FINITE MÖBIUS NEAR-PLANES

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ABSTRACT. We introduce finite Möbius near-planes of order n and show that these planes uniquely extend to Möbius planes of the same order if $n \geq 5$. Furthermore, Möbius near-planes of order $n \leq 4$ are discussed and the situation for the other two types of finite circle near-planes, Laguerre and Minkowski near-planes, is reviewed.

1. Introduction and result

A *Möbius* or *inversive plane* $\mathcal{M} = (P, \mathcal{C})$ consists of a set P of points and a set $\mathcal{C}$ of circles such that the following three axioms are satisfied:

(J) *Joining*: Three mutually distinct points can be uniquely joined by a circle.

(T) *Touching*: For every circle $C \in \mathcal{C}$ and any two points p, q, where $p \in C$ and $q \notin C$, there is precisely one circle D passing through q which touches C at p, i.e. $C \cap D = \{p\}$.

(R) *Richness*: Each circle contains at least three points and there are at least two circles.

The *Miquelian Möbius plane* is obtained as the geometry of non-trivial plane sections of an elliptic quadric in 3-dimensional projective space over some field. Geometrically, the Miquelian Möbius planes can be characterized by Miquel's theorem, cf. [1]. Miquel's configuration has eight points and six circles. If the points are identified with the vertices of a cube, then one considers the six faces of the cube. If the points on five of the faces are all contained in circles, then the points of the sixth face also are on a circle, see Figure 1. Labelling the points from 1 to 8 each quadruple $\{1, 2, 3, 4\}$, $\{1, 2, 6, 7\}$, $\{1, 4, 5, 6\}$, $\{2, 3, 7, 8\}$, $\{3, 4, 5, 8\}$ defines a circle; then also 5, 6, 7 and 8 must be on a circle.

Generalizing the notion of an elliptic quadric, one defines an *ovoid* to be a subset of points of a 3-dimensional projective space such that no line has more than two points in common with it and such that the collection of all tangents at a point fills

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a plane, called the tangent plane at that point. Then the model for the Miquelian Möbius plane can be generalized to an *ovoidal* (or *egglike*) *Möbius plane* where one takes an ovoid instead of an elliptic quadric. These ovoidal Möbius planes obviously comprise the Miquelian planes.
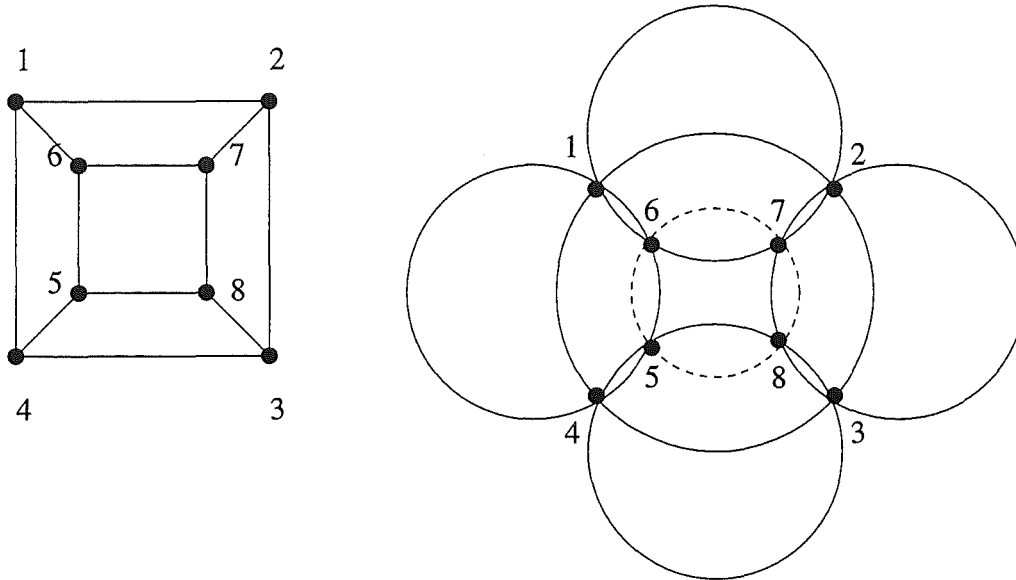

Figure 1

The *internal incidence structure* at a point p of a Möbius plane consists of all points different from p and, as lines, all circles passing through p (without the point p). This is an affine plane, called the *derived affine plane* $\mathcal{A}_p$ at p. The axioms of a Möbius plane are equivalent to each internal incidence structure being an affine plane.

A circle K not passing through the point of derivation p induces an oval in the derived affine plane. A Möbius plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of ovals. (The derived affine planes of the Miquelian Möbius planes are Desarguesian and the ovals are conics that are obtained from a given conic via translations and homotheties.) This planar description of a Möbius plane, which is the most commonly used, is then extended by one point which is adjoined to all the lines of the affine plane.

The spatial description of an ovoidal Möbius plane as the geometry of plane sections of an ovoid is related to the planar description in one derived plane by stereographic projection from one point of the ovoid onto a plane not passing through the point of projection. In this description all points of the Möbius plane except the point of projection are covered.

If P is finite, any two circles have the same number $n + 1$ of points, and n is called the *order* of $\mathcal{M}$. If $\mathcal{M}$ has order n, then every derived affine plane also has order n. There are $n^2 + 1$ points and $n(n^2 + 1)$ circles in a Möbius plane of order n. It is an easy excercise to show that in the finite case the axioms (T) and (R) can

be replaced by requiring that there are $n^2 + 1$ points and each circle contains $n + 1$ points. Hence, the Möbius planes of order n are exactly the $3 - (n^2 + 1, n + 1, 1)$ designs.

All known finite Möbius planes of odd order are Miquelian. Every finite Möbius plane of even order is ovoidal, cf. [3]. Furthermore, if a finite Möbius plane of odd order admits a derived affine plane that is Desarguesian, then the Möbius plane must be Miquelian, cf. [17]. The Möbius planes of order at most 7 and the Möbius planes of orders 9 and 16 are unique, cf. [2], [4], [5], [16, Theorem C] and [12, Corollary 2.4]. There are two Möbius planes of orders 8 and 32, the Miquelian planes and the ovoidal planes over a Tits ovoid, see [6] and [13, Theorem 4.2]. For a recent survey on ovoids (and thus ovoidal Möbius planes) in 3-dimensional projective spaces over finite fields see [11].

In [14] Olanda considered finite *seminversive planes*, that is, incidence structures (P, C) that satisfy the axioms (J) and (R) from above and axiom (T) is replaced by

(T') For every circle $C \in \mathcal{C}$ and any two points p, q, where $p \in C$ and $q \notin C$, there are precisely one or two circles passing through q which intersect C only at p.

Olanda showed that such a seminversive plane of order $n > 5$ is either a Möbius plane of order n or a Möbius plane of order n with one point deleted.

In this note we consider the restriction of a finite Möbius plane of order n to one of its derived affine planes. When verifying the axioms of a Möbius plane in such a planar representation one always has to consider special cases involving the extra point. Given a planar representation we therefore ask whether this already determines the Möbius plane. More precisely, we have a set A of points and a set $\mathcal{B}$ of circles and there is an integer $n \geq 2$ such that the following two axioms are satisfied:

(C) There are n^2 points and each circle contains $n + 1$ or n points.
(J) Three mutually distinct points can be uniquely joined by a circle.

