \leq \top$. $\bot$ is the *least element* of L; $\top$ is the corresponding *greatest element*.

Now let enclosure by '(' and ')' denote *complementation* as in the BA, with $(a)=a'$ denoting the *complement* of a. Let the members of L include $\bot$ and $\top$. We can the define:

3.3.5. Definition. A bounded lattice L is *complemented* iff:

- L is closed under complementation;
- For all $a \in L$, the following axioms hold:

L6. $(a)a=\top$.
L7. $\bot=(\top)$ and $(\bot)=\top$.

The next theorem shows how L5 is redundant in the presence of L6 and L7.

3.3.6. Theorem. A complemented lattice is *bounded*.

Proof. $\top a$ [L6] = $(a)aa$ [L4] = $(a)a$ [L6] = $\top$.

$[\bot a]$ [L7; L6] = $[(a'a)a]$ [L3a] = $[(a'a)[a(a'a)]]$ [OI] = $[(a'a)[(a'a)a]]$ [L3b] = $[(a'a)]$ [L6; L7] = $[\bot]$ [L4] = $\bot$. □

$[\bot a]$ [L7; L6] = $[\bot a[a\bot]]$ $(a'a)a]$ [L3a] = $[(a'a)a[a(a'a)]]$ [OI] = $[(a'a)[(a'a)a]]$ [L3b] = $[(a'a)]$ [L6; L7] = $[\bot]$ [L4] = $\bot$. □

Remark. The converse of 3.3.6 does not hold; a bounded lattice is not necessarily complemented.

3.3.7. Definition. A *distributive* lattice obeys the axiom:

L8. *a*[*bc*] = [[*ab*][*ac*]].

3.3.8. Definition. A *Boolean algebra* is a complemented and distributive lattice (Donnellan 1968: §27).

3.3.9. Theorem. The pa is a complemented distributive lattice and hence a Ba.

Proof.[31] Let *L*=*B*, ()=T, and let (*a*′*b*′) in the pa translate lattice meet and *ab* join. (The opposite translation is equally valid. In §4.1, we shall see that (*a*′*b*′) and *ab* constitute a *dual pair*.) I now verify that the pa satisfies L0–L8.

L0. pa concatenation and enclosure always yield a pa formula (T1). A pa formula has a simplification (T3), and that simplification is necessarily a member of *B*.
OI. pa concatenation commutes and associates in precisely the way lattice concatenation does.
L1. ((*ab*))=*ab* is an instance of C1.
L2. *ab*=*cd* iff (*ab*)=(*cd*) follows from the demonstration of $\mathbf{C_2}$, $\mathbf{C_3}$ in §A.5.
L5. [⊥*a*] ⇔ ((⊥)*a*) [A2] = (()*a*) [C3] = ⊥.
L7. If T⇔(), L7 is simply A2.

Table 3-3. Completing the Proof of 3.3.9.		
	ab* ⇔ *ab	***ab* ⇔ (*a*′*b*′)**
L3	*ab*=*a* ⇔ *b*≤*a*. (*a*′*b*′)=*b* ⇔ *a*′*b*′=*b*′ ⇔ *a*′≤*b*′ ⇔ *b*≤*a*.	(*a*′*b*′)=*a* ⇔ *a*′*b*′=*a*′ ⇔ *b*′≤*a*′ ⇔ *a*≤*b*. *ab*=*b* ⇔ *a*≤*b*.
L4	*a*(*a*′*b*′) [C2,2x] = *a*((***ab*′**)*ab*′) [J1] = *a*.	(*a*′(*ab*)) [C2,2x] = (*a*′((***ab***)*ab*)) [J1] = (*a*′) [C1] = *a*.
L6	Let T⇔(). Then L6⇔J0.	Let ⊥⇔(()). Then L6⇔J1. □
L8	J2	(*a*′(*bc*)) [C1,2x] = (*a*′((*b*′)(*c*′))) [J2] = (((*a*′*b*′)(*a*′*c*′))) [C1] = (*a*′*b*′)(*a*′*c*′).

Dilworth (1938: Postulate 1) showed that *ab.c*=*b.ac* and *a*⊥=*a* (taken as an axiom) sufficed to demonstrate that juxtaposition commutes and associates. Byrne (1946: Ths. 2′, 3′) proved that fact from the axiom *abc*=*bca* and the consequence *aa*=*a*; see §A.1. Note that the repeated application of OI to the formulae *abc* and *cba* generates all six possible arrangements of the letters *a*, *b*, and *c*. Henceforth, OI refers to *abc*=*bca*.

A Ba also requires that equality '=' be a congruence relation, defined as follows:

31. *LoF* (Appendix 1), too, proves that Ba is a model of the pa, in the following curious way. It begins with Sheffer's (1913) postulate set for Ba (Table 6-2), containing a single binary functor, the *Sheffer stroke*, whose pa representation is (*ab*) (dually, *a*′*b*′; Table 4-2, bottom row). Sheffer's first two postulates are effectively C1 and J1. His third postulate is an easy pa consequence: *Dem.* **((a(bc)))** [C1,2x] = ((*a*((*b*′)(*c*′)))) [J2] = ((((*b*′***a***)(*c*′***a***)))) [C1] = **((b′a)(c′a))** □. Engineers know the Sheffer stroke as NOR; its dual as NAND. The pa representations of NAND and NOR make it easy to see how they commute but do not associate: e.g., *a*′*b*′= *b*′*a*′ and (*ab*)=(*ba*), but ((*ab*)*c*)=(*a*(*bc*)) is not always the case. In group theory, by contrast, the sole binary operation associates but does not necessarily commute.

3.3.10. Definition. A Ba *congruence relation* is an equivalence relation R also satisfying $aRb \rightarrow a'Rb' \wedge \forall c[c \in B \wedge acRbc]$. (Stoll 1963: 261). For a proof that '=' is a congruence relation, see §A.5.

A Boolean algebra is any algebra defined over a finite poset B, such that B is closed under:

- A binary operation '·' that commutes and associates;[32]
- A unary operation ''', such that $\forall c \in B$, $c' \cdot c = \inf B$ or $\sup B$,

and such that '=' is a binary dyadic relation. Moreover, that $B=\{\perp,()\}$ is partially ordered emerges if the required ordering relation is taken to be $(a)b$. From Table 2-1 we may conclude that $(a)b=()$ for all possible values of a and b except $a=()$ and $b=\perp$,[33] as desired if $\perp \leq ()$ is to be the case. Table 2-1 further implies that juxtaposition commutes and associates, and that B is closed under both juxtaposition and enclosure. If $a \cdot b \Leftrightarrow ab$, $a' \Leftrightarrow (a)$, and T $\Leftrightarrow$ (), the pa can be seen as a minimalist notation for Ba.

We can now see that L3 [L4] is analogous to J2 [J0] of Ba. Hence J0 and J2 nicely distill why a lattice is not necessarily a Ba. As argued above, L4 (and hence J0) is independent of L0–L2. But J2 and C2 are not independent of OI, L2, and J0, by virtue of the following argument. §A.1 derives L2 from J1, C1, and C2. §A.1, in turn, derives J1 from J0 alone, and C1 from J0 and C2. Moreover, *LoF* derives C1 and C2 from J1 and J2. Hence L2 is redundant, given J0 and one of J2 or C2. Finding an axiom independent of OI, L2, and J0 that, when added to these, yields Ba would be an interesting exercise.

Because B has only two members, $c'c$ designates the greatest element of B. Let a, b, and c range over B. Then $ab=a$ and $ab'=c'c$ both imply that B is partially ordered. Specifically, let $\perp \leq ()$. Then $b \leq a$, $ab=a$, and $ab'=c'c$ are all equivalent. More generally, the following theorem shows that $b'a$, $ab=a$ and $(a'b')=b$ all assert the same thing:

3.3.11. Theorem (*Consistency Principle*). $a \leq b$, $a \cup b=b$, and $a \cap b=a$ are equivalent Ba statements, and these in turn have the pa equivalents $a'b=()$, $ab=b$ and $(a'b')=a$.

Proof. See §A.3.

Unlike Ba complementation, pa enclosure can have an empty scope; the result is a lattice bound. By virtue of A2, the other lattice bound is, in effect, the blank page. Bricken (personal communication) argues that the mere presence of ⊥ (or of any other symbol with the same semantics) renders BA indistinguishable from Ba. To fasten on ⊥ in this manner, however, overlooks the signal contribution of BA to our understanding of **2**: the carrier need be no more than empty complementation and the blank page.[34] The improper symbol ⊥ can be seen as just a convenient way to denote the blank page as a lattice bound (§4.1 discusses another justification for ⊥). In all other respects, 3.3.9-11 and the adjacent discussion show that BA and **2** are isomorphic. This fact is independent of the

32. That this connective associates can be demonstrated; see §A.1.

33. If $a=\perp$ or $b=()$, $(a)b=()$ by C3 and A2. Substituting the only remaining valuation, $a=()$ and $b=\perp$, into $(a)b$ yields $(())\perp=\perp\perp=\perp$.

34. In this respect, *LoF* was only walking a trail blazed by Peirce's graphical logic; see §6.1.

ontological status of ⊥ or its denotatum, and does not make BA redundant. Nor does arguing that BA and Ba are isomorphic demean BA, because Ba is a rich subject; see the references under "Boolean Algebra" in the Bibliographic Postscript.

3.3.12. Definition. Let $a,b \in B$ where B is some carrier. a is an *atom* iff $b \leq a \rightarrow (b=\bot \vee b=a)$. If for any $b \neq \bot$, there exists an atom a such that $a \leq b$, then Ba is *atomic*. A Ba is *complete* if every nonempty subset of B has a least upper bound.

We now state a simple version of a famous deep result about Ba, the *Stone Representation Theorem* (SRT):

3.3.13. Theorem (Stoll 1963: §6.5). A Ba is isomorphic to the algebra of all subsets of its set of atoms iff the Ba is complete and atomic.[35]

Every finite B is trivially complete and atomic, simply because B is a bounded lattice. The SRT has the following very interesting consequence:

3.3.14. Theorem. A tautological equivalence holds in every possible Ba iff it holds in **2**.

Proof: Koppelberg (1989: Prop. 2.19 (a) and (d)). The 'if' part of the proof follows easily from **2** being a subalgebra of every Ba whose carrier has a cardinality of two or more. Her concise proof of the converse invokes the SRT and notions beyond the scope of this essay, namely homomorphism and ultrafilter.

Since **2** is a model for BA, a consequence of 3.3.14 is that the advantages of BA as a calculation tool apply to all nontrivial Boolean algebras. Unfortunately, BA as it stands cannot serve as a notation for nontrivial Boolean algebras other than **2**, because $x=() \vee x=\bot$ evaluates to ().

3.3.15. Theorem. The cardinality of B in BA is necessarily 2.

Proof: see §A.3.

Modifying BA so that its models include any Ba with a finite carrier would be a worthy endeavour. Models of such "large" Boolean algebras include certain forms of *mereology*, (Casati and Varzi 1999) the formal theory of part and whole.

3.4. Boundary Algebra and Groupoids.

The BA can be seen as the culmination of a fact pointed out in Huntington (1933): Ba requires but two operations, one binary and one unary. Hence the seldom-noted fact that Boolean algebras are magmas (a.k.a. groupoids). To see this, note that the BA is a commutative:

- *Semigroup* because BA juxtaposition is order-irrelevant;
- *Monoid* with *identity element* ⊥, by virtue of the elementary BA consequence $a\bot = a$.

Groups further require a unary operation, called *inverse*, and an *inverse element*. By J0, the BA inverse element is (). Hence the BA is a ⟨--,(-),()⟩ algebra of type ⟨2,1,0⟩. It would also be an commutative (Abelian) group were the identity and inverse elements equal

35. For a more topologically flavoured exposition of the SRT, see Cori and Lascar (2000: §2.6).

rather than mutual inverses; that is, if one of (*a*)*a*=⊥ or *a*()=*a* were a BA consequence.[36] C2 demarcates BA from other magmas, because C2 enables demonstrating the absorption and the distributive laws central to lattice theory but irrelevant to magmas.

4. pa Semantics: From BA to Boundary Logic.

"Every logical notation hitherto proposed has an unnecessary number of signs."

Peirce (4.12, 1880).

"Yet logic is nothing more than the properties of the act of distinction!" Kauffman (2001: 90).

LoF (pp. 113-17) shows how the CTV and the elementary Boolean algebra of sets are possible interpretations (models) of the pa. Before showing how the pa translates the CTV, I first sketch some facts about the key players in the CTV, the truth functors (functors for short). A functor has an arity $n \in N$. Because *B* has cardinality 2, there are 2^{2^n} possible functors with arity *n*; in particular, there are 16 binary functors. Six of these map *a* and *b* into one of {*a*,~*a*,*b*,~*b*,T,F} and will not detain us. The remaining 10 binary functors are {∧,∨,→,←,↔} and their negations (see Table 4-2). There are 2^{2^1}=4 possible unary functors; of these, only ~*a* need be considered. There are 2^{2^0}=2 0-ary functors, T and F by convention. All functors of arity>2 are redundant, because any formula employing such functors is tautologically equivalent to a formula whose functors all have arity≤2 (Epstein 1995: §II.J.3).

It would seem that there are 5+1+2=8 essential truth functors. In fact, there is ample redundancy among these, in that starting from 2 or 3 functors, the remaining 5 or 6 can be defined. If for any CTV formula, there exists an equivalent CTV formula in which only a subset of these 8 functors appears, the members of that subset are termed *expressively adequate* (abbreviated EA) or *truth-functionally complete* (Bostock 1997: §§2.7, 2.9).[37]

For the purpose at hand, the primitive basis of CTV (e.g., DeLong 1971: 107) consists of the primitive values T and F, and any EA set of functors. *Boundary logic* results from a one-to-one correspondence between the BA and an EA set of functors. Among the binary functors, ∨, ∧, and ↔, commute and associate, just as juxtaposition does in the pa. Denial, ~, is a unary functor whose scope is set by brackets, which is exactly the way (·) works in the pa. We shall see in §4.3 that {∨,~} and {∧,~} are EA; the upshot is two of the three interpretations of the pa shown in Table 4-1.

Table 4-1. Some Interpretations of the pa.

Interpretation	*Primal*		*Dual*
Key Binary Functor	Alternation	Conditional	Conjunction
Implied EA Functor Pair	{∨,~}	{→,F} or {→,~}	{∧,~}
pa *Equivalent*	{((··)),(·)}	{(·)·,(())} or {(·)·,(·)}	{((··)),(·)}
Representation of:			

36. Magma, semigroup, monoid, and (Abelian) group are defined in Burris and Sankappanavar (1981: §2.1).

37. For more on connectives, axiomatics, etc., see the references under *Calculus of Truth Values* in the Bibliographic Postscript.

Alternation	*ab*		(*a′b′*)
Conjunction	(*a′b′*)		*ab*
Conditional	*a′b*		(*ab′*)
Antecedents	1910: *PM*, 6.11	1879: Frege, 1.1 1885: Peirce, 3.11 1956: Church, 1.4c, $\mathbf{P_1}$	1892: Johnson 1897: Peirce EG
Recent Examples	Halmos & Givant (1998: §§8,13)	Machover (1996: §7.6) Bostock (1997: §5.2)	Quine (1982: §1)
Note. *m.n* refers to a numbered system in Prior's (1962) Appendix I.			

I now establish a correspondence between the PA and conventional logic, beginning with the assumption ()⇔T. Table 2-1b immediately reveals that the semantics of (α) are identical to those of ~α, namely ~T=F (A2) and ~F=T. Thus emerges the most salient fact about boundary logic: just as an empty boundary denotes a BA primitive value, *a negation with an empty scope denotes a truth value*. ()() = () (A1) and ⊥⊥ = ⊥ in Table 2-1a imply that juxtaposition is idempotent. The two remaining cells of Table 2-1a reveal that juxtaposition commutes, as discussed in §3.2. Hence by virtue of the PA, αβ can interpret either α∨β or α∧β and the road to a CTV translation of Table 2-1 is now clear.[38]

Since Frege and Peirce, the predominant primitive CTV connective has been the conditional, for which the current notation is '→'.[39] The well-known equivalence *a*→*b* ⇔ ~*a*∨*b* suggests the translation *a*→*b* ⇔ *a′b*. Then note that *a*→F ⇔ *a′*⊥ = *a′*; thus the pa also translates the EA set {→,F}. Table 2-1b now translates as T→⊥ and its converse, ⊥→T. Other possible translations of the pa into the CTV are discussed in §4.2. Table 4-1 summarizes this section. Throughout this paper, "Prior *m.n*" refers to axiom set *m.n* in Prior (1962: Appendix I). Epstein (1995: 407-9) is a more recent and limited survey.[40]

4.1. Duality.

"Algebra includes many formal calculations drawing consequencs from axioms, so the notation should be chosen to make these calculations efficient. The device of juxtaposing two letters… is so efficient that it is used in many different senses…" Birkhoff and MacLane (1998: 70).

Let *S* be a set partially ordered by the relation *R*, and let *a*,*b*∈*R*. Then there exists a relation *R′* that also partially orders *S*, such that *bR′a*=T ↔ *aRb*=T; this is the *principle of duality* for posets (Donnellan 1968: Th. 13; Stoll 1974: 193-94). Let *S*=*B*, [{T,F}] *Rab* ⇔ *a*≤*b* [→], and *R′ab* ⇔ *a*≥*b* [↚], and the duality of Ba [CTV] follows. Ba is typically formulated

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38. 3.3.4 and 3.3.5 also suggest that the CTV with primitive {∨,¬} is a model for the pa.

39. Quine (1982: §3) rightly prefers 'conditional' to the 'material implication' of *PM*, because *a*→*b* does *not* translate "*a* implies *b*", but arguably does translate "if *a* then *b*". I prefer reading *a*→*b* as a synonym for *a*≤*b*, where *a* and *b* are members of some ordered set.

40. For more on systems with {→,F} primitive, cf. Prior (1962: §I.III.1, 3.11-13). Systems based on {→,¬} are quite standard, e.g., Prior 1.1-5, Church's (1956: §20) $\mathbf{P_2}$, Epstein's (1995: 408) **PC**, and Mendelson's (1997: 35) **L**. Systems with {∧,¬} primitive include that of Johnson (1892) discussed in §6.2 below, those of Rosser and of Sobocinski (Prior 6.3), and the modal logics of C. I. Lewis (Prior 11.1). Peirce's existential graphs are the subject of §6.1. For more on historical axiom systems, see §6.2-3 below and Prior (1962: Appendix I).

such that B={0,1}, 1′≡0, and 0≠1. BA duality follows from (())≠() being a trivial consequence of A2, and from (())≠() ⇔ 0≠1.

By 3.3.2, B is partially ordered. B must also be connected, since B has only two members, so that one of ⊥≤() or ()≤⊥ must be the case. ()=⊥ leads to triviality; hence the inequalities must hold strictly. Thus far, I have tacitly assumed ⊥<(). The nearest *LoF* gets to the content of this paragraph is the first complete paragraph on p. 113.

4.1.1. Definition. The BA semantics that flow from assuming ⊥<() [()<⊥] make up the *primal* [*dual*] *reading*. Each reading is the *semantic dual* of the other. *Duality* refers to the fact that Ba, BA, and logic can all be carried out under either reading. To switch from one reading to the other, *mutatis mutandis*, is known as *dualization*.

Duality is little more than an interesting consequence of B's being an ordered set. By an interpretation of BA I mean a one-to-one correspondence between B and another set, and there are two possible such correspondences between B and {T,F}. Thus far, I have assumed T⇔(). The dual reading begins with T⇔⊥. Conjunction now interprets juxtaposition, and the conditional interprets (ab'). Thus the dual of {∨,~} is {∧,~}.

Under the dual reading, Table 2-1a is now the table for Boolean multiplication, and Boolean and numerical multiplication yield the same result when the carrier is assumed to be {0,1}. Perhaps surprisingly, Table 2-1 and T1-T7 hold under both interpretations. Likewise, the rules defining and simplifying PA and pa formulae do not change. Hence the syntax of BA is invariant under dualization. PA duality is merely a semantic affair, namely a switch from one mapping of B onto {T,F} to the other.

Matters are a bit more involved for the pa, because dualization alters the semantics of juxtaposition. The semantics of a formal system are known as its *truth definition* or *Boolean valuation* (Smullyan 1968: §I.2, Def. 1). Recall that an atomic valuation (3.1.3) assigns one of () or ⊥ to every variable. The pa then has the following trivial truth definition:

4.1.2. Definition. *A Boolean valuation for* BA. Let φ,δ be metalogical notation for BA formulae, and let the value of φ be |φ|, given some atomic valuation. All molecular formulae then evaluate to either () or ⊥ by recursive application of two elementary rules: |(φ)|=(|φ|), and |δφ|=max[|φ|,|δ|], where max[(),⊥]=() [=⊥] under the primal [dual] reading.[41]

This truth definition follows trivially from Table 2-1. A tautology can now be defined as a formula whose value is invariant to the choice of atomic valuation. Moreover, φ=δ is a tautological equivalence if |φ|=|δ| holds for all atomic valuations. Some further definitions:

4.1.3. Definition (adapted from Halmos and Givant 1998, §22): Let $\alpha = \alpha\langle a_1,\ldots,a_n\rangle$ be a formula containing the atomic formulae $a_1,\ldots,a_n$. α is the *primal*, $(\alpha\langle a_1,\ldots,a_n\rangle)$ the *complement*, $\alpha\langle a_1',\ldots,a_n'\rangle$ the *contradual*, and $\alpha^D = (\alpha\langle a_1',\ldots,a_n'\rangle)$ the *syntactic dual* (*dual* for short). The dual of the dual is the primal; hence a primal and its dual are known as a *dual pair*.

41. For more on truth definitions, see Bostock (1997: §§2.4, 3.4) and Hodges (2001: §3).

4.1.4 answers the following question: if the equation $\alpha = \phi$ holds under some or all atomic valuations, what is true of α^D and ϕ^D?

4.1.4. Duality Theorem. $\alpha = \phi \leftrightarrow \alpha^D = \phi^D$.

Proof. See §A.9.

The Duality Theorem is the basis for the *duality principle* characterising Ba and CTV, which says that the dual of a tautology is also a tautology. Keeping in mind that α in 4.1.3 is a tautology if α=() or ⊥ for all possible valuations of $a_1,...,a_n$, 4.1.4 says more. If a formula or equation is tautologous under some interpretation, then its contradual and dual are also tautologies under that same interpretation.[42]

Duality, about which *LoF* is silent, is another compelling reason for an explicit symbol denoting the unmarked state. There is no syntactical or proof-theoretic ground for preferring one interpretation over the other. Spencer-Brown preferred the primal reading, claiming that *a'b* is a more economical represention of the conditional than (*ab*') (*LoF*, p. 113-14).[43] There is, however, a pragmatic reason for conjunction, not disjunction, as the preferred reading of concatenation. I agree with Prior when he wrote:

> "…'and' and 'not' are the only operators which are quite unambiguously truth functional in ordinary speech; truth functional interpretations of other ordinary-speech connectives all wear at times an air of artificiality."
>
> Prior (1962: 254).

In §6.1, we shall see that Peirce too came to prefer the dual reading.

4.2. Boundary Logic.

"*…everything in pp. 98-126 of* Principia Mathematica *can be rewritten without formal loss in the one symbol* '()'*… Allowing some 1500 symbols to the page, this represents a reduction of the mathematical noise-level by a factor of more than 40,000.*" *LoF*, p. 117.

Table 4-2 translates the ten nontrivial CTV binary connectives into the pa, assuming the primal reading. Each row of Table 4-2 contains a dual pair; hence the connectives can be grouped into two groups of five, I and II, with each group being the dual of the other. Connectives sharing the same numerical identifier (shown in the two middle columns) can be derived from each other via negation. Let *a** stand for either *a* or *a'*; *a** is a *literal*. The *simple* connectives are those that can be described by *a***b** or the duals thereof; these are *ab*, *a'b*, *ab'*, *a'b'*, and their duals.

Note that the assignment of () to either T [⊥≤()] or F [()≤⊥] is arbitrary. But once a choice is made, the pa representation of all connectives is determined. Table 4-2 translates *ab* as *a*∨*b*, and its dual, (*a'b'*), as *a*∧*b*, both as per the first column of Table 4-1. Likewise, either

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42. Quine (1982: §12) states five "laws of duality." The first follows from semantic duality; the second, proved in §A.9, defines syntactic duality. His third law is α=() ⇔ α^D=⊥; the fourth, $(\alpha \to \phi) \leftrightarrow (\phi^D \to \alpha^D)$; the fifth, 4.1.4. On duality, also see Bostock (1997: §2.10).

43. Spencer-Brown makes too much of this, especially if one downplays the conditional in favor of conjunction/alternation. Moreover, while (*ab'*) has more symbols than *a'b*, (*ab'*)=(()) is equivalent to *b'a*=(), which has no more symbols than *a'b*=().

$a'b'$ or (ab) translates the Sheffer stroke, $a|b$. These translations render obvious that | can be read as “not and” and as “if a then not b”; the latter reading suggests more strongly, perhaps, the peculiar expressive power of the Sheffer stroke. The duality of $a'b'$ and (ab) points to “not or” as the semantic dual of the Sheffer stroke.[44]

Table 4-2.
The 10 Nontrivial Binary Connectives (Functors).

Primal									**Dual**
Name	*Logic*	*Sets*	**pa**			**pa**	*Sets*	*Logic*	*Name*
Alternation	$a\vee b$	$a\cup b$	ab	1	5	$(a'b')$	$a\cap b$	$a\wedge b$	Conjunction
Conditional	$a\rightarrow b$	$a\subseteq b$	$a'b$	2	4	(ab')	$b\sim a$	$a\nleftarrow b$	Difference
Symmetric Difference	$a\nleftrightarrow b$	$a\Delta b$	$(a'b)(ab')$	3a	3a	$((a'b)(ab'))$	$a\subseteq b\subseteq a$	$a\leftrightarrow b$	Biconditional
			$((a'b')(ab))$	3b	3b	$(a'b')(ab)$			
Converse	$a\leftarrow b$	$a\supseteq b$	ab'	4	2	$(a'b)$	$a\sim b$	$a\nrightarrow b$	Difference
Sheffer stroke	$a\mid b$	$\overline{a\cap b}$	$a'b'$	5	1	(ab)	$\overline{a\cup b}$	$a\downarrow b$	NOR

Note. Each row contains a *dual pair*. Items with the same number are negation pairs. The six remaining binary connectives are uninteresting as they map $\{a,b\}$ into one of a, b, $\sim a$, $\sim b$, **T**, and **F**.

The meaning of CTV duality should now be clear: for any statement α, there exists an equivalent statement α^D derived by interchanging $\wedge$ and $\vee$, $\rightarrow$ and $\nleftarrow$, $\leftrightarrow$ and $\nleftrightarrow$, | and $\downarrow$, and T and F. More generally, under either interpretation, the pa representation of conjunction is the dual of the pa representation of alternation, and the same dual relation holds for the conditional and the negation thereof.

Duality reveals that the conventional syntax for Ba and CTV are uneconomical. A given pa formula enjoys a multiplicity of CTV translations, revealing the ample redundancy inherent in the CTV. For instance, take the well-known De Morgan’s laws, $\sim(a\vee b)\leftrightarrow(\sim a\wedge\sim b)$ and $\sim(a\wedge b)\leftrightarrow(\sim a\vee\sim b)$. The pa translation of these laws, $(ab)=(ab)$ and $a'b'=a'b'$, immediately reveals that these laws are trivially true and form a dual pair.

Because the algebra of sets is a model for **2**, it is also a model for the pa. Let U be the universal set, $a,b\in$U, and $\varnothing$ be the null set. Then the columns headed by “Sets” show how the algebra of sets and the pa are equivalent.

