Hereditary Convexity for Harmonic Homeomorphisms

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ABSTRACT. We study hereditary properties of convexity for planar harmonic homeomorphisms on a disk and an annulus. A noteworthy class of examples with the hereditary property arises from energy-minimal diffeomorphisms of an annulus, whose existence was established in [9, 11]. An extension of a result by Hengartner and Schober [8] to an annulus is used to deduce the boundary behavior of a harmonic mapping from an annulus into a doubly-connected region bounded by two convex Jordan curves.

1. INTRODUCTION

Harmonic mappings, which are complex-valued orientation-preserving univalent functions satisfying Laplace's equation $\Delta f = 0$ on their respective domains in $\mathbb{C}$, bear some curious features. For example, while harmonic mappings of hyperbolic regions generally do not decrease either the Euclidean metric or the hyperbolic metric (because a result of Heinz [6, Lemma] is optimal—see, e.g., [4, p. 77] or [12, p. 91]), it was shown in [12, Theorem 1.1] that harmonic mappings preserving the unit disk $\mathbb{D}$ decrease the Lebesgue area measure of concentric disks $\mathbb{D}_r = \{z \in \mathbb{C} \colon |z| \le r < 1\}$.

If the image of the unit disk under a conformal mapping is a convex region Ω, then the image of every disk in $\mathbb{D}$ is also convex (see, e.g., [3, proof of Theorem 2.11] or [16]). On the other hand, the situation for harmonic mappings is markedly different. The harmonic mapping

$$f(z) = \operatorname{Re} \frac{z}{1-z} + i \operatorname{Im} \frac{z}{(1-z)^2}$$

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maps $\mathbb{D}$ onto the half plane $\{w : w > -\frac{1}{2}\}$, which is convex, but $f(\mathbb{D}_r)$ is not convex for $\sqrt{2} - 1 < r < 1$ (see, e.g., [2, Example 5.5] or [4, pp. 46–48]). This is related to the fact that unidirectional convexity is not a hereditary property of holomorphic univalent functions (see, e.g., [2, Theorem 5.3; 5; 7]). Hence, convexity is not a hereditary property of harmonic mappings in general. Nevertheless, we obtain sufficient conditions for this hereditary property to be present in harmonic mappings. We also study a related hereditary property of harmonic mappings between doubly-connected regions, which is the main subject of this paper.

2. Connection with the Doubly-Connected Case

For $0 < r < 1$, let $\mathbb{A}_r$ denote the annulus $\{z \in \mathbb{C} : r < |z| < 1\}$, let $\overline{\mathbb{A}_r}$ denote its closure, and let $\mathbb{T}_r$ denote the circle $\{z \in \mathbb{C} : |z| = r\}$. We will use $\mathbb{T}$ to represent the unit circle $\partial\mathbb{D}$. Harmonic diffeomorphisms will refer to harmonic mappings that are diffeomorphisms.

At first glance, our following result may appear somewhat surprising.

Theorem 2.1. *Let h be a harmonic diffeomorphism of $\mathbb{D}$ into a bounded convex region Ω_0 in the finite plane such that the radial limit $\lim_{r\to 1} h(re^{i\theta})$ lies on $\partial\Omega_0$ for almost all θ. Suppose that, on $\mathbb{A}_{\sqrt{2}-1}$,*

$$\Delta \operatorname{Im} \log\left(1 - \frac{\bar{z}h_{\bar{z}}}{zh_z}\right) = 0, \tag{2.1}$$

where $h_z = \partial h/\partial z$, $h_{\bar{z}} = \partial h/\partial \bar{z}$, and Δ represents the Laplace operator. Then, $h(\mathbb{D}_r)$ is a strictly convex region for $0 < r < 1$.

Remark. If h is conformal, then $1 - \bar{z}h_{\bar{z}}/(zh_z) \equiv 1$. Hence, its argument function $\operatorname{Im}\log(1 - \bar{z}h_{\bar{z}}/(zh_z))$ is a constant integer multiple of 2π.

In view of condition (2.1), we have stated Theorem 2.1 for harmonic diffeomorphisms in place of harmonic mappings. This is nonetheless hardly a restriction, since a result of Lewy (see, e.g., [4, p. 20] or [14, Theorem 1]) shows that the Jacobian of a harmonic mapping does not vanish at any point.

A harmonic mapping f of $\mathbb{D}$ into a bounded region has bounded real and imaginary parts. By Fatou's Theorem [17, Theorem IV.6], the angular limits of f exist almost everywhere on $\partial\mathbb{D}$. Hence, the radial limit assumption in Theorem 2.1 is apparently weaker than customarily requiring either the convexity of $h(\mathbb{D})$ or the surjectivity of h.

Given a harmonic mapping g of $\mathbb{D}$, where $g(\mathbb{D})$ is a convex region, it is known that $g(\mathbb{D}_r)$ is convex for $r \in (0, \sqrt{2} - 1]$ (see, e.g., [4, p. 46] or [15, Theorem 1]). This explains the focus on $\mathbb{A}_{\sqrt{2}-1}$ in Theorem 2.1. More generally, we prove the following.