In analogy to the terminology for Minkowski planes adopted in [15], see section 4, we call an incidence structure $(A, \mathcal{B})$ satisfying axioms (C) and (J) a *Möbius near-plane of order n*. With this notation we prove the following.

Main Theorem. *A finite Möbius near-plane of order $n \geq 5$ can be uniquely extended to a Möbius plane of order n by adjoining one point.*

We further determine all Möbius near-planes of orders 2 and 3 and construct models of Möbius near-planes of order 4. We finally give a brief review of what can be said about finite Minkowski and Laguerre near-planes.

2. Proof of the Main Theorem

Let $\mathcal{M} = (A, \mathcal{B})$ be a finite Möbius near-plane of order n. In view of the characterization of finite Möbius planes of order n as the $3 - (n^2 + 1, n + 1, 1)$ designs it is

clear how one has to extend $\mathcal{M}$ in order to obtain a Möbius plane if this is possible at all. We adjoin to A a new point ∞ to obtain the new point set

$$P = A \cup \{\infty\}$$

with $n^2 + 1$ points. To each circle of length n we adjoin the new point ∞. The circles of length $n + 1$ remain unchanged. Thus

$$\mathcal{C} = \{B \in \mathcal{B} \mid |B| = n + 1\} \cup \{B \cup \{\infty\} \mid B \in \mathcal{B}, |B| = n\}$$

is the new set of circles. Then every new circle contains $n + 1$ points and $(P, \mathcal{C})$ is a finite Möbius plane of order n if and only if ∞ and any two distinct points of A determine a unique circle passing through them, i.e., if and only if there is a unique circle of length n through any two distinct points of A.

We always assume that $n \geq 3$ in this section. We call a circle of length n *short* and a circle of length $n + 1$ *long*. We denote the number of short and long circles by σ and λ, respectively. The total number $\sigma + \lambda$ of circles is denoted by τ. For points p and q we then denote the number of short and long circles and of all circles through p and q by $\sigma_{p,q}$, $\lambda_{p,q}$ and $\tau_{p,q}$, respectively.

Proposition 1. *Let p and q be two distinct points of a finite Möbius near-plane $(A, \mathcal{B})$ of order $n \geq 3$. Then there is at least one short circle and at most n long circles through p and q. Furthermore, the numbers λ and σ of long and short circles, respectively, satisfy*

$$(*) \qquad\qquad (n + 1)\lambda + (n - 2)\sigma = n(n + 1)(n^2 - 2).$$

Proof. Since the circles through p and q partition $A \setminus \{p, q\}$ by axiom (J), we obtain

$$(n - 2)\sigma_{p,q} + (n - 1)\lambda_{p,q} = n^2 - 2$$

and thus

$$1 + (n - 2)\sigma_{p,q} = (n - 1)(n + 1 - \lambda_{p,q}).$$

Since the left-hand side is positive, we find $\lambda_{p,q} \leq n$. Furthermore,

$$(**) \qquad\qquad (n - 2)(\sigma_{p,q} - 1) = (n - 1)(n - \lambda_{p,q}).$$

Since the right-hand side of (**) is non-negative, we now find $\sigma_{p,q} \geq 1$.

Counting the triples of mutually distinct points that are on short or long circles one readily obtains

$$\binom{n + 1}{3} \lambda + \binom{n}{3} \sigma = \binom{n^2}{3}.$$

Simplifying yields the stated equation (*) for λ and σ.

Corollary 2. *Let $\mathcal{M}$ be a finite Möbius near-plane of order $n \geq 3$. Then 3σ is divisible by $n + 1$ and*

$$n(n+1) \leq \sigma \leq \frac{n(n+1)(n^2 - 2)}{n - 2},$$

$$0 \leq \lambda \leq n^2(n - 1),$$

$$n(n^2 + 1) \leq \tau \leq \frac{n(n+1)(n^2 - 2)}{n - 2},$$

i.e., σ and τ are bound from below and λ is bound from above by the corresponding values in a Möbius near-plane obtained from a Möbius plane of the same order by deleting a point. Furthermore, τ uniquely determines σ and λ.

$\mathcal{M}$ is obtained from a Möbius plane of order n by deleting a point if and only if one of the following statements is satisfied.

(1) *$\mathcal{M}$ has at most $n(n + 1)$ short circles;*
(2) *$\mathcal{M}$ has at least $n^2(n - 1)$ long circles;*
(3) *$\mathcal{M}$ has at most $n(n^2 + 1)$ circles.*

Proof. On the one hand, there are $\binom{n}{2} \sigma$ triples (x, y, C) with $x, y \in C$, $x \neq y$, and a short circle C. On the other hand, there is at least one short circle through any two points. Therefore there are at least $\binom{n^2}{2}$ such triples. Solving for σ yields $\sigma \geq n(n + 1)$. Equality holds if and only if there is exactly one short circle through any two points. Moreover, this inequality and (*) imply that $\lambda \leq n^2(n - 1)$.

From equation (*) in Proposition 1 we have $n(n+1)(n^2-2) = (n+1)\lambda+(n-2)\sigma = (n + 1)\tau - 3\sigma$. This equation implies that $n + 1$ divides 3σ. Furthermore,

$$(n + 1)\tau = n(n + 1)(n^2 - 2) + 3\sigma$$
$$\geq n(n + 1)(n^2 - 2) + 3n(n + 1)$$
$$= n(n + 1)(n^2 + 1).$$

Hence $\tau \geq n(n^2 + 1)$.

Since $\lambda \geq 0$, equation (*) gives us the upper bound for σ and $n(n+1)(n^2 - 2) = (n + 1)\lambda + (n - 2)\sigma = (n - 2)\tau + 3\lambda$ yields the upper bound for τ.

The system of linear equations

$$\sigma + \lambda = \tau$$
$$(n - 2)\sigma + (n + 1)\lambda = n(n + 1)(n^2 - 2)$$

has determinant 3. Therefore σ and λ are uniquely determined by n and τ.

Since a Möbius plane of order n has $n(n^2 + 1)$ circles and through every point there are $n(n + 1)$ circles it readily follows that a Möbius near-plane of order n that comes from a Möbius plane of order n by deleting a point has $n(n^2 + 1)$ circles of

which $n(n+1)$ are short and $n^2(n-1)$ are long. Conversely, we now suppose that $\sigma \leq n(n+1)$. Then, in fact, $\sigma = n(n+1)$ and there must be precisely one short circle through any two distinct points as seen before. Hence $\mathcal{M}$ can be completed to a Möbius plane of order n by adjoining a point.

Suppose that $\lambda \geq n^2(n-1)$. Then $\sigma \leq n(n+1)$ by (*) and $\mathcal{M}$ can be completed to a Möbius plane of order n by (1).

Finally, suppose that $\tau \leq n(n^2+1)$. Then, in fact, $\tau = n(n^2+1)$. Solving the above system of linear equations we find $\sigma = n(n+1)$ and $\lambda = n^2(n-1)$. Hence $\mathcal{M}$ comes from a Möbius plane of order n by deleting a point. $\square$

From now on we always assume that $n \geq 5$ in the remainder of this section. Under this assumption the possible numbers of long and short circles through two points are severely restricted.