Table 4-3 translates the *LoF* consequences into CTV notation, using Table 4-2 as the key. For each *LoF* consequence, Table 4-3 also supplies a name, if the conventional literature provides one, and the number of the corresponding tautology in Kalish et al (1980: §II.11) (KMM), an unusually comprehensive list of tautologies. *LoF* says very little about how J1-C9 relate to the extant literature on logic and Ba. J2, C1, C3, and C5 should be very familiar. J1 is the Law of Excluded Middle (LEM); C2, Johnson’s (1892: 342) Law of Exclusion; C3 means that () is a lattice upper bound, cf. 3.3.5; C4, the biconditional corresponding to an axiom in conditional form proposed by Peirce (W5: 162-90, 1885). C6 is the Law of Elaboration or Development, so called by Bostock (1997: 41). C7 and C9 are

44. The misconception that the pa is little more than a new notation for the Sheffer stroke (Grattan-Guiness 2000: 557; Wolfram 2002: 1173) may stem from hasty readings of *LoF*’s Appendix 1.

not well known. If *a* and *b* on either side of C9 were to trade places, the two sides of C9 would then form a dual pair.[45]

Table 4-3. The Standard Reading of the *LoF* Initials and Consequences.			
LoF	**Conventional Notation**	**Name**	**KMM**
J1	$\sim(a\to a) \leftrightarrow \bot$	Law of Contradiction	59
J2	$(a\vee r)\wedge(b\vee r) \leftrightarrow (a\wedge b)\vee r$	Distributive Law	62
C1	$\sim(\sim a) \leftrightarrow a$	Law of Involution	110
C2	$(b\vee a)\to a \leftrightarrow b\to a$	Law of Exclusion	73
C3	$\top\vee a \leftrightarrow \top$	*B* has an upper bound.	
C4	$((a\to b)\to a) \leftrightarrow a$	Peirce's Law	23
C5	$a\vee a \leftrightarrow a$	Law of Tautology	47
C6	$(a\wedge b)\vee(a\wedge\sim b) \leftrightarrow a$	Law of Elaboration	68
C7	$[(a\to b)\wedge\sim c] \leftrightarrow \sim[(a\vee c)\wedge(b\to c)]$		
C9	$[(a\to r)\wedge(r\to\sim b)] \leftrightarrow \sim[(a\vee r)\wedge(r\to b)]$		

LoF invokes J1-C3 104x, C4-C9 15x, and C5-C8 a mere 6x. *LoF* invokes C2 more often than any other consequence, C1 excepted. C2 allows a subformula to be copied into and erased from any subspace deeper than the shallowest instance of itself (with the proviso that a subformula cannot be copied into a part of itself). While not a standard part of elementary logic, C2 is at once a trivial corollary of the Consistency Principle, 3.3.11, and a powerful tool for BA demonstrations (§§5.0, 5.2). I say more about C2 in §A.7.

Perhaps all I have done thus far is to employ the following elementary reasoning to eliminate all explicit truth functors from the syntax. Alternation and conjunction commute and associate; hence mere juxtaposition suffices to notate either. Brackets are then free to notate negation. It is well known that negation and one of conjunction or alternation are EA. Hence brackets are the only explicitly truth functional notation required. QED. Equivalently, recall that $\{\to,\bot\}$ is EA and that $(a)b \Leftrightarrow a\to b$ and $(a)\bot \Leftrightarrow \sim a$. Hence a single two place functor, (−)−, and the constant ⊥ also suffice to express all truth functors. To express juxtaposition, note that *ab* [C1] = **((*a*))*b*** [C2] = ((*a*)***b***)*b*.[46]

With the pa and its CTV interpretation in hand, and given our definition of a Ba, we can speak to the algebraic structure of the CTV (cf. Stoll 1963: 267-76). Let S_0 be a set of CTV atomic formulae, and let S be the set whose members are all possible formulae constructed from members of S_0 by conjunction (or alternation) and denial. Let logical equivalence '=' be the congruence relation (cf. 3.3.10) Ba requires. A congruence relation partitions its field into equivalence classes; let $S/=$ be the set of equivalence classes resulting from '='. Define ⊥ as $\sim a\wedge a$, $\forall a\in S$, and T as $\sim\bot$. Then $\langle S/=,\wedge,\sim,\bot\rangle$ is a Ba,

45. *PM* (*2–*5) proves many tautologies including the following ones from *LoF*: J1, 3.24; J2, 4.41; C1, 4.13; C2, 2.621 & 2.67; C5, 4.25; C6, 4.42; C8, 4.4. Other lists of tautologies include Rosser (1953: Theorem VI.6.1), Carnap (1958: T8-2, T8-6), Wolf (1998: Appendix 3), and Cori & Lascar (2000: §1.2.3). In Ba, J1 is known as "complementarity"; C3, "union"; C5, "idempotence"; De Morgan's laws, "dualization". For more on the relation between J1-C5 and conventional logic and Ba, see §A.6.

46. In the language of universal algebra (Abbott 1969: §2-5), the pa is a $\langle B,(-)-,\bot\rangle$ algebra of type $\langle 2,0\rangle$. A model of this algebra is Church's (1956) $\mathbf{P_1}$; see Table 4-1.

specifically the "free Boolean algebra generated by S_0 under =," more commonly known as a *Tarski-Lindenbaum algebra*.

A Historical Digression on Notation.

What I enclose in parentheses, Spencer Brown places under ⏋, the 'mark'; e.g., (*a*)*b* and (*ab*) correspond to $\overline{ab}|$ and $\overline{a}|b$ in *LoF*. (Martin Gardner (*Scientific American* 1980 (2): 14) deemed *LoF*'s notation "eccentric.") *LoF*'s notation has antecedents. In a paper written 1880 but not published until 1933, Peirce (4.12-20) proposed to notate Ba with concatenation, interpreted as NAND, and brackets. This notation is that of this paper, but for his limitation of concatenation to a binary scope. Kauffman (2001), citing Peirce (1976: 106-15), an excerpt from a manuscript titled "Qualitative Logic," written in 1886 but not published in full until 1993 (*W*5: 323-71), points out that Peirce later fused the overbar (denoting Boolean complementation) to the Boolean '+' (OR) to create a symbol Peirce called the "sign of illation," closely resembling the '⏋' of *LoF* and having the same semantics.[47] Peirce saw that his sign of illation sufficed for Ba and syllogistic logic. Kauffman also notes that the '·⌈··' notation of Nicod (1917) likewise has the functionality of '·⏋·', but does not mention that Nicod on occasion commuted '$a\overline{|b}$' to '$\overline{b|}a$'. Many authors, as well as earlier versions of this paper, denote the denial of *a* by ¬*a* rather than ~*a*. I prefer to reserve '¬' for intuitionist negation.

4.3. The Enigmatic Degeneracy of BA.

The expressively adequate (EA) subsets of the functors in common use are {→, F/~} and {∧/∨, ~}. Table 4-1 shows how these four EA functor sets map into BA. This mapping reveals a curious detail: corresponding to each EA functor set is a pair of BA formulae, one involving one boundary, the other two. The question naturally arises as to whether this is true of the Sheffer stroke and its dual, and all nine EA functor sets with two members (Wernick 1942: 132), consisting of four dual pairs and the self-dual set, {→,↛}. Row 1 of Table 4-4 shows how the Sheffer stroke follows from the pair {(--),(())}. Rows 2 through 5 show how seven of the nine two-member EA functor sets can be derived by inserting 1 or 2 letters into (), and 0 or 2 letters into (()). The sixth row reveals that the dual pair of EA functors involving ↔ cannot be represented in this manner, suggesting that ↔ should be seen as a tacit conjunction of conditionals.

Table 4-4.
Building the Nine EA CTV Functor Pairs from () and (()).

Pa		*EA CTV Functor Pairs*			
		primal		*dual*	
(*ab*), (())	2,0	a↓b	F	a | b	T
(*a*)*b*, (())	2,0	a→b	F	~(b→a)	T
((*a*)*b*)*b*=*ab*=((*ab*)), (*a*)	2,1	a∨b	~a	a∧b	~a
((*b*)*a*), (*a*)	2,1	b↛a	~a	a→b	~a
(*a*)*b*, ((*b*)*a*)	2,2	a→b, b↛a			
(*a*)*b*, ((*a*)*b*)((*b*)*a*)	2,4	a→b	a↮b	b↛a	a↔b
Source for EA Functor Pairs: Wernick (1942: 132).					

47. Peirce's manifold contributions to mathematics, logic, and semiotics inform Kauffman's discussion in other ways.

The first five rows of Table 4-4 capture the essence of the correspondence between Ba and the CTV. The reader is welcome to explore further the symmetries inhering in the two leftmost columns of Table 4-4.

The pa suggests that expressive adequacy requires two capabilities, namely a way of:

- Concatenating subformulae. Let these ways be *a*b* and (*a*b*);
- Enclosing subformulae. We may create *a′* in one of three ways:
  - Invoke it outright;
  - Given (*ab*), set *a*=*b* so that (*aa*) [C5] = *a′*;
  - Given *a′b*, set *b*=(()) so that *a′*(()) [C4a] = *a′*.

Note how (()) follows from (*b′a*): either erase *a* and *b*, or let *a*=*b*, then J1. Note that the Sheffer stroke is EA by itself. As () and (()()) denote distinct primitive values, () alone suffices for all of truth functional logic.

Table 4-5. How *a*b*, (*a*b*), and *a′* Yield
The Sheffer Stroke and the Nine EA Functor Pairs.

	Interpretation:		*Needed to Obtain* (*a*):			
	primal	*dual*	*Assume*	*Interpretation*		(*a*) =
Commute						
ab	a∨b	a∧b	*a′*	~a		---
(*ab*)	a↓b	a|b	---	---		(*aa*)
Do not Commute				*primal*	*dual*	
a′b	a→b	b↛a	*a′*	~a		---
"	"	"	(())	**F**	**T**	*a′*(())
"	"	"	(*c′a*) †	c↛a	a→c	*a′*(*a′a*)
"	"	"	(*c′a*)(*c′b*)	a↮c	a↔c	*a′*(*a′a*)(*a′a*)
† Self-dual row.						

Tables 4-4 and 4-5 reveal that all possible EA functor pairs can be obtained by inserting letters in certain ways into () alone, or into () and (()). Hence there is a sense in which the two members of *B* encapsulate all EA functor pairs. The members of *B* can be seen as the operators (-)- and ((--)-), where '-' indicates a possible location of a letter. BA does not syntactically demarcate operators from operands; only in context can the operators (-)- and ((--)-) be distinguished from the operands () and (()), the primitive values.

4.4. pa: Metatheory.

Every pa formula has a *normal form*, a fact repeatedly invoked in proofs of pa metatheory.

4.4.1. Definition. Let the pa formula α contain *n* variables so that $\alpha = f(a_1,...,a_n)$. The *normal form*, *NF*, is a formula tautologically equivalent to α having the form:

(#) $(a_i^* ...)_j \Leftrightarrow \bigvee_j [\bigwedge_i a_{ij}^*]$.

All variables in (#) appear as literals.

$(a_i^* ...)_j$ is the *jth disjunct*. The ranges of the indices *i* and *j* begin with 1 and are finite; otherwise, these ranges are deliberately unspecified, if only because the NF is not unique. Also, either *i* or *j* may in some cases not exceed 1. If the *j*th disjunct is (⊥), then the entire NF degenerates to (); if it is (()), the *j*th disjunct vanishes. The NF can be seen as the analog of a polynomial in ordinary algebra. It is easier to parse a NF if the variables in each disjunct appear in lexicographic order, moving from left to right. This reordering is allowed because the variables in any disjunct can be reordered at will, but is not a mathematical imperative.

Given any Ba/CTV formula, there exists an equivalent formula resembling the rhs of (#), namely a series of subformulae linked by alternation. Each of these subformulae in turn consists of literals linked by conjunction. This is the disjunctive normal form (DNF), whose dual is the conjunctive normal form (CNF). *LoF* is silent about the well known Ba/CTV result that there exists a CNF/DNF dual pair equivalent to any formula. In the pa, the distinction between the DNF and the CNF is merely semantic.[48]

4.4.2-7 lay down the metatheory of the pa. The corresponding proofs are in §A.10.

4.4.2 (T14). Let α be a formula such d_α^* >2. Then α can be transformed, by taking steps, into an equivalent formula β such that d_β^* =2.

Remarks.

1. In *LoF*, T14 only serves to help prove T15 and T17.

2. Crucial to the proof of T14 is the ability of C7 to transform any subformula of depth 3 into an equivalent sub-formula of depth 2. Invoking C7 repeatedly, beginning at each point in α with depth=d_α^*−3, transforms α into an equivalent formula with maximum depth≤2. The Appendix proof views a pa formula as an ordered tree; the *LoF* proof does not.

3. Read from left to right, both C4 and C9 can also be seen as depth reduction tools. C4 [C9] reduces a subformula of depth 2 [3] to one of depth 0 [2].

4. Note that no formula in J1-C9 is more than two parentheses deep, the left side of C7 and C9 excepted.

4.4.3 (T15). Let the pa formula α⟨*v*⟩ contain more than two instances of the variable *v*. Then α⟨*v*⟩ can be transformed, by taking steps, into an equivalent formula β⟨*v*⟩, such that β⟨*v*⟩ contains at most two instances of *v*.

Remark. In *LoF*, T15 only serves to prove T17. T15 is essentially a simple form of the following well-known Ba theorem (Hohn 1966: 229, Lemma 2), recast into pa notation as follows:

Let f be a truth function whose arguments are $x_1,...,x_n$. *Then* $f(x_1,...,x_i,...,x_n) = (f(x_1,...,(),...,x_n)'x_i')\ (f(x_1,...,\bot,...,x_n)'x_i)$, 1≤*i*≤ *n*.

---

48. For more on the CNF and DNF see, e.g., Quine (1982: §10), Bostock (1997: §2.6), Halmos & Givant (1998: §38), and Cori & Lascar (2000: §1.3.2). Bostock defines the DNF so that each disjunct includes all *n* variables, in which case *i* in (#) necessarily ranges over 1 to *n*. He does this so that the truth table corresponding to α can be easily recovered from the DNF. This stipulation is unnecessary here, because truth tables play no essential role in the pa.

T14 and T15 together guarantee that every pa formula has an NF equivalent, whose depth does not exceed 2 and that contains at most two instances of any given variable.

4.4.4 (T16). If two or more formulae are equivalent in every case of one variable, they are equivalent.

Remarks.

1. I propose to restate this enigmatic theorem as "Let the variable *v* appear in one or both of the formulae α and β, and let $v_i = |v|$. Let $\alpha\langle v = v_i \rangle$ and $\beta\langle v = v_i \rangle$ be $\alpha\langle v \rangle$ and $\beta\langle v \rangle$ with *v* set to v_i. If $\alpha\langle v = () \rangle = \beta\langle v = () \rangle$ and $\alpha\langle v = \perp \rangle = \beta\langle v = \perp \rangle$, then α=β." The converse is also true.

2. *LoF* maintains that T16 justifies the decision procedure described in §5.1.[49]

3. Erasing every instance of a variable is equivalent to setting that variable equal to ⊥. Hence T16 has another implication, heretofore unmentioned: a tautology remains a tautology when every instance of any variable is erased.

4. Prior (1962: §I.III.4) shows that the CTV can be derived from a single metalogical axiom. Its pa translation, $(\alpha\langle v = () \rangle)(\alpha\langle v = \perp \rangle)\alpha\langle v \rangle = ()$, reveals that it is an instance of a clause (§5.3) and is equivalent to T16.

4.4.5 (T17). The pa is complete.

Remark. A logic is *complete* if for any tautology α, one of α or ~α can be proved from the axioms/initials, using the inference rules. The pa inference rules are in fact R1 and R2, although *LoF* does not make this explicit. Moreover, the completeness asserted by T17 is of the strong sort (*LoF*, p. 119), because adding an initial that cannot be proved from the existing initials would render the pa unsound. Finally, if a formula simplifies to a member of *B*, T17 also implies that there exists a corresponding tautology in the pa.

While T17 is arguably the most important (meta)theorem of the BA, it is not an unexpected result because, as we shall see, the CTV and **2** are both models of the pa, and the completeness of these models is well established. The *LoF* proof of T17 requires all *LoF* consequences except C4 and C5; hence T17 can be seen as the culmination of chapters 1-10 of *LoF*. The proof of T17 resembles Quine's (1938) proof (which *LoF* cites) that the CTV is complete, in that both proofs proceed by strong induction on the total number of variables in hypothetical pa formulae in normal form; this is the only explicit instance of an inductive proof in *LoF*. Crucial to this proof are two facts:

- Every pa formula has, by virtue of T14 and T15, an equivalent in normal form;
- A1 [A2] is a tautological equivalence because it is an instance of C3 [C1].[50]

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49. T16 is Cole's (1968: 346) rule R2. *LoF* (p. xvii) states that T16 resembles a lemma in Quine's (1938) proof that the CTV is complete. *LoF* neglects to mention Quine's later invention of TVA, which is essentially identical to the *LoF* decision procedure for which T16 is the main justification. Prior's (1962: 53, (3); 58-60) re-exposition of Quine's proof includes proving a lemma that is essentially T16. The *LoF* proof of T16 (restated in §A.10) is vastly easier than either Quine's or Prior's.

50. §A.1 demonstrates every consequence the proof of T17 in §A.10 requires. §A.10 also includes a pa version of Kneebone's (1963: 48) proof that the CTV is complete, perhaps the simplest proof extant. Post (1921) was the first to prove the CTV complete (for a summary,

If a logic is *sound*, then there does not exists a formula α such that both α and ~α can both be proved in that logic. If both α and α′ were provable in the pa, then (α′α)=(), contradicting J1. Hence the soundness of the pa follows from its completeness. If there exists a formula α such that both α=() and ~α=(), then all formulae can be equated to ().[51] In short, if a logic is both sound and complete, then for any statement α of that logic, α *is a tautology* ↔ α *is provable*. A simple direct proof of soundness goes as follows:

4.4.6. Theorem. The pa is sound.

Proof: The pa initials are tautologies. When R1 is applied to a tautology, the result is a tautology. The rule R2 is valid only when applied to tautologies, in which case it yields a tautology. R1 and R2 are the sole rules of inference. Hence any demonstration results in a tautology. □

Remark. Any formal system for which a proof of this nature goes through is said to possess the *hereditary property* (DeLong 1971: 134).

4.4.7 (T18). The initials J1 and J2 are independent.

Remark. That is, neither initial can be proved from the other alone. The very concise *LoF* proof of T18 is wholly syntactic and predicated on there being only two initials. Given that J1 and J2 can be demonstrated from C6 alone (cf. §6.2 and §A.4), and that I prefer to make OI explicit, so that there are in fact three initials, T18 loses some of its luster. On axiom independence, also see Hunter (1971: §36) and Bostock (1997: 195-99).

5. Proof and the pa.

"*As a material machine economises the exertion of force, so a symbolic calculus economises the exertion of intelligence. …the more perfect the calculus, the smaller the intelligence compared to the results*." Thus begins Johnson (1892).

What is conventionally termed a proof, *LoF* calls a *demonstration*, meaning a sequence of steps showing that two pa formulae, e.g., ϕ and γ, are equivalent. The consequence ϕ=γ results. Each step invokes an axiom, initial, or previously demonstrated consequence. R1 or R2 are seldom explicitly invoked in demonstrations. A demonstration is carried out entirely *within* an object language, the pa or other formal system. The correctness of a

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see Hunter 1971: §30). For the completeness proofs of Hilbert and Ackerman, and of Quine (1938), see Prior (1962: §§I.II.3, I.III.2). These proofs require over 20 tautologies apiece. Hunter (1971: §§31, 32) restates the proofs of Kalmar and Henkin. Kalmar's proof is the basis for those of Stoll (1963: Th. 9.2.3), Epstein (1995: §§II.L.2-3, II.M.2), and Mendelson (1997: 1.14). These proofs require the Deduction Theorem and at least a dozen lemmas. The proof of T17 merely requires J1-C7 and C9. Henkin's proof has the advantage of yielding the Compactness and Interpolation theorems as corollaries. Finally, there is the cryptic proof of Anderson and Belnap (1959), restated less tersely in Hunter (1971: §37.4). Nowadays, the preferred approach to proving the CTV complete relies on refutation trees (e.g., Bostock 1997: §§4.6-7; Smullyan 1968: chpt. II). The reader is welcome to peruse this literature and to draw a conclusion about the economy of the pa.

51. The proof is in §A.8. On soundness, see Hunter (1971: §§24, 25a,b, 28) and Hunter's references to Church (1956). Also see Inconsistency in Table 5-3.

demonstration can be verified algorithmically, at least in principle. In *LoF*, *proof* applies only to (meta)theorems. A proof is necessarily metalinguistic, may draw on any device from mathematics or logic, and cannot be verified by algorithm.[52]

A *calculation* is a demonstration that methodically eliminates all variables from a pa formula. If this elimination is successful, by 2.2.3 and T3 the resulting PA formula simplifies to a primitive value.[53] The following algorithm may be helpful:

5.0.1. Algorithm.

1. Alter α in a series of steps, justifying each step using one or more initials and consequences, with the objective of finding a formula *β* equivalent to α, from which all variables have been eliminated;

   Remark. C1, C2, and J1 are espfecially powerful here. But C4-C6 too may be seen as tools for eliminating redundant variables. C7 can be useful as a last resort.

2. If *β* exists, it will be a PA formula, and by T3-T4, any PA formula can be simplified using A1 and A2;

3. Hence if *β* exists, α is a tautology. If *β* does not exist, α is satisfiable.

End of Algorithm

Cal. Signals the beginning of a calculation. To verify a pa equation of the form ϕ=γ, first calculate ϕ′γ, denoted LR (*left* to *right*), then calculate γ′ϕ, denoted RL (*right* to *left*). If both conditionals reduce to the same formula, ϕ=γ holds by T7. Equivalently, ϕ=γ holds if a calculation reduces the biconditional ((ϕ′γ)(γ′ϕ)) to ().

A demonstration of ϕ=γ consists of a sequence of formulae, beginning with ϕ. Each formula in the sequence results from a *step*, inferred from one or more preceding formulae in a manner to be discussed shortly. The demonstration terminates when a step results in the formula γ. *Hilbert demonstration* is the rather pedantic name I propose for an exercise of this nature. A Hilbert demonstration is a variant of common-garden mathematical proof. By virtue of the completeness of the pa (T17), there exists a Hilbert demonstration for any tautology. But T17 gives us no clue on how to find that demonstration; the proof of T17 suggests restating ϕ=γ in normal form. Hence if both ϕ and γ are hypothesized from the outset, it is usually easier to verify ϕ=γ by calculation.

Time was, Hilbert demonstrations were the only verification technique. During the past 50-odd years, however, the reigning fashion among logicians (in contrast to mathematicians doing logic) has been, first natural deduction and sequent calculi, both derived from Gentzen's work in the 1930s, then refutation trees (Bostock 1997: §§4.1-4, 6.2, 7.4).

5.1. A Decision Procedure.

"An operand in the primary algebra is merely a conjectured presence or absence of an operator." (*LoF*, p. 88)

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52. The distinction between proof and demonstration is not peculiar to *LoF*. See Quine (1951: 319-22), Machover (1996: 120), and Mendelson (1997: 36, fn †).

53. "Calculation", a word not appearing in *LoF*, is my shortening of Dijkstra and Scholten's (1990: 21) *calculation proof*, meaning a series of steps that transform a given Boolean expression into *True*.

A different proof procedure, very much in the spirit of the PA, follows from T16: *If formulae are equivalent in every case of one variable, they are equivalent, and conversely*. Let $f\langle a\rangle$ and $g\langle a\rangle$ be formulae containing the variable a. Let $\phi\langle()/a\rangle$ denote the uniform replacement of a by (), and so on. If $f\langle()/a\rangle=g\langle()/a\rangle$ and $f\langle\bot/a\rangle=g\langle\bot/a\rangle$, T16 concludes that $f=g$, regardless of the values of any other variables appearing in f and g. Hence $f=g$ is a tautological equivalence.

Consider the following algorithm for determining the satisfiability of f. Evaluate $f\langle()/a\rangle$ and $f\langle\bot/a\rangle$; let these be two *branches*. Then note the following facts:

- Setting an unprimed [primed] variable to ⊥ [()] makes the variable vanish;
- Setting an unprimed [primed] variable to () [⊥] results in ();
- If both branches result in the same formula, they terminate;
- If a branch simplifies to () or ⊥, it terminates.

At any stage, a branch may be simplified by invoking a consequence; in this regard, J0, J1, and C3 are especially useful. A branch containing a recognisable tautology terminates; set it to whichever of () or ⊥ is applicable. Repeat this procedure, each time selecting the remaining variable with the most instances so as to save labor. I find it useful to notate in the left margin of a row the variable being instantiated in that row. The structure of this algorithm is that of a tree; the algorithm terminates when all branches of the tree have terminated.

If all branches of the tree terminate with the same formula, the original formula is a tautology. If the branches terminate in a mixture of () and ⊥, the formula is satisfiable, with the pattern of () and ⊥ indicating the satisfying atomic valuations. This algorithm sufficiently resembles Quine's (1982: §5) *truth value analysis* (TVA) that I appropriate the name.[54]

Fig. 2 gives, by way of example, a TVA proof of Leibniz's (1969: 244) *Praeclarum Theorema*, $[(p\rightarrow r)\wedge(q\rightarrow s)]\rightarrow[(p\wedge q)\rightarrow(r\wedge s)]$:

Fig. 2.

Verifying Leibniz's *Praeclarum Theorema* via Truth Value Analysis.

		$(((p'r)(q's)))((p'q'))(r's')$	
p	$(((r)(q's)))((q'))(r's')$		$((((\,)r)(q's)))((()q'))(r's')$
q	$((r's'))(())(r's')$	$(((r)(()s)))((()))(r's')$	$((((\,)r)(q's)))()(r's')$ [C3; C4a]
	$((r's'))(r's')$ [C4a]	$(((r)(()s)))()(r's')$ [A2]	() [C3]
	() [J0]	() [C3]	

□

Section 6 below will demonstrate the *Theorema* in three ways.

Verifying both $\alpha'\beta$ and $\beta'\alpha$ by TVA amounts to a TVA verification of the equation $\alpha=\beta$. Iterate the TVA until the set of formulae terminating the branches of α is the same as the

54. TVA first saw the light of day in the 1950 first edition of Quine (1982). I owe my discovery of TVA to Bostock's (1997: §2.11) elegant treatment thereof, to my knowledge unique among contemporary texts. For a fine example of a TVA proof in tree form, see Prior (1962: 17). N.B: the "truth value analysis" in Kalish et al (1980: §§II.8-9) is an unrelated concept.

set terminating the corresponding branches of β, in which case the equation if verified. In all other cases, the equation does not hold.

Given a formula with n distinct variables, the construction of the corresponding truth table requires evaluating 2^n PA formulae. This is always a tedious affair, but not an impractical one when n does not exceed three or four. Moreover, T3 assures us that a truth table and a pa demonstration must yield the same result. But thanks to TVA, any such resort to brute force is unnecessary. One round of the above algorithm, applied to the variable with the most instances, often suffices.[55]

5.2. J0 and C2 as Initials.

Because the pa is a Ba, the many postulate sets proposed for Ba (Rudeanu 1963: chpt. 5) are also possible sets of pa initials. We shall see that J1, J2, and OI are not necessarily the best initials among those known.

Bricken (1986) demonstrated J1 and J2 from C1-C3; hence C1-C3 can serve as initials. They appeal mainly because C1 and C2 alone suffice to justify most calculation steps. Also, C1-C3 are very easily verified by a decision procedure, and C2 is much shorter than J2. Bricken (2002) improved on C1-C3 by taking the complement of C3, obtaining (a())=⊥, and replacing C2 with a notational variant of T13 (see §3.1). I will henceforth refer to T13 as C2. From C2 and the complement of C3 he then calculated C1. I replace Bricken's (a()) =⊥ with J0, and derive J1 and J2 from J0 and C2 in §A.1. Thus J0, C2, and OI form a very economical basis for the CTV and **2**. T18a in §A.10 proves these initials independent. For a derivation of J0 from C2 and C3, see §A.1.