Theorem 2.2. *Let h be a harmonic diffeomorphism of $\mathbb{A}_\rho$ into a doubly-connected region Ω bounded by two convex Jordan curves in the finite plane such that the* *radial limits* $\lim_{r\to 1} h(re^{i\theta})$ *and* $\lim_{r\to\rho} h(re^{i\varphi})$ *lie on* $\partial\Omega$ *for almost all* θ *and* φ. *If* (2.1) *holds on* $\mathbb{A}_\rho$, *then* $h(\mathbb{T}_r)$ *is a strictly convex curve for* $\rho < r < 1$.

Suppose f is a bounded harmonic mapping of $\mathbb{A}_\rho$. The compact set $\partial\mathbb{A}_\rho$ may be covered by a finite number of simply-connected neighborhoods R_1, R_2, ..., R_n in $\overline{\mathbb{A}_\rho}$ whose boundaries are Jordan curves. For each integer $k \in [1, n]$, let g_k be a conformal mapping of $\mathbb{D}$ onto the interior of R_k. By Fatou's Theorem, the harmonic mapping $f \circ g_k$ has angular limits almost everywhere on $\mathbb{T}$. The isogonality of g_k at each boundary point of $\mathbb{T}$ (see, e.g., [17, Theorem IX.5 and the subsequent paragraph]) implies that f has angular limits almost everywhere on $\partial\mathbb{A}_\rho \cap R_k$. It follows that f has angular limits almost everywhere on $\partial\mathbb{A}_\rho = \bigcup_{k=1}^{n}(\partial\mathbb{A}_\rho \cap R_k)$. Hence, the radial limit assumption in Theorem 2.2 may appear to be weaker than requiring either the boundary components of $h(\mathbb{A}_\rho)$ to be convex Jordan curves or h to be surjective. While the latter comparison is correct, it will follow from Corollary 4.2 in Section 4 that the radial limit assumption in Theorem 2.2 implies that the boundary components of $h(\mathbb{A}_\rho)$ are convex Jordan curves.

3. Illustrative Examples

Definition 3.1. An orientation-preserving homeomorphism $h\colon \mathbb{A}_\rho \longrightarrow \Omega$ is said to be energy-minimal if h minimizes the quantity

$$E(f) = \iint_{\mathbb{A}_\rho} |f_z|^2 + |f_{\bar z}|^2 \tag{3.1}$$

among all orientation-preserving homeomorphisms $f\colon \mathbb{A}_\rho \longrightarrow \Omega$ with $E(f) < \infty$.

Remark 3.2. An energy-minimal homeomorphism $h\colon \mathbb{A}_\rho \longrightarrow \Omega$ exists as long as the conformal modulus of $\mathbb{A}_\rho$ does not exceed that of Ω (see, e.g., [9, Theorem 1.1] or [11, Theorem 1.1]).

Remark 3.3. Energy-minimal homeomorphisms are diffeomorphisms [10, Theorem 1.2]. This can also be seen as a consequence of their harmonicity (see, e.g., [14, Theorem 1] and the next remark). Henceforth, we refer to them as energy-minimal diffeomorphisms.

Remark 3.4. Since Poisson modification decreases the quantity in (3.1) (see, e.g., [1, proof of Lemma 7] or [9, Lemma 4.2]), energy-minimal diffeomorphisms are necessarily harmonic [9, Proposition 8.1 and Theorem 2.3].

It turns out that an energy-minimal diffeomorphism $h\colon \mathbb{A}_\rho \longrightarrow \Omega$ satisfies (2.1). Since h is an orientation-preserving diffeomorphism,

$$|h_z| > |h_{\bar z}| \tag{3.2}$$

on $\mathbb{A}_\rho$. It was shown in [9, Lemma 6.1] that

$$h_z\overline{h_{\bar z}} = \frac{m}{z^2}, \tag{3.3}$$

where m is a real constant. We can rewrite (3.3) as

$$\frac{\bar{z}h_{\bar{z}}}{zh_z} = \frac{m}{|z|^2\,|h_z|^2},$$

which, in view of (3.2), yields

$$-1 < \frac{\bar{z}h_{\bar{z}}}{zh_z} < 1,$$

and thus the function $1 - \bar{z}h_{\bar{z}}/(zh_z)$ is real and positive. Consequently, its argument function $\operatorname{Im}\log(1 - \bar{z}h_{\bar{z}}/(zh_z))$ is a constant integer multiple of 2π. Theorem 2.2 now yields the following result.