Proposition 3. *Let p and q be two distinct points of a finite Möbius near-plane $\mathcal{M}$ of order $n \geq 5$. Then there are either one short circle and n long circles or n short circles and two long circles of $\mathcal{M}$ passing through p and q.*

Proof. Since $n \geq 3$, the integers $n-2$ and $n-1$ are relatively prime. Equation (**) in the proof of Proposition 1 then shows that $n - \lambda_{p,q}$ must be a multiple of $n-2$. Since $n \geq 5$ this implies $\lambda_{p,q} = 2$ or $\lambda_{p,q} = n$. The respective values for $\sigma_{p,q}$ then are $\sigma_{p,q} = n$ and $\sigma_{p,q} = 1$ from equation (**). $\square$

By Proposition 3 two points are on either $n+1$ or $n+2$ circles. Accordingly we call a pair of points (p,q) *small* (i.e., if $\tau_{p,q} = n+1$) or *large* (i.e., if $\tau_{p,q} = n+2$). Note that Proposition 3 implies that there may be one, two or three circles passing through a point $q \notin C$ and intersecting the circle C only in a given point p depending on whether C is short or long and on the number of circles through p and q. Hence axiom (T') is not necessarily satisfied and the Möbius near-plane may not be a seminversive plane. Thus we cannot directly apply Olanda's result [14].

We now fix a point p and consider the internal incidence structure $\mathcal{I} = \mathcal{A}_p$ at p. It follows from axiom (J) and $n \geq 5$ that $\mathcal{I}$ is a linear space. We therefore call the circles through p (but punctured at p) *lines* of $\mathcal{I}$. Again, we call a line of length $n-1$ *short* and a line of length n *long*. It follows from Proposition 3 that a point is on either $n+1$ or $n+2$ lines. Accordingly we call a point q *small* or *large*. Hence, a small point is on a unique short line and on n long lines whereas a large point is on exactly two long lines and on n short lines. Note that a point is small if and only if there are at least three long lines through it; likewise, a point is large if and only if it is on at least two short lines.

Proposition 4. *The linear space $\mathcal{I}$ has at least two small points and $2t$ is divisible by n where t is the number of large points. In particular, there are either no or at least three large points in $\mathcal{I}$.*

Proof. Let $s = s_p$ and $t = t_p$ be the number of small and large points of $\mathcal{I}$, respectively. Obviously, $s + t = n^2 - 1$. Counting the flags with a long line we obtain

$$nl = ns + 2t$$

where l is the number of long lines of $\mathcal{I}$. This shows that $2t$ is divisible by n. In particular, $2t = 0$ or $2t \geq n$, that is, $t = 0$ or $t \geq \frac{n}{2} \geq \frac{5}{2}$. Furthermore, $2t = 2n^2 - 2(1 + s)$. Hence $2(1 + s) > 0$ is divisible by n. Therefore $s \geq \frac{n}{2} - 1 \geq \frac{3}{2}$ and we have $s \geq 2$. $\square$

Let (q, L) be an anti-flag of $\mathcal{I}$. Then there are $|q| - |L|$ parallels to L through q where $|q|$ and $|L|$ denotes the number of lines through q and the number of points on L, respectively. Depending on whether we have a small or a large point and a short or a long line, we obtain 1, 2 or 3 parallels.

We want to show that there are no large points in $\mathcal{I}$. We suppose to the contrary that there is at least one large point in $\mathcal{I}$. Then any combination of small/large point and short/long line occurs and consequently there are anti-flags (q, L) that admit 1, 2 or 3 parallels to L through q. Hence, in the notation of [9], the linear space $\mathcal{I}$ is a proper $\{1, 2, 3\}$-semiaffine plane. By [9, Proposition 4.1] such a plane admits at most one small point if $n \geq 12$. (Note that $\mathcal{I}$ has order $n + 1$ in the definition of [9] and the small points are the points of degree $n - 1$.) This contradicts Proposition 4.

A closer examination of the proof of [9, Proposition 4.1] shows that only the last two cases where $\lambda = 1$ are relevant to our situation and that the conclusion is still valid for $n \geq 7$. (At one stage one needs at least six points on a short line. Note that λ has a different meaning in [9].) We basically follow Lo Re and Olanda's proof but make simplifications and some subtle modifications in order to include the cases $n = 5$ and $n = 6$. We do this in a series of lemmata.

Lemma 5. *If a short line L contains more than two small points, then every point off L is small and L contains all large points.*

Proof. Suppose that the short line L contains three small points a, b and c. Let d be a point not on L. Then the lines joining d with a, b and c are long, since the unique short line through these points is L. Hence d is on at least three long lines, and therefore must be small. $\square$

For the following lemmata 6 to 9 we always assume that $\mathcal{I}$ contains large points, i.e., $t \geq 3$ by Proposition 4. The next two lemmata are to show that two small points cannot be joined by a short line.

Lemma 6. *A short line contains at most two small points.*

Proof. Suppose that the short line L contains three small points. Since $\mathcal{I}$ contains large points and because $n \geq 5$, there must be a large point q on L by Lemma 5. Let $M \neq L$ be a short line through q. By Lemma 5 all points on M but q are small. Thus all points in $A \setminus M$ are small and q is the only large point of $\mathcal{I}$. However, this contradicts $t \geq 3$ from Proposition 4. $\square$

Lemma 7. *A short line contains at most one small point.*

Proof. Suppose that the short line L contains two small points a and b, cf. Figure 2. Then every point of $L \setminus \{a, b\}$ is large by Lemma 6. Let q be a large point on L and let M be a long line through q. Every point in $M \setminus \{q\}$ is small since the

line M and the lines joining it to a and b are long. Let s be a small point on M. Then there are two parallels to L through s. In particular, there is at least one long parallel L'. Now, every point x on L' is on at least three long lines (L' and the two lines joining it to a and b). Therefore, L' has only small points.
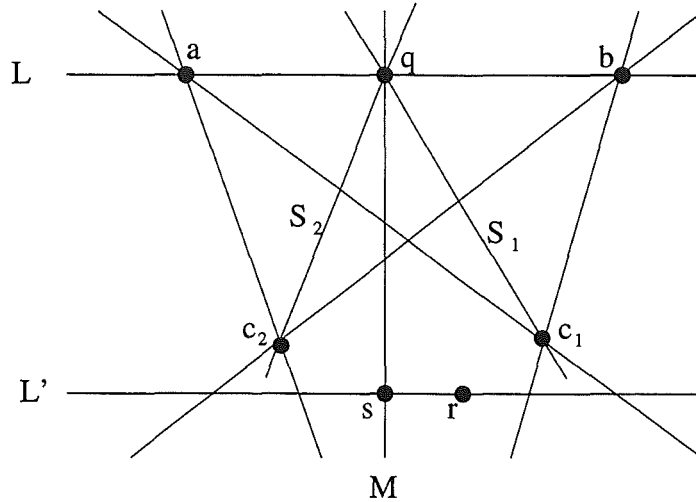

Figure 2

Since $n \geq 5$, there are at least two short lines $S_i \neq L$, $i = 1, 2$, through q. By Lemma 6, each line S_i contains at most two small points. Thus there must be a large point $c_i \neq q$ on S_i. The two long lines through c_i are the lines joining c_i to a and b. All other lines through c_i are short. These four long lines through c_1 or c_2 intersect L' in at most four points. Hence there is a point r on L' which is not on any of the long lines through c_1 or c_2. But then r is on the two short lines joining it to c_1 and c_2.