J0 and J1 govern the primitive value (). Table 5-1 shows how the initials J0,C2 can be seen as insertion/cancellation rules whose tacit goal is to make all variables vanish:

- *J0 Insert*. Anything may be written on both sides of an empty boundary.
- *J0 Cancel.* If the *entire* content of a boundary also occurs in the pervasive space, both instances of that content may be erased, leaving an empty boundary.
- *C2 Insert*. Anything outside a boundary may be copied into a boundary.
- *C2 Cancel*. If any *part* of the content of a boundary also occurs in the pervasive space, that part may be erased. When a given subformula appears on both sides of a boundary, the interior instance is always redundant.

J0 can be seen as the sole axiom of natural deduction, and as akin to the rule for closing a branch in a tableau (cf. Bostock 1997: chpts. 4,6). C2 can be seen as a paired insertion and elimination (Fitch's term was *intelim*) rule for ¬ and ∨/∧ of the sort that is typical of natural deduction. Likewise, A1 [A2] can be seen as an intelim rule for () [⊥].

55. Ascertaining the satisfiability of a formula with n distinct variables via truth tables requires evaluating 2^n interpretations. Hence the truth table decision procedure for CTV satisfiability is said to require *exponential time*. Whether there exists a decision procedure for satisfiability that is merely a polynomial function of n, (i.e., a procedure executable in *polynomial time*) is a major unsolved problem in computational mathematics; see Hodges (2001: 23-24) and references cited therein. While I submit that TVA is quicker and easier than truth tables, especially when n is not large, I cannot claim that executing TVA on a computer would require less than exponential time for any n.

Table 5-1. The pa in a Nutshell.				
Initial	*Content of* ()	*Action*	*Notation*	*Antecedents*
OI	Na	Reorder at will the content of any subspace.	*abc=bca*	Dilworth (1938) Byrne (1946)
Let there be a boundary, and let a *appear outside it.*				
J0	*a*	Write ($a'a$) anywhere, and erase it at will.	$(a)a = ()$	Natural deduction
C2	*ab*	Only the shallowest instance of *a* is nonredundant.	$a(b\langle a\rangle)=a(b\langle\bot//a\rangle)$	Rule of (De)Iteration in Peirce's Existential Graphs†
† See **2i-e** in Table 6-1.				

It should now be clear why pa demonstrations here and in *LoF* work J0–C2 very hard. Moreover, consequences beyond these are required in only a few situations. Demonstrating C9 requires C6; calculating it requires C7 as well. The proofs of T14-T18 invoke C9 twice and C7 once. The pa demonstrations in §3.3 and §5.4 invoke C5 twice and C4 once.

J2 is one of Huntington's (1904) Ba postulates (cf. §6.2), and all demonstrations of J2 I have encountered are nontrivial. Is this why Spencer-Brown chose J2 as an initial? In any event, by deeming J2 an initial, all *LoF* demonstrations, those of C1 and C9 excepted, are near-trivial. While demonstrating J2 from J0,C2 is a bit involved, this is amply offset by an easy derivation of C1 and by the calculating power, revealed in §5.4, that J0, C1, and C2 afford.

All CTV tautologies can be verified by TVA. Furthermore, the axioms of the CTV are a small subset of the set of all tautologies. From these undisputed facts, Quine (1951: *100; 1982: §13), citing Herbrand, argues that all CTV tautologies are equally deserving of the honorific title of axiom. (Concurring voices include Smullyan 1968: 81, Wolf 1998: 79, and Cori & Lascar 2000: §4.1.1.) This commendably egalitarian view, however, fails to distinguish the context of *verification*, i.e., determing the satisfiability of formulae, for which decision procedures are indeed adequate, from the context of *discovery*, one requiring proof from axioms or rules, trial, error, and inspiration. We may make the axioms as economical as desired.[56]

Boundary logic is classical simply because $(a\vee{\sim}a)\leftrightarrow\mathsf{T}$ interprets J0. *Intuitionist logic* is based on axioms (Kneebone 1963: §9.4) from which equivalents of J0 and C1 cannot be demonstrated. Moreover, in intuitionist logic, no subset of $\{\wedge,\vee,\rightarrow,\sim\}$ is EA. C2 has the following intuitionist interpretation. $[(a\rightarrow c)\wedge(b\rightarrow c)]\rightarrow(a\vee b)\rightarrow c$ is an intuitionist axiom. The substitution *b/c* yields $[(a\rightarrow b)\wedge(b\rightarrow b)]\rightarrow[(a\vee b)\rightarrow b]$. Now $b\rightarrow b$ evaluates to T, and $(a\rightarrow b)\wedge\mathsf{T}$ evaluates to $a\rightarrow b$. Hence $(a\rightarrow b)\rightarrow[(a\vee b)\rightarrow b]$, one half of C2, is an intuitionist tautology but the converse, $[(a\vee b)\rightarrow b]\rightarrow(a\rightarrow b)$, is not. Devising a boundary syntax and proof theory for intuitionist logic would be an interesting exercise.

I now derive another boundary representation of the Ba and CTV. Consider the following trivial syntax: If *a,b,c* are formulae, then *ab*, (*ab*)*c*, and *a*(*bc*) are formulae. Given the

56. For a defense of Hilbert proof in contexts where a decision procedure is available, see Epstein (1995: §II.K.1).

semantics $ab \Leftrightarrow a \rightarrow b$ and $(ab)c \Leftrightarrow (a \rightarrow b) \rightarrow c$, this syntax suffices for CTV statements whose sole connective is →. The axioms PIA1, $(ab)b=(ba)a$, and PIA2, $a(bc)=b(ac)$, result in *positive implication algebra* (PIA), in which all intuitionistically valid tautological equivalences whose sole connective is the conditional are demonstrable. Parentheses are required because the conditional does not associate. Introduce the symbol ⊥ with intended reading *false*. $a\bot$ then defines intuitionist negation, $\neg a$ (Prior 12.3); no axioms are required. The path to classical logic begins by adding $(ab)a=a$ [C4] to PIA, yielding *implication algebra* (IA; Abbott 1969: §7-4; Wolfram 2002: 803).[57] IA stands to the implicational calculus as BA stands to the CTV. $a\bot$ now defines classical negation. Adding the axiom $\bot a=bb$ [C3] to IA yields the CTV (Prior 3.12) and renders C1, $(a\bot)\bot=a$, and J0, $aa=bb$, demonstrable. Hence IA+C3 is equivalent to Ba, interpretable as classical logic.

5.3. The Usual Inference Rules of Logic.

"…if one could find characters or signs appropriate for expressing our thoughts as neatly and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, one could accomplish in all subjects in so far as they are amenable to reasoning *all that can be done in Arithmetic and Geometry. For all investigations depending on reasoning would be performed by the transposition of characters and by a sort of calculus, which would render very easy the invention of beautiful results. Hence we would not need to worry our heads as much as we do at present, yet we would be sure that we could execute anything feasible. Moreover, we could convince all of what we had found or concluded, since it would be easy to verify the calculation… And were someone to doubt what I was proposing, I would say to him 'Sir, let us calculate' and thus… soon settle the question."* Leibniz (1903: 155-56).[58]

The only inference rules mentioned so far are R1 and R2, which suffice for equational logics. All other truth functional inference rules, including those characterizing ponential logics, are special cases of the inference rule Wolf (1998: §§3.5, 4.2) calls *propositional consequence*, PC. First some definitions:

5.3.1. Definition. An *argument* consists of one or more formulae called *premises*, and a formula called a *conclusion*. A *clause* (Cori & Lascar 2000: §§1.3.2, 4.4.1) links the conjoined premises to the conclusion via the conditional. PC asserts that an argument is *valid*, meaning that the conclusion follows from the premises, iff the corresponding clause is a tautology.

Let the premises be $\phi_1 \ldots \phi_n$, and let χ be a conclusion. Then PC can be stated as:

(1) $\phi_1,\ldots,\phi_n \vdash \chi \Leftrightarrow \vdash [\phi_1 \wedge \ldots \wedge \phi_n] \rightarrow \chi \Leftrightarrow (((\phi_1)\ldots(\phi_n)))\chi$ [C1] $= (\phi_1)\ldots(\phi_n)\chi = ()$.

The BA clause associated with an argument simply encloses each premise, then juxtaposes the conclusion and the enclosed premises. While I have tacitly assumed that a

57. When stated in IA notation, C4 exactly resembles the absorption law of lattice theory (L2 in Table 3-3).

58. Emphasis in original. From "Préface a la Science Génerale," included in the volume of Leibniz texts first published by Couturat. When I compared Wiener's translation (Leibniz 1951: 15) to the original (Leibniz 1903: 155), I saw that the former left something to be desired. Hence the translation is mine.

clause contains but one conclusion, doing so entails no loss of generality: a clause can contain multiple conclusions, all juxtaposed.

Ascertain the satisfiability of a clause as follows:

- Translate every premise and every conclusion into the pa;
- Enclose each premise, then concatenate the premises and conclusions;
- Invoke C5 to erase all duplicate instances of a subformula within a given subspace;
- Invoke C1 to erase redundant parentheses;
- Invoke C2 to erase redundant subformula instances at different depths;
- Invoke J1 to erase any subformulae of the form (α′α).

If the above results in a primitive value, the clause or its negation is always valid. If the result is a formula, the clause is valid under those atomic valuations satisfying that formula.

Table 5-2.
Some Common Instances of Propositional Consequence.

Name	ϕ_1	ϕ_2	ϕ_3	χ	Source
Contrapositive	α′	β		α′β	88
Conjunction	α	β		(α′β′)	88
modus ponens	α	α′β		β	79
modus tollens	α′β	β′		α′	88
Biconditional	α′β	β′α		((α′β)(β′α))	85
Syllogism[59]	α*β	β′γ*		α*γ*	---
Proof by cases	αβ	α′γ	β′γ	γ	84

Source: Corresponding page number in Wolf (1998).

The inference rule *modus ponens* (from α and α→β, infer β; also known as the *rule of detachment*) of conventional ponential logic is a special case of PC. Table 5-2 includes modus ponens and several other inference rules discussed in Wolf (1998). Greek letters are metalogical devices, standing for anything that can be assigned a truth value.

Table 5-3 presents the usual inference rules of contemporary logic, taken from Machover (1996) and Bostock (1997), along with their boundary justifications. The boundary translation of the syntactic and semantic turnstiles is: prime all objects to the left of the turnstile, then concatenate everything on both sides. Given this translation, the inference rules in Table 5-3 are all trivial pa consequences.

Basic Sequents and *Inconsistency* are all J0 in another guise; ditto for INT and OI, CON and C5, and *Indirect Proof* and C1. The Cut Rule is the only rule whose demonstration invokes C2. Its meaning is simpler than may appear: if both φ and φ′ appear in the premises, φ is irrelevant to the conclusion.[60]

59. *Cal*. (α*β)(β′γ*)α*γ* [C2,2x] = (β)(β′)α*γ* [C2; OI] = (β′α*γ*)β′α*γ* [J0] = (). □ Validity requires that the letter not appearing in the conclusion (β in this case) appear primed in one premise, unprimed in the other.

60. This is the import of *LoF*'s (p. 123) unproved Interpretive Theorem 1. The Cut Rule is nowhere mentioned.

The molecular subformulae making up the BA representation of a clause can be permuted at will. Hence all partitions of these molecular subformulae into premises and conclusions have the same BA translation and hence are equivalent. In particular, (ϕ_n) and ⊢ in (1) can be transposed; the result is the boundary logic equivalent of the Deduction Theorem. Moreover, the validity of a clause does not depend on whether any particular molecular subformula is included among the premises or the conclusion, as long as any formula moved from one side of the turnstile to the other is first enclosed. This is presumably why boundary logic dispenses with all turnstiles.

Table 5-3. Some Common Logical Rules and Their Boundary Derivations.

Name	*Formal Version*	pa *Derivation*	*Source*†
Bostock's (1997) *Structural Rules*			
Basic Sequents‡	Γ,φ ⊨ φ,Δ	Γ′φ′φΔ [J0] = ().	p. 285
INTerchange, L	[Γ,φ,ψ,Δ⊨Θ] → [Γ,ψ,φ,Δ⊨Θ]	(Γ′φ′ψ′Δ′Θ)Γ′ψ′φ′Δ′Θ [OI; J0] = ().	§7.1
" , R	[Γ⊨Δ,φ,ψ,Θ]→[Γ⊨Δ,ψ,φ,Θ]	(Γ′ΔφψΘ)Γ′ΔψφΘ [OI; J0] =()	"
CONtraction, L	[Γ,φ,φ ⊨ Δ] → [Γ,φ ⊨ Δ]	(Γ′φ′φ′Δ)Γ′φ′Δ [C5; J0] = ().	"
" , R	[Γ ⊨ φ,φΔ] → [Γ ⊨ φ,Δ]	(Γ′φ′Δ)Γ′φ′φ′Δ [C5; J0] = ().	"
CUT	[Γ ⊨ φ,Δ] ∧ [Φ,φ ⊨ Θ] → [Γ,Φ ⊨ Δ,Θ]	(((Γ′φΔ)(Φ′φ′Θ)))Γ′Φ′ΔΘ [C1; C2,4x] = (φ)(φ′)Γ′Φ′ΔΘ [J0] =()	§2.5.C
Machover's (1996) *Inference Rules*			
Indirect proof, *reductio*	Γ,~α ⊢ ⊥ ↔ Γ ⊢ α	Γ′((α))⊥ [C1] = Γ′α .	§7.8.9, 15
Deduction Theorem	Γ,α ⊢ β ↔ Γ ⊢ α→β	Γ′α′β = Γ′α′β.	§7.7.2
Inconsistency[61]	[Γ ⊢ ⊥] ⊢ [Γ ⊢ β]	(Γ′⊥)Γ′β [C4a; J0] = ().	§7.8.6

† Section of Bostock (1997) or Machover (1996) where the rule in question is introduced and discussed. The formal versions are from Bostock (1997: 385).
‡ Replacing Bostock's (§2.5) ASSumptions, and THINning from the left, right.
Note: L=left; R=right. A lower [upper] case Greek letter denotes a single formula [set of formulae]. A primed upper case letter signifies that each constituent formula is primed.

5.3.2. The pa Recapitulated. The primitive basis of the primary algebra (pa) consists of:

- The PA;
- Variables (statement letters), with or without subscripts ranging over the natural numbers, inserted anywhere in a PA formula. '′' and '…' are improper symbols;
- The initial (3.1.5) set $(a)a=()$, $a(ab) = a(b)$, and $abc=bca$;
- The usual inference rules for equational logics, the substitution of equivalents (R1), and the uniform replacement of variables (R2).

Juxtaposition is a tacit connective that commutes and associates. Hence the contents of a boundary and its pervasive space can be rearranged at will. The pa is well-suited to a decision procedure resembling Quine's truth value analysis. That decision procedure verifies the initials. Other tautologies may be demonstrated, Hilbert-style, or verified by calculation. The pa is sound and complete, and has two intended interpretations: **2** and

61. Cf. §A.8 and text related to fn 50.

the CTV. Boundary logic follows from the interpretation () ⇔ T [F], then αβ ⇔ α∨β [α∧β]. In either case, (α) ⇔ ~α.[62]

5.4. Some Worked Examples.

"Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments." Whitehead (1948: 61).

"…standard university logic problems, which the calculus published in this text renders so easy that we need not trouble ourselves further with them..." *LoF*, p. viii.

I now give a number of worked examples showing that CTV proofs in undergraduate textbooks can be greatly simplified if the problem is first translated into BA notation, as per Table 4-2, and the proof executed as a pa demonstration, as per §5.0 above. This procedure requires the following metatheorem. "MP" abbreviates the rule *modus ponens*.

5.4.1. Theorem. The BA and the CTV have the same expressive power. In symbols, BA ⊢ *CTV* and *CTV* ⊢ BA.

Proof. The proof is in two parts. The first derives in BA the rule MP and a set of three axioms for sentential logic, PC1-PC3 below (Prior 1.4c) and hereinafter PC1-3. Hence BA ⊢ CTV. I then derive OI, J0, and C2 (simple form) in the CTV, and note that R1 and R2 are CTV metatheorems—see the references in §3.1. Hence CTV ⊢ BA.

BA ⊢ *CTV*. The basis PC1-3 and the rule MP are all easy BA consequences:

PC1: *Cal*. φ→ξ→φ ⇔ (φ)(ξ)φ [C2; OI] = (ξ′φ)ξ′φ [J0] = (). □
PC2: *Cal*. [φ→ξ→ν]→[(φ→ξ)→(φ→ν)] ⇔ ((φ)(ξ)ν)((φ)ξ)(φ)ν [OI; C2] = ((ξ)(φ)ν)(ξ)(φ)ν [J0] = (). □
PC3: *Cal*. [~φ→~ξ]→ξ→φ ⇔ (((φ))(ξ))(ξ)φ [C1; OI] = (φ(ξ))φ(ξ) [J0] = (). □
MP: *Cal*. (α)(α′β)β [OI] = (α′β)α′β [J0] = (). □

CTV ⊢ BA. The CUT rule (Table 5-3) is derivable in the CTV. Let Γ=PC1-3+MP, φ=*PM*, Θ= BA, and Δ,Φ = ∅. Then CUT results in the clause [PC1-3+MP ⊨ *PM*] ∧ [*PM* ⊨ BA] → [PC1-3+MP ⊨ BA]. PC1-3+MP ⊨ *PM* holds simply because PC1-3 is a CTV basis and MP is *1.11. To show that *PM* ⊨ BA, translate BA concatenation as alternation and '=' as the biconditional. Then note that J0 is *2.08 in *PM*, and that the two halves of C2, viewed as a biconditional, are *2.621 and *2.67. Recall that the only purpose of OI is to prove that alternation commutes and associates, facts which the *PM* axioms *1.4 and *1.5 assure. Hence PC1-3+MP ⊨ BA. □

Remark. The calculations for PC1 and PC2 reveal that these are C2 in another guise; ditto for PC3 and C1. Note that all four calculations invoke J0 in the final step. PC1-3 neither are, nor claim to be, "obvious" and "elementary." PC3 is one of many possible axioms

62. Kauffman (2001) and Bricken (2002) exposit BA and boundary logic in a manner arguably more in philosophical sympathy with *LoF*. Other possible approaches, not pursued here, to a deeper understanding of BA include elementary topology (Rosser 1969: 12-20), mereology and mereotopology (Simon 1987; Casati and Varzi 1999), cognition and mathematics (Lakoff and Núñez 2001), and semiotics (Merrell 1995).

governing negation. J1 fills that role in *LoF*; C1 is arguably the simplest. PC1 and PC2 are popular as axioms because they facilitate the proof of the Deduction Theorem (Machover 1996: 7.7.3), stated in Table 5-3 above.[63]

Many of the demonstration in the remainder of this paper are in columnar form, with the annotations written to the right of each step. Text about to be deleted in the next step is underlined. When subformulae or nested parentheses first appear, or change depth, they are shown in bold, as before. Demonstrations very seldom require A1, and J1 usually can do what A2 does.

If a step invokes one of C6-C9, the annotation may be more complicated, building on the fact that BA formulae can be taken as schemata, in which case they are stated using upper case letters. E.g., C6 is assumed to take the form (A′B′)(A′B)=A. R2 allows the uniform replacement of any upper case statement letter by a subformula. Substitutions are notated as per the following example. If the subformulae α and β are substituted for A and B in C6, the annotation is 'C6, α/A, β/B', with the actual values of α and β written using lower case letters.

Example 1. I now calculate six rather involved tautologies taken from standard texts. The first two are from Nolt et al (1998: 4.46, 109).

Dem. $(p\rightarrow q) \leftrightarrow \sim(p\wedge\sim q) \Leftrightarrow (p)q = \underline{((}(p)\underline{((}q\underline{))))}$ [C1, 2x] $= (p)q$. □

I have taken the liberty of translating '↔' as '='. To someone experienced in the pa, the tautological equivalence of $(p\rightarrow q)\leftrightarrow\sim(p\wedge\sim q)$ is evident at a glance.

The next problem is to verify the clause:

$\sim s\leftrightarrow(\sim p\vee\sim v),\ v\wedge p \vdash s$	From the conjunction of everything to the left of '⊢', infer the alternation of everything to the right of '⊢'.
$\Leftrightarrow \underline{((((}(sp'v')((p'v')s')\underline{))((}v'p'\underline{))))}s$	
$(sp'v')((p'v')s')v'p's$	C1, 3x
$\underline{(sp'v')sp'v'}((p'v')s')$	OI
$()$	J0. □

I chose the two preceding examples because the corresponding demonstrations in Nolt *et al* are the longest purely sentential proofs in that text, respectively 18 and 21 lines long. The next two examples are from Kalish et al (1980: 417, 66f).

$(a\rightarrow b) \rightarrow [(a\wedge b) \leftrightarrow a)]$	
$\Leftrightarrow (a'b)((\underline{((}a'b'\underline{))a})(a'(a'b')))$	
$(a'b)(\underline{(a'ab')}(a'(a'b')))$	C1; OI
$(a'b)((a'(\underline{a}'b')))$	J1
$(a'b)((a'\underline{(}b'\underline{)}))$	C2

$[(\sim a\rightarrow r)\wedge(b\rightarrow r)] \leftrightarrow [(a\rightarrow b)\rightarrow r]$	
$\Leftrightarrow (\underline{((}(ar)(b'r)\underline{))}((a'b)r))(((a\underline{r})(b'\underline{r}))(a'b)r)$	
$((ar)(b'r)((a'b)r))((a'\underline{(}b'\underline{)})(a'b)r)$	C1; C2,2x
$((ar)(b'r)((a'b)r))((a'b)\underline{(a'b)}r)$	C1
$((ar)(b'r)\underline{((a'b)r)})((a'b)r)$	C5

63. PC1, PC2, and the converse of PC3 are three of the six axioms in Frege's *Begriffschrifft* (Prior 1.1). Lukasiewicz found in 1930 that Frege's three other axioms could be derived from PC1-3, a basis commanding pride of place in Church (1956: 119) and Bostock (1997: 387).

$(a'b)((a'b))$ C1
$()$ J0. □

$((a\underline{r})(b'\underline{r}))((a'b)r)$ C2
$\mathbf{r}\underline{((a'b))}(a'b)r$ J2
$()$ C1; J0. □

I chose these examples because the corresponding demonstrations in Kalish et al are 32 and 27 lines long, respectively. The former fills an entire page and is preceded by five pages of discussion. The demonstrations in Nolt et al and Kalish et al are not reproduced here because they require 102 lines in all and invoke natural deduction techniques that are beyond the scope of this paper, typographically as well as logically. The above four pa calculations require a mere 22 steps.

The next example, from MacKay (1989: exercise 9m.4), requires determining the satisfiability of $((p\leftrightarrow\sim q)\leftrightarrow\sim p)\leftrightarrow\sim q$. I translate $\alpha\leftrightarrow\sim\beta$ as $((\alpha'\beta')(\alpha\beta))$. To avoid working with a single very long formula, I break up the rightmost biconditional into two conditionals.

$((p\leftrightarrow\sim q)\leftrightarrow\sim p)\rightarrow\sim q$

$\Leftrightarrow (((p'\underline{q'})(\underline{p}q)p)(((p'q')(pq))p'))q'$
$(((p')(\underline{q})p)(\underline{((p'q')\mathbf{p'q'}(pq))}p'))q'$ C2,4x
$(((\underline{p'})p)(\ p'))q'$ J1; C2
$\underline{((p)(\ p'))}q'$ C2
q' J1

$\sim q\rightarrow((p\leftrightarrow\sim q)\leftrightarrow\sim p)$

$\Leftrightarrow q((p'q')(pq)p)(((p'q')(p\underline{q}))p')$
$q\underline{((p'q')(pq)p\mathbf{q})}(((\underline{p'q'\mathbf{q}})(p))p')$ C2,3x
$q\underline{(((p))p')}$ J1,2x
q J1. □

As one conditional simplifies to q' and the other to q, their conjunction evaluates to ⊥ by C1 and J1. While this calculation is a bit involved (14 steps), mainly because '↔' lacks a concise pa representation, it requires only J1 and C2. By contrast, MacKay's (pp. 368-69) proof is 43 lines long and invokes 11 natural deduction rules.

The most spectacular example of this nature I have saved for last. Leblanc and Wisdom's (1976: 395) proof of $[p\vee(q\rightarrow r)]\leftrightarrow[(p\vee q)\rightarrow(p\vee r)]$ is 42 lines long and invokes eight natural deduction rules. If the single instance of '↔' is taken as '=', the pa demonstration is utterly trivial: *Dem*. $pq'r$ [OI] = $q'pr$ [C2] = $(\mathbf{p}q)pr$. □

Example 2. Quine (1982: 69) introduces the DNF as a method for determining satisfiability, and builds his exposition of the DNF around a six page discussion of the formula (1), one he deems "forbidding":

(1) $\sim(((p\rightarrow(\sim s\wedge q))\rightarrow\sim((s\wedge q)\rightarrow p))\wedge\sim(\sim(r\wedge p)\wedge\sim(p\rightarrow s)))$.

Because (1) includes five instances of conjunction and none of alternation, I translate it using the dual reading. Hence $x\wedge y$ translates as xy rather than $(x'y')$.

$\Leftrightarrow (((p(s'q))\underline{((}(sq)p'\underline{))})((rp)(ps')))$
$(((p(s'q))(sq)p')\chi)$ C1; let $\chi=((rp)(ps'))$
$(((\underline{\mathbf{p'}p(s'q)})(sq)p')\chi)$ C2; OI
$(((sq)p')\chi)$ J0
$(((sq)p')((r\underline{p})(\underline{p}s')))$ Expand χ
$(((sq)p')\mathbf{p}(r'\underline{(s')}))$ J2
$((\underline{p'\mathbf{p}(sq)})p(r's))$ C2; OI; C1; OI

$(p(sr'))$ J1. □

Conclusion: (1) is satisfied when $p \rightarrow s \rightarrow r$ is the case. Note how this technique easily reveals the irrelevance of q, even of all of (1) to the left of the third '$\wedge$'.

Example 3. Now for two examples from texts with a contemporary following. Hurley (2000: 415, exercise 19) asks students to verify the clause:

$$a \rightarrow (nn') \rightarrow s \vee t,\ t \rightarrow (f \wedge \sim f)\ \therefore\ a \rightarrow s.$$

$\Leftrightarrow (a'\underline{(nn')}st)(t'\underline{(ff')})a's$

$(a'st)\underline{(}t'\underline{)}a's$ J1,2x

$(a'st)a'st$ C1; OI

$()$ J0. □

Hurley's natural deduction proof (p. 653) requires 19 steps and invokes 13 rules.

Lepore's (2003: 131) exercise 8.5.2 asks whether $((p \wedge q \wedge k) \vee \sim r)$ and $(r \rightarrow (\sim q \rightarrow (p \wedge \sim (v \vee \sim j))))$ are equivalent. Following Lepore, I breake up the problem into two conditionals and calculate each:

$((p \wedge q \wedge k) \vee \sim r) \rightarrow (r \rightarrow (\sim q \rightarrow (p \wedge \sim (v \vee \sim j))))$

$\Leftrightarrow ((p'q'k')\underline{r'})r'q(p'vj')$

$\underline{(}(p'q'k')\underline{)}r'q(p'vj')$ C2

$\underline{q'q}p'k'r'(p'vj')$ C1; OI

$()$ J0.

$(r \rightarrow (\sim q \rightarrow (p \wedge \sim (v \vee \sim j)))) \rightarrow ((p \wedge q \wedge k) \vee \sim r)$

$\Leftrightarrow (\underline{r'}q(p'vj'))(p'q'k')r'$

$(q(\underline{p'}vj'))(\underline{p'}q'k')r'$ C2

$(q(vj'))(q'k')p'r$ J2.

$(qv')(qj)(q'k')p'r$ C7.