Theorem 3.5. *Let $h\colon \mathbb{A}_\rho \longrightarrow \Omega$ be an energy-minimal diffeomorphism, where Ω is a doubly-connected region bounded by two convex Jordan curves in the finite plane. Then, $h(\mathbb{T}_r)$ is a strictly convex curve for $\rho < r < 1$.*

We conclude this section with a family of examples for which

$$\operatorname{Im}\log\left(1 - \frac{\bar{z}h_{\bar{z}}}{zh_z}\right)$$

is not constant. Define $h\colon \mathbb{A}_\rho \longrightarrow \mathbb{C}$ by

$$h(z) = \frac{z+a}{1+az} - b\log|z|,$$

where $a \in (0,1)$ and $b = a(1-\rho^2)/((1-\rho^2a^2)\log\rho) < 0$. Then, h is harmonic, and

$$1 - \frac{\bar{z}h_{\bar{z}}}{zh_z} = \frac{2(1-a^2)z}{2(1-a^2)z - b(1+az)^2},$$

which is meromorphic on $\mathbb{A}_\rho$. Another elementary computation shows that the Jacobian

$$|h_z|^2 - |h_{\bar{z}}|^2 = \frac{(1-a^2)^2}{|1+az|^4}\left(1 - \frac{b}{1-a^2}\operatorname{Re}\frac{(1+az)^2}{z}\right),$$

whose last factor on the right-hand side has its minimum on $\overline{\mathbb{A}_\rho}$ at $z = -\rho$. Hence, h will be a harmonic diffeomorphism of $\mathbb{A}_\rho$ onto $\mathbb{A}_\sigma$ satisfying (2.1) for values of a and ρ such that this minimum is positive, where we have that $\sigma = \rho(1-a^2)/(1-\rho^2a^2) \in (0,\rho)$. An instance of this occurs at $a = \rho = \frac{1}{2}$.

4. AUXILIARY RESULTS ON BOUNDARY BEHAVIOR

Let θ and φ denote polar angles. In this section, we study the boundary behavior of a harmonic mapping between an annulus $\mathbb{A}_\rho$ and a doubly-connected region Ω bounded by two Jordan curves that is not necessarily surjective, but whose radial limits at $\partial\mathbb{A}_\rho$ are contained in $\partial\Omega$.

Proposition 4.1. *Let h be a harmonic mapping of $\mathbb{A}_\rho$ into a doubly-connected region Ω bounded by two Jordan curves C_1 and C_ρ in the finite plane such that the radial limits $\lim_{r\to 1} h(re^{i\theta})$ and $\lim_{r\to\rho} h(re^{i\varphi})$ lie on C_1 and C_ρ, respectively, for almost all θ and φ. Then, there is a countable set $W \subset \partial\mathbb{A}_\rho = \mathbb{T} \cup \mathbb{T}_\rho$ such that the unrestricted limits*

$$H(e^{i\theta}) = \lim_{z\to e^{i\theta}} h(z), \quad H(\rho e^{i\varphi}) = \lim_{z\to\rho e^{i\varphi}} h(z)$$

exist on $\partial\mathbb{A}_\rho \setminus W$, and are contained in C_1 and C_ρ, respectively. Moreover, we have the following:

(a) *H is both continuous and orientation-preserving on $\mathbb{T} \setminus W$ and $\mathbb{T}_\rho \setminus W$.*

(b) *For each $e^{i\theta} \in W$, the one-sided limits*

$$H(e^{i\theta-}) = \lim_{\sigma\to\theta^-,\, e^{i\sigma}\notin W} H(e^{i\sigma}),$$
$$H(e^{i\theta+}) = \lim_{\sigma\to\theta^+,\, e^{i\sigma}\notin W} H(e^{i\sigma})$$

exist, belong to C_1 and are distinct.

(c) *For each $\rho e^{i\varphi} \in W$, the one-sided limits*

$$H(\rho e^{i\varphi-}) = \lim_{\sigma\to\varphi^-,\, \rho e^{i\sigma}\notin W} H(\rho e^{i\sigma}),$$
$$H(\rho e^{i\varphi+}) = \lim_{\sigma\to\varphi^+,\, \rho e^{i\sigma}\notin W} H(\rho e^{i\sigma})$$

exist, belong to C_ρ and are distinct.

(d) *The cluster sets of h at the points $e^{i\theta} \in W$ and $\rho e^{i\varphi} \in W$ are the line segments joining $H(e^{i\theta-})$ to $H(e^{i\theta+})$ and $H(\rho e^{i\varphi-})$ to $H(\rho e^{i\varphi+})$, respectively.*

A version of the above result for harmonic mappings between $\mathbb{D}$ and bounded simply-connected regions with locally-connected boundary was given by Hengartner and Schober [8, Theorem 4.3]. Since the conclusions concern local properties of h, we may obtain Proposition 4.1 by covering $\partial\mathbb{A}_\rho$ with a finite number of simply-connected neighborhoods $R_1, R_2, \dots, R_n$ in $\overline{\mathbb{A}_\rho}$ whose boundaries are Jordan curves, and applying Hengartner and Schober's result to the harmonic mapping $h \circ g_k$, where g_k is a conformal mapping of $\mathbb{D}$ onto the interior of R_k for each integer $k \in [1, n]$. A noteworthy consequence of Proposition 4.1, besides Corollary 4.2 below, is that h may be extended continuously to $\partial\mathbb{A}_\rho$ outside the countable set W, and that each boundary point of $h(\mathbb{A}_\rho)$ corresponds to a nonempty "pre-image" (with infinitely many points on a boundary line segment of $h(\mathbb{A}_\rho)$ associated with a "pre-image" point in W from (d)). These facts will prove their worth in Section 5.