This shows that r is large in contradiction to the earlier observation that all points on L' must be small. $\square$

The next two lemmata are to show that two small points cannot be joined by a long line either if there are large points. This result then is the crucial step in the proof of the Main Theorem.

Lemma 8. *A long line contains at most two small points.*

Proof. Suppose that the long line L contains three small points a, b and c. Let A, B and C be the unique short lines through a, b and c, respectively. By Lemma 7 each point on A, B or C but a, b and c is large. We distinguish three cases according to how these three short lines intersect, cf. Figure 3.

Case 1: A, B and C pass through a common point q. Since $n \geq 5$, there is a short line $S \neq A, B, C$ through the large point q. Since each point in $S \setminus \{q, S \cap L\}$ is on the three large lines joining it to a, b and c, each such point must be small. However, this fact contradicts Lemma 7 as there are at least 2 such points.
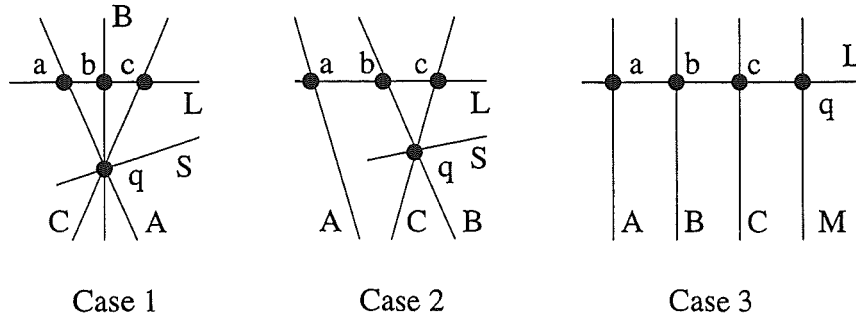
Figure 3

Case 2: A, B and C do not pass through a common point but at least two of the lines intersect. Assume that $B \cap C = \{q\}$ and that the point q is not on A. Since q is large and A is short, there are three parallels to A through q. In particular, there is a short parallel $S \neq C$ to A through q. As before, every point in $S \setminus \{q, S \cap L\}$ is small in contradiction to Lemma 7.

Case 3: A, B and C are mutually parallel. Let q be a point in $L \setminus \{a, b, c\}$ and let $M \neq L$ be a long line through q. This line cannot intersect any of A, B or C, since such a point of intersection would be small as it is on at least three long lines, namely, M and two of the lines joining it to a, b or c. Now every point on M but q is small so that there are $n - 1$ short lines intersecting M in a point other than q. On the other hand, each of the $n - 2$ large points in $A \setminus \{a\}$ is on at least $n - 3$ short lines intersecting M in a point other than q. (A is a short parallel to M and there may be another short parallel; furthermore, the line joining the given point to q may be short. This leaves $n - 3$ short lines.) Hence, $n - 1 \geq (n - 2)(n - 3)$ and therefore $1 < n < 5$ in contradiction to our general assumption $n \geq 5$.

Since we obtained a contradiction in each of the above three cases, the assertion of the lemma is verified. $\square$

Lemma 9. *A long line contains at most one small point.*

Proof. Suppose that the long line L contains two small points a and b. Let M be a long line intersecting L in a point other than a or b and let c be a point in $M \setminus L$ and not on the short lines containing a or b, see Figure 4. Then c is small as it is on three long lines, namely M and the lines joining it to a or b.

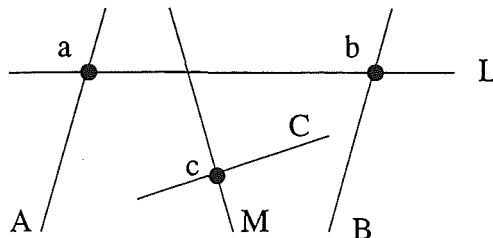Figure 4

Let A and C be the unique short lines through a and c, respectively. Since there are two parallels to A through b, there is such a long parallel $B \neq L$. Now every point in $B \setminus (B \cap C)$ and not on the line joining a and c is small as it is on the long line B and the long lines joining it to a or b. Since there are at least three such points, we obtain a contradiction to Lemma 8. $\square$

Proposition 10. $\mathcal{I}$ *contains no large points.*

Proof. From Proposition 4 we know that $\mathcal{I}$ contains at least two small points. Two such points are joined by a line which may be either short or long. The former case is not possible by Lemma 7 and the latter case by Lemma 9 unless there are no large points. $\square$

We are now in a position to prove the Main Theorem. Proposition 10 is true for every point p of A. Hence there is precisely one short circle through any two distinct points of the Möbius near-plane. As remarked at the beginning of this setion this means that we can extend the Möbius near-plane to a Möbius plane. Clearly, such an extension is unique.

Remark 11. Propositon 10 in fact shows that a linear space with $n^2 - 1$ points and lines with n or $n - 1$ points where $n \geq 5$ contains no large points. There therefore are exactly $n + 1$ lines through every point and each point is on precisely one short line. Hence the short lines partition the point set and there are $n + 1$ short lines. Furthermore, given an antiflag (q, L) there are precisely 1 or 2 parallels to L through q depending on whether L is long or short. Hence we have a biaffine plane. In the notation of [10] this plane is of type I and by [10, Satz 14] such a plane is obtained from an affine plane by deleting a point.

3. Möbius near-planes of order at most 4

Having determined Möbius near-planes of order at least 5 we now turn to Möbius near-planes of smaller orders. So let $\mathcal{M} = (A, \mathcal{B})$ be a finite Möbius near-plane of order $n = 2, 3, 4$. Note that equation (**) in the proof of Propositon 1 remains valid. Möbius near-planes of order 2 are characterized by a lack of short circles whereas in order 3 we may have an abundance of short circles.

Möbius near-planes of order 2. The case of Möbius near-planes of order 2 is somewhat pathological since we may have circles of length 2, which are not covered by axiom (J). We have four points and axiom (J) implies that every 3-subset of A is a circle. This yields four long circles. Beyond this, we can throw in any number of 2-subsets of A. Obviously, every Möbius near-plane of order 2 is obtained in this way. However, only if we take every possible 2-subset as a circle we can extend the Möbius near-plane to the unique Möbius plane of order 2 by adjoining to the 2-subsets a new point ∞. It is easily verified that there are eleven possible configurations for a collection of 2-subsets in a 4-set. In summary, one obtains the following.