The conditional on the right cannot be simplified any further. Hence the two halves of the biconditional do not simplify to the same formula, and the two statements are not equivalent. Note the use of C7 on the right to obtain the NF, which is more nakedly revealing of the inability to proceed further. Lepore's worked answer using refutation trees is 25 lines long, invokes 6 rules, and fills all of his p. 389.

Example 4. I now give a detailed example of how the pa simplifies clausal reasoning, by reworking Stoll's (1963: 184) Example 4.4.3. Unlike the case with the other examples in this section, reproducing Stoll's proof is a manageable affair; thus Table 5-4. A lone 'p' in the third column identifies a row containing a premise. A 't' in the same column signifies that an unspecified tautology has been invoked. The numbers in the rightmost column are the row numbers of the premises upon which the formula in a given row depends. The conclusion is in the bottom row.

Table 5-4. Stoll's (1963) Example 4.4.3.			
1	$\sim C \wedge \sim U$	p	1
2	$\sim U$	1,t	1
3	$S \rightarrow U$	p	3
4	$\sim S$	2,3,t	1,3
5	$\sim C$	1,t	1
6	$\sim C \wedge \sim S$	4,5,t	1,3

7	$\sim(C\vee S)$	6,t	1,3
8	$(W\vee P)\rightarrow I$	p	8
9	$I\rightarrow(C\vee S)$	p	9
10	$(W\vee P)\rightarrow(C\vee S)$	8,9,t	8,9
11	$\sim(W\vee P)$	7,10,t	1,3,8,9
12	$\sim W\wedge\sim P$	11,t	1,3,8,9
13	$\sim W$	12,t	1,3,8,9

Readers unversed in natural deduction need take away from Table 5-4 only the relative opacity of its content. Stoll's example can be recast as the following clause:

Premises: $(CU) \Leftrightarrow \sim C\wedge\sim U$ *Conclusion*: $(W) \Leftrightarrow \sim W$
$(S)U \Leftrightarrow S\rightarrow U$
$(WP)I \Leftrightarrow (W\vee P)\rightarrow I$
$(I)CS \Leftrightarrow I\rightarrow(C\vee S)$

The pa calculation verifying this clause goes as follows:

$\underline{(}(CU)\underline{)}((S)U)((WP)I)((I)CS)W'$	Enclose premises, concatenate all.
$CU((S)\underline{U})((WP)I)((I)\underline{C}S)W'$	C1
$CU\underline{(}(S)\underline{)}((WP)I)((I)S)W'$	C2,2x
$CUS((WP)I)((I)\underline{S})W'$	C1
$CUS((WP)I)\underline{(}(I)\underline{)}W'$	C2
$CUS((WP)\underline{I})IW'$	C1
$CUS\underline{(}(WP)\underline{)}IW'$	C2
$CUS\underline{W}PI\underline{W'}$	C1
$\underline{W'W}CUSPI$	OI
$()$	J0. □

Stoll proof's staggers the introduction of the four premises; deciding what premise should be invoked where requires nontrivial reflection. The pa calculation introduces all premises at the outset, then proceeds mechanically, either invoking C2 to prune a redundant instance of a variable, or C1 to eliminate redundant boundaries. When a primed and unprimed instance of the same variable appears in the pervasive space, J0 terminates the calculation. I submit that the above calculation is vastly simpler than Stoll's proof. The pa also reveals that a valid argument from Stoll's premises requires that at least one variable appearing in the premises also appear primed in the conclusion.
End of Examples

These examples reveal that pa calculations are much easier than conventional proofs. The simplicity of pa calculation stems from:

- A notation that fully embodies the expressive adequacy of $\{\vee/\wedge,\sim\}$;
- Working very hard a mere five rules, OI, J0, J1, C1, and C2.

The 10 demonstrations in Examples 1-4 employ all other resources of the pa a mere four times: J2 twice, and C5 and C7 once apiece. That the pa accomplishes so much with so little reveals that in practice, the pa is more than just a new notation for the CTV and **2**.[64]

5.5. Syllogisms as Clauses.

"If, as I hope, I can conceive all propositions as terms, and hypotheticals as categoricals... this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance." Leibniz (1966: 66).[65]

The *syllogism* of traditional logic is the oldest and most intensively studied clausal form. That logic, founded in ancient Greece, restricted each premise and conclusion to a statement of the form "[All/Some] α are [Not] β." Such statements are *categorical forms*, where α and β are metavariables standing for *terms*. Linguistically and intensionally, a term is a *common noun* or a noun phrase. Mathematically and extensionally, a term is a *set*, in which case 'all α are β' may be seen as shorthand for 'all members of set α are also members of set β', i.e., $\alpha \subseteq \beta$. A *syllogism* is a clause consisting of two premises and a conclusion, each in categorical form. The clause contains three terms, each term appearing in two categorical forms. For a modern overview of the syllogism, see Kneebone (1963: 8-22).

Table 5-5 shows how to interpret the pa as a logic of terms and categorical forms, if letters are reinterpreted as term names. Letting, as before, a '*' after a variable denote a literal, the pa notation (α'β*)* captures all possible categorical forms. Monadic logic works as follows. Let A*a*=() if it is indeed the case that *a* is a member of set α; likewise for B*b* and the set β. Quine (1982: §§18-20) designed his *Boolean term schemata* (BTS in Table 5-5) so as to embody the Boolean structure common to the syllogism, the logic of terms, and the monadic predicate calculus. The resulting notation is (unwittingly) very similar to that of the pa.[66]

Table 5-5.
Alternative Notations for the Four Categorical Forms.

*	*Categorical Form*	pa	*BTS*	*Monadic Logic*	*Set Algebra*
(1)	(2)	(3)	(4)	(5)	(6)
A	All α are β	α'β	-αβ'	$\forall x[Ax \rightarrow Bx]$	$\alpha \subseteq \beta$
E	No α are β	α'β'	-αβ	$\forall x \neg[Ax \wedge Bx]$	$\alpha \cap \beta = \varnothing$
I	Some α are β	(α'β')	αβ	$\exists x[Ax \wedge Bx]$	$\alpha \cap \beta \neq \varnothing$

64. I invite the reader to compare the demonstrations in *LoF* and here with those in Nidditch (1962), a book comparable to *LoF* in size and time of writing, also intended for undergraduate instruction, but far more conventional in approach. Deferring to intuitionist logic, Nidditch posits 11 algebraic axioms and the rule *modus ponens*, then proves 4 lemmas and 58 theorems (a category that lumps together what are here called (meta)theorems and consequences).

65. Original in Leibniz (1903: 377, §75).

66. Leibniz algebraized the categorical forms in a manner closely related to the one set out in the text. In paragraphs 83-87 of a paper written in 1686 but published only in 1903, Leibniz (1966: 67-68) wrote αβ*=α and αβ*≠α where I write α'β* and (α'β*). Also see Leibniz (1966: xlvii, Scheme III).

O	Some α are not β	(α′β)	αβ′	$\exists x \neg[Ax \rightarrow Bx]$	$\alpha \cap \beta' \neq \varnothing$
* These abbreviation are from medieval logic.					

The clause (α′β)(β′γ)α′γ corresponds to the syllogism medieval tradition named 'Barbara'.[67] This clause is an instance of the more general clause (α*β)(β′γ*)α*γ* (cf. Table 5-2), which admits of 24 possible permutations, and tradition indeed asserts the validity of 24 syllogistic forms. Unfortunately, not all of those forms correspond to one of the 24 permutations of (α*β)(β′γ*)α*γ*. Determining the number and form of the possible variants of (α*β)(β′γ*)α*γ* admitted by tradition is a nontrivial combinatoric exercise. There are three meaningful permutations of (α′β)(β′γ)α′γ, one for each possible position of α′γ. There are also six possible permutations of (α′β)(β′γ′)α′γ′. Given any of the latter, if both terms in a premise are primed, the terms may commute. Thus (α′β)(β′γ′)α′γ′ gives rise to three permutations, for a total of 15 valid syllogisms thus far.

If we assume that at least one of α, β, or γ is nonempty, i.e., that at least one of $\alpha\neq\bot$, $\beta\neq\bot$, or $\gamma\neq\bot$ holds, then variants of the clause (α′β)(β′γ*)(α′γ*)* can be valid, where each '*' stands for the presence or absence of a '′'. (β′γ′)(α′β)(α′γ) is valid if $\alpha\neq\bot$ or $\gamma\neq\bot$. Moreover, permuting β′γ′ does not affect validity; result, four valid syllogisms. (α′β)(β′γ′)(α′γ) is valid if $\alpha\neq\bot$; β′γ′ can again be permuted, resulting in two valid syllogisms. (α′β)(β′γ)(α′γ′) is valid if at least one of $\alpha\neq\bot$, $\beta\neq\bot$, or $\gamma\neq\bot$ is the case, result ing three valid syllogisms. The approach of this paragraph yields nine more valid syllogisms, for a total of 24. These 24 include five pairs whose members differ only in that where one has "all" in the conclusion, the other has "some."

Hence Appendix 2 of *LoF* is mistaken when it asserts that Barbara (by which it means (α*β)(β′γ*)α*γ*) nests *all* 24 valid syllogisms. Barbara nests only the 15 syllogisms not requiring that one or more terms be assumed nonempty. (α′β)(β′γ*)(α′γ*) is not an instance of Barbara, but can be valid given suitable nonemptiness assumptions. Deriving the necessary and sufficient conditions for a syllogism to be valid, I leave to future research.

The above approach is essentially that of Lukasiewicz, who axiomatized the syllogism by adding term variables to the CTV, extending the scope of R2 to such variables, and then introducing four axioms. Translated into the pa notation of this section, these axioms are:

- α′α=(). J0 holds for term variables;
- (α′α′) [C5; C1] = α = (). This has the effect of assuming all terms nonempty;
- Two axioms equivalent to asserting that (α′β)(β′γ*)α′γ*=() holds for term variables.

On this and other attempts to algebraize the syllogism, see Prior 10.11-6. None of these alternatives are as simple as (α′β)(β′γ*)(α′γ*)*, or reveal that elementary Ba suffices to test syllogisms if all terms are assumed nonempty. At any rate, the pa nicely trivializes what had been a rather involved subject for over 2000 years.

67. Barbara follows from the CUT rule (Table 5-3) when both Δ and Φ are empty.

6. Historical Antecedents and More Axiomatics.

6.1. Peirce's Existential Graphs.

"[*A thorough understanding of mathematical reasoning*] *is the purpose for which my logical algebras were designed but which, in my opinion, they do not sufficiently fulfill. The system of existential graphs is far more perfect in that respect...*" Peirce (4.429, 1903)

Kauffman (2001) discusses how the pa resembles the graphical logic to which Peirce devoted much of his last 20 years. Peirce's logical graphs are planar representations of logic formulae, consisting of ovals, called *seps*, that may be nested, and atomic formulae written anywhere. (Peirce usually referred to seps as *cuts*, a term I avoid because of possible confusion with the Cut Rule of conventional logic, discussed in §5.3.) The graphical logic has but one very simple syntactic rule: seps cannot intersect. Seps and BA boundaries are functionally identical and share a common logical interpretation as denial.[68]

Peirce devised two systems of graphical logic, the *entitative* and *existential* graphs. In the former, the blank page denotes *falsity*, so that juxtaposition denotes alternation. *LoF* (p. 5) unwittingly concurs with this entitative interpretation of the blank page. Hence the entitative graphs and the primal reading of the pa share the same semantics. Peirce developed the existential graphs (EG) at far greater length, even subtitling them "My Chef d'Oeuvre" (4.347-529, 1903); hence I will not linger over the entitative graphs. In the EG, the blank page denotes *truth* and juxtaposition denotes conjunction. Hence the EG are dual to the entitative graphs. The EG are of three kinds, the first of which, *alpha* (hereinafter *alpha graphs*), is isomorphic to the pa. The scope and power of Peirce's graphical logic did not become clear until Roberts (1973).[69] Nevertheless, that logic is a major precursor to boundary logic.[70]

The alpha graphs are governed by six *conventions* (i.e., definitions, roughly), and four *rules of transformation*, akin to natural deduction rules. These are shown in Table 6-1,

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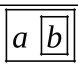
68. An advantage of the *alpha* graphs is that they dispense with formula definitions such as 2.1.4: any (finite) nesting or juxtaposition of nonintersecting seps is well-formed.

69. Roberts (1973) evolved out of his 1963 PhD thesis. Zeman's (1964) thesis, never published, likewise saw that *alpha* is isomorphic to CTV, and went beyond Roberts by proving that the *beta* graphs are isomorphic to first order logic with identity. However, Roberts was the first to give this fact wide currency. Shin (2002) includes a thorough exposition of *alpha* and *beta*, and discusses (§§2.4, 2.5) Peirce's view of logic as semiotic. This section has not benefited from Hilpinen's (2004) survey of Peirce's logic.

70. By pointing out parallels between BA and Peirce's graphical logic, I do not wish to suggest plagiarism. One should keep in mind that *LoF* is more of a text than a scholarly tract. Moreover, *LoF* predates the publication of Peirce (1976), containing the crucial excerpt from Peirce's 1886 paper on the "sign of illation" (cf. §4.2 above). *LoF* cites Volume IV of Peirce's *Collected Papers* (Peirce 1933), which includes 115pp on the logical graphs, but Spencer Brown could easily have overlooked this part of Peirce's *oeuvre*, as it was dismissed or ignored until Roberts (1973). Roberts also made extensive use of Peirce's unpublished papers, not accessible to scholars before 1956 and not catalogued until 1967. In time and place where *LoF* was written, the 1960s UK, the secondary literature on Peirce's graphs, cited in Roberts, was sparse and hard to access.

where CVn denotes the nth convention, and Rn the nth rule of transformation. The third column of this Table proposes BA counterparts to the *alpha* rules and conventions.[71]

Table 6-1.
Peirce's Existential Graphs and Boundary Logic

Name in Roberts (1973)	*Remarks*	BA
CV0 (4.394, 1903)	What is not forbidden is permitted. *This contradicts the* PA *Convention of Intention*.	
CV1	A blank surface asserts truth. Sometimes referred to as SA, the "sole axiom" of the alpha graphs.	T1 Dualreading
CV2	A graph asserts some truth about the domain.	T2
CV3	$ab \Leftrightarrow a \wedge b$.	Dualreading
CV4	 $\Leftrightarrow (a(b))$.	"
CV5	 $\Leftrightarrow (a)$;  $\Leftrightarrow () \Leftrightarrow$ *false*.	"
1i. Insert Odd	Any subgraph may be written in an odd depth.	6.1.1
1e. Erase Even	Any evenly enclosed subgraph may be erased.	"
2i. Iteration	$a(b) \rightarrow a(ab)$; $a \rightarrow aa$.	T13, C5
2e. Deiteration	$a(ab) \rightarrow a(b)$; $aa \rightarrow a$.	"
3i. Insert double cut	$a \rightarrow ((a))$. Write (()) anywhere	C1, A2
3e. Delete double cut	$((a)) \rightarrow a$. Erase any instance of (()).	"

SA stands for "sheet of assertion," the blank surface on which graphs are to be written. The blank page tacitly asserts *truth.* For Roberts (1973: 32, 119), this assertion, which he names SA, is the sole axiom of the alpha graphs. SA follows from A2, which asserts that the blank page denotes a primitive value; cf 2.2. CV1 and CV2 can be seen as defining "graph" and "domain". I state CV3-CV5 using *alpha* as the object language and the pa as the metalanguage. CV3 defines conjunction; CV4, the conditional; CV5, denial. For a comparison of Peirce's graphs with other notations for logic, see Roberts (1973: 136).

The BA is an uninterpreted formal system. Meanwhile, CV1-CV5 reveal that the *alpha* graphs and the dual reading of the pa share the same semantics. Hence the *alpha* graphs, unlike BA, are not self-dual, although it would be trivial to make them so. Peirce failed to see that the *alpha* graphs could be self-dual; this is the main way in which they differ from BA.

The rules of transformation **1i**-**3e** operationalize "step" in the context of *alpha*; in the terminology of Roberts, a step *preserves tautologies*. Roberts (1973: §3.2) shows that **1i**-**3e** are CTV consequences. **2i**-**e** can be seen as analogs of T13 and C5; **3i**-**e** are C1 and A2 in new guises. **2i**-**e** and **3i**-**e** are bidirectional. If **1i** is invoked only in contexts where it is equivalent to $(a())=\perp$ (the complement of C3), **1i** too becomes bidirectional. Any step invoking a bidirectional rule is analogous to a BA equational step. In EG demonstrations below, I indicate this bidirectionality via a double-headed arrow. However, a step

71. *LoF* includes a few diagrams in the spirit of Peirce's graphs; see chapter 12, the notes thereto, and p. 115.

invoking **1e** cannot be retraced; **1e** is thus unredeemedly non-bidirectional and alien to boundary logic. Fortunately, we do not require this rule.

If the inserted /erased (sub)formula evaluates to ⊥, **1i-e** follow trivially from A2. More generally, we have:

6.1.1. Theorem. **1i-e** preserve tautologies.

Proof. Let γ be a subformula of the formula α. It follows from T14 that there exists a formula $\beta=\alpha$, also containing γ, such that the depth of γ in β does not exceed 2. Alternatively, if γ is to be inserted in α, the result is equivalent to inserting γ in some β whose depth also does not exceed 2. Hence only three cases need be considered: γ has depth 0, 1, or 2. As the EG map very naturally into the dual reading of the pa, the objective is to reduce β to ⊥.

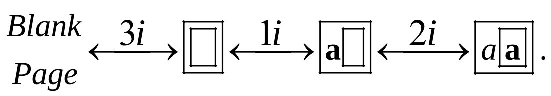
Erase γ at depth 0. *Cal.* $a\gamma b'\to ab' \Leftrightarrow (a\gamma b'(ab'))$ [OI] = $((ab')ab\gamma')$ [J1] = ⊥.
Insert γ at depth 1. *Cal.* $(ab')\to(a\gamma b') \Leftrightarrow ((ab')\underline{(}(a\gamma b')\underline{)})$ [C1; OI] = $((ab')ab'\gamma)$ [J1] = ⊥.
Erase γ at depth 2. *Cal.* $(a(b\gamma))\to(ab') \Leftrightarrow ((a(b\gamma))((ab')))$ [C1] = $((\underline{a}(b\gamma))ab')$ [C2] = $(\underline{(}(b\gamma)\underline{)}ab')$ [C1] = $(b\gamma ab')$ [OI] = $(b'b\gamma a)$ [J1] = ⊥ .

The depth=0,2 cases justify *Erase Even*. The case depth=1 justifies *Insert Odd*.

I now show that the three remaining possibilities do not reduce to tautologies:
Insert γ at depth 0. *Dem.* $ab'\to a\gamma b' \Leftrightarrow (ab'(a\gamma b'))$ [OI; C2,2x] = $(ab'\gamma')$.
Erase γ at depth 1. *Dem.* $(a\gamma b')\to(ab') \Leftrightarrow ((a\gamma b')\underline{(}(ab')\underline{)})$ [C1] = $((\underline{a}\gamma\underline{b'})ab')$ [C2,2x] = $(ab'\gamma')$.
Insert γ at depth 2. *Dem.* $(ab')\to(a(b\gamma)) \Leftrightarrow ((ab')\underline{(}(a(b\gamma))\underline{)})$ [C1] = $((\underline{a}b')a(b\gamma))$ [C2] = $(\underline{(b')}a(b\gamma))$ [C1] = $(ba(b\gamma))$ [C2; OI] = $(ab\gamma')$. □

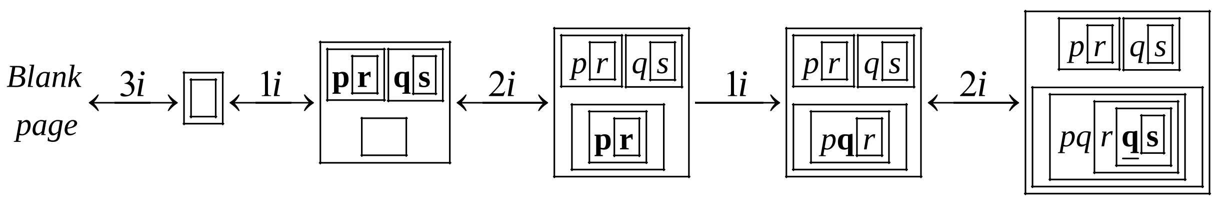
Remark. The six demonstrations making up 6.1.1 require only C1 (**3i-e**), C2 (**2i-e)**, OI (tacit in the EG), and J1. I infer that J1 in effect plays the same role in BA that **1i-e** play in EG.

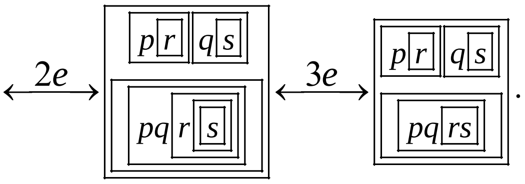
Since **2i-e** is analogous to C2, the converse of 6.1.1—that the pa is derivable from *alpha*—merely requires an *alpha* demonstration of J0, to wit:



Hence **1i**-**3e** form a basis for the pa. There is a sense in which J0 does the work of **1i-e**. I submit, however, that J0 is more intuitive than **1i**-**e**, and eliminates any need to keep track of depth parity.

I now present, by way of example, Sowa's (2002) demonstration of Leibniz's *Praeclarum Theorema*, verified by TVA in §5.1. Variables inserted by **1i** or duplicated by **2i** first appear in bold; variable instances about to be eliminated by **2e** are underlined.



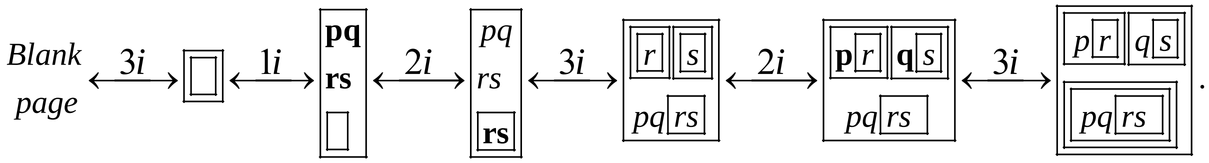


The first use of 1i is bidirectional as discussed above **6.1.1**, but the second is not, as this step has no pa analog.

I now verify the *Theorema* via a pa calculation. In keeping with the spirit of the *alpha* graphs, the pa translation invokes the dual reading.

Cal. $[(p\to r)\wedge(q\to s)]\to[(p\wedge q)\to(r\wedge s)] \Leftrightarrow ((pr')(qs')\underline{((}pq(rs)\underline{))})$ [C1] $= ((\underline{p}r')(\underline{q}s')pq(rs))$ [C2,2x] $= (\underline{(}r'\underline{)}\underline{(}s'\underline{)}pq(rs))$ [C1,2x; OI] $= ((rs)rspq)$ [J1] $= \bot$.

Reading this calculation in reverse suggests the following *alpha* demonstration:



The preceding demonstration has one less step than Sowa's, a minor advantage. Much more significant is that my demonstration invokes **1i** but once, at the outset, to insert, in any order, one instance of each variable appearing in the *Theorema*. **1i** employed in this manner is analogous to C3 and hence bidirectional. Since **2i-e** and **3i-e** are analogous to C2 and C1, respectively, and C1-C3, OI form a pa basis (Table 6-2), **1e** is redundant. Since the remaining *alpha* rules are all bidirectional, *alpha* can be recast as an equational system. Sowa's demonstration, on the other hand, invokes **1i** twice, each time requiring careful thought about what to insert. Sowa claims that the demonstration of the *PM* counterpart of the *Theorema*, *3.47, involves 43 steps and five axioms. Having demonstrated the *Theorema* via TVA, *alpha* (twice), and a pa calculation, I invite the reader to decide which method is most perspicuous and easiest to learn.

6.2. Some Ba Postulate Sets.

"*Any finite… selection of statements* (*preferably true ones, perhaps*) *is as much a set of postulates as any other. …'postulate' is significant only relative to an act of inquiry; we apply the word to a set of statements… to which we have seen fit to direct our attention*."Quine (1982: 35)

Table 6-2 includes a variety of postulate sets (bases) which are one or more of: important benchmarks, relevant to an evaluation of *LoF*, little known, or have otherwise piqued my curiosity. The first eight rows of Table 6-2 consist of CTV bases to be discussed in §6.3. The remainder of the Table consists of Ba and pa bases, none of which are mentioned in Prior (1962) or Epstein (1995: 407-9); logicians, evidently, are not in the habit of delving into the Boolean algebra literature. Boolean algebraists do not all sin reciprocally; see, e.g., the references in Huntington (1933) and Bernstein (1934). If a basis includes a pair of axioms asserting that a connective commutes and associates, I have replaced the pair with OI. I have added OI to all pa bases, even though no author did so. The "length" of a basis is the number of BA symbols required to express it. For other details of how I operationalize the "length" of a basis, see the Note to Table 6-2. The Ba and pa bases (excepting those of Bernstein, McCune, and Schröder) seem simple and intuitive.

Leibniz. In two brief memoranda, written in 1690, published in 1903, and translated into English as chapters 9 and 10 of Leibniz (1966), Leibniz set out a 'logical calculus' with primitive conjunction and denial, respectively denoted by juxtaposition and 'non-*a*'. I shall cite passages in these memoranda as (9.*m*) and (10.*n*), where *m* and *n* are paragraph numbers. I also have taken the liberty of reordering Leibniz's axioms (undemonstrated propositions, actually) and restating them in pa notation.

Leibniz effectively devised **2** because:

- His axioms are tantamount to a CTV basis, called L in §6.3 and due to Lukasiewicz;
- He postulated that '=' is a congruence relation.

The latter is the easier of the two to show. (9.3) is *a*=*a*; hence '=' is tacitly reflexive. (10.5) reads '*a*=*b* means that one can be substituted for the other…[*a* and *b*] are equivalent', which I take as tantamount to R1. The symmetry and transitivity of '=' can be derived from R1 and reflexivity, so that '=' is an equivalence relation (2.3.8). By virtue of (9.8) and (9.11), '=' is also a congruence relation (3.3.10).

I derive Lukasiewicz's CTV axioms as follows. Leibniz's (9.6) is C5, and (10.9) is J0 and C3. Two of Lukasiewicz's axioms are immediate: *Cal.* (*a*$\underline{a}$)*a* [C5] = (*a*)*a* [J0] = (); (*a*)*ab* [J0] = (). I thus include J0, C3, and C5 in Leibniz's "basis." Lukasiewicz's remaining axiom is what I call *Syll1*, (*a'b*)(*b'c*)*a'c*. At this point the reader would do well to refer to §5.5, as Leibniz intended letters in his formalism to stand for terms. Syll1 can be read as asserting the validity of the syllogism Barbara. While *Syll1 per se* cannot be found in Parkinson (1966: chpts. 9,10), Leibniz (1966: 33, 42) freely assumed Barbara, its equivalent in categorical form. He (p. 105) purports to derive Barbara from his version of the medieval logicians' *dictum de omni et nullo* taken as an axiom. Hence I take Leibniz as granting *Syll1*. The upshot is a reading of Leibniz at once generous and novel, that makes him the inventor of Ba, interpreted as a logic of terms. Leibniz failed to see that alternation as well as conjunction could interpret concatenation. Thus he missed duality and De Morgan's laws.[72]

Grassmann. In 1872, Robert Grassmann (brother of the better known Hermann) published a curious book titled *Die Formenlehre* ("The Theory of Forms"), setting out a theory of *magnitudes*. He defined a magnitude as ""anything that is or can be the subject of

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72. After writing this section, I discovered Lenzen (2004: 3,4,§4) and Hailperin (2004: 324-37). Lenzen, reviewing papers he published in German in the 1980s, concludes that his system L1, extracted from Leibniz's work, is isomorphic to sentential logic and Boolean algebra. It is not evident whether Lenzen based his conclusion in part on the original texts underlying chpts. 9,10 of Leibniz (1966). Hailperin does base his discussion on these texts, but reaches no conclusion about the strength of the implied system. The true value of Leibniz's work was not appreciated before the 1980s because both Couturat, the editor of the Latin originals (Leibniz 1903), and Parkinson, the editor and translator of Leibniz (1966), failed to appreciate the strength of Leibniz's system. Moreover, when Leibniz (1966) was published, the two 20th century logicians with the strongest interest in the history of the subject were either dead (Lukasiewicz) or about to be (Prior). According to Lenzen, Rescher (1954) was the first to see that the power of Leibniz's formal systems had been seriously underestimated.

thought, insofar as it has one value, not more" (Grassman 1966: Be-6).[73] The primitive values of the Theory of Magnitudes are *stems*, denoted by an *e* (which may or may not have numerical subscripts) and defined as follows:

> "…a magnitude that is initially posited, and which therefore does not result from [a combination] of other magnitudes… The initial stems of the universe are the God-given properties of particles, the ether, and the spirit, of whose synthesis the entire universe consists." Grassmann (1966: Be-6).