Suppose f is a harmonic mapping of $\mathbb{A}_\rho$ into a doubly-connected region Ω bounded by two convex Jordan curves in the finite plane such that the radial limits $\lim_{r\to 1} f(re^{i\theta})$ and $\lim_{r\to\rho} f(re^{i\varphi})$ lie on $\partial\Omega$ for almost all θ and φ. The radial limits $\lim_{r\to 1} f(re^{i\theta})$ and $\lim_{r\to\rho} f(re^{i\varphi})$ are contained in distinct boundary components of $f(\mathbb{A}_\rho)$ by virtue of the fact that f is a homeomorphism (see, e.g., [13, p. 11]), and thus they lie on distinct boundary components of $\partial\Omega$. It follows from Proposition 4.1 that the boundary of $f(\mathbb{A}_\rho)$ consists of two Jordan curves. Since any inner boundary line segment of $f(\mathbb{A}_\rho)$ has to be a subset of the inner boundary of Ω, the inner boundaries of $f(\mathbb{A}_\rho)$ and Ω must coincide. On the other hand, replacing any outer boundary sub-arc γ of $\partial\Omega$ with a line segment joining the endpoints of γ results in a convex Jordan curve. Hence, we have the following result.

Corollary 4.2. *Let h be a harmonic mapping of $\mathbb{A}_\rho$ into a doubly-connected region Ω bounded by two convex Jordan curves in the finite plane such that the radial limits $\lim_{r\to 1} h(re^{i\theta})$ and $\lim_{r\to\rho} h(re^{i\varphi})$ lie on $\partial\Omega$ for almost all θ and φ. Then, the boundary of $h(\mathbb{A}_\rho)$ consists of two convex Jordan curves, of which the inner boundary curve coincides with the inner boundary curve of Ω.*

5. SECANT BEHAVIOR NEAR THE BOUNDARY

Suppose h is a harmonic diffeomorphism of $\mathbb{A}_\rho$ into a doubly-connected region Ω bounded by two convex Jordan curves in the finite plane such that the radial limits $\lim_{r\to 1} h(re^{i\theta})$ and $\lim_{r\to\rho} h(re^{i\varphi})$ lie on $\partial\Omega$ for almost all θ and φ. For each $\tau > 0$, let

$$f_\tau(re^{i\theta}) = \frac{h(re^{i(\theta+\tau)}) - h(re^{i\theta})}{\tau},$$

and let $\psi_\tau(z) = \arg f_\tau(z)$ for all $z = re^{i\theta} \in \mathbb{A}_\rho$. We will establish the following result.

Lemma 5.1. *The period of $\psi_\tau - \theta$ is 2π, and the single-valued harmonic functions $\psi_\tau - \theta$ are uniformly bounded on $\mathbb{A}_\rho$ for sufficiently small τ.*

Since the radial limits $\lim_{r\to 1} h(re^{i\theta})$ and $\lim_{r\to\rho} h(re^{i\varphi})$ are contained in distinct boundary components of $\partial\Omega$, it follows from Proposition 4.1 that h has a continuous extension to $\overline{\mathbb{A}_\rho}\setminus W$ for some countable set $W \subset \partial\mathbb{A}_\rho$. The orientation-preserving feature of h carries over to $\mathbb{T} \setminus W$ and $\mathbb{T}_\rho \setminus W$, and by Corollary 4.2, the boundary components of $h(\mathbb{A}_\rho)$ are convex Jordan curves, one of which is a curve Γ containing a point a' such that

$$|a'| = \sup_{z\in\mathbb{A}_\rho} |h(z)|.$$

We may suppose, without loss of generality, that

$$h(\mathbb{T} \setminus W) \subseteq \Gamma. \tag{5.1}$$

The line L through a' that makes an angle of $\arg a' + \pi/2$ with the positive real axis is a supporting line of Γ. We let c' be a point on Γ that has a parallel supporting line distinct from L, and pick distinct points b' and d' on Γ that have supporting lines parallel to the line segment $a'c'$. The points a', b', c', and d' are chosen so that they follow one another in the positive direction around Γ, and we denote the angles made by their supporting lines with the positive real-axis by α, β, $\alpha + \pi$, and $\beta + \pi$, respectively, such that

$$\alpha < \beta < \alpha + \pi < \beta + \pi < \alpha + 2\pi.$$

By virtue of the results obtained in Section 4 (see, e.g., Proposition 4.1 and the subsequent paragraph), we may choose on $\mathbb{T}$ four associated "pre-image" points a, b, c, and d of a', b', c', and d', respectively. Let A, B, C, and D be the overlapping open arcs on $\mathbb{T}$ from a to c, from b to d, from c to a, and from d to b, respectively. We then cover $\mathbb{T}$ with a finite number of sufficiently small open disks $D_1, D_2, \ldots, D_n$ such that the following hold:

(1) Each disk D_k intersects $\mathbb{T}$ in an arc A_k that is contained within at least one of the arcs A, B, C, or D.