Proposition 12. *Every Möbius near-plane of order 2 is obtained from the Miquelian Möbius plane of order 2 by deleting a point from the Möbius plane and some (including none or all) circles through this point. Furthermore, there are, up to isomorphism, exactly eleven Möbius near-planes of order 2.*

Möbius near-planes of order 3. In a Möbius near-plane of order 3 we have 9 points and a collection of 4- and 3-subsets as circles. Equation (**) in the proof of Proposition 1 yields $\sigma_{p,q} + 2\lambda_{p,q} = 7$. Hence the possible values for $\lambda_{p,q}$ are 0, 1, 2 and 3. The respective values for $\sigma_{p,q}$ then are 7, 5, 3 and 1. Therefore, in this case, we may have pairs of points with 7, 6, 5 or 4 circles through them.

From Corollary 2 we obtain that σ is divisible by 4 and that

$$12 \leq \sigma \leq 84, \ 0 \leq \lambda \leq 18 \ \text{ and } \ 30 \leq \tau \leq 84.$$

Rewriting equation (*) as $\tau + 3\lambda = 84$ it follows that the total number $\tau = \sigma + \lambda$ of circles must be divisible by 3. Conversely, such an integer between 30 and 84 uniquely determines σ and λ. Furthermore, $\tau = 30$ implies $\sigma = 12$ and $\lambda = 18$ and the Möbius near-plane extends to a Möbius plane of order 3 by Corollary 2.

In order to construct a Möbius near-plane of order 3 we can choose any collection of 4-subsets of A subject to the condition that these sets mutually intersect in at most two points. We then add all 3-subsets that are not contained in any of the 4-subsets. For example, we can have no 4-subset; then we take all possible 3-subsets. We can choose just one 4-subset; then we take all possible 3-subsets except the four 3-subsets contained in the unique 4-subset; etc. Obviously, every Möbius near-plane of order 3 is obtained in this way. However, only if we take a maximal collection of 18 long circles we can extend the Möbius near-plane to the unique Möbius plane of order 3 by adjoining to the 3-subsets a new point ∞.

The above method for constructing Möbius near-planes of order 3 works well for such planes with few long circles. For Möbius near-planes with many long circles we may start off with the Möbius plane of order 3 and delete one point. We can then replace a long circle by the four 3-subsets contained in it. Indeed, this method of replacing a long circle by 4 short circles can be applied to any Möbius near-plane of order 3 and yields again a Möbius near-plane of order 3. In this way we can obtain Möbius near-planes of order 3 with any given integer $0 \leq \lambda \leq 18$ as the number of long circles. Consequently, all integers τ between 30 and 84 that are divisible by 3 actually occur as the total number of circles in Möbius near-planes of order 3.

It should be noted that the process of replacing four short circles by one long circle cannot be reversed in general. More precisely, we say that a *Möbius near-plane of order 3 admits reduction* if it contains 4 short circles C_1, C_2, C_3, C_4 such that $C_1 \cup C_2 \cup C_3 \cup C_4$ consists of precisely 4 points. In this case, we can replace these 4 short circles by the 4-set $C_1 \cup C_2 \cup C_3 \cup C_4$ which becomes one long circle and obtain again a Möbius near-plane of order 3 whose total number of circles has decreased by 3. (The number of short circles has decreased by 4 and the number of long circles has increased by 1.) Once we have reached 18 long circles the Möbius near-plane can be completed to a Möbius plane of order 3 by adjoining a point, see Corollary 2. However, there are Möbius near-planes of order 3 that do not admit

reduction but still have less than 18 long circles. In particular, this means that not every Möbius near-plane of order 3 is obtained from the Miquelian Möbius plane of order 3 by deleting one point and converting some long circles into short circles as above.

In order to give an example for a Möbius near-plane that does not admit any further reduction we label the 9 points from 1 to 9. Then we have the following 14 long circles

$$\{1,2,3,4\}, \{1,2,6,7\}, \{1,4,5,6\}, \{2,3,7,8\}, \{3,4,5,8\}, \{5,6,7,9\},$$
$$\{1,2,5,8\}, \{1,3,5,7\}, \{1,3,8,9\}, \{1,4,7,8\}, \{2,3,5,6\}, \{2,4,5,7\},$$
$$\{2,4,6,8\}, \{3,4,6,7\}$$

and 28 short cirles

$$\{1,2,9\}, \{1,3,6\}, \{1,4,9\}, \{1,5,9\}, \{1,6,8\}, \{1,6,9\}, \{1,7,9\},$$
$$\{2,3,9\}, \{2,4,9\}, \{2,5,8\}, \{2,5,9\}, \{2,6,9\}, \{2,7,9\}, \{2,8,9\},$$
$$\{3,4,9\}, \{3,5,9\}, \{3,6,8\}, \{3,6,9\}, \{3,7,9\}, \{4,5,9\}, \{4,6,9\},$$
$$\{4,7,9\}, \{4,8,9\}, \{5,6,8\}, \{5,7,8\}, \{6,7,8\}, \{6,8,9\}\{7,8,9\}.$$

It is easily verified that axiom (J) is satisfied so that the above circles define a Möbius near-plane of order 3. It also readily follows by inspection that no further reduction is possible. Note that the points from 1 to 8 and the first five of the above long circles form the eight points and five of the six circles in Miquel's configuration, see Figure 1. If Miquel's theorem is to be true, then $\{5,6,7,8\}$ must be a circle. As $\{5,6,7,8\}$ is not a circle, the Möbius near-plane cannot come from the (Miquelian) Möbius plane of order 3. In fact, the above long circles were obtained from Miquel's configuration by choosing the first five circles and then altering the sixth circle so that the configuration does not close. The remaining 8 circles were then found by reduction.

Möbius near-planes of order 4. In a Möbius near-plane of order 4 we have 16 points and a collection of 5- and 4-subsets as circles. Equation (**) yields $2\sigma_{p,q} + 3\lambda_{p,q} = 14$. In particular, $\lambda_{p,q}$ must be even and so the possible values for $\lambda_{p,q}$ are 0, 2 and 4. The respective values for $\sigma_{p,q}$ then are 7, 4 and 1. Therefore, in this case, we may have large pairs of points with 7 circles (all 7 short), medium pairs of points with 6 circles (4 short and 2 long) and small pairs of points with 5 circles (1 short and 4 long) through them.

From Corollary 2 we obtain that σ is divisible by 5 and that

$$20 \leq \sigma \leq 140, \; 0 \leq \lambda \leq 48 \; \text{and} \; 68 \leq \tau \leq 140.$$

Rewriting equation (*) as $3\lambda + 2\tau = 280$ it is easily seen that the total number $\tau = \sigma + \lambda$ of circles must be congruent to 2 modulo 3. Conversely, such an integer between 68 and 140 uniquely determines σ and λ. Furthermore, $\tau = 68$ implies $\sigma = 20$ and $\lambda = 48$ and the Möbius near-plane can be extended to the unique (Miquelian) Möbius plane of order 4 by adjoining to the 4-subsets a new point ∞, see Corollary 2.

Let s, m and l be the number of small, medium or large pairs of points, respectively. Then $s + m + l = 120$. Counting the triples (x, y, C) with $x, y \in C$, $x \neq y$, and a long circle C and those with a short circle, we find the following equations.

$$10\lambda = 4s + 2m \qquad \text{and}$$
$$6\sigma = s + 4m + 7l.$$

Hence $6\sigma = 120 + 3m + 6l$. The two equations also confirm $5\lambda + 2\sigma = 280$ from Corollary 2. Furthermore, m and λ must be even.