He then applied that theory to four subjects: numbers, 'combinations,' 'externals,' and 'concepts' (i.e., logic); only this last will concern us here.

Grassmann wrote before Peano, Hilbert, Huntington, and others formulated the current understanding of an axiomatic theory. Hence 'axiom' and 'postulate' do not appear in his work. The axioms I propose below for the Theory of Concepts are BA translations of propositions Grassmann states without proof. The part of the Theory of Magnitudes Grassmann deemed applicable to his Concepts consists of two primitive binary operations, denoted by '+' and '·', governed by the following laws (Be-7: 2). I have taken the liberty of modernizing Grassmann's terminology:

a) Magnitudes are closed under '+' and '·';
b) These operations commute and associate;
c) The identity elements for '+' and '·' are 0 and 1, respectively;
d) Each operation distributes over the other.

The closure property (a) I take as tacit throughout this paper; OI captures the essence of (b). $a\perp=a$ nicely summarizes the dual pair (c); ditto for J2 and (d).

Grassman (Be-7: 3) then introduced two laws peculiar to the Theory of Concepts. They warrant quotation in full:

> "1. The sum and… product of two equal stems gives the same stem again, and
> 2. The product of two different stems is zero."

The Theory of Concepts is a model of BA. Let the 'stems' be () and ⊥, and let $a+b \Leftrightarrow ab$ and $a\cdot b \Leftrightarrow ((a)(b))$. Then the first law is equivalent to the PA equations ()()=(), ⊥⊥=⊥, ((())(()))=(), and ((⊥)(⊥))=⊥; Table 6-2 retains the first two. I propose to translate the second law as ()⊥=(), true by virtue of A2. Thus it would seem that Grassmann unknowingly anticipated the PA in 1872. He defined complementation in context (Be-14: 29) via a pair of equations that are notational variants of the dual pair J0 and J1. Table 6-2 retains J0. With the postulation of J1, the Theory of Concepts becomes isomorphic to BA.

Schröder (1966), vol. 1 of the *Vorlesungen* (originally published in 1890 and discussed in Brady 2000) is, I would argue, the first systematic presentation of Ba. To my knowledge,

73. I am very grateful to Lloyd Kannenberg, the translator of Hermann Grassmann's *Ausdenungslehre*, for having taken up my invitation that he translate the *Formenlehre*, and for making his unpublished translation available to me. Page numbers refer to this unpublished translation. I know of the *Formenlehre* thanks only to Grattan-Guiness's (2000: 157-60) discussion thereof. Grassmann (G-9) mentions Leibniz but no work on logic more recent than Hegel's. In particular, Grassmann appears to have had no knowledge of Boole's work.

Lejewski (1960: 23) is the only mention of this possibility. The basis shown in Table 6-2 appears nowhere in the *Vorlesungen*, but rather is my distillation of Lejewski's S1-S9, which he derived from 3 axioms and 6 definitions spread over 140pp of the *Vorlesungen*. Translating S1-S9 into the pa reveals that S1, S8, and S9 are effectively J0, and that S3 and S4 are effectively C3. Neither Schröder nor Lejewski noticed that S4 and S5 in effect form a dual pair; I dropped the longer of the two, S4, as redundant. The result is the five axioms shown in Table 6-2. Comparing Schröder's basis with that of Sheffer (1913) reveals that the two bases have (*ab*)*c*=((*a*′*c*)(*b*′*c*)), a variant of J2, in common. But Sheffer requires only two other postulates, his versions of C1, C5, and J1. Hence Schröder's basis is amply redundant. It is likewise prolix because his notation includes equivalents of the ∪, ∩, ⊂, overbar, **U**, and ∅ of set theory, and of the truth functors ∧, →, and ↔.

Johnson (*1892*). In a three-part article in the leading journal *Mind*, the British logician W E Johnson set out a system whose syntax—juxtaposed letters with and without overbars—translates trivially into BA: if α, β are formulae, $(\alpha\beta)$ translates Johnson's $\overline{\alpha\beta}$. He interpreted juxtaposition as conjunction, the overbar as complementation. His axioms were C1, C5, axioms equivalent to OI, and the Law of Dichotomy, Johnson's name for (*ab*)(*ab*′)=*a*′. Since Dichotomy is but the contradual of C6, I will refer to it as C6. §A.4 includes a demonstration of J1, C1, C2, and C5 from C6 and OI, thus proving that Johnson's axioms form a pa basis, and that his C1 and C5 are redundant. Johnson's verbose exposition falls short of the taut understanding of Ba and CTV that later emerged. He did claim Peirce as an important influence. To my knowledge, the only study of Johnson's system is Meredith and Prior (1968). Prior (1962) repeatedly cites Johnson (1892) but does not mention Johnson's system.

Huntington (*1904*). *LoF* rightly cited this paper, the wellspring of self-aware Boolean axiomatics. My summary of Huntington's axioms follows Stoll (1974: §4.1) and Eves (1990: 216, 257). Huntington defined Boolean algebra as a set *B* with at least two members and closed under two binary and one unary operations. The binary operations are dual to each other, so that his remaining eight axioms are grouped into four dual pairs. The binary operations commute (B1), have distinct identity elements (B2), distribute over each other (B3), and have inverses defined in terms of the unary operation and the identity elements (B4). Given B1-B4, associativity is a theorem (Eves 1990: 217-19). Since the pa has both a primal and dual reading, a pa initial contains the same information as a dual pair of Huntington axioms. Interpreting Huntington's two binary operations as *ab* and (*a*′*b*′), BA satisfies Huntington's basis:

B1. *ab*=*ba* and ((*a*)(*b*))=((*b*)(*a*)) are both true by OI;
B2. *a*⊥=*a* has already been established. ((*a*)(())) [C4a] = ((*a*)) [C1] = *a*;
B3. J2. ((*a*)(*bc*)) [C1,2x] = (*a*′(**(***b*′**)(***c*′**)**)) [C2,2x] = (((***a*′***b*′)(***a*′***c*′))) [C1] = (*a*′*b*′)(*a*′*c*′);
B4. J0 and J1.

Bernstein (1916) combined B2 and B4 into *a*(*b*′*b*)=*a* (J1, in effect) and its dual.[74]

Huntington (*1933, 1933a*) derived his 1904 Ba basis from OI andC6, a nontrivial exercise. Like Johnson, he (1933) at first thought that C5 had to be postulated, but very soon (1933a) corrected himself. Kauffman (1990), using the pa, considerably simplified Hun-

74. Also see Wolfram (2002: 773). Montague and Jan Tarski (1954) later showed that given either of *ab*=*ba* and (*a*′*b*′)=(*b*′*a*′), the other is redundant.

tington's result; see §A.4 for details. That J1,J2, J0,C2, and C6 each form a pa basis requires demonstrating each of J1,J2 ⊢ C6 (*LoF*), C6 ⊢ J0,C2 (§A.4), and J0,C2 ⊢ J1,J2 (§A.1).

Robbins conjectured that C6 could be replaced by its dual. That OI and the dual of C6 constitute a Ba basis eluded proof until McCune (1997) brought powerful theorem proving software to bear on the question.

Table 6-2.
Selected CTV/Ba Axioms, pa Initials, Reexpressed in pa Notation.

Year	*Author*		*Axioms/Initials* *Syll1: (a′b)(b′c)a′c. Syll2: (ab′)(bc)ac*	*Diff.*	*Length*
1885	Peirce[75]	CTV	J0, C3, *((a′b)a)a*, *Syll1*, *(a′b′c)b′a′c*	3	55
1917	Nicod	CTV	J0, *Syll2*	4	20
1924	Lukasiewicz-Bernays	CTV	*A′ab*, *(aa)a*, *Syll2*, *(ab)ba*	4	36
1929	Lukasiewicz	CTV	*a′ab*, *(aa)a*, *Syll1*	4	32
1942	Rosser[76]	CTV	*a′ab*, *(aa)a*, *Syll2*	4	28
1948	Lukasiewicz-Wajsberg	CTV	C3, *((a′b)r)(r′a)s′a*	5	25
1956	Church	CTV	PC1, PC2, *((a′⊥)⊥)a*	4	42
1964	Mendelson	CTV	PC1, PC2, *(a′b)(ab)b*	4	43
1690	Leibniz	na	J0, C3, C5, *Syll1*	2	32
1872	Robert Grassmann	na	J0, J2, ()()=(), ⊥⊥=⊥, ()⊥=(), OI	1	45
1890	Schröder-Lejewski	Ba	J0, C3, *Syll1*, *(ab)c = ((a′c)(b′c))* *a′(bc)(a′b′)(a′c′) = ()*	2	73
1892	Johnson	na	C1, C5, contradual of C6, OI	2	32
1904	Huntington	Ba	J1, J2, *a⊥=a*, *ab=ba*	3	37
1913	Sheffer[77]	Ba	*((aa)(aa))=a*, *a(b′b)=a*, *(ab)c=((a′c)(b′c))*	3	41
1916	Bernstein	Ba	*a(b′b)=a*, J2, *ab=ba*	3	34
1933	Huntington	Ba	C6, OI	2	23
1933	Robbins-McCune	Ba	C6 (dual), OI	6	21
1969	Abbott	Ba	C3, C4, *ab=ba*, *(ab)c=(ba)c*	2	31
1969	*LoF*	Pa	J1, J2, OI	1	35
1986	Bricken	Pa	C1-C3, OI	1	28
2000	Veroff	Ba	*((ab)(a(bc)))=a*, *ab=ba*	5	20
2002	McCune et al	Ba	*((b((ab)b))(a(cb)))=a*	6	21
2002	Bricken	Pa	C2, C3, OI	1	21

75. This is Prior's 3.11, his reading of Peirce (W5: 162-90, 1885).

76. Eves 1990: 256, L′; Prior 6.3. The pa translation invokes the dual reading, as Rosser's primitive connective is '∧'.

77. *LoF* discusses this basis then asserts (p. 107), without proof or citation, that *(ab)c* = *((c′a)(b′a))* and the dual of *a′(b′b)=a* form a Ba basis. Since *((aa)(aa))=a* is the Sheffer stroke equivalent of C1, Spencer Brown in effect alleged that replacing *a(b′b)=a* with its dual enables a proof of C1. Bernstein (1934: 880) proved that *(ab)c* = *((a′c)(b′c))* and an axiom equivalent to the Robbins axiom form a Ba basis.

Note. PC1, PC2 are defined in §5.4. *Length* = Number of BA symbols required to state the axioms in BA notation. A primed variable counts as 3 symbols, '()' as 1. (*ab*), not *a*′*b*′, translates the Sheffer stroke. A CTV axiom is treated as an equation ending in '=()', adding 2 to its length. I have shortened axioms of the form (ϕ)=() [(ϕ)=(γ)] to ϕ=⊥ [ϕ=γ]. A dual pair of axioms in the source is stated as a single pa axiom here. I have: (i) eliminated axioms requiring that the cardinality of the carrier be at least 2; (ii) replaced Ba axioms asserting commutativity and associativity with OI; and (iii) added OI to every pa basis. To date, no published basis includes OI. *Diff.*: A purely subjective assessment of the ingenuity required to derive J1-C6 in *LoF* from the given basis, with 6 requiring the most ingenuity.

McCune et al. Employing computer-intensive search methods they devised, McCune et al (2002: Sh1) found a new single axiom for Ba based on the Sheffer stroke, (((*bc*)*a*)(*b*((*ba*)*b*)))=*a*,[78] requiring only 21 symbols. This is the shortest known single axiom for Ba/CTV, whether based on the Sheffer stroke (Prior 6.4) or not (Prior 1.5, 3.13, 6.14; McCune et al 2002: DN1). They proved (Th. 3) that there can be no shorter single axiom for Ba whose sole connective is the Sheffer stroke. Note that the Robbins-McCune basis, and Bricken's (2002) basis with ()*a*=() replacing his ((()*a*)=⊥, are also 21 symbols long.

The literal pa translation of Veroff's (McCune et al 2002: 2) basis is ((*ab*)(*a*(*bc*)))=*a* and (*ab*)=(*ba*). The four parentheses in the second axiom are needed only because Veroff's sole primitive operation is the Sheffer stroke, and thus can be eliminated at no cost. The resulting Ba basis is only 20 symbols long, the shortest in Table 6-2, contradicting Wolfram's claim that his basis is the shortest possible.

Single axiom bases make for very difficult proofs of elementary results. For instance, Wolfram's (2002: 810-11) derivation of Sheffer's (1913) Ba axioms from the single axiom of McCune et al requires 343 steps, 81 lemmas, and expressions with as many as 128 operators. Merely proving that the Sheffer stroke commutes requires 42 lemmas! Wolfram (p. 1175) also ran the following computer "horse race," where the contestants were the eight bases named on his p. 808. For each basis, he counted how many steps were required to prove, using proprietary software, each of 582 consequences, each containing no more than 2 variables and 6 instances of the Sheffer stroke. His results, reported only as graphical summaries, revealed large differences across the eight bases in the number of proof steps required. Sheffer's 1913 and Veroff's basis were more or less tied for fewest proof steps, averaged over the 582 consequences. The Robbins-McCune basis did poorly; its dual, Huntington's (1933a) basis, was not a contestant. The single axiom bases of McCune and of Wolfram fared worst of all.[79]

6.3. Other Historical Systems Related to the pa.

A pa basis can serve as a CTV basis and vice versa. CTV axioms take the form 'α→β' or can be re-expressed as such. pa initials are of the form 'α=β', and are easier to work with, especially for those whose mathematical habits are those of elementary algebra.

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78. Verified as follows. Let *a* | *b*⇔(*ab*). *Dem*. (((*bc*)*a*)(*b*((*ba*)*b*))) [C2,2x] = (((*bc*)*a*)(*b*((*a*)))) [C1] = (((*bc*)*a*)(*ba*)) [J2] = ((((*bc*))*b*′)*a* [C1; OI] = (*b*′*bc*)*a* [J1] = *a*. □

79. This computer 'horse race' suggests a relatively objective way of ranking the various bases in Table 6-2 by difficulty.

The distinction is not essential, however, because any axiom of the form $\alpha\rightarrow\beta$ is equivalent to the equational form $(\alpha)\beta=()$. OI is absent from the CTV bases in Table 6-2, because no CTV basis contains a pair of postulates asserting that one of $\wedge$ or $\vee$ both commutes and associates. Instead, these bases were designed so that this can be demonstrated.

Nicod (1917: 34) proposed a two axiom basis for the CTV, formulated using only the Sheffer stroke, read as NAND. The longer of these axioms is more easily understood when re-expressed using the conditional as well as the stroke (*PM*, p. xviii). Invoking the dual reading, so that $a|b \Leftrightarrow (ab)$ and $a\rightarrow b \Leftrightarrow (a(b))$, and treating any outermost parentheses as redundant, Nicod's axioms are $a'a$ and $(ab')(bc)ac$, the shortest (20) CTV basis in Table 6-2. The substitutions a'/a and b'/b reveal that the latter axiom is an instance of *Syll1*. Because it is shorter by four symbols, I give it a distinct name, *Syll2*. Nicod then condensed his two axioms into one (*PM*, p. xix). From this single axiom and a variant of *modus ponens*, he derived the axioms of *PM*, thereby proving that his single axiom was a CTV basis. Lukasiewicz later simplified Nicod's single axiom into something (not shown in Table 6-2 but well-known: cf. Prior 6.4; Quine 1982: 87) whose dual pa translation has length 23.

Lukasiewicz (Prior 1.4a; Quine 1982: 85) proposed in 1929 a basis I call L, one with a straightforward pa interpretation. $(a\underline{a})a$ is one half of C5. $a'ab$ is J0 and is the other half of C5 when b/a. *Syll1* asserts the transitivity of the conditional and the validity of Barbara. *Syll1* with a=() yields another hoary chestnut, *modus ponens*. Nicod's two axiom basis, just discussed, is, in effect, L with $a'ab$ omitted. (Note also the similarity of Nicod's basis to Rosser's.) Hence a slight modification of *Syll1* renders $a'ab$ redundant; I have not seen this fact mentioned in print.

Lukasiewicz and Bernays, working independently, earlier proposed a basis including *Syll2* and $(ab)ba$ in place of *Syll1*, yielding a revised version (Prior 6.11) of the truth-functional axioms of *PM*.[80] The literature is silent about the similarity between Rosser and Lukasiewicz-Bernays, as well as about how L and Rosser render *PM*'s $(ab)ba$ redundant. In 1948, Lukasiewicz proved that $((a'b)r)(r'a)s'a$ is the shortest possible single axiom from which all formulae involving '$\rightarrow$' alone can be demonstrated (Prior 2.15d). Wajsberg (Prior 3.12) had shown in 1937 that adding $\mathsf{T}\rightarrow a$ (C3, in effect) and $\sim a =_{\mathbf{df}} a\rightarrow\mathsf{F}$ to any axiom system adequate for '$\rightarrow$' alone results in a CTV axiom system.

The radically "arithmetical" foundation BA proposes for Ba and the truth functors has a curious precedent. Van Horn (1917), writing in ignorance of Sheffer (1913), purported to derive *PM* 's CTV axioms from a single axiom giving the semantics of the Sheffer stroke (here notated by $\uparrow$), restated as follows: $[|a|=|b|]\rightarrow[|a\uparrow b|\neq|a|]$, and $[|a|\neq|b|]\rightarrow[|a\uparrow b|=\mathsf{T}]$. This axiom follows trivially from A1, A2, keeping in mind that $[a\uparrow b \Leftrightarrow (a)(b)] \leftrightarrow[()\Leftrightarrow\mathsf{T}]$ and $[a\uparrow b \Leftrightarrow (ab)]\leftrightarrow[(())\Leftrightarrow\mathsf{T}]$. The axiom also follows from 4.1.2 above. Nicod (1917: 40) praised Van Horn's paper, but claimed that its derivations of the *PM* axioms were flawed, because Van Horn freely invoked what is here called T16, without being aware that he was invoking a metatheorem needing proof. Even though Van Horn's and Nicod's papers were published side by side, the only citation of Van Horn I have en-

80. For a (nontrivial) demonstration of J2 from these axioms, see Halmos & Givant (1998: 37-8).

countered is Grattan-Guiness (2000: 434). While no one, to my knowledge, has revisited Van Horn's paper, the pa reveals that its intuition was sound.[81]

Byrne's (1946) Ba notation is based on juxtaposition as sole connective and the unary prime. It translates into the pa as follows: (1) remove parentheses not immediately followed by a prime; (2) if a pair of parentheses is primed, remove the prime; (3) replace subformulae of the form $\alpha\alpha'$ by (). Hence only the manner in which parentheses are used distinguishes Byrne's notation from that of the pa. Curiously, six of Byrne's eight theorems are C1-C6. Byrne proves his algebra Boolean by deriving Huntington's (1933a) basis, C6 and OI. A virtue of Byrne's system is that it deliberately leaves juxtaposition uninterpreted, and dispenses with Boolean 0 and 1. Even though Byrne's paper is well-known, to my knowledge his notation has no imitators. Likewise, the boundary mathematics literature, such as it is, does not seem aware of Byrne's work.[82]

The system AB of Anderson and Belnap (1959) features one unary operation, denial, denoted by an overbar, and one connective, disjunction, denoted by $\vee$. Erasing all instances of $\vee$, and enclosing in parentheses that which lies under an overbar, results in the equivalent pa formula. AB is semantically identical to the primal reading of the pa.[83] The sole axiom of AB is $b(a)ac$ = (), true by J0. The rules of inference of AB are (1) from *bac*, infer $b((a))c$, true by C1, and (2) from $(a)c$ and $(b)c$, infer $(ab)c$. *Cal.* $(a'\underline{c})(b'\underline{c})(ab)c$ [C2,2x] = $\underline{(}a'\underline{)}\underline{(}b'\underline{)}(ab)c$ [C1,2x] = $\underline{ab}\underline{(ab)}c$ [J0] = (). □

The creators of AB founded relevance logic, a fact consistent with AB's disavowal of *modus ponens* and silence re substitution. Hunter (1971: §37.6) sets out an effective proof

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81. The way *LoF* grounds Ba in a bit of Boolean arithmetic is not without precedent. Shannon (1938) gives the following basis for **2**: (a) his arithmetical postulates (1)-(3), isomorphic to Table 2-1a; (b) his assumption (4) that *B* has two members; and (c) a loose definition of complementation. Table 2-1b is isomorphic to his "theorems" (7a) and (7b). Prior (1962: 4-13), following Polish practice, grounds his exposition of sentential logic in the Boolean arithmetic of 0 and 1. Cole (1968) derives Ba from analogues of Table 2-1, R1, and T16. Malmstadt et al (1973: 281, §3-4.1) sketch a derivation of **2** from Boolean addition (dually, multiplication) and complementation, taken as axiomatic. Rudeanu (1974) asserts (Example 1.1), but does not show, that his axioms for Ba (namely OI, absorption, Boolean versions of J1, J2, C3, and the duals of all the preceding, as per his Definition 1.1) can, in the case of **2**, "be easily established by direct verification" from the operation tables for Boolean addition, multiplication, and negation. (Shannon 1938 makes a similar assertion about his (1a)-(8).) Most of Rudeanu's axioms are redundant, in that J1, J2, and OI suffice as a pa basis, the pa is a Ba by 3.3.8, and is complete by T17.

82. Byrne's Ba axioms are four: *B* has at least 2 members, OI, $\forall xyz[xy{=}x \rightarrow y'x{=}z'z]$, and $\forall xy\exists z[y'x{=}z'z \rightarrow xy{=}x]$. The latter two axioms, often misquoted, are essentially the Consistency Principle, 3.3.10. I do not grant Byrne's axioms the pride of place I granted them in Meguire (2003), because existentially quantified variables in algebraic axioms now strike me as a *faute de beauté*.

83. To Hunter (1971: §37) I owe my discovery of the system AB. In a striking feat of bravado, Anderson and Belnap (1959) lay out AB and prove it sound, complete, decidable, and axiom independent in less than 300 words. AB thus enjoys the dubious distinction of being the tersest version of sentential logic devised to date. Hunter (§§37.1-5) expands the proofs of these metatheorems to all of three whole pages. The alternative proof of T17 in §A.10 makes possible an equally terse statement of the pa.

procedure for AB based on refutation trees, but the resulting proofs are a good deal more complicated than ones based on J0-C2. For example, Hunter requires 16 lines to verify that $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$. The corresponding pa calculation is trivial: *Cal.* $(\underline{p}'q'\underline{r})(\underline{p}'q)p'r$ [C2,3x] = $\underline{(q')(q)}p'r$ [J0] = (). □

Translated into BA, Schütte's (1977: 17) basis for the CTV is J1 and (⊥) [A2] = (). His rules I∧, I∨, and I→ are all instances of J0. I cannot determine whether an equivalent to C2 resides somewhere in Schütte's system.

7. Why the Indifference?

"[The pa] *is a very beautiful version of the propositional calculus, and I cannot understand why it has not become a standard method in logic text-books… Spencer-Brown's theory has gained great popularity among various people, but logicians have taken little interest in it.*"

Grattan-Guiness (1982: §5.1).

Notwithstanding the merits I and others have claimed for it, the pa remains unadopted. I begin exploring the reasons behind this fate by setting out what this curious book reveals about its origins. Spencer-Brown worked out many of the ideas in *LoF* while teaching an introductory course in logic (pp. xii; the page references below without citation are to *LoF*); *LoF* may have begun as lecture notes for this course. He derived his version of BA (namely, one using '⏋' instead of '()', and with no analogue to '⊥') by working backwards from **2** and CTV (p. 112). He (p. xii) attributed some key insights to his having designed electronic circuits during the 1960s, but did not cite Shannon's (1938) celebrated result that the algebra of switching circuits is a model for the CTV (and thus also for the pa).

LoF does not sufficiently disclose the extent to which it builds on Boolean algebra. While rightly citing some classics. Huntington (1904, 1933) and Sheffer (1913), as well as Boole and Peirce, it cites no text on Boolean algebra extant at the time of writing (e.g., the 1958 ed. of Hohn 1966; Whitesitt 1961; Arnold 1962; Goodstein 1963). The only formal logic text cited is Prior (1962), invoked near the end to make a minor point about syllogisms. *LoF* fails to cite any of the following texts, all standard at the time of writing: Hilbert and Ackermann (1950), the 1950 ed. of Quine (1982), Rosenbloom (1950), Quine (1951), Rosser (1953), Church (1956), Suppes (1957), Carnap (1958), Nidditch (1962), and Kneebone (1963). Worst of all, *LoF* appears oblivious to the centrality of first order logic and quantification.

Spencer-Brown argues (chpt. 11) that certain infinite pa formulae with a finite recursive representation have an "imaginary" truth value, arising in a manner analogous to the way complex numbers arise from the roots of certain polynomial equations with real coefficients, also having a recursive interpretation.[84] He alleges that such truth values

84. If a Boolean equation has a pa representation that is not recursive, *LoF* (p. 57) says that is "of the first degree". Recursive equations are said to be of "degree higher than one." If neither () nor (()) solve an equation of higher degree, then *LoF* (pp. viii-x, 58) argues that it has an "imaginary" solution. Relating imaginary Boolean values to extant work on recursive arithmetic and functions (e.g., Mendelson 1997: chpts. 3,5; Kneebone 1963: chpt. 10) and to Bochvar's (1981) paradox logic are all possible directions for future research.

have momentous implications for mathematics, philosophy, and engineering. For instance, they supposedly extinguish the paradoxical character of self-reference[85]:

> "All we have to show is that the self-referential paradoxes, discarded with the Theory of Types, are no worse than similar self-referential paradoxes, which are considered quite acceptable, in the ordinary theory of equations. The most famous such paradox is… 'This statement is false.'" (p. ix)

> "[Recursive Boolean] equations have hitherto been excluded from the subject matter of ordinary logic by the Russell-Whitehead theory of types" (p. xviii)[86]

The Theory of Types is meaningful only if the ground logic is of order greater than zero, and such is never the case in *LoF*. In fact, *LoF* is innocent of polyadic predicates, as well as of all but a few trivial bits of naïve set theory (the Boolean algebra of classes makes a very brief and casual appearance in Appendix II). Spencer-Brown asserts that his imaginary truth values render the well-known limitative theorems of Gödel and Church (Stoll 1963: §§9.9, 9.10; Mendelson 1997: Ths. 3.37-3.54) "…less destructive than was hitherto supposed" (p. xvii), but gives no details.[87] Meanwhile, *LoF* is silent about nonclassical and infinitary logics and the theory of recursive functions, conventional topics which render chpt. 11 less radical than it would seem at first blush.

While *LoF* does not claim that BA suffices to ground all of mathematics, others stride boldly where angels fear to tread: "…the propositional calculus…develops naturally from [A1 and A2]. Thus the act of severance leads inexorably to logic and through *PM* to the whole of mathematics." (Croskin 1978: 187) This statement would be true if BA could model the quantification and second order logic *PM* requires, and if *PM* had succeeded in its aims. (A classic critique of *PM* is Quine 1995: 3-36, first published in 1941.)