(2) The endpoints a_k, b_k of each A_k do not coincide with any of the points a, b, c, or d.

By so doing, we obtain a finite number of simply-connected sets

$$R_1 = \overline{\mathbb{A}_\rho} \cap D_1,\ R_2 = \overline{\mathbb{A}_\rho} \cap D_2,\ \ldots,\ R_n = \overline{\mathbb{A}_\rho} \cap D_n$$

in $\overline{\mathbb{A}_\rho}$ whose respective boundaries are Jordan curves, and $R_k \cap \mathbb{T} = A_k$ for each k. Fix $\delta \in (0, \pi/2)$. For each integer $k \in [1, n]$, let g_k be a conformal mapping of $\mathbb{D}$ onto the interior R_k° of R_k such that the harmonic measure

$$\omega(g_k(0), A_k, R_k^\circ) = 2\delta \quad \text{and} \quad A_k = \{g_k(e^{is}) \colon -\delta < s < \delta\}.$$

This may be achieved by extending g_k to a homeomorphism of $\bar{\mathbb{D}}$ and making appropriate choices of $g_k(0)$ and $g_k(e^{is})$ for one particular s. If

$$J = \{s \colon -\pi < s < -2\delta\} \cup \{s \colon 2\delta < s \le \pi\},$$

then for all $s \in J$ and $t_k \in (-\delta, \delta)$, we have

$$\delta < |s - t_k| < \frac{3\pi}{2}, \quad \cos(s - t_k) < \cos\delta,$$

and thus

$$\begin{aligned} 1 - 2r_k\cos(s - t) + r_k^2 &> 1 - 2r_k\cos\delta + r_k^2 \\ &= \sin^2\delta + (r_k - \cos\delta)^2 \ge \sin^2\delta. \end{aligned}$$

We fix

$$\tau \in \Big(0, d\Big(\{a,b,c,d\}, \bigcup_{k=1}^{n} \{a_k, b_k\}\Big)\Big), \tag{5.2}$$

where $d(X,Y) = \inf\{d(x,y)\colon x \in X,\ y \in Y\}$, and $d(x,y)$ denotes the Euclidean distance between the points x and y. The harmonicity of

$$u_\tau = \operatorname{Im}(e^{-i\alpha} f_\tau) \tag{5.3}$$

on $\mathbb{A}_\rho$ implies that the compositions $u_\tau \circ g_k$ are harmonic on $\mathbb{D}$ for all k, and thus

$$\begin{aligned} u_\tau(g_k(r_k e^{it_k})) &= \int_{-\pi}^{\pi} u_\tau(g_k(e^{is})) \cdot P_k(e^{is})\,\mathrm{d}s, \\ &= \Big(\int_{-2\delta}^{2\delta} + \int_J\Big) u_\tau(g_k(e^{is})) \cdot P_k(e^{is})\,\mathrm{d}s, \end{aligned} \tag{5.4}$$

where

$$P_k(e^{is}) = \frac{1 - r_k^2}{2\pi(1 - 2r_k\cos(s - t_k) + r_k^2)}, \quad 0 \le r_k \le 1.$$

If $A_k \subseteq A$, then the first integral on the second line in (5.4) is non-negative by virtue of the fact that the restriction of u_τ to A_k is non-negative. The second integral, on the other hand, may be estimated as follows. If $\varepsilon > 0$, then on each set

$$S_k = \left\{ r_k e^{it_k} \in \mathbb{D} \colon 1 - \frac{\varepsilon \sin^2\delta}{4|a'|} < r_k \le 1,\ -\delta < t_k < \delta \right\},$$

we have

$$\left| \int_J u_\tau(g_k(e^{is})) \cdot P_k(e^{is})\,\mathrm{d}s \right| \le \frac{2|a'|(1 - r_k^2)}{\sin^2\delta} < \frac{4|a'|(1 - r_k)}{\sin^2\delta} < \varepsilon.$$

Hence, for each $\varepsilon > 0$, there is a set S_k containing $g_k^{-1}(A_k)$ such that

$$(u_\tau \circ g_k) > -\varepsilon. \tag{5.5}$$

A similar argument applied to each of the other cases

$$A_k \subseteq B,\ A_k \subseteq C,\ A_k \subseteq D$$

(with α replaced by β, $\alpha + \pi$, $\beta + \pi$, respectively) also yields $S_k \supset g_k^{-1}(A_k)$ such that (5.5) is valid.