As for the upper bound for the total number of circles one finds that $\tau = 140$ implies $\sigma = 140$ and $\lambda = 0$. Therefore $m = s = 0$ from the first of the above equations for σ and λ. Indeed, one can construct a Möbius near-plane of order 4 with 140 short circles and no long circles so that every pair of points is large. A simple example for such a Möbius near-plane to which the referee drew my attention is as follows. The point set is the point set of the 4-dimensional affine space over the field $\mathbb{F}_2 = GF(2)$ with two elements; circles are the affine planes (i.e., the 2-dimensional subspaces) of this space. In particualr, given $a, b, c \in \mathbb{F}_2^4$, then $\{a, b, c, a + b + c\}$ is a circle.

We give a different description of this model which will in turn be generalized. Let $\mathbb{F}_4 = GF(4)$ denote the Galois field of order 4. We write the point set A as $A = \mathbb{F}_4 \times \mathbb{F}_4$. There a three kinds of circles. First, there are the four verticals

$$\{(c, y) \mid y \in \mathbb{F}_4\}$$

for $c \in \mathbb{F}_4$. Second, there are the circles contained in two verticals; they are of the form

$$\{(a, y) \mid y \in U\} \cup \{(b, y) \mid y \in V\}$$

where $a, b \in \mathbb{F}_4$, $a \neq b$, $U \subseteq \mathbb{F}_4$ is a 2-subset of $\mathbb{F}_4$ and either $V = U$ or $V = \mathbb{F}_4 \setminus U$. This yields $\binom{4}{2} \cdot \binom{4}{2} \cdot 2 = 72$ circles. Third, there are the graphs of polynomials of degree at most 2, i.e., these circles are of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_4\}$$

for all $a, b, c \in \mathbb{F}_4$; this yields 64 circles. Each circle has length 4 and there are 7 circles through any pair of points. Hence every pair of points is large and every circle is short. To verify axiom (J) note that given three distinct points (x_i, y_i), $i = 1, 2, 3$, a connecting circle is of the first kind if and only if $x_1 = x_2 = x_3$; it is of the second kind if and only if exactly two of the x_i are equal, and it is of the third kind if and only if the x_i are mutually distinct. In each case it then readily follows that there is a unique circle of the respective type. Note that $\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_4\}$ is a 2-dimensional affine subspace of the 4-dimensional affine space $\mathbb{F}_4^2 \cong \mathbb{F}_2^4$ over $\mathbb{F}_2$ since $x \mapsto x^2$ is additive. Hence we have the same model of points and planes from above.

This model generalizes to a whole family of models of Möbius near-planes of order 4 with only short circles and large pairs of points. Circles of the first and

third kind represent generators and circles, respectively, of a Laguerre near-plane of order 4 obtained from a Laguerre plane by deleting one generator, see the following section. Clearly, we can replace circles of the third kind by circles of any Laguerre near-plane of order 4. Since, in general, circles of a Laguerre near-plane of order 4 cannot be represented as 2-dimensional affine spaces in 4-dimensional affine space over $\mathbb{F}_2$, one obtains models of Möbius near-planes of order 4 that are not isomorphic to the model of points and planes of $\mathbb{F}_2^4$.

But there are more Laguerre near-planes hidden in the above model. The horizontals $\{(x,c) \mid x \in \mathbb{F}_4\}$ for $c \in \mathbb{F}_4$ can be regarded as the generators of another Laguerre near-plane. Circles then are circles of the third kind that are graphs of permutations and circles of the second kind for V being the complement of U. Again, these circles can be replaced by circles of any Laguerre near-plane of order 4. Finally, the family of circles of the second kind can be modified as follows. For each $a,b,c \in \mathbb{F}_4$ we choose a permutation $f_{a,b}$ of $\mathbb{F}_4$ of order at most 2. These permutations induce permutations of the collection of 2-subsets of $\mathbb{F}_4$. Then the new circles of the second kind are of the form

$$\{(a,y) \mid y \in U\} \cup \{(b,y) \mid y \in V\}$$

where $a,b \in \mathbb{F}_4$, $a \neq b$, $U \subseteq \mathbb{F}_4$ is a 2-subset of $\mathbb{F}_4$ and either $V = f_{a,b}(U)$ or $V = \mathbb{F}_4 \setminus f_{a,b}(U)$.

It is not known at present whether or not the above two kinds of models of Möbius near-planes of order 4 with only large pairs of points or only small pairs of points comprise all possible Möbius near-planes of order 4. There seems to be no straightforward way to convert two long circles into five short circles (or multiples of these numbers) as we did for Möbius near-planes of order 3 (one long circle could be converted into four short circles, all its four 3-subsets). We just mention that the third pure case $s = l = 0$, i.e., every pair of points is medium, is not possible. (In the case of a Möbius near-plane of order 4 that comes from a Möbius plane we have $m = l = 0$ so that every pair of points is small.) To see this we look at the internal incidence structure $\mathcal{I}_p$ at a point p. Let λ_p denote the number of long circles through p. Since there are no small or long pairs of points, every point q of $\mathcal{I}_p$ must belong to a medium pair (p,q). Counting the triples (p,q,C) with $q \in \mathcal{I}_p$ and a long circle C through p and q we obtain $4\lambda_p = 2 \cdot 15 = 30$ which is not possible.

4. The other circle planes

We conclude this paper with a brief look at the other two kinds of circle planes, Laguerre planes and Minkowski planes, and at what can be said about their restrictions to derived affine planes. To begin with, a finite *Minkowski plane of order $n-1$* where $n \geq 3$ is an integer consists of a set P of n^2 points and a set C of circles such that the following three axioms are satisfied:

(G) There are two classes $\mathcal{G}_1$ and $\mathcal{G}_2$ of generators each of which partitions P. Every generator contains n points and two generators of different classes intersect in precisely one point.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

Every Minkowski plane of order $n - 1$ can be represented on a set S of size n in the following way. The point set P is the cartesian product $S \times S$; the generators in $\mathcal{G}_1$ are the verticals $\{(a, y) \mid y \in S\}$ for $a \in S$ and the generators in $\mathcal{G}_2$ are the horizontals $\{(x, b) \mid x \in S\}$ for $b \in S$. Circles are the graphs of certain permutations of S. In this representation, $\mathcal{C}$ corresponds to a sharply 3-transitive set of permutations of S. If $\mathbb{F} = \mathrm{GF}(p^m)$ is the Galois field of order p^m, p a prime, then $\mathrm{PGL}(2, p^m)$, the projective linear group over $\mathbb{F}$, is a sharply 3-transitive group of permutations of $S = \mathbb{F} \cup \{\infty\}$. In this way one obtains the *Miquelian Minkowski plane of order* p^m.