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85. On self-reference, logically and philosophically contemplated, see Bartlett (1992).

86. Before touching on Russell's paradox and the like, Spencer-Brown should have read Prior (1962: §III.3.3) closely. Spencer-Brown cites the 1958 edition of Fraenkel, Bar-Hillel, and Levy (1973: chpt. III) in support of his contention that there have been prior attempts to "…rehabilitate, on a logical rather than on a mathematical basis, something of what was discarded with the Theory of Types…" (*LoF*, p. xix, fn 8). This is a very curious reading of Fraenkel et al, who survey type theory in the context of set theory, but never claim that type theory ever was standard. On type theory, see Quine (1969: §§34-38) and Hatcher (1982: chpt. 4).

87. Church's theorem states that first order logic, a system quite distinct from the pa and Ba, is undecidable. Those doubting the classic limitative theorems of Gödel should consult the short simple proofs in Boolos (1998: 383-88) and in Smullyan (1991: chpts 1,2). Boolos's proof is grounded in Berry's paradox; Smullyan's proofs require little more than a formal language capable of enough self-reference to sustain the Diagonal Lemma. Neither author requires any of recursion, the Chinese Remainder theorem, a specific Gödel numbering, and Peano arithmetic. Spencer-Brown would have done well to study Tarski's theorem (a finitely axiomatized formal system strong enough for mathematics cannot define its own truth predicate), which is easier to prove than Gödel's and deserves to be better known. Smullyan also shows that the self-referential paradoxes of the sort *LoF* deems "quite acceptable" (see quote in text) are not so easily dismissed.

LoF also indulges in philosophical speculation (pp. v, vi, xix-xxii, 85, 89-96, 101-106) and invokes dubious etymologies (pp. 93, 101, 105, 106, 109, 126). Spencer-Brown claims (p. ii) to have studied under Wittgenstein[88] (whom he cites four times) and R D Laing, but is silent on how he learned mathematics and logic. Elsewhere, he claims to have worked with Lord Cherwell in the 1950s and the mathematician J C P Miller in the 1960s.

Spencer-Brown is especially guilty of falsified predictions about the future course of mathematics. Writing in 1967, he claimed that:

> "...if we confine our reasoning to an interpretation of Boolean equations of the first degree only, we should *expect* to find theorems which will always defy decision, and the fact that we do seem to find such theorems in common arithmetic may serve, here, as a practical confirmation of this obvious prediction. To confirm it theoretically, we need only to prove (1) that such theorems *cannot* be decided by reasoning of the first degree, and (2) that they *can* be decided by reasoning of a higher degree. (2) would of course be proved by providing such a proof of one of the theorems.
>
> "I may say that I believe that at least one such theorem will shortly be decided by the methods outlined in [*LoF*]. In other words, I believe that I have reduced their decision to a technical problem which is well within the capacity of an ordinary mathematician who is prepared, and who has the patronage or other means, to undertake the labour." (pp. 99-100; emphasis in original)

More specifically:

> "…I found evidence, in unpublished work undertaken in 1962-65, suggesting that the four-colour map theorem [*sic*] and Goldbach's conjecture are undecidable with a proof structure confined to Boolean equations of the first degree, but decidable if we are prepared to avail ourselves of equations of higher degree." (p. xix)[89]

Regarding Fermat's Last Theorem (FLT), Spencer-Brown wrote:

> "…it is my guess that Fermat (who was apparently too excellent a mathematician to make a false claim to a proof) used [imaginary truth values] in the proof of his great theorem, hence the 'truly remarkable' nature of his proof, as well as its length." (p. 99).

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88. Whose spectre haunts *LoF*, as well as much of British philosophy of the mid-20th century. I leave to others the pleasure of tracing the specific influence of Wittgenstein's *oeuvre*, the *Tractatus* in particular, on *LoF*. A similar pleasure undoubtedly awaits the Peirce or Whitehead expert willing to give *LoF* a close reading.

89. Spencer-Brown repeated his claim of a proof of the Four Colour Map theorem in a letter to the editor of *Nature*, dated 17.12.76.

Spencer-Brown was asserting that certain very well-known mathematical conjectures (as of the time he wrote) were unprovable using standard mathematics grounded on classical bivalent logic, but could be proved using mathematics grounded in the 3-valued logic (i.e., one incorporating an imaginary truth value) he introduced in chpt. 11 of *LoF*. In the nearly 40 years that have elapsed since *LoF* first appeared, nothing of this sort has eventuated. Instead, seven years after *LoF*'s first publication, Haken and Appel announced their proof of the Four Colour Map Theorem, one based on conventional mathematics albeit supplemented by a large amount of machine computation (for a definitive treatment see Haken and Appel 1989). Wiles (1995) finally proved FLT using difficult, albeit thoroughly standard, mathematics. All this may help explain why Spencer-Brown's work has been ignored, and why it need not be swallowed whole.[90]

8. Conclusion.

"Logical laws are the most central and crucial statements of our conceptual scheme, and for this reason the most protected from revision by the forces of conservatism; but… they are the laws an apt revision of which might offer the most sweeping simplification of our whole system of knowledge." Quine (1982: 3).

Let there be a blank surface upon which marks may be written. The mark can be the '⏋' of *LoF* (proposed by Peirce in 1886), a simple closed curve (proposed by Peirce from 1896 onwards), or Croskin's (1978) '()', adopted here. The symbol () is the sole primitive constant and is both operator and operand. Whatever the mark is taken to be, it is essential that it have a distinguishable "interior" and "exterior," as the mark serves as the boundary between the interior and exterior. The mark and the blank page are the boundary primitive values. Interpreting the blank page as one of 'true' or 'false' gives rise to boundary logic.

The only means we have, at this stage, to distinguish anything is to write another mark on the state we wish to distinguish. To mark the exterior, we write ()(); to mark the interior, (()). The Law of Calling says that ()() is indistinguishable from simple (). The Law of Crossing says that (()) cannot be distinguished from the blank page. Hence the exterior and interior of a mark are distinguished simply by the way each interacts with another mark. The exterior of a mark is idempotent; the interior, nilpotent. Let Calling and Crossing be the sole axioms. *LoF* shows, in a rather cryptic fashion, that the primary arithmetic (PA) emerges from these notions plus logical equivalence, an equivalence relation. Thus the PA can be seen as Boolean arithmetic notated so as to lay bare its tree structure. For another recapitulation of the PA, see 2.3.4.

Inserting letters anywhere in a PA formula yields a pa (primary algebra) formula. Combine the PA and pa to obtain boundary algebra (BA). Letting the blank page interpret Boolean 0, and () interpret Boolean 1 (or vice versa), yields **2**. The set *B*={(),(())} corresponds to the carrier of **2**, and a letter (a.k.a. variable) can assume any value of *B*. BA and boundary logic are equational rather than ponential, i.e., they privilege tautological

90. Lest I seem too critical, I should add that *LoF* is an instance of a worthy genre, namely introductions to formal logic intended for nonspecialists. A fine instance of the genre is Hodges (1977), built around refutation trees and linguistic examples.

equivalence rather than tautology, and invoke the substitution of equals for equals instead of *modus ponens*.

An initial is a tautological equivalence verified by a decision procedure. Any set of CTV axioms or Boolean algebra basis translates into a set of pa initials. The initials *a′a*=() and (*ba*)*a*=*b′a*, and the well-known consequences *aa*=*a*, (*a′*)=*a*, and (*a′b′*)*r*=((*ar*)(*br*)) enable *calculation*, a proof method similar to, but easier than, that of Peirce's *alpha* existential graphs. pa calculations are much easier than the proofs taught in standard texts, especially natural deduction proofs. The pa facilitates clausal reasoning, and trivializes the derivations of the inference rules of conventional logic. The pa and the CTV share a common metatheory.

The pa is, at minimum, a simple yet powerful notation for the truth functors and Boolean algebra, revealing the unity and simplicity underlying the seeming diversity of truth functors. Moreover, because the CTV and **2** are models of the pa, BA highlights the seldom mentioned axiomatic role of Boolean arithmetic for these systems. BA also suggests that mathematical logic, set theory, theoretical computer science, and probability[91] all share a common source: the mental act of making a distinction, for which the marker is the boundary sign (). I invite others to explore whether BA may be seen as a nominalist grounding for **2** and sentential logic, i.e., one abstaining from the notion of set.

95 years after Boole's first book and 30 years after *PM*, Berkeley (1942) noted that formal logic and Boolean algebra had been little applied.[92] With the exception of computer science, electrical engineering, and formal philosophy, this appears to be the case down to the present day (Hehner 2004 speaks to this). BA, treated as a demotic version of the hieratic languages **2** and the CTV, could facilitate the application of logic and Boolean methods. BA could be taught in secondary schools: pragmatically, as an introduction to the abstractions underlying information technology; intellectually, as a gentle introduction to logic and discrete math. Likewise, incorporating BA into Hehner's (2004, 2005) Unified Algebra, an integrated notation for Ba and numerical mathematics, warrants exploration.

Ba (and hence BA) may yet play a significant role in theoretical physics. The notion of boundary appears to be making inroads into physical theory. In a talk titled "It from Bit," given at the Santa Fe Institute in 1989, John Wheeler said:

> "The boundary of a boundary is zero. This central principle of algebraic topology, identity, triviality, tautology though it is, is also the unifying theme of Maxwell's electrodynamics, [general relativity], and almost every version of modern field theory. That one can get so much out of so little, almost everything from almost nothing, inspires hope that we will someday complete the mathematization of physics and derive everything from nothing, all law from no law." Wheeler (1996: 302)

91. Discrete probability can be given a Boolean foundation by taking mathematical expectation as primitive, then defining probability as the expectation of a Boolean random variable. See Lad (1996: §2.2).

92. Some of the reasons Berkeley gave for why this might be the case do not apply to BA. By the way, Berkeley, who was employed in insurance at the time he wrote, neither cited Shannon (1938) nor mentioned the possibility of electronic computation, on the brink of discovery.

Less speculatively, the BA points to a streamlined unified treatment of Boolean algebra, he truth functors, and monadic logic. This treatment should include a more rigorous re-statement of BA, one dispensing with the enigmatic "canons" of *LoF*. The ontological grounding and semantics of the PA need firming, and the boundary analogue to refutation trees should be explored. I have indicated the applicability of boundary methods to mereology and lattice theory, the latter being a gateway to a variety of nonclassical (e.g., intuitionistic, relevant) logics. Boundary methods should prove fruitful for other formal systems near to Boolean algebra, such as modal logic and relation algebra, and may prove applicable to combinatory logic and to more general algebraic structures.[93]

93. Meguire (2004) proposes boundary versions of quantification theory, normal modal logic, the theories of sets, lattices, categories, groupoids, and ringoids. On connections between Boolean and other algebras, see Rudeanu (1974: §§12.3-7) and Burris et al (1981: §II.1). Smullyan (1985) is a humorous introduction to combinatory logic. On non-classical propositional logic, see Restall (2000) and Epstein (1995).

Bibliographic Postscript.

BA lies at the intersection of four disciplines: mathematics, philosophy, computer science, and electrical engineering. I included a reference below either because I found it useful, or it was extant and relevant at the time *LoF* was written. The refences, grouped by broad topic, are listed in order of increasing perceived difficulty.

- There are two distinct perspectives on *Boolean algebra*:
  - *Mathematical*: Arnold (1962: §§2,5), Hohn (1966: §§5.1-5), Goodstein (1963: §§2,3), Halmos and Givant* (1998: §§19-39), Stoll* (1963: §6), Rosenbloom* (1950: §§I.1-3, II.3), Abbott (1969: §§6,7), Cori & Lascar* (2000: §2), Koppelberg (1989), Burris et al (1981: §§II.1, IV.1-4). Algebras more general than **2** are typically assumed, and are developed in either a *set theoretic* or *algebraic* manner. Starred references discuss the close connection between Ba and CTV called *Tarski-Lindenbaum algebra*.
  - *Engineering/Computer Science*: Whitesitt (1961: §§1-3), Hohn (1966: §1), Rudeanu (1974: esp. §1; many references).
- *Lattice theory*. Arnold (1962: §§3,4), Donnellan (1968), Curry (1963: §4 is very relevant for logic), Davey and Priestley (2002), Burris et al (1981: §§I, II.1).
- *Calculus of Truth Values*. *Mathematical*: Arnold (1962: §1), Hohn (1966: §§3.1-12), Goodstein (1963: §4), Kneebone (1963: §§2,6), Halmos and Givant (1998: §§8-18), Epstein (1995: §§II.J-M), Stoll (1974: §§2.1-5,3.5), Nidditch (1962), Machover (1996: §7), Mendelson (1997: §1), Hunter (1971: §§15-36), Smullyan (1968: Part I), Cori & Lascar (2000: §1), Curry (1963: §§5,6), Schütte (1977: §I). *Philosophical*: Girle (2002: Part One), Quine (1982: Part I), Suppes (1957: §§1,2), Bostock (1997: §2), Prior (1962: 1-71, 301-6, 318-19), Hodges (2001: §§1-7), Carnap (1958: §§2-8, 12a, 22), Zeman (1973: 1-76), Segerberg (1982).
- *Calculus of Quantified Individuals*. *Mathematical*: Stoll* (1974: §2.6-9, §3.6), Machover* (1996: §8), Quine* (1951: §2), Hunter (1971: §§38-59), Pollock (1990: §2.1), Smullyan (1968: Part II), Mendelson* (1997: §2), Schütte (1977: §II), Cori & Lascar (2000: §§3,4). Starred references include axiomatic set theory. *Philosophical*: Girle (2002: §§12-14), Quine (1982: Parts II, III), Bostock (1997: §§3,5), Hodges (2001: §§8-18), Carnap (1958: §§1, 9-14, 21-25). Bostock, Hodges, and Machover are best for current terminology.

For gentle introductions to logic as part of elementary mathematics, see Wolf (1998: Unit 1); to metamathematics and axiomatic thinking, see Stoll (1974: §3); to the philosophy of mathematics, see Lucas (1999).

On Peirce's role in the early history of Ba, see Brady (2000: 1-142). The first systematic treatment of Ba is vol. 1 of Schröder (1966), written in 1890. Ba came of age as serious mathematics in the 1930s, thanks to Marshall Stone, Tarski, and others. MacColl (Frege) invented the CTV (FOL) in 1877 (1879). Mathematical logic came into its own with *Principia Mathematica* and the work of Hilbert and his students between the Wars. On the history of logic and related mathematics, see Curry (1963), Grattan-Guiness (2000), Gabbay and Wood (2004), and Kneebone (1963).

There is a fair secondary literature on *LoF*; see the bibliography at http://www.lawsofform.org/bib/index.html, in which the names Bricken, Kauffman, and Varela stand out, as do a number of articles in the *International Journal of General Systems*. Another URL bearing on *LoF* is http://www.enolagaia.com/GSB.html . Both sites reveal that *LoF*'s love of paradox and enigma has attracted a nonmathematical following.

Appendix: The Controverted Ontology of the Null Individual.

This Appendix reviews a controversy in the foundations of *mereology*, a body of first order theories about the part-whole relation described in Simons (1987: chpts. 1,2) and Casati and Varzi (1999: chpt. 3). Mereology begins with a domain of individuals, and a primitive dyadic predicate P*xy*, read as '*x* is part of *y*.' P is assumed transitive and can be proved a partial order. Let the fusion *b* of any number of inviduals *a* be such that *a*P*b* comes out true for all *a*. An axiom asserts that the *fusion* of the members of any nonull set [of those individuals satisfying any monadic predicate] exists.

Nearly all mereological systems deny the existence of a *null individual*, one that is part of every individual; the main exception is a system advocated by R. M. Martin. For present purposes, the null individual can be deemed the mereological analogue of the null set. In this Appendix, I argue that the denotation I propose for ⊥ is controversial in a manner analogous to the controversies aroused by Martin's null individual.

To my knowledge, the null individual, under the name *null entitity*, made its first public appearance in the following passage from Martin (1943: 3): "In order to develop an unrestricted Boolean algebra… it is desirable to admit the existence of a null entity… We shall retain then the interpretation of this system as a calculus of individuals and also admit the null entity." Carnap (1956: 36), citing Martin (1943), postulated the existence of a *null thing* as one of seven possible things named by a nonunique description. He wrote: "…a natural solution offers itself if we construct the system in such a way that the spatiotemporal part-whole relation is one of its concepts. …it is possible, although not customary in the ordinary language, to count among the things also the null thing, which corresponds to the null class of space-time points. …it is characterized as that thing which is part of every thing."

Geach (1972: 200), written in 1949 in response to the 1947 edition of Carnap (1956), wrote: "There is a well-known convention in mathematics whereby 'the least' or 'the only' number fulfilling a condition is deemed to be zero if there is in fact *no* number thus uniquely described. This has technical advantages… Carnap proposes an allegedly similar convention for language about physical objects [, the null thing]. Further, [Carnap] describes the null thing as corresponding 'to the null class of spacetime points'—or, in plain English, as existing nowhen and nowhere!"

The following long quotation is taken from the opening paragraphs of Martin's vigorous defense of the null individual, first published in 1965 and reprinted as Martin (1979). I trust that my substitution of ⊥ for 'null individual' has not traduced Martin's meaning.

"Is there such a thing as ⊥? Well, as an actual or concrete entitity, certainly not. There is no such actual entity, there never has been, and there never will be. If this were the whole story, one could end therewith. As a convenient technical fiction and useful notational device, however, introducing ⊥ into [first order logic] is not without interest. ⊥ can be given important roles to perform and it can be made to perform them well, so well in fact as to lend strong support to regarding its theory as a suitable appendage to logic.

"One speaks of *the* ⊥ in the sense of there being one and only one ⊥. Could there be two or more? Possibly, but there is no need for such, and anyhow it is desirable to keep traffic with the ghostly at a minimum.

“Attitudes differ as to the feasibility of introducting ⊥... Lejewski… explicitly admits a ‘nonreferential name… meant to be a name that does not designate anything.’ Such a name is to be read ‘object which does not exists.’

“That the notion of ⊥ is no better or worse than that of the null set seems likely. Refusal to postulate one should perhaps go hand in hand with refusal to postulate the other. The null set… is a useful mathematical notion that has been with us with impunity for some time. Set theory… would be impoverished without it and technical inconveniences would result. These are perhaps not insurmountable, but little would be gained if one were to reject it. And mathematics abounds with other convenient technical fictions that by parity of reasoning would have to be forsworn, many of these depending definitionally on the null class. The more reasonable course then seems to be to admit not only the null set but also such additional ‘fictions’ as are feasible if strong technical reasons can be given on their behalf.” (Martin 1979: 82-83)

Martin went on to cite Carnap (1956) with approval.

Bunt (1985: 56-7), nowhere mentioning Martin or Carnap, wrote of the *empty ensemble* as follows: “*emptiness*… is defined as the property of having no other parts than itself…From the transitivity of the part-whole relation it follows that all parts of an empty ensemble are empty. …it can be proved that there exists an empty ensemble, and that an empty ensemble is part of every ensemble [individual].” [emphasis in original]

Simons (1987: 13), citing Geach, summarily dismisses the null individual as follows: “Most mereological theories have no truck with the fiction of a null individual which is part of all individuals… The chief culprit in propounding this absurdity is R M Martin.”

Finally, Lewis (1991: 11) writes: “If we accepted the null individual, no doubt we would identify the null set with it, and so conclude that the null set is part of every class. But it is well nigh unintelligible how anything could behave as the null individual is said to behave. It is a very queer thing indeed, and we have no good reason to believe in it. Such streamlining as it offers in formulating mereology [e.g., closure under intersection] can well be done without. Therefore, reject the null individual; look elsewhere for the null set.”

Casati and Varzi (1999: 45) also distance themselves from Martin, whom they relegate to a footnote, but in a more cautious way: “…few authors have gone so far as to postulate the existence of a ‘null individual’ that is part of everything. Without such… (which one could hardly countenance except for algebraic reasons), the existence of an [intersection] is not always guaranteed. Likewise… complements may not be defined, e.g., relative to the universe.”

Given how contentious the null individual has proved to be, despite its seeming ontological innocence, those who doubt the innocence of ⊥ can be forgiven.

Appendix: Demonstrations, Proofs, etc.

Throughout this Appendix:

- 'OI' means that order irrelevance is invoked;
- The symbol '☐' signals the end of a proof/demonstration/calculation.

A.1. The Core Demonstrations and Calculations.
LoF's notrivial deduction of C1 from J1 and J2 enables the following quick demonstration of J0: *Dem:* **a′a** [C1] = **((*a′a*))** [J1] = (), which generalizes as follows: *Dem*. (*a*)*ab* [OI] = (*a*)*ba* [C2; OI] = (*a**b***)*ab* [J0] = (). ☐

C2,C3 ⊢ J0. *Proof*. Erasing all instances of any letter from a tautology results in a tautology, because letters stand indifferently for () and ⊥, and A2 always sanctions erasing ⊥. Erasing *b* in C2 yields *a*(*a*)=*a*() [C3] = (). ☐ Bricken (2002) invokes C2 to justify *a*(*a*)=*a*() without discussion.

The demonstrations below differ from those in *LoF* mainly in that the initials here are J0 and C2. The calculations of C1 and C5 are original. For the time being, I mention each invocation of R1.

C5. **aa=a**. *Cal*. *LR*: (*aa*)*a* [C2] = (*a*)*a* [J0] = (). *RL*: (*a*)*aa* [C2] = (*a**a***)*aa* [J0] = (). ☐
J1. *Dem.* **((a)a)** [J0; R1] = **(())**. ☐
C3. *Dem.* **()a** [J0; R1] = **(a)a***a* [C2] = (*a**a***)*aa* [J0] = **()**. ☐
C4. *Dem*. **(a′b)a** [C2] = ((*a*)*b**a***)*a* [C2; TR] = ((*a**b***)*ab*)*a* [J1] = (())*a* [A2] = **a**. ☐
C4a. *Corollary*. *Dem*. **(())a** [J0; R1] = ((*a*)*a*)*a* [C4] = **a**. ☐ C4a can do most of what A2 does.

C1. ((a))=a. *Cal*. *LR*: (((*a*)))*a* [C2,2x] = (((*a*)*a*)*a*)*a* [C4; R1] = (*a*)*a* [J0] = ().
RL: (*a*)((*a*)) [TR; J0] = (). ☐

I now demonstrate the remaining *LoF* consequences. I adapted the demonstration of J2 from Bricken (1986). The calculation of J2 is original. R1 will henceforth be invoked without mention.

J2. a((b)(c)) = ((ab)(ac)).
Dem. **a((b)(c))** [C2,3x] = *a*((***a**b*)(***a**c*)) [C1; χ=((*ab*)(*ac*))] = **(*a′*)**χ [C2] = (*a*′**χ**)χ [C2,3x] = (*a*′(***a*′***a*′*ab*)(***a*′***ac*)))χ [J1,2x] = (*a*′(*a*′(())(())))χ [C4a,2x] = (*a*′(*a*′))χ [J1] = (())χ [C4a] = **((ab)(ac))**. ☐
Cal. *LR*: (*a*(*b*′*c*′))((*ab*)(*ac*)) [C2,2x] = (*a*((***ab***)(***ac***)))((*ab*)(*ac*)) [C2] = (*a*)((*ab*)(*ac*)) [C2,3x] = *a*′(***a*′**(*a*′*ab*)(***a*′***ac*)) [J1,2x] = *a*′(*a*′) [J0] = ().
RL: (((*ab*)(*ac*)))*a*(*b*′*c*′) [C1] = (*ab*)(*ac*)*a*(*b*′*c*′) [C2,2x; OI] = *ab*′*c*′(*b*′*c*′) [J0] = (). ☐

The demonstrations of C6 and C7 follow *LoF*.

C6. *Dem*. **(a′b′)(a′b)** [C1] = **(((*a*′*b*′)(*a*′*b*)))** [J2] = (*a*′((*b*′)*b*′)) [J1] = (*a*′) [C1] = **a**. ☐
C7. *Dem*. **((a′b)c)** [C1] = (((*a*)**((*b*))**)*c*) [J2] = (((*ac*)(*b*′*c*))) [C1] = **(ac)(b′c)**. ☐

C9 is simpler than its *LoF* counterpart, and its demonstration is new:

C9. *Dem.* $\mathbf{((b'r)(a'r'))}$ [J2] = $\underline{\underline{(((\boldsymbol{b'r})a)((\boldsymbol{b'\underline{r}})r))}}$ [C1; C2] = $((b'r)a)(\underline{(b')}r)$ [C1; OI] = $(a(b'r))(br)$ [C2] = $(a(b'r)\mathbf{(br)})(\ br)$ [OI; C1,2x] = $(a(\underline{\mathbf{(r')}b'}\underline{(\mathbf{(r')}b)}))(br)$ [C6, r'/A, b/B] = $\mathbf{(ar')(br)}$. □

The calculation of a consequence is usually easier than its demonstration. C9 is an exception; the *RL* part of the following calculation is surprisingly difficult.

Cal. *LR*: $\underline{\underline{((b'r)(a'r'))}}(ar')(br)$ [C1; OI] = $(rb')(rb)(r'a')(r'a)$ [C1,2x] = $(\mathbf{(r')}b')(\mathbf{(r')}b)(r'a')(r'a)$ [C6,2x] = $r'r$ [J0] = ().

RL: $((ar')(br))((b'r)(a'r'))$ [C7,2x] = $(r(b\underline{r}))(a'(br))(b(a'r'))(r'(a'\underline{r'}))$ [C2,2x] = $(r(b))(a'(br))(b(a'r'))(r'\underline{(a')})$ [C1; OI] = $(rb')(a'(rb))(b(r'a'))(r'a)$ [C2,2x] = $(rb')(a'\underline{\mathbf{(rb')}}\underline{(rb)})(b\underline{(r'a')}\underline{\mathbf{(r'a)}})(r'a)$ [C6,2x] = $(rb')(a'r')(br)(r'a)$ [OI; LR] = (). □

The *RL* part of the above calculation is much more easily verified by TVA:

r=(): $((a)\underline{(b())})(\underline{(b'())}(a'))$ [C3; C4a] = $\underline{\underline{(a)}}\underline{\underline{(a')}}$ [C1,2x] = aa' [J0] = ().

r=⊥: $(\underline{(a())}(b))((b')\underline{(a'())})$ [C3; C4a] = $\underline{\underline{(b)}}\underline{\underline{(b')}}$ [C1,2x] = bb' [J0] = ().

Absorption (**L3**).

Dem. Primal: $\mathbf{a(a'b')}$ [C2] = $a((a\mathbf{b'})b')$ [C2; OI] = $a\underline{((ab')\mathbf{a}b')}$ [J1] = $\mathbf{a}$.

Dual: $\mathbf{(a'(ab))}$ [C2] = $(a'(\mathbf{a'}ab))$ [C2; OI] = $(a'\underline{((a\mathbf{b})ab)})$ [J1] = (a') [C1] = $\mathbf{a}$. □

L3, $a(ab)=a$, the lattice version of Absorption follows, since the pa translation of $a(ab)$ is the dual pair $a(a'b')$ and $(a'(ab))$.

Commutativity and Associativity

Demonstrating that juxtaposition commutes and associates, while contrary to the planar spirit of the pa, is nevertheless possible. First note that erasing c from OI yields $ab=ba$; call this TR. Now let OI be read, here only, as $ab.c = bc.a$. Then:

Associativity. *Dem.* **ab.c** [OI] = $bc.a$ [TR] = **a.bc**. □

That the dual of juxtaposition associates also follows trivially from C1: *Dem.* $(\underline{\underline{(a'b')}}c')$ [C1] = $(a'b'c')$ [C1] = $(a'\mathbf{((}b'c'\mathbf{))})$. □ That juxtaposition itself associates then follows from 4.1.3. Hence that juxtaposition associates requires only what the proof of 4.1.3 and the demonstration of C1 require. *Proof*: C1, R1 ⊢ 4.1.3. J0, C1, C2 ⊢ R1. J0, C2, TR ⊢ C1. Hence J0, C2, TR ⊢ 4.1.3. □ This proof is much simpler than Huntington's (1904; reproduced in Eves 1990: 217-19) proof of associativity from the basis {J1, J2, a⊥=a, TR} (see Table 6-2).