Fix r sufficiently close to 1 such that

$$\mathbb{T}_r \subset \bigcup_{k=1}^{n} g_k(S_k) \setminus \mathbb{T}. \tag{5.6}$$

Since f_τ is non-zero on the compact set $\mathbb{T}_r$, there exists $m_r > 0$ such that

$$|f_\tau| \geq m_r \tag{5.7}$$

on $\mathbb{T}_r$. Pick $k_r > 0$ satisfying

$$\varepsilon_r = \arcsin \frac{k_r}{m_r} < \frac{1}{2} \min\{\beta - \alpha, \alpha + \pi - \beta\}. \tag{5.8}$$

Since

$$u_\tau = \operatorname{Im}(e^{-i\alpha} f_\tau) = |f_\tau| \sin(\psi_\tau - \alpha),$$

it follows from (5.7) that on $g_k(S_k) \cap \mathbb{T}_r$,

$$\sin(\psi_\tau - \alpha) > -\frac{k_r}{m_r}$$

if $A_k \subseteq A$, and thus

$$-\varepsilon_r < \psi_\tau - \alpha < \pi + \varepsilon_r. \tag{5.9}$$

Likewise, on $g_k(S_k) \cap \mathbb{T}_r$,

$$\begin{cases} -\varepsilon_r < \psi_\tau - \beta < \pi + \varepsilon_r & \text{if } A_k \subseteq B; \\ -\varepsilon_r < \psi_\tau - (\alpha + \pi) < \pi + \varepsilon_r & \text{if } A_k \subseteq C; \\ -\varepsilon_r < \psi_\tau - (\beta + \pi) < \pi + \varepsilon_r & \text{if } A_k \subseteq D. \end{cases} \tag{5.10}$$

In particular, on $\mathbb{T}_r$, we obtain

$$\psi_\tau(re^{i(\theta+2\pi)}) = \psi_\tau(re^{i\theta}) + 2\pi,$$

though this may also be seen from the fact that $h(\mathbb{T}_r)$ is a Jordan curve. Hence, $\psi_\tau - \theta$ is a single-valued harmonic function on $\mathbb{A}_\rho$.

In view of (5.8), (5.9), and (5.10), we see that $\psi_\tau(re^{i\theta}) - \theta$ is uniformly bounded for all τ and r satisfying (5.2) and (5.6), respectively. A similar argument could be applied to $\mathbb{T}_\rho$ to obtain a corresponding result when r is sufficiently close to ρ. This proves Lemma 5.1, since $\psi_\tau - \theta$ is continuous on $\mathbb{A}_\rho$.

6. Proof of Theorem 2.2

Let $\psi(z) = \arg(\partial/\partial\theta)h(z)$ for all $z = re^{i\theta} \in \mathbb{A}_\rho$. As a consequence of (5.1), the convexity of $h(\mathbb{T}_r)$ will follow from the inequality

$$\frac{\partial\psi}{\partial\theta} \geq 0 \tag{6.1}$$

on $\mathbb{A}_\rho$, with $h(\mathbb{T}_r)$ being strictly convex if the inequality is strict. Since $h(\mathbb{T}_r)$ is a smooth (or, more precisely, real-analytic) Jordan curve, we obtain

$$\psi(re^{i(\theta+2\pi)}) = \psi(re^{i\theta}) + 2\pi, \tag{6.2}$$

Observe that

$$\frac{\partial h}{\partial\theta} = i(zh_z - \bar{z}h_{\bar{z}}) = izh_z\left(1 - \frac{\bar{z}h_{\bar{z}}}{zh_z}\right). \tag{6.3}$$

By (3.2), the quantity in parentheses has positive real part and hence, by (2.1), its argument is a single-valued harmonic function. Since h_z is holomorphic and non-zero on $\mathbb{A}_\rho$, it follows from (6.2) and (6.3) that $\psi - \theta$ is a single-valued harmonic function on $\mathbb{A}_\rho$. Moreover, it is bounded by virtue of Lemma 5.1 since $\psi = \lim_{\tau\to 0}\psi_\tau$ on $\mathbb{A}_\rho$.