The *internal incidence structure* at a point p of a Minkowski plane has the collection of all points not on a generator through p as point set and, as lines, all circles passing through p (without the point p) and all generators not passing through p (without their points of intersection with the generators through p). Again, this is an affine plane. A circle K not passing through the point of derivation p induces an oval in the projective extension of the derived affine plane at p which intersects the line at infinity in the two points corresponding to lines that come from generators of the Minkowski plane; in $\mathcal{A}_p$ one has a *hyperbolic curve*. (The derived affine planes of the Miquelian Minkowski planes are Desarguesian and the hyperbolic curves are hyperbolae whose asymptotes are the horizontals and verticals, i.e., the lines that come from generators of the Minkowski plane.) A Minkowski plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of hyperbolic curves. This planar description of a Minkowski plane, which is the most commonly used representation of a Minkowski plane, is then extended by the points of two intersecting generators; the set of points on the first and the second of these generators, minus their intersection, is in bijective correspondence with the set of horizontal generators and the set of vertical generators, respectively. The point of intersection of the two new generators point is adjoined to all the lines of the affine plane. A hyperbolic curve is extended by two points corresponding to the two generators in $\mathcal{A}_p$ with which it does not have a point in common. Again one can ask whether the description in a derived affine plane already determines the Minkowski plane. This question has been answered by Rinaldi in [15], although the introduction of the so-called Minkowski near-planes of order n is motivated differently. More precisely, a *Minkowski near-plane of order* $n \geq 3$ is an incidence structure of points and circles satisfying the axioms (G) and (J) from above but where axiom (C) is replaced by

(C') There are n^2 points and each circle intersects each generator in at most one point and contains either n or $n - 1$ points.

Rinaldi showed that a Minkowski near-plane of order $n \geq 5$ is either a Minkowski plane of order $n - 1$ or a Minkowski plane of order n with two intersecting generators deleted. Hence a *proper Minkowski near-plane of order* $n \geq 5$, that is, circles of lengths n and $n - 1$ actually occur, extends to a Minkowski plane of order n. Clearly,

such an extension is unique.

Theorem 13. *A finite proper Minkowski near-plane of order $n \geq 5$ can be uniquely extended to a Minkowski plane of order n. The only other Minkowski near-planes of order $n \geq 5$ are the Minkowski planes of order $n - 1$.*

Minkowski near-planes of order 3 and 4 were also described by Rinaldi in [15]. The picture one obtains is like the one for Möbius near-planes of order 2 and 3, respectively. Let $\mathbb{F}_3 = GF(3)$ and $\mathbb{F}_4 = GF(4)$ denote the Galois fields of order 3 and 4, respectively. A Minkowski near-plane of order $n = 3$ or 4 has point set $A_n = \mathbb{F}_n \times \mathbb{F}_n$. The two classes of generators are the verticals $\mathcal{G}_1 = \{a \times \mathbb{F}_n \mid a \in \mathbb{F}_n\}$ and the horizontals $\mathcal{G}_2 = \{\mathbb{F}_n \times b \mid b \in \mathbb{F}_n\}$. Let $\mathcal{C}'_n$ be the collection of all graphs of permutations of $\mathbb{F}_n$. A Minkowski near-plane of order 3 then has circle set $\mathcal{C}_3 = \mathcal{C}'_3 \cup \mathcal{P}$ where $\mathcal{P}$ is any collection of 2-subsets of A_3 not contained in a generator. Note that $\mathcal{P} = \emptyset$ yields the Miquelian Minkowski plane of order 2. A Minkowski near-plane of order 4 has circle set $\mathcal{C}_4 = (\mathcal{C}'_4 \setminus \mathcal{C}^0_4) \cup \mathcal{T}$ where $\mathcal{C}^0_4$ is any subset of $\mathcal{C}'_4$ and $\mathcal{T}$ is the collection of 3-subsets of A_4 obtained by converting each circle in $\mathcal{C}^0_4$ into its four 3-subsets. In particular, $\mathcal{C}^0_4 = \mathcal{T} = \emptyset$ yields the Miquelian Minkowski plane of order 3.

Note that unlike Möbius near-planes of order 3 Minkowski near-planes of order 4 always admit reduction unless there are no short circles left. (Here reduction is defined similarly as for Möbius near-planes of order 3 where, of course, no two of the four points involved can be on the same generator.) This means that starting with a Minkowski near-plane of order 4 we can convert 4 appropriate short circles into one long circle until we have only long circles. But then we must have the Miquelian Minkowski of order 3. Hence we obtain the above method for constructing Minkowski near-planes of order 4. We can now geometrically summarise the above constructions of Minkowski near-planes of order 3 or 4 as follows.

Proposition 14. *Every Minkowski near-plane of order 3 is obtained from the Miquelian Minkowski plane of order 3 by deleting the points on the two generators through a particular point p and removing some (including none or all) circles not passing through p.*

Every Minkowski near-plane of order 4 is obtained from the Miquelian Minkowski plane of order 3 by converting some (including none or all) of its circles into all the 3-subsets contained in them.

The situation for the third kind of circle planes, Laguerre planes, is more intriguing and not as clear as for Möbius and Minkowski planes. A finite *Laguerre plane of order n* where $n \geq 2$ is an integer consists of a set P of $n(n + 1)$ points, a set $\mathcal{C}$ of circles and a set $\mathcal{G}$ of generators (subsets of P) such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains n points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

From this definition it readily follows that a Laguerre plane of order n has $n + 1$ generators and n^3 circles and that every circle contains exactly $n + 1$ points.

All known models of finite Laguerre planes are of the following form. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, p^m)$, p a prime. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, p^m)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order p^m*. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order p^m*. All known finite Laguerre planes of odd order are Miquelian.

The *internal incidence structure* at a point p of a Laguerre plane has the collection of all points not on the generator through p as point set and, as lines, all circles passing through p (without the point p) and all generators not passing through p. Again, this is an affine plane. A circle K not passing through the point of derivation p induces an oval in the projective extension of the derived affine plane at p which intersects the line at infinity in the point corresponding to lines that come from generators of the Laguerre plane; in $\mathcal{A}_p$ one has a *parabolic curve*. (The derived affine planes of the Miquelian Laguerre planes are Desarguesian and the parabolic curves are parabolae whose axes are the verticals, i.e., the lines that come from generators of the Laguerre plane.) A Laguerre plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of parabolic curves. This planar description of a Laguerre plane, which is the most commonly used representation of a Laguerre plane, is then extended by the points of one generator where one has to adjoin a new point to each line and to each parabolic curve of the affine plane.

Again one can ask whether the description in a derived affine plane already determines the Laguerre plane. To our knowledge this problem has not been solved so far. To be more precise, a *Laguerre near-plane of order* $n \geq 3$ is an incidence structure of n^2 points, circles and generators satisfying the axioms (G), (C) and (J) from above. Clearly, there are n generators, n^3 circles and every circle contains exactly n points. One obviously obtains a Laguerre near-plane of order n by deleting a generator from a Laguerre plane of order n.