I now reproduce Byrne's (1946: 271) demonstration of TR from OI and C5 alone. Note that the calculation of C5 above requires neither OI nor TR, and only J0 and C2. OI justifies each step below, unless otherwise noted.

TR. *Dem.* **ab** [C5] = $ab.ab$ = $b.ab.a$ = $ab.a.b$ = $ba.a.b$ = $aa.b.b$ = $bb.aa$ [C5, R1, 2x] = **ba**. □

Having shown that juxtaposition associates, we have no further need for notations such as $ab.c$.

A.2. Proof of 2.3.10.

2.3.10. **Theorem**. R is an equivalence relation iff R is *reflexive* and *Euclidian*.

Proof. I assume that the uniform replacement of letters ranging over the field of a relation is allowed (the analogous BA property is R2). Since formulae of the form *xRy* have truth values, they can be treated as BA atomic formulae. The BA version of *Euclidian* is (*aRc*)(*bRc*)*aRb* = (). Then:

(*aRc*)(*bRc*)*aRb* [*a*/*c*] = (*aRa*)(*bRa*)*aRb* [*reflexive*; C4a] = (*bRa*)*aRb* ⇔ *bRa*→*aRb*.
(*aRc*)(*bRc*)*aRb* [*a*/*b*; *b*/*a*; *c*/*b*] = (*bRb*)(*aRb*)*bRa* [*reflexive*; C4a] = (*aRb*)*bRa* ⇔ *aRb*→*bRa*. Hence *aRb* ↔ *bRa*, i.e., *R* is *symmetric*.
(*aRc*)(*bRc*)*aRb* [*c*/*b*; *b*/*c*] = (*aRb*)(*cRb*)*aRc* [*symmetric*] = (*aRb*)(*bRc*)*aRc* ⇔ (*aRb* ∧ *bRc*) → *aRc*. Hence *R* is *transitive*. □

Remark. Using ponential methods, Lukasiewicz (1967: 97-98) derives the three properties of equivalence relations from a variant of *Euclidian*, (*cRb*)(*cRa*)*aRb*, 1 on p. 97. Reflexivity and symmetry are *6 and *7 on p. 97; transitivity is 5 on p. 98.

A.3. Proofs Required for §3.3.

3.3.11. **Theorem** (Consistency Principle). *a*≤*b*, *a*∪*b*=*b*, and *a*∩*b*=*a* are equivalent Ba statements, and these in turn have the pa equivalents *a′b*=(), *ab*=*b* and (*a′b′*)=*a*.

Dem: ***a∪b=b*** ⇔ *ab*=*b* ⇔ $(((a\underline{b})b)(b(a\underline{b})))$ [C2,2x] = $(((a)b)\underline{(b(a))})$ [OI; C5] = $\underline{(((a)b))}$ [C1] = ***a′b***.
a∩b=a ⇔ (*a′b′*)=*a* ⇔ $((\underline{((a'b'))}a)((\underline{a'}b')(a)))$ [C1; C2] = $((a'b'a)(\underline{(b')}a'))$ [C1] = ((*a′b′a*)(*ba′*)) [OI] = $(\underline{(a'ab')}(a'b))$ [J1] = $\underline{((a'b))}$ [C1] = ***a′b***.

Moreover, *a′b* ⇔ *a*≤*b* because *a′b* satisfies the three criteria for a partial ordering:
Reflexivity: *a′a* [J0] = (). *Antisymmetry*: *a*≤*b* ∧ *b*≤*a* ⇔ ((*a′b*)(*b′a*)) [Def. of =] ⇔ *a*=*b*.
Transitivity: (*a*≤*b* ∧ *b*≤*c*) → *a*≤*c* ⇔ $\underline{(((a'b)(b'c)))}a'c$ [C1] = $(\underline{a'}b)(b'\underline{c})a'c$ [C2,2x] = $\underline{(b)(b')}a'c$ [J0] = ()*a′c* [C3] = (). □

3.3.15. Theorem. The cardinality of *B* in BA is necessarily 2.
Cal. *x*=() ∨ *x*=⊥ ⇔ (*x′*(()))(*x*())(*x′*⊥′)(*x*⊥) [C4a,3x] = (*x′*)(*x*())(*x′*())(*x*) [C3,2x] = (*x′*)(())(())*x′* [C4a,2x] = (*x′*)*x′* [J0] = (). □

A.4. Demonstrating J0 and C2 from C6 and OI.

Huntington (1933), cited on p. 88 of *LoF*, showed that C5, C6, and OI form a basis for Ba. LoF was not aware that Huntington (1933a) showed C5 redundant by deriving it from C6 and OI. The following demonstration that C6 and OI form a Ba basis, adapted from Kauffman (1990) and simpler than Huntington's,[94] also implies that Johnson's axioms C1 and C5 are redundant. I now derive J0 and C2 from OI and C6.

J0. *Dem*. **a′a** [C6,2x] = **((*a′*)(*b′*))((*a′*)*b′*)(*a′*(*b′*))(*a′b′*)** [OI] = ((*a′*)(*b′*))(*a′*(*b′*))((*a′*)*b′*)(*a′b′*) [OI] = $\underline{((b')(a'))((b')a')(b'(a'))(b'a')}$ [C6,2x] **= b′b**. Hence *a′a* has the same value for any *a*, so that *a′a* can be equated to either primitive value. I go with *a′a*=(), but *a′a*=⊥ is equally valid and gives rise to the dual reading. A trivial variant of this demonstration yields J1. □

C1. *Dem*. **(a′)** [C6] = $(((a'))a')\underline{(((a'))(a'))}$ [J1, (*a′*)/A, *a′*/B] = (((*a′*))*a′*)((*a′*)*a′*) [C6] = **a**. □
C5. *Dem*. **aa** [C1] =**((*aa*))** [J1] =(**(a(a))**(*aa*)) [C1,2x] =(((**(*a*)**)(*a*))(**(*a*)**)*a*)) [C6] =$\underline{((a))}$ [C1] = **a**. □

94. See http://www.lawsofform.org/logic.html. This URL also includes Bricken's (1986) demonstrations.

C2. *Dem*. **a′b** [C6,2x] = $(ab)\underline{(ab')}(b'a)(b'a')$ [OI; C5] = $(ab)\underline{(b'a)(b'a')}$ [C6] = **(ab)b**. □

A.5. α=β ⇔ α↔β ⇔ ((α′β)(β′α)) Because '↔' Is an Equivalence/Congruence Relation. In this section, I employ '⇔' in place of '=' because I am temporarily suspending belief in '=' as an equivalence relation.

Symmetric: [α=β]⇔[β=α] ⇔ [α↔β]↔[β↔α].
LR: *Cal*. $\underline{((}(\alpha'\beta)(\beta'\alpha)\underline{))}((\beta'\alpha)(\alpha'\beta))$ [C1] ⇔ $(\alpha'\beta)(\beta'\alpha)((\beta'\alpha)(\alpha'\beta))$ [C2,2x] ⇔ ().
RL: Trivial, and also evaluates to (). □

Transitive: Let χ stand for (α′β)(β′α)(β′δ)(δ′β). 'α=β and β=δ implies α=δ' trnslates as (α′β)(β′α)(β′δ)(δ′β)((α′δ)(δ′α)) ⇔ χ((α′δ)(δ′α)) [J2] ⇔ ((χα′δ)(χδ′α)).
Cal. χα′δ ⇔ $(\underline{\alpha'}\beta)(\beta'\alpha)(\beta'\underline{\delta})(\delta'\beta)\alpha'\delta$ [C2,2x] ⇔ $(\beta)(\beta'\alpha)\underline{(\beta')}(\delta'\beta)\alpha'\delta$ [C1] ⇔ $\underline{\beta'}(\beta'\alpha)\underline{\beta}(\delta'\beta)\alpha'\delta$ [OI; J0] ⇔ ().
χδ′α ⇔ () by a strictly parallel reasoning. □

Reflexive: *Cal*. [α=β] ⇔ $(\underline{(\alpha'\alpha)(\alpha'\alpha)})$ [J1,2x] ⇔ (). □

By virtue of satisfying conditions C_1 and C_2 below, '=' is also a *congruence relation* (cf. 3.3.10 in the text, and Stoll 1963: 259-61). I now demonstrate this fact, $\forall a,b,c \in B$, again translating '=' as '↔':

C_1. $a=b \rightarrow ac=bc$. *Cal*. $(a'b)(b'a)(((a\underline{c})bc)((b\underline{c})ac))$ [C2,2x] ⇔ $(a'b)(b'a)(((a)b\underline{c})((b)a\underline{c}))$ [J2] ⇔ $(a'b)(b'a)((a'b)(b'a))\mathbf{c}$ [OI] ⇔ $\underline{((a'b)(b'a))(a'b)(b'a)}c$ [J0] ⇔ ().

C_2. $a=b \rightarrow a'=b'$. *Cal*. $(a'b)(b'a)(((\underline{a'})b')((\underline{b'})a'))$ [C1,2x] ⇔ $(a'b)(b'a)((ab')(ba'))$ [OI] ⇔ $((a'b)(b'a))(a'b)(b'a)$ [J0] ⇔ (). □

C_3. $a'=b' \rightarrow a=b$. *Cal*. $((\underline{a'})b')((\underline{b'})a')((a'b)(b'a))$ [C1,2x] ⇔ $(ab')(ba')((ab')(ba'))$ [OI] ⇔ $((a'b)(b'a))(a'b)(b'a)$ [J0] ⇔ (). □

Remark. Stoll does not mention C_3, which enables replacing C_2 with $a'=b' \leftrightarrow a=b$, a corollary of which being that either of J0 or J1 does duty for the other. Note how the calculations reveal that C_2 and C_3 reduce to the same thing, which differs only slightly from C_1.

A.6. J1-C5 Are Standard in Logic and Boolean Algebra.

Table A-1.

	Algebra of Propositions	BA
1	Idempotent	C5
2	Associative	OI
3	Commutative	OI
4	Distributive	J2
5	Identity	A2
6	"	C3
7b	Complement	J1
8a	"	C1
8b	"	A2
9	De Morgan's	Transcriptional triviality

Table A-1 shows the correspondence between Lipschutz's (1964: 195) "Laws of the Algebra of Propositions" and BA. Each Law is actually a dual pair, 8a and 8b excepted; 8a is self-dual. Corresponding to each law is a law for the algebra of sets (p. 104). When *LoF* was written, Lipschutz was a standard elementary treatment of set theory and sentential logic. Only two basic BA notions are missing from Table A-1: A1, a trivial consequence of 5 and 6, *and C2*.

A.7. More on C2.

First a quick TVA proof. Given $(ba)a = (b)a$, let:

- $b=\perp$: Simply erase b. The lhs becomes $(a)a$ [J0] = (), and the rhs becomes $()a$ [C3] = ().
- $b=()$: The lhs becomes $(()a)a$ [C3] = $(())a$, the rhs, $(())a$.

Invoke T7 twice to complete the proof. □

Equivalents of C2 include Johnson's (1892: 342) Law of Exclusion, the third consequence proved in his system, Th. 7 in Byrne (1946), (48) in Rosser (1953: 113), T8-6.j4 in Carnap (1958), T12 in Stoll (1963: 257), and exercise 3.12.a in Hohn (1966). C2 is a trivial corollary of $(b\rightarrow a)\leftrightarrow[(b\vee a)\leftrightarrow a]$, *4.72 in *PM*, T73 in Kalish et al (1980: §II.11), and (38) in Cori and Lascar (2000: 32). C2 also follows trivially from (31) in Suppes (1957: 204), once it is understood that $a\sim b$ (Suppes) $\Leftrightarrow$ $(a'b)$. C2 can be obtained from $[(a\rightarrow b)\wedge (c\rightarrow b)]\leftrightarrow[(a\vee c)\rightarrow b]$, *4.77 in *PM*, (18) in Stoll (1974: 85), and (57) in Cori and Lascar (2000: 33), via b/c and noting that $[(a\rightarrow b)\wedge(b\rightarrow b)]\leftrightarrow(a\rightarrow b)$.

One half of C2, viewed as a biconditional, is C2′, $(a\rightarrow b)\rightarrow[(a\vee b)\rightarrow b]$. C2′ is (22) in Grassmann (1966: Be-13), *2.621 in *PM* (the other half is *2.67), Zeman (1973: 2.20), and Leblanc and Wisdom (1976: 99, Example 21). Zeman shows that C2′ is merely a substitution instance of a tautology equivalent to *modus ponens*. The converse of C2′ is likewise a substitution instance of his 2.12. Zeman derives C2′ from that part of the implicational calculus intuitionists accept. C2′ is also an axiom in a system Hilbert set out in 1922 (system 1.3) and in four other CTV axiom systems set out in Epstein (1995: 408-9). A trivial substitution turns Reichenbach's (1947: 39) (8h)[95], $((b\vee c)\rightarrow a)\rightarrow(b\rightarrow a)$, into the converse of C2′. Even though C2′ is a substitution instance of Mendelson's (1997: 35) axiom A3, his proof of the converse (Th. 1.11.g) requires 43 lines and the Deduction Theorem! C2′ can even be viewed as an analogue to the special case $\phi=\Delta$ of the "left" version of Bostock's (1997: §2.5) structural rule THIN.

Another related tautology is C2″, $(b\rightarrow a)\rightarrow[(b\vee c)\rightarrow(a\vee c)]$, axiom *1.6 of *PM* (Prior 6.11). Hence C2″ is included in the *PM* axiom system as modified by Lukasiewicz and Bernays (Table 6-2; Prior 6.11 net of (4)) and commonly used since (e.g., Carnap 1958: 86, P1-P4; Kneebone 1963: 43; Mendelson 1997: 45, system L1; Halmos and Givant 1998: 22, T1-T4). To obtain C2′, substitute a for c in C2″ and note that this axiom system trivially implies $(a\vee a)\leftrightarrow a$, C5 in the pa. As best as I can determine, however, the only proof in the sources I cite that invoke any of C2, C2′, or the converse of C2′ is Zeman's proof of his 2.21. I conclude that extant expositions of the CTV are unnecessarily complicated.

A.8. From Any Contradiction, Anything Can Be Proved.

Proof. By C2, $b(ba)=b(a)$ for any a,b. Now let b be any pa formula whatsoever, and let a be a formula such that both $a=()$ and $a=\perp$ are demonstrable by hypothesis. The lhs of C2

95. *Cal*. $((b\vee c)\rightarrow a)\rightarrow(b\rightarrow a) \Leftrightarrow ((bc)\underline{a})b'a$ [C2] = $\underline{((bc))}b'a$ [C1; OI] = $b'bac$ [J0] = (). □

evaluates as $b(ba)$ [let a=()] = $b(\underline{b}())$ [C3] = $b(())$ [C4a] = b. The rhs of C2 evaluates as $b(a)$ [let a=⊥] = $b(\bot)$ [C4a] = $b()$ [C3] = (). Hence b=(), the desired absurd result. □

A.9. Proof of Theorem 4.1.3.

The proof is by *induction on formula length*, a standard technique well-explained in Bostock (1997: §2.8). The proof follows Bostock closely, except for notation.

Definition: Given a formula α, its *length* $l(\alpha)$ = number of variables in α less 1, plus the number of left parentheses and primes in α. If $l(\beta) < l(\alpha)$, then β is *shorter* than α.

Notation: Given the pa formula $\alpha = \alpha\langle x_1, \ldots, x_n \rangle$, its contradual is $\overline{\alpha} =_{\mathbf{df}} \alpha\langle x_1', \ldots, x_n' \rangle$.

Lemma: $\alpha^D = (\overline{\alpha})$.

Proof. The hypothesis of strong induction is:

For all formulae β such that $l(\beta) < l(\alpha)$, $\beta^D = (\overline{\beta})$.

There are three cases to consider.

α *is atomic*: $\alpha^D = \alpha$ [C1] = **((**α**))** [Def. of overbar] = $(\overline{\alpha})$.

α *is enclosed*: α = (β).

$\alpha^D = (\beta)^D$	Substitute (β) for α [R1]
$= (\beta^D)$	Def. of duality
$= ((\overline{\beta}))$	Inductive hypothesis
$= (\underline{(\beta)})$	Def. of overbar
$= (\alpha)$	Substitute α for (β) [R1].

α *is a concatenate*: α = βχ for some formula χ.

$\alpha^D = [\beta\chi]^D$	Substitute βχ for α [R1]
$= ((\beta^D)(\chi^D))$	Def. of duality
$= (((\overline{\beta}))((\overline{\chi})))$	Inductive hypothesis
$= (\underline{\overline{\beta}\overline{\chi}})$	C1, 2x
$= (\beta\underline{\chi})$	Def. of overbar
$= (\alpha)$	Substitute α for βχ [R1]. □

Theorem 4.1.3: $\alpha = \varphi \leftrightarrow \alpha^D = \varphi^D$.

Proof. If $\alpha = \varphi$, then $\overline{\alpha} = \overline{\varphi}$ [R2], so that $(\overline{\alpha}) = (\overline{\varphi})$. Hence $\alpha^D = \varphi^D$ by the lemma above. □

The lemma [4.1.3] is Bostock's (1997: §2.10) First [Third] Duality Theorem and Quine's (1982: §12) second [5th] law of duality.

A.10. The pa metatheorems.

T10. J2 extends to any finite number n of divisions (2.1.7) of the subspace of depth 1.

Proof (*LoF*, pp. 38-39). T10 with n=0 is simply C3. T10 with n=1 yields $((a))r$ [C1] = ar [C1] = $((ar))$. The following demonstration verifies the case n=3.

Dem. $r(a_1'\ a_2'\ a_3')$ [C1,2x] = $r(a_1'\,\mathbf{((}a_2'\,\mathbf{((}a_3'\mathbf{))))})$ [J2] = $((ra_1)(r(a_2'((a_3')))))$ [J2] = $((ra_1)\underline{((}(ra_2)(r\underline{(}a_3'\underline{)))}))$ [C1,2x] = $((a_1r)(a_2r)(a_3r))$.

It should be evident that the case n=3 generalizes to any finite n, if each instance of '2x' in the preceding demonstration is replaced by '(n-1)x'. □

T14. Let d_α^*>2 for some formula α. Then α can be transformed, by taking steps, into an equivalent formula β such that d_β^* =2.

Proof. (The repeated use of C7 is from *LoF*; the proof is otherwise new.) α can be seen as an *ordered tree*, one or more of whose *branches* terminate at some maximal depth d_α^*. Let d_α^*>2, and let β, χ, and ϕ be subformulae appearing at depths d_α^*, d_α^*-1, and d_α^*-2, respectively, of any longest branch of α. Let γ denote all of α not accounted for by β, χ, and ϕ, so that α=(((β)χ)ϕ)γ. By C7, (((β)χ)ϕ)γ=(βϕ)(χ′ϕ)γ. The maximum depth of (βϕ)(χ′ϕ)γ is 1 less than that of (((β)χ)ϕ)γ. This depth-reducing procedure based on C7 can be repeated, each time suitably redefining β, χ, ϕ, and γ. When one branch of α is exhausted, switch to the longest remaining branch. Continue until no branch of α has depth>2. □

Remark: Each application of C7 to the *terminus* of any branch of α reduces that branch's depth by 1. This fact enables the following, perhaps simpler, algorithm. Beginning at the terminus of any branch, apply C7 repeatedly until a *node* is encountered, then switch to the terminus of any other branch. Continue until no further applications of C7 are possible, at which time no branch will have depth>2. For more on ordered trees, see Smullyan (1968: §§I.0-1). Ordered trees are also models of bounded semilattices.

T15. Let the pa formula α⟨*v*⟩ contain more than 2 instances of the variable *v*. Then α can be transformed, by taking steps, into an equivalent formula β⟨*v*⟩, such that β⟨*v*⟩ contains at most 2 instances of *v*.

Proof (Adapted from *LoF*). By C1 and T14, there exist a subformula *f*⟩*v*⟨, and the sequences of subformulae, a_i, p_i, and x_j, such that

α⟨*v*⟩	$= ((va_i)p_i)\ldots f(vx_i)\ldots$	
	$= (((\mathbf{(}v'\mathbf{)(}a_i'\mathbf{)})p_i)\ldots f(vx_i)\ldots$	Apply C1 twice for each value of *i*.
	$= (v'\mathbf{p_i})(a_i'\mathbf{p_i})\ldots f(vx_i)\ldots$	Apply J2 and C1 once for each value of *i*.
	$= (v'p_i)\ldots(a_i'p_i)\ldots f(vx_i)\ldots$	By OI, the disjuncts $(v'p_i)$ can grouped together, to the left of the $(a_i'p_i)$.
	$= (v'p_i)\ldots g(vx_j)\ldots$	Let $g = (a_i'p_i)\ldots f$.
	$= \mathbf{(((}\underline{v}'p_i)\ldots\mathbf{))(((}\underline{v}x_j)\ldots\mathbf{))}g$	C1, 2x; OI *g*.
	$= ((p_i'\ldots)v')((x_j'\ldots)v)g$	T10, 2x. □

T16. Let the variable *v* appear in at least one of the formulae α and β. Let $v \in B$ be a possible value of *v*, and α⟨ v ⟩ be α⟨*v*⟩ with *v* set to v. Then if $\forall \upsilon[\alpha\langle\upsilon\rangle = \beta\langle\upsilon\rangle]$, then α=β.

Proof (Adapted from *LoF*, pp. 47-49). Let *v vary* between ⊥ and (). Now consider the following two mutually exclusive and exhaustive cases:

1. Variation in *v* either alters or does not alter *both* α⟨*v*⟩ and β⟨*v*⟩. If α⟨*v*⟩=β⟨*v*⟩ after a change in *v*, then α⟨*v*⟩=β⟨*v*⟩ must have been the case before the change. Hence α=β in this case.

2. Variation in *v* causes the value of one of α⟨*v*⟩ or β⟨*v*⟩ to change, but *not both*. If both were to change, there would exist a v such that α⟨ v ⟩≠β⟨ v ⟩, contrary to hypothesis. Hence if varying the value of *v* causes one of α⟨*v*⟩ or β⟨*v*⟩ to change, the other must change as well, so that the reasoning of case 1 applies.

Hence α⟨*v*⟩ and β⟨*v*⟩ are equivalent in all cases of *v*, so that α=β in any case. □

Remark. If only one of $\alpha\langle v\rangle$ and $\beta\langle v\rangle$ is the case, then variation in v cannot affect the value of the formula in which it appears.

T17. The pa is complete.

Proof (adapted from *LoF*, pp. 50-52). Let α_k, β_k be formulae each containing some variable v. Let k-1 be the number of distinct variables other than v appearing in either α_k or β_k. The proof proceeds by strong induction on k, with the inductive hypothesis being $\forall k<n$ $[\alpha_k=\beta_k]$ for some integer $n>0$. T15 assures us that there exist formulae α_n^* and β_n^*, with v appearing at most twice in each, such that $\alpha_n^* = \alpha_n$ and $\beta_n^* = \beta_n$. Moreover, T14 assures us that the depths of α_n^* and β_n^* need not exceed 2. Since α_n^* and β_n^* are the normal forms of α_n and β_n, they can be written as:

(E1) $\alpha_n^* = (v'a_1)(va_2)a_3$ (E2) $\beta_n^* = (v'b_1)(vb_2)b_3$,

where a_i and b_i, i=1,2,3, are suitable sub-formulae whose depth do not exceed 1.

Now let v=() and $v=\perp$ in turn, with each case resulting in formulae for α_{n-1}^* and β_{n-1}^*:

v=(): $\alpha_{n-1}^* = ((())a_1)(()a_2)a_3 = (a_1)a_3$; $\beta_{n-1}^* = ((())b_1)(()b_2)b_3 = (b_1)b_3$.

$v=\perp$: $\alpha_{n-1}^* = ((\perp)a_1)(\perp a_2)a_3 = (()a_1)(a_2)a_3 = (a_2)a_3$; $\beta_{n-1}^* = ((\perp)b_1)(\perp b_2)b_3 = (b_2)b_3$.

The inductive hypothesis asserts that $\alpha_{n-1}^* = \beta_{n-1}^*$ can be proved. Hence:

(E3) $\alpha_{n-1}^* = \beta_{n-1}^* \rightarrow (a_1)a_3 = (b_1)b_3$, and

(E4) $\alpha_{n-1}^* = \beta_{n-1}^* \rightarrow (a_2)a_3 = (b_2)b_3$,

are provable. I now demonstate that $\alpha_n^* = \beta_n^*$. N.B. C9: $((A'R')(B\,R)) = (AR')(BR)$.

$(v'a_1)(va_2)a_3$	E1, transcription of α_n^*.
$((v'(a_1))(v(a_2)))a_3$	C9, a_1/A, a_2/B, v/R
$((v'(a_1)a_3)(v(a_2)a_3))$	J2, a_3/R
$((v'(b_1)b_3)(v(b_2)b_3))$	E3, E4
$((v'(b_1))(v(b_2)))b_3$	J2, b_3/R
$(v'b_1)(vb_2)b_3$	C9, b_1/A, b_2/B, v/R
β_n^*	E2.

All that remains to be shown is that there exists a value of n for which the inductive hypothesis, $\alpha_k=\beta_k$ $\forall k<n$, holds. Suppose that n to be 1, in which case only k=0 need be considered. Now if k=0, then α_0 and β_0 are PA formulae. $\alpha_0=\beta_0$ can be proved in the pa if A1 and A2 are pa consequences, which I proceed to show: A1 is C5, ()/A. A2 is C1, $\perp$/A, and *Dem*. $\perp\perp$ [C5, $\perp$/A] = $\perp$; ($\perp$) [A2] = **((()))** [C1, ()/A] = (); $\perp$()= ()$\perp$ [C3, $\perp$/A] =(). □

T17. Alternate Proof (following Kneebone 1963: 48).

I take completeness to mean: *all tautologies are demonstrable from the* pa *initials*. By T14, any formula α has an NF representation $\beta=(a_i^*...)_j...$. Keep in mind that for some j, i may equal 1. Every step in the derivation of β from α is justified by invoking one of C7, C1, or C2, all initials or consequences derivable from the initials. Hence if β is derivable from the initials, then so is α. Also if β is a tautology, α is as well, because $\beta=\alpha$ by T14. To demonstrate that β is a tautology, it suffices to consider two cases:

Case 1. $\forall j\,(a_i^*\ldots)_j=\perp$. If every disjunct contains some variable in both primed and unprimed form, then each disjunct (and hence β as well) simplifies to ⊥ by J1.
Case 2. $\exists j\,(a_i^*\ldots)_j=()$. This is demonstrable if there exists a variable x such that one disjunct is simply (x) and another is $((x))$. Then β=() by J0.

J0, J1, C1, C2, and C7 are either initials, or can be derived from the initials. Hence if α is a tautology, the initials suffice to verify that fact. □

T18. The initials J1 and J2 are independent. ({J1, J2} is an independent basis.)
Proof. (Adapted from *LoF*). J1: $(a'a)=\perp$. J2: $((ar)(br))=(a'b')r$.
In J1, no variable instance crosses a boundary. In J2, r moves from depth 2 to depth 0. Hence J2 cannot be proved using J1 alone. Meanwhile, J1 creates or eliminates the variable a. J2 does not create or eliminate any variable. Hence J1 cannot be proved using J2 alone. □

T18a. {J0, C2, OI} and {C6, OI} each form an independent basis.
Proof. J0 [C6] creates or eliminates a variable; C2 and OI do not. C2 creates or eliminates some but not all instances of a variable. J0 [C6] creates and destroys all instances of a variable; C2 and OI do not. OI has no boundary while J0 and C2 [C6] each have 1 [2] boundaries. Hence J0, C2, [C6] and OI are mutually independent. □

A Précis of Mathematical Logic.