Let $G_z(\zeta)$ denote the Green's function for $\mathbb{A}_\rho$ with singularity at $z = re^{i\theta}$, and let $n = n_w$ be the inward normal at $w = Re^{i\varphi} \in \partial\mathbb{A}_\rho$. We may rotate $\mathbb{A}_\rho$ together with the singularity $z = re^{i\theta}$ about the origin through an angle σ to obtain

$$G_{re^{i\theta}}(Re^{i\varphi}) = G_{re^{i(\theta+\sigma)}}(Re^{i(\varphi+\sigma)}), \tag{6.4}$$

from which the definition of partial differentiation implies

$$\begin{aligned}
\frac{\partial}{\partial\theta}G_z(w) &= \lim_{\sigma\to 0}\frac{1}{\sigma}\Big(G_{re^{i(\theta+\sigma)}}(Re^{i\varphi}) - G_{re^{i\theta}}(Re^{i\varphi})\Big)\\
&= \lim_{\sigma\to 0}\frac{1}{\sigma}\Big(G_{re^{i\theta}}(Re^{i(\varphi-\sigma)}) - G_{re^{i\theta}}(Re^{i\varphi})\Big) \qquad \text{by (6.4)}\\
&= \lim_{\sigma\to 0}\frac{1}{-\sigma}\Big(G_{re^{i\theta}}(Re^{i(\varphi+\sigma)}) - G_{re^{i\theta}}(Re^{i\varphi})\Big)\\
&= -\frac{\partial}{\partial\varphi}G_z(w).
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial n}G_z(w) = \frac{\partial}{\partial n}\frac{\partial}{\partial\theta}G_z(w) = -\frac{\partial}{\partial n}\frac{\partial}{\partial\varphi}G_z(w) = -\frac{\partial}{\partial\varphi}\frac{\partial}{\partial n}G_z(w). \tag{6.5}$$

Let $T = \{t \in \mathbb{R} : (\partial/\partial t)\Psi_1(t) \text{ and } (\partial/\partial t)\Psi_\rho(t) \text{ both exist}\}$, where

$$\Psi_1(\theta) = \arg \frac{\partial}{\partial\theta} h(e^{i\theta}), \quad \Psi_\rho(\theta) = \arg \frac{\partial}{\partial\theta} h(\rho e^{i\theta}).$$

Since the boundary components of $h(\mathbb{A}_\rho)$ are convex Jordan curves by Corollary 4.2, it follows that $\mathbb{R} \setminus T$ is countable. Recall that the harmonic function $\psi - \theta$ has the integral representation (see, e.g., [17, Theorem I.21])

$$\begin{aligned} 2\pi(\psi(z) - \theta) = &\int_0^{2\pi} [\Psi_1(\varphi) - \varphi] \frac{\partial}{\partial n} G_z(e^{i\varphi})\, \mathrm{d}\varphi \\ &+ \int_0^{2\pi} [\Psi_\rho(\varphi) - \varphi] \frac{\partial}{\partial n} G_z(\rho e^{i\varphi}) \rho\, \mathrm{d}\varphi, \end{aligned}$$

where the integrals are taken over $[0, 2\pi] \cap T$. Partial differentiation with respect to θ followed by an application of (6.5) yields

$$\begin{aligned} &2\pi\left(\frac{\partial}{\partial\theta}\psi(z) - 1\right) \\ &= \int_0^{2\pi} [\Psi_1(\varphi) - \varphi] \frac{\partial}{\partial\theta}\frac{\partial G_z}{\partial n}\, \mathrm{d}\varphi + \int_0^{2\pi} [\Psi_\rho(\varphi) - \varphi] \frac{\partial}{\partial\theta}\frac{\partial G_z}{\partial n} \rho\, \mathrm{d}\varphi \\ &= -\int_0^{2\pi} [\Psi_1(\varphi) - \varphi] \frac{\partial}{\partial\varphi}\frac{\partial G_z}{\partial n}\, \mathrm{d}\varphi - \int_0^{2\pi} [\Psi_\rho(\varphi) - \varphi] \frac{\partial}{\partial\varphi}\frac{\partial G_z}{\partial n} \rho\, \mathrm{d}\varphi \quad \text{by (6.5)} \\ &= \int_0^{2\pi} \frac{\partial G_z}{\partial n}\, \mathrm{d}[\Psi_1(\varphi) - \varphi] + \int_0^{2\pi} \frac{\partial G_z}{\partial n} \rho\, \mathrm{d}[\Psi_\rho(\varphi) - \varphi]. \end{aligned}$$

Hence (see, e.g., [17, Theorem I.20]),

$$2\pi \frac{\partial}{\partial\theta}\psi(z) = \int_0^{2\pi} \frac{\partial G_z}{\partial n}\, \mathrm{d}\Psi_1(\varphi) + \int_0^{2\pi} \frac{\partial G_z}{\partial n} \rho\, \mathrm{d}\Psi_\rho(\varphi). \tag{6.6}$$

It follows from (5.1) that $\Psi_1(\varphi)$ and $\Psi_\rho(\varphi)$ are non-decreasing functions of φ. Since $\partial G/\partial n$ is positive on $\partial\mathbb{A}_\rho$, it follows from (6.6) that $\partial\psi/\partial\theta$ is also positive on $\mathbb{A}_\rho$. Hence, $h(\mathbb{T}_r)$ is strictly convex for $\rho < r < 1$, which concludes our proof of Theorem 2.2.