In contrast to both former situations, it is not clear how to extend circles in order to construct a Laguerre plane from a Laguerre near-plane since all circle have the same length. Even worse, if an extension exists, it may not be unique. In the Laguerre near-planes that come from ovoidal Laguerre by deleting one generator this phenomenon occurs in the case of even order. For example, consider the ovoidal Laguerre plane over an oval $\mathcal{O}$ in $\mathrm{PG}(2, 2^m)$. The tangents of $\mathcal{O}$ pass through a common point ν, the *nucleus of $\mathcal{O}$*, so that $\mathcal{O} \cup \{\nu\}$ becomes a *hyperoval*; cf. [8, Lemma 12.10] or [7, §8.1]. We can now remove any point of $\mathcal{O} \cup \{\nu\}$ and obtain again an oval. Hence, if we delete a generator from the ovoidal Laguerre plane over $\mathcal{O}$ we obtain a Laguerre near-plane of order 2^m. But now we can either add the deleted generator or a generator formed from the line through the vertex and the nucleus of $\mathcal{O}$. In both cases we obtain a Laguerre plane. In general, the two Laguerre planes are not isomorphic. Substituting a point of a conic by its nucleus yields a

translation oval which is not a conic unless $m \leq 2$. Hence one extension is the Miquelian Laguerre plane whereas another extension is an ovoidal non-Miquelian Laguerre plane. In coordinates, let $\mathbb{F}_{2^m} = \mathrm{GF}(2^m)$ be the Galois field of order 2^m. We consider the following Laguerre near-plane of order 2^m with point set $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, generators being the verticals $\{c\} \times \mathbb{F}_{2^m}$ for $c \in \mathbb{F}_{2^m}$ and circles being of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_{2^m}\}$$

for $a, b, c \in \mathbb{F}_{2^m}$. We extend this Laguerre near-plane by a generator $\{\infty\} \times \mathbb{F}_{2^m}$. A circle described by $a, b, c \in \mathbb{F}_{2^m}$ as above is adjoined the point (∞, a). This yields the Miquelian Laguerre plane of order 2^m. If we adjoin the point (∞, b) however, we obtain an ovoidal non-Miquelian Laguerre plane of order 2^m if $m \geq 3$. (For $m = 1$ or 2 we obtain again the Miquelian Laguerre plane.)

Although we leave the investigation of finite Laguerre near-planes to later, we want to take a brief look at Laguerre near-planes of order 3 and 4 since 5 was the critical order for Möbius and Minkowski near-planes and because Laguerre near-planes of order 4 were used to construct Möbius near-planes of order 4. To begin with, we call a triple of points *admissible* if and only if no two of the points are on the same generator. In a Laguerre near-plane of order 3 every circle corresponds to an admissible triple of points and each admissible triple of points must occur by axiom (J). This is exactly the same as in the Miquelian Laguerre plane of order 3 with one generator deleted. Hence we have the following result.

Proposition 15. *A Laguerre near-plane of order 3 is obtained from the Miquelian Laguerre plane of order 3 by deleting one generator.*

As for Laguerre near-planes of order 4 we begin with a description of the model obtained from the Miquelian Laguerre plane of order 4 by deleting one generator. As before, let $\mathbb{F}_4 = \mathrm{GF}(4) = \{0, 1, \omega, \omega + 1\}$ be the Galois field of order 4 where $\omega^2 + \omega + 1 = 0$. The field $\mathbb{F}_4$ admits a unique involutory automorphism defined by $x \mapsto x^2$ for $x \in \mathbb{F}_4$. The point set of the affine part of the Miquelian Laguerre plane of order 4 is $\mathbb{F}_4 \times \mathbb{F}_4$, generators are the verticals $\{c\} \times \mathbb{F}_4$ for $c \in \mathbb{F}_4$ and circles are of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_4\}$$

for $a, b, c \in \mathbb{F}_4$. These circles fall into three classes. First, there are the graphs of the four constant polynomials obtained for $a = b = 0$. Then there are the graphs of the 24 permutation polynomials obtained for $a = 0$, $b \neq 0$ and $a \neq 0$, $b = 0$. Third, there are the graphs of the remaining 36 polynomials obtained for $a, b \neq 0$; these polynomials take on exactly two values and each of these values occurs exactly twice. Note that the same picture emerges if we delete a different generator from the Miquelian Laguerre plane of order 4 because the automorphism group of this plane is transitive on the point set.

We now modify the above model to obtain a new Laguerre near-plane. To this end we consider the circles that are entirely contained in

$$W = \{0, 1\} \times \{0, 1\} \cup \{\omega, \omega + 1\} \times \{\omega, \omega + 1\},$$
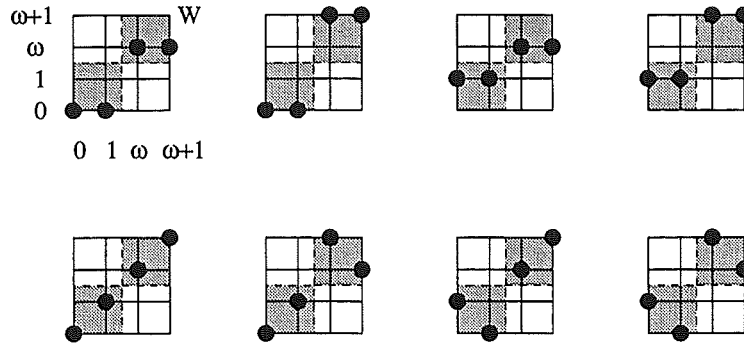
see the shaded area in Figure 5.

Figure 5

There are 8 such circles, four of the second and third kind each, see Figure 5 for a schematic representation of these circles. These 8 circles cover 32 admissible triples of points. We now replace these circles by 8 new circles covering the same 32 admissible triples of points. From this property it will be clear that we again obtain a Laguerre near-plane of order 4. The new circles are obtained as the images of the 8 old circles under the map

$$\Phi : (x, y) \mapsto \begin{cases} (x, y), & \text{if } x \neq \omega + 1, \\ (x, y^2), & \text{if } x = \omega + 1; \end{cases}$$

that is, the points $(\omega + 1, \omega)$ and $(\omega + 1, \omega + 1)$ are swapped and all other points remain unchanged, see Figure 6 for a schematic representation of these new circles.
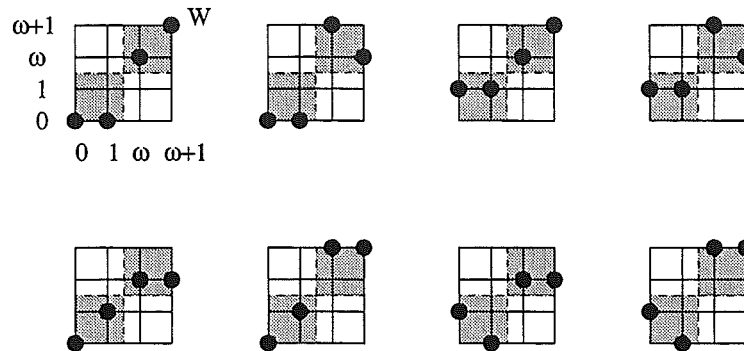

Figure 6

However, since we still have circles of all three types and the new circles, this Laguerre near-plane cannot be obtained from a Laguerre plane of order 4 by deleting one generator.

Cearly, the above construction generalises as follows. Let U and V be two 2-subsets of $\mathbb{F}_4$ and let U' and V' be the complement of U and V, respectively. We then replace all the 8 circles that are entirely contained in $W = U \times V \cup U' \times V'$ by the circles obtained thereof by swapping two points on one fixed generator. We can even repeat the construction for different sets U and V and obtain a Laguerre near-plane of order 4 in each step.

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