I now review some notions from that combination of the CTV and the CQI known as *First Order Logic* (**FOL**). For a masterly précis of FOL and its extension to axiomatic set theory, see Fraenkel, Bar-Hillel, and Levy (1973: §V.2). For more leisurely expositions, consult the references cited under “Quantifier Logic” in the Bibliographic Postscript especially Bostock (1997), the *Encyclopedia Britannica* articles “Logic” and “Metalogic.” For a treatment more sophisticated than the one below, see http://plato.stanford.edu/entries/logic-classical/ .

A *string* consists of a single symbol, or of concatenated symbols. Absent semiotic theory, *symbol* is undefined. $B=\{\mathsf{T},\mathsf{F}\}$ is the set of possible *truth* or *primitive values*.

Calculus of Truth Values, CTV. A *statement* (*sentence*, *proposition*) is a string that can be assigned a truth value. Statements include *formulae*, i.e., strings that satisfy a formation rule. The definitions of *subformula* and *atomic formula* in 2.1.3 carry over to CQI in the obvious way. A *statement letter* (*sentential variable*) stands for any member of some set of statements. (Truth) *functor*, *connective*, and *operator* are defined in 3.1.2; these relate to mappings from B^n onto B, $n\in N$. The constants T and F are 0-*ary* functors by assumption. Common functors include the prefix ~ “not”, and the infix connectives ∧ “and”, ∨ “or”, → “if”, ↔ “iff”, | “NAND”.

An *atomic valuation* (i) assigns an element of B to every atomic formula, and (ii) completely describes the mapping f_k: $B^n \to B$ for every n-ary operator k. A statement consisting of $m\in N$ statement variables (*arguments*) and constants, linked by connectives, is a *truth function* from B^m onto B. The *truth value* of a statement is the image of its truth function under some atomic valuation. If the image is T [F], the statement is *valid* [invalid]. If a statement is valid under [all/some] atomic valuations, the statement is [*tautologous*/*satisfiable*]. If a statement is not valid under any atomic valuation, its denial is a tautology. If all atomic valuations satisfying α also satisfy β, and vice versa, α and β are *tautologically equivalent*, denoted α↔β.

Proof (adapted from Halmos and Givant 1998: §13). An *axiom* is a statement asserted true without proof. The *rule of detachment* is: If α and α→β are both tautologies, then β is also a tautology. Let $i,j,k,n \in N$. A *formal proof* (*demonstration* in *LoF*-speak, or simply *proof*) is an ordered sequence of *n* statements with typical statement α_k $1 \le k \le n < \infty$. A *step* transforms α_k into α_{k+1}. For each α_k, $i,j<k$, α_k is either (i) a substitution instance of some definition, axiom, or already proved consequence, resulting from the application, often tacit, of R1 and R2 (§3.1), or (ii) the result of applying detachment to some pair α_i and α_j. (i) alone suffices for equational logics (e.g., boundary logic), for which detachment is just a special case of *propositional consequence* (§5.3). If there exists a proof whose last statement is α_n, α_n is *provable* and a *theorem*.

CTV is *sound* (all provable formulae are valid), *complete* (all valid formulae are provable), and *decidable* (there exist algorithms, e.g. TVA, for determining whether any finite formula is valid). The *primitive basis* (DeLong 1971: 91) of a formal system consists of its primitive symbols, defined constants, rules of formula formation, axioms and rules of inference, and a truth definition. A *model* is an interpretation of a formal system under which its formulae all come out true; see Suppes (1957: §4.2) or Mendelson (1997: §2.2).

Calculus of Quantified Individuals, CQI. An *variable* stands for any member of a nonempty collection (*domain* [*of interpretation*]) of physical or abstract *individuals*, each having a *name*. A *term*, denoted by a lower case letter, is a name, variable, or a function thereof. The uniform replacement of a term letter by another is permitted. *Predicate letters* are upper case. Associated with each predicate and function letter is a nonnegative integer called its *arity*. A predicate [function] letter with an arity of 0 is a statement variable [*constant*]. An *atomic formula* (aka predicate) consists of a predicate letter followed by *arity* terms. An atomic formula with [more than] one term is [*polyadic*] *monadic*. ∀ [∃] is the *universal* [*existential*] *quantifier*. A quantifier operates on the variable that immediately follows it; ∀*x* [∃*x*] translates as "for all [for some] *x*". A CQI formula consists of quantifiers, and atomic formulae linked by truth functors.

Let α,β be arbitrary CQI formulae. $\exists x\alpha =_{\mathbf{df}} \sim\forall x[\sim\alpha]$, so that there is in fact only one quantifier. Let ***x*** and ***y*** be vectors of variables, of unspecified dimension. Let Q_i be one of ∀ or ∃, and let Q(***x***) be a string of the form $Q_1x_1Q_2x_2...$ known as a *prefix*. Let a *matrix* M(***x***,***y***) consist of atomic formulae and truth functors but no quantifiers, with each x_i and y_i appearing at least once. Q_i *binds* x_i, and x_i is a *bound* variable; the y_i are *free*. An atomic formula has a truth value only if its variables are all bound. If ***y*** has dimension 0 (≥0), then α is *closed* (*open*). CTV formulae are the special case when both ***x*** and ***y*** have dimension 0. The *prenex* form of α is then Q(***x***)M(***x***,***y***), The *scope* of Qx_i is M(***x***) by default, or overridden by parentheses. If ***x*** has dimension 1, and if α is closed and does not lie within the scope of another quantifier, then α is an *elementary quantification*. Writing $\forall y_1\forall y_2...$ to the left of an open formula results in its *universal closure*. α is *valid* (is a "law of logic") if it evaluates to T for all nonempty domains.

CQI requires three axioms in addition to any basis sufficient for the CTV: UI, ∀*x*α⟨*x*⟩ →α⟨*a*/*x*⟩, α⟩*x*⟨→∀*x*α⟩*x*⟨, and ∀*x*[α→β]→(∀*x*α→∀*x*β) (Bostock 1997: 236). Fitch devised these axioms and Quine 1951 popularized them; they enable dispensing with the rule of generalization. 'α⟩*x*⟨' means that any appearances of *x* in α are bound. CQI *with identity* includes a primitive dyadic predicate, denoted by infix '=', which is *reflexive* (*x*=*x*), and obeys the axiom schema (*x*=*y*)→(*F*⟨*x*⟩↔*F*⟨*y*//*x*⟩), where *F* is any atomic formula.

CQI is provably sound (Hunter 1971: §42), complete (§46), and undecidable. Some necessary conditions for a CQI formula to be undecidable include a domain with infinitely many individuals, a prefix with at least one ∃ preceding one or more ∀, and a matrix that is not a substitution instance of a monadic formula. The necessary and sufficient conditions for decidability are not known.

Table of Cross-references between *LoF* and this paper.										
LoF	*Here*		*LoF*	*Here*		*LoF*	*Here*		*LoF*	*Here*
A1	Table 2-1		R1	3.1.7		T5	2.3.6		T15	4.4.3
A2	"		R2	3.1.8		T6	2.3.7		T16	4.4.4
C1-C9	Table 3-1		T1	2.3.2		T7	2.3.9		T17	4.4.5
J0	5.2, A.1		T2	2.3.3		T10	3.1.9		T18	4.4.7
J1	3.1.6		T3	2.3.4		T13	3.1.10			
J2	"		T4	2.3.5		T14	4.4.2			

References

Abbott, J.C. (1969) *Sets, Lattices, and Boolean Algebras* (Allyn and Bacon, Boston).

Anderson, A.R. and Belnap, N.D. (1959) "A simple treatment of truth functions," *Journal of Symbolic Logic 24*: 301-02.

Angell, R.B. (1960) "A logical notation with two primitive signs" (abstract), *Journal of Symbolic Logic 25*: 385.

Arnold, B.H. (1962) *Logic and Boolean Algebra* (Prentice Hall, Englewood Cliffs NJ).

Bartlett, S. J., ed. (1992) *Reflexivity: A Source Book in Self-Reference* (North-Holland, Amsterdam).

Berkeley, E.C. (1942) "Conditions Affecting the Application of Symbolic Logic," *Journal of Symbolic Logic 7*: 160-68.

Bernstein, B.A. (1916) "A Simplification of the Whitehead-Huntington Set of Postulates for Boolean Algebra," *Bulletin of the American Mathematical Society 22*: 458.

——— (1934) "A Set of Four Postulates for Boolean Algebra in Terms of the 'Implicative' Operation," *Transactions of the American Mathematical Society 35*: 876-84.

Birkhoff, G., and Mac Lane, S., (1998) *Algebra*, 2nd ed. Chelsea.

Bochvar, D.A. (1981) "A 3-Valued Logical Calculus," *History & Philosophy of Logic 2*: 87-112.

Boolos, G., 1998. *Logic, Logic, and Logic* (Harvard Univ. Press, Cambridge MA).

Bostock, D. (1997) *Intermediate Logic* (Oxford Univ. Press, London).

Brady, Geraldine (2000) *From Peirce to Skolem* (North-Holland, Amsterdam).

Bricken, W. (1986) "A deductive mathematics for efficient reasoning." Technical Report HITL-R-86-2, Human Interface Technology Laboratory of the Washington Technology Center (University of Washington, Seattle).

——— (2001) *Lecture Notes for SE 502: Mathematical Foundations*. Unpublished.

——— (2002) "Boundary Logic from the Beginning." Unpublished ms.

Bunt, H. (1985) *Mass Terms and Model Theoretic Semantics* (Cambridge Univ. Press, London).

Burris, S. and Sankappanavar, H.P. (1981) *A Course in Universal Algebra* (Springer-Verlag, Berlin). http://www.thoralf.uwaterloo.ca/htdocs/ualg.html.

Byrne, L. (1946) "Two brief formulations of Boolean algebra," *Bulletin of the American Mathematical Society 52*: 269-72.

Carnap, R. (1956) *Meaning and Necessity* (Univ. of Chicago Press, Chicago).

——— (1958) *Introduction to Symbolic Logic and Its Applications* (Dover, New York).

Casati, R., and Varzi, A. (1999) *Parts and Places* (MIT Press, Cambridge MA).

Church, A. (1956) *Introduction to Mathematical Logic. Vol. 1* (Princeton Univ. Press, Princeton NJ).

Cole, R. (1968) "Definitional Boolean Calculi," *Notre Dame Journal of Formal Logic 9*: 343-50.

Cori, R. and Lascar, D. (2000) *Mathematical Logic*: *Part I* (Oxford Univ. Press, London).

Croskin, C. (1978) "Ways of Knowing," *Cybernetica 21*: 185-92.

Curry, H.B. (1963) *Foundations of Mathematical Logic* (McGraw-Hill, New York). Dover reprint 1977.

Davey, B.A. and Priestley, H.A. (2002) *Introduction to Lattices and Order*, 2nd ed. (Cambridge Univ. Press, London).

Davis, M.D., Sigal, R., and Weyuker, E.J. (1994) *Computability, Complexity, and Languages*, 2nd ed. (Morgan Kauffmann, San Diego).

DeLong, H. (1971) *A Profile of Mathematical Logic* (Addison-Wesley, Reading MA). Dover reprint 2004.

Dijkstra, E.W. and Scholten, C.S. (1990) *Predicate Calculus and Program Semantics* (Springer Verlag, Berlin).

Dilworth, R. P. (1938) "Abstract Residuation over Lattices," *Bulletin of the American Mathematical Society 44*: 262-68. Also in Bogart, K, Freese, R., and Kung, J., eds. (1990) *The Dilworth Theorems: Selected Papers of R. P. Dilworth* (Birkhäuser, Basel): 309-16.

Donnellan, T. (1968) *Lattice Theory* (Pergamon, London).

Epstein, R.L. (1995) *Propositional Logics*, 2nd ed. (Oxford Univ. Press, London).

Eves H. (1990) *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, 3rd ed. 1st ed., 1958. Dover reprint 1997.

Ewald, W., ed., (1996) *A Source Book in the History of Mathematics. Vol. 1* (Oxford Univ. Press, London).

Fraenkel, A. Bar-Hillel, J. and Levy, A. (1973) *Foundations of Set Theory*, 2nd ed. (North-Holland, Amsterdam). 1st ed., 1958.

Frege, G. (1879) *Begriffschrifft*. Translated in Bynum (1972), 101-203.

———— (1882) "Über die wissenschaftliche Berechtigung einer Begriffschrifft," *Zeitschrift für Philosophie und philosophische Kritik 81*: 48-56. Translated in Bynum (1972), 83-89.

———— (1972) *Conceptual Notation and Related Articles*. Bynum, T.W., translator and ed. (Oxford Univ. Press, London).

Gabbay, D., and Wood, J. eds., (2004) *Handbook of the History of Logic, Vol. 3: The Rise of Modern Logic from Leibniz to Frege*. (North-Holland, Amsterdam).

Geach, P.T. (1972) *Logic Matters* (Blackwell, London).

Girle, R.A. (2002) *Introduction to Logic* (Pearson Education, Auckland NZ).

Goldblatt, R. (1984) *Topoi: The Catgorial Analysis of Logic*, 2nd ed. (North-Holland, Amsterdam). Available gratis from the author: Rob.**Goldblatt**@vuw.ac.nz .

Goodstein, R.L. (1963) *Boolean Algebra* (Pergamon, London).

Grassmann, Robert (1966) *Die Formenlehre oder Mathematik* (Georg Olms, Hildesheim). First publication, 1872. Unpublished translation by Lloyd Kannenberg.

Grattan-Guiness, I. (1982) "Psychology in the foundations of logic and mathematics: the cases of Boole, Cantor, and Brouwer," *History and Philosophy of Logic 1*: 61-93.

———— (2000) *The Search for Mathematical Roots 1870-1940* (Princeton Univ. Press, Princeton NJ).

Hailperin, T. (1986) *Boole's Logic and Probability*, 2nd ed. (North-Holland, Amsterdam).

———— (2004) "Algebraical Logic 1685-1900" in Gabbay and Wood (2004): 323-88.

Haken, W. and Appel, K. (1989) *Every Planar Map is Four Colorable*, Contemporary Mathematics, Vol. 98. (American Mathematical Society, Providence RI).

Halmos, P.R. and Givant, S.R. (1998) *Logic as Algebra,* Dolciani Mathematical Expositions No. 21. (Mathematical Association of America, New York).

Hatcher, W. (1982) *The Logical Foundations of Mathematics* (Pergamon, London).

Hehner, E.C.R. (2004) "From Boolean Algebra to Unified Algebra," *The Mathematical Intelligencer 26*(*2*): 3-19. http://www.cs. toronto.edu/~hehner/BAUA.pdf .

———— (2005) "Unified Algebra." http://www.cs.toronto.edu/~hehner/UA.pdf.

Hilbert, D. and Ackermann, W. (1950) *Principles of Mathematical Logic* (Chelsea, New York). A translation of 1938 German edition, subsequent editions remaining untranslated.

Hilpinen, R. (2004) "Peirce's Logic" in Gabbay and Wood (2004): 611-58.

Hodges, W. (1977) *Logic* (Penguin, London).

———— (2001) "Elementary Predicate Logic" in Gabbay, D. M., and Guenthner, F., eds., *Handbook of Philosophical Logic*: *Vol. 2*, 2nd ed. (Kluwer, Dordrecht), 1-129.

Hohn, F.E. (1966) *Applied Boolean Algebra*, 2nd ed (Macmillan, New York).

Hunter, G. (1971) *Metalogic: An Introduction to the Metatheory of Standard First Order Logic* (Macmillan, London).

Huntington, E.V. (1904) "Sets of independent postulates for the algebra of logic," *Trans. of the American Mathematical Society 5*: 288-309.

———— (1933) "New sets of independent postulates for the algebra of logic," *Trans. of the American Mathematical Society 33*: 274-304. Errata (1933a) *33*, 557-58, 971.

Hurley, P.J. (2000) *A Concise Introduction to Logic*, 7th ed (Wadsworth, Belmont CA).

Ishiguro, H. (1990) *Leibniz's Philosophy, Logic, and Language*, 2nd ed. (Cambridge Univ. Press, London).

James, J.M. (1993) "A calculus of number based on spatial forms," M.Sc. Thesis, University of Washington. http://www.hitl.washington.edu/publications/th-93-2/th-93-2.pdf

Johnson, W.E. (1892) "The Logical Calculus," *Mind 1* n.s.: 3-30, 235-50, 340-57.

Kalish, D., Montague, R., and Mar, G. (1980) *Logic*: *Techniques of Formal Reasoning*, 2nd ed. (Harcourt Barce Jovanovich, New York). 1st ed., 1964.

Kauffman, L.H. (1990) "Robbins Algebra," in *Proceedings of the 20th Annual Symposium on Multi-Valued Logic* (IEEE Computer Society Press, Charlotte NC), 54-60.

———— (2001) "The Mathematics of Charles Sanders Peirce," *Cybernetics and Human Knowing 8*, 79-110. http://www2.math.uic.edu/~kauffman/CHK.pdf.

Kneebone, G.T. (1963) *Mathematical Logic and the Foundation of Mathematics* (Van Nostrand, Princeton NJ). Dover reprint 2001.

Koppelberg, S. (1989) *Handbook of Boolean Algebra, Vol. I.* (North-Holland, Amsterdam).

Lad, F. (1996) *Operational Subjective Statistical Methods* (Wiley Interscience, New York).

Lakoff, G., and Núñez, R. (2001) *Where Mathematics Comes From* (Basic Books, New York).

Le Lionnais, F., ed., (1948) *Les Grands Courants de la Pensée Mathématique* (Cahiers du Sud, Marseille).

LeBlanc, H., and Wisdom, W.A., *Deductive Logic*, 2nd ed (Prentice Hall, Englewood Cliffs NJ).

Leibniz, G.W. (1903) *Opuscules et Fragments Inédits de Leibniz*. Couturat, L., ed. (Georg Olms, Hildesheim).

———— (1951) *Leibniz Selections*. Wiener, P., ed. and translator (Scribner's, New York).

———— (1966) *Leibniz*: *Logical Papers*. Parkinson, G. H. R., ed. and translator (Oxford Univ. Press, London).

———— (1969) *Philosophical Papers and Letters*, 2nd ed. Loemker, L. E., ed. and translator (Reidel, Dordrecht).

———— (1996) *New Essays on Human Understanding*. Remnant, P., and Bennett, J. eds. and translators (Cambridge Univ. Press, London).

Lejewski, C. (1960) "Studies in the axiomatic foundations of Boolean Algebra. I," *Notre Dame Journal of Formal Logic 1*: 23-47.

Lenzen, W. (2004) "Leibniz's Logic" in Gabbay and Wood (2004): 1-84. Also http://www.philosophie.uni-osnabrueck.de/Woods.htm .

Lepore, E. (2003) *Meaning and Argument*, 2nd ed. (Blackwell, Oxford UK).

Lewis, C.I. (1918) *A Survey of Symbolic Logic* (University of California Press, Berkeley CA). Dover reprint 1960.

Lewis, D.K. (1991) *Parts of Classes* (Blackwell, London).

Lipschutz, S. (1964) *Set Theory and Related Topics* (McGraw-Hill, New York).

Lucas, J.R. (1999) *The Conceptual Foundations of Mathematics* (Routledge, London).

Lukasiewicz, J. (1967) "The Equivalential Calculus" in McCall, Storrs, ed., *Polish Logic 1920-1939* (Oxford University Press, London): 88-115.

Machover, M. (1996) *Set Theory, Logic, and Their Limitations* (Cambridge Univ. Press, London).

MacKay, T.J. (1989) *Modern Formal Logic* (Macmillan, New York).

Malmstadt, H.V. Enke, C.G. and Crouch, S.R. (1973) *Digital and Analog Data Conversions* (W A Benjamin, Menlo Park CA).

Martin, R.M. (1943) "A homogeneous system for formal logic," *Journal of Symbolic Logic 8*: 1-23.

———— (1978) *Events, Reference, and Logical Form* (Catholic Univ. Press of America, Washington).

———— (1979) *Pragmatics, Truth, and Language* (Reidel, Dordrecht).

———— (1979a) *Semiotics and Linguistic Structure* (SUNY Press, Albany NY).

McCune, W. (1997) "Solution to the Robbins Problem," *Journal of Automated Reasoning 19*: 263-76.

————, Veroff, R., Fitelson, B., Harris, K., Feist, A., and Wos, L. (2002) "Short Single Axioms for Boolean Algebra," *Journal of Automated Reasoning 29*: 1-16.

Meguire, P.G. (2003) "Discovering Boundary Algebra: A Simple Notation for Boolean Algebra and the Truth Functors," *International Journal of General Systems 32*: 25-87.

———— (2004) "A Prolegomenon to Algebra and Set Theory from a Boundary Viewpoint." Unpublished ms; contact author.

Mendelson, E. (1997) *Introduction to Mathematical Logic*, 4th ed. (Chapman and Hall, London). 1st ed., 1964.

Meredith, C. A., and Prior, A. N. (1968) "Equational Logic," *Notre Dame Journal of Formal Logic 9*: 212-26.

Merrell, F. (1995) *Semiosis in the Postmodern Age*. Purdue University Press.

Montague, R, and Tarski, J. (1954) "On Bernstein's Self-Dual Postulates for Boolean Algebra," *Proceedings of the American Mathematical Society* 5: 310-11.

Nicod, J. (1917) "A reduction in the number of primitive propositions of logic," *Proceedings of the Cambridge Philosophical Society 19*: 32-41.

Nidditch, P. H. (1962) *Propositional Calculus* (Routledge & Kegan Paul, London).

Nolt, J. Rohatyn, D. and Varzi, A. (1998) *Logic*, 2nd ed. (McGraw-Hill, New York).

Peirce, C.S. (1931-35) *Collected Papers of Charles Sanders Peirce*. Hartshorne, C., and Weiss, P., eds. (Harvard Univ. Press, Cambridge MA). A citation of the form *x.y,z* refers to section *y* in volume *x* of this edition, first published or written in year *z*.

——— (1976) *The New Elements of Mathematics*: *Vol. IV, Mathematical Philosophy*. Eisele, Carolyn, ed. (Mouton, The Hague).

——— (1982–) *Writings of Charles S. Peirce: A Chronological Edition*. Kloesel, C. J. W. et al, eds. (Indiana Univ. Press, Indianapolis). A citation of the form *Wx*:*y*-*z* refers to volume *x*, pages *y* to *z* of this edition.

Pollock, J.L. (1990) *Technical Methods in Philosophy*. Westview Press.

Post, E.L. (1921) "Introduction to a general theory of elementary propositions," *American Journal of Mathematics 43*: 163-85. Also in Van Heijenoort, J., ed., (1967) *A Source Book in Mathematical Logic*: *1879-1931* (Harvard Univ. Press, Cambridge MA): 265-83.

Prior, A.N. (1962) *Formal Logic*, 2nd ed. (Oxford Univ. Press, London).

Quine, W.V. (1938) "Completeness of the Propositional Calculus" in Quine (1995), 159-63.

——— (1951) *Mathematical Logic*, 2nd ed. (Harvard Univ. Press, Cambridge MA).

——— (1969) *Set Theory and Its Logic*, 2nd ed. (Harvard Univ. Press, Cambridge MA).

——— (1980) *From a Logical Point of View* (Harvard Univ. Press, Cambridge MA). 1st ed., 1953.

——— (1981) *Theories and Things* (Harvard Univ. Press, Cambridge MA).

——— (1982) *Methods of Logic*, 4th ed. (Harvard Univ. Press, Cambridge MA). 1st ed., 1950.

——— (1995) *Selected Logic Papers*, 2nd ed. (Harvard Univ. Press, Cambridge MA). 1st ed., 1966.

Reichenbach, H. (1947) *Elements of Symbolic Logic* (Macmillan, New York).

Rescher, N. (1954) "Leibniz's Interpretation of his Logical Calculus," *Journal of Symbolic Logic 19*: 1-13.

Restall, G. (2000) *An Introduction to Substructural Logics* (Routledge, London).

Roberts, D.D. (1973) *The Existential Graphs of Charles S Peirce* (Mouton, The Hague).

Rosenbloom, P. (1950) *The Elements of Mathematical Logic* (Dover, New York).

Rosser, J.B. (1953) *Logic for Mathematicians* (McGraw-Hill, New York).

——— (1969) *Simplified Independence Proofs* (Academic Press, San Diego CA).

Royce, J. (1917) "Order" in *Encyclopedia of Religion and Ethics.* Hasting, J., ed. (Scribner's, New York), 533-40.

Rudeanu, S. (1963) *Axiomale Laticilor şi ale Algebrelor Booleene* (Axioms of Lattices and of Boolean Algebras) (Editura Academia RPR, Bucharest).

——— (1974) *Boolean Functions and Equations* (North-Holland, Amsterdam).

Schröder, Ernst (1966) *Vorlesungen über die Algebra der Logik, 3 Vols*. (Chelsea, New York).

Schütte, K. (1977) *Proof Theory*, 2nd ed. (Berlin, Springer-Verlag). Crossley, J N, translator.

Shannon, C. (1938) "A symbolic analysis of relay and switching circuits," *Transactions of the American Institute of Electrical Engineers 57*: 713-23.

Sheffer, H.M. (1913) "A set of five independent postulates for Boolean algebras, with application to logical constants," *Trans. of the American Mathematical Society 14*: 481-88.

Shin, S.-J. (2002) *The Iconic Logic of Peirce's Graphs* (MIT Press, Cambridge MA).

Simons, P. (1987) *Parts*: *A Study in Ontology* (Oxford Univ. Press, London).

Smullyan, R.M. (1968) *First Order Logic* (Springer-Verlag, Berlin). Dover reprint 1995.

——— (1985) *To Mock a Mocking Bird*.

——— (2001) "Gödel's Incompleteness Theorems" in Goble, Lou, ed., *The Blackwell Guide to Philosophical Logic*. Blackwell: 72-89.

Sowa, J.F. (2002) "Existential Graphs." www.jfsowa.com/peirce/ms514.htm.

Spencer-Brown, G. (1969) *Laws of Form* (Allen & Unwin, London). Currently stocked by E P Dutton (USA).

Stoll, R.R. (1963) *Set Theory and Logic* (W H Freeman, San Francisco). Dover reprint 1979.

——— (1974) *Sets, Logic, and Axiomatic Method*, 2nd ed. (W H Freeman, San Francisco). 1st ed., 1961.

Suppes, P. (1957) *Introduction to Logic* (Van Nostrand, Princeton NJ). Dover reprint 1999.

——— (1960) *Axiomatic Set Theory* (Van Nostrand, Princeton NJ). Dover reprint 1972.

Van Horn, C.E. (1917) "An axiom in symbolic logic," *Proceedings of the Cambridge Philosophical Society 19*: 22-31.

Wernick, W. (1942) "Complete Sets of Logical Functions," *Transactions of the American Mathematical Society 51*: 117-32.

Wheeler, J.A. (1996) *At Home in the Universe*. American Institute of Physics Press.

Whitehead, A.N. (1948) *An Introduction to Mathematics*, 2nd ed. (Oxford Univ. Press, London). 1st ed. 1911.

———, and Russell, B. (1925) *Principia Mathematica Vol. 1*, 2nd ed. (Cambridge Univ. Press, London). Reprinted through *56, 1962.

Whitesitt, J.E. (1961) *Boolean Algebra and Its Applications* (Addison-Wesley, Reading MA). Dover reprint 1999.

Wiles, A. (1995) "Modular Elliptic Curves and Fermat's Last Theorem," *Annals of Mathematics 142*: 443-551.

Wolf, R.S. (1998) *Proof, Logic, and Conjecture: The Mathematician's Toolbox* (W H Freeman, San Francisco).

Wolfram, S. (2002) *A New Kind of Science* (Wolfram Media, Champaign IL).

Zeman, J.J. (1964) “The Graphical Logic of C. S. Peirce.” Ph.D. thesis, University of Chicago. http://www.clas.ufl.edu/users/jzeman/.htm .

——— (1973) *Modal Logic*. (Oxford Univ. Press, London). Also http://www.clas.ufl.edu/users/jzeman/ .