Remark. The proof would have been much simpler if $h \in C^2(\overline{\mathbb{A}_\rho})$, for it follows from (6.3) that (6.1) is then equivalent to

$$\begin{aligned} 1 + \frac{\partial}{\partial\theta}&\left\{\arg h_z + \arg\left(1 - \frac{\bar{z} h_{\bar{z}}}{z h_z}\right)\right\} \\ &= 1 + \operatorname{Re}\left\{\frac{z h_{zz}}{h_z} + \frac{\bar{z}(z h_{\bar{z}} h_{zz} + 2 h_z h_{\bar{z}} + \bar{z} h_z h_{\bar{z}\bar{z}})}{h_z(z h_z - \bar{z} h_{\bar{z}})}\right\} \geq 0. \end{aligned}$$

Since this holds on $\partial\mathbb{A}_\rho$, the maximum principle yields the same inequality on $\mathbb{A}_\rho$. The desired conclusion then follows from the observation that $\partial\psi/\partial\theta$ cannot be identically zero on $\mathbb{A}_\rho$, as $h(\mathbb{T}_r)$ is a Jordan curve for $\rho < r < 1$.

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REFERENCES

[1] K. ASTALA, T. IWANIEC, and G. MARTIN, *Deformations of annuli with smallest mean distortion*, Arch. Ration. Mech. Anal. **195** (2010), no. 3, 899–921. http://dx.doi.org/10.1007/s00205-009-0231-z. MR2591976 (2011c:30059)

[2] J. CLUNIE and T. SHEIL-SMALL, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25. http://dx.doi.org/10.5186/aasfm.1984.0905. MR752388 (85i:30014)

[3] P. L. DUREN, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 259, Springer-Verlag, New York, 1983. MR708494 (85j:30034)

[4] ———, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, vol. 156, Cambridge University Press, Cambridge, 2004. http://dx.doi.org/10.1017/CBO9780511546600. MR2048384 (2005d:31001)

[5] A. W. GOODMAN and E. B. SAFF, *On univalent functions convex in one direction*, Proc. Amer. Math. Soc. **73** (1979), no. 2, 183–187. http://dx.doi.org/10.2307/2042288. MR516461 (80e:30006)

[6] E. HEINZ, *On one-to-one harmonic mappings*, Pacific J. Math. **9** (1959), no. 1, 101–105. http://dx.doi.org/10.2140/pjm.1959.9.101. MR0104933 (21 #3683)

[7] W. HENGARTNER and G. SCHOBER, *A remark on level curves for domains convex in one direction*, Applicable Anal. **3** (1973), no. 1, Collection of articles dedicated to Eberhard Hopf on the occasion of his 70th birthday, 101–106. http://dx.doi.org/10.1080/00036817308839059. MR0393450 (52 #14260)

[8] ———, *Harmonic mappings with given dilatation*, J. London Math. Soc. (2) **33** (1986), no. 3, 473–483. http://dx.doi.org/10.1112/jlms/s2-33.3.473. MR850963 (87j:30037)

[9] T. IWANIEC, N. T. KOH, L. V. KOVALEV, and J. ONNINEN, *Existence of energy-minimal diffeomorphisms between doubly connected domains*, Invent. Math. **186** (2011), no. 3, 667–707. http://dx.doi.org/10.1007/s00222-011-0327-6. MR2854087

[10] T. IWANIEC, L. V. KOVALEV, and J. ONNINEN, *Hopf differentials and smoothing Sobolev homeomorphisms*, Int. Math. Res. Not. IMRN **14** (2012), 3256–3277. http://dx.doi.org/10.1093/imrn/rnr144. MR2946225

[11] D. KALAJ, *Energy-minimal diffeomorphisms between doubly connected Riemann surfaces*, Calc. Var. Partial Differential Equations **51** (2014), no. 1-2, 465–494. http://dx.doi.org/10.1007/s00526-013-0683-8. MR3247397

[12] N. T. KOH and L. V. KOVALEV, *Area contraction for harmonic automorphisms of the disk*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 91–96. http://dx.doi.org/10.1112/blms/bdq083. MR2765553 (2012a:31001)

[13] O. LEHTO and K. I. VIRTANEN, *Quasiconformal Mappings in the Plane*, 2nd ed., Die Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 126, Springer-Verlag, New York-Heidelberg, 1973. Translated from the German by K.W. Lucas. MR0344463 (49 #9202)

[14] H. LEWY, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), no. 10, 689–692. http://dx.doi.org/10.1090/S0002-9904-1936-06397-4. MR1563404

[15] S. RUSCHEWEYH and L. C. SALINAS, *On the preservation of direction-convexity and the Goodman-Saff conjecture*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), no. 1, 63–73. http://dx.doi.org/10.5186/aasfm.1989.1427. MR997971 (90h:30041)

[16] E. STUDY, *Vorlesungen ber ausgewhlte Gegenstnde der Geometrie*, Teubner, Leipzig, 1913. Heft II.

[17] M. TSUJI, *Potential Theory in Modern Function Theory*, Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original. MR0414898 (54 #2990)

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