LAGUERRE NEAR-PLANES OF ORDER FOUR

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No. 175 March, 1999

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ABSTRACT. We investigate Laguerre near-planes of order 4 and classify all such planes. We further develop various descriptions of these planes and characterise those Laguerre near-planes that can be extended to the Miquelian Laguerre plane. We also determine the automorphism groups of these planes and give characterisations of some of the planes in terms of their automorphism groups.

1. Introduction and result

A finite *Laguerre plane of order n* where $n \geq 2$ is an integer consists of a set P of points, a set $\mathcal{C}$ of circles and a set $\mathcal{G}$ of generators (subsets of P) such that the following four axioms are satisfied:

(P) P contains $n(n+1)$ points.
(G) $\mathcal{G}$ partitions P and each generator contains n points.
(C) Each circle intersects each generator in precisely one point.
(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

From this definition it readily follows that a Laguerre plane of order n has $n+1$ generators, that every circle contains exactly $n+1$ points and that there are n^3 circles.

All known models of finite Laguerre planes are of the following form. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, p^m)$, p a prime. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, p^m)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* p^m. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order* p^m. All known finite Laguerre planes of odd order are Miquelian.

1991 *Mathematics Subject Classification.* 51E25, 51B15.

The *internal incidence structure* $\mathcal{A}_p$ at a point p of a Laguerre plane has the collection of all points not on the generator through p as point set and, as lines, all circles passing through p (without the point p) and all generators not passing through p. This is an affine plane, the *derived affine plane at* p. A circle K not passing through the point of derivation p induces an oval in the projective extension of the derived affine plane at p which intersects the line at infinity in the point corresponding to lines that come from generators of the Laguerre plane; in $\mathcal{A}_p$ one has a *parabolic curve.* (The derived affine planes of the Miquelian Laguerre planes are Desarguesian and the parabolic curves are parabolae whose axes are the verticals, i.e., the lines that come from generators of the Laguerre plane.) A Laguerre plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of parabolic curves. This planar description of a Laguerre plane, which is the most commonly used representation of a Laguerre plane, is then extended by the points of one generator where one has to adjoin a new point to each line and to each parabolic curve of the affine plane. It follows from [8] that every parabolic curve in a finite Desarguesian affine plane of odd order is in fact a parabola. Furthermore, using a simple counting argument it was shown in [1] that a finite Laguerre plane of odd order that admits a Desarguesian derivation is Miquelian.

The spatial description of an ovoidal Laguerre plane as the geometry of plane sections of an oval cone is related to the planar description in one derived plane by stereographic projection from one point of the cone onto a plane not passing through the point of projection. In this description all points of the Laguerre plane except the points on the generator through the point of projection are covered.

In this note we consider the restriction of a finite Laguerre plane to one of its derived affine planes. When verifying the axioms of a Laguerre plane in such a planar representation one always has to consider special cases involving the extra points. We now ask to what extend the description in a derived affine plane determines the Laguerre plane. A partial solution to this problem was given in [11] in the case of odd order and under the assumption that a point exists at which the internal incidence structure (defined in exactly the same way as for Laguerre planes) can be extended to a Desarguesian affine plane. To be more precise, a *Laguerre near-plane of order* $n \geq 3$ is an incidence structure of n^2 points, circles and generators satisfying the axioms (G), (C) and (J) from above. This definition extends the terminology for Minkowski planes and Möbius planes adopted in [6] and [10], respectively. Laguerre near-planes occur as special Laguerre semi-planes in [7] but have not been further investigated there.

Clearly, there are n generators, every circle contains exactly n points and there are n^3 circles. One obviously obtains a Laguerre near-plane of order n by deleting a generator from a Laguerre plane of order n. Conversely, it is not clear how to extend circles in order to construct a Laguerre plane from a Laguerre near-plane since all circles have the same length. Even worse, if an extension exists, it may not be unique, see Section 2.

In [11] all Laguerre near-planes of order at most seven, except order 4 are covered. Furthermore, Laguerre near-planes of order 4 were used in [10] to construct Möbius near-planes of order 4. In this paper we investigate Laguerre near-planes of order 4. We develop a representation of such planes in terms of a single map. We determine, up to isomorphism, all Laguerre near-planes of order 4 and characterize those planes that can be extended to Laguerre planes. The results obtained in this note can be summarized as follows.

Theorem. *Let $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ where $\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$, $\omega^2 = \omega + 1$, denotes the Galois field of order 4 be a map such that for each $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $x \mapsto f(x, y_0, z_0)$, $y \mapsto f(x_0, y, z_0)$ and $z \mapsto f(x_0, y_0, z)$ are permutations of $\mathbb{F}_4$. Such a map describes a Laguerre near-plane $\mathcal{L}(f)$ of order 4 as follows. The point set is $\mathbb{F}_4 \times \mathbb{F}_4$ and generators are the verticals $\{c\} \times \mathbb{F}_4$ for $c \in \mathbb{F}_4$. Circles are of the form*

$$\{(u, (x, y, z, f(x, y, z)) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & \omega & \omega + 1 & 0 \\ 1 & \omega + 1 & \omega & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix}) \mid u \in \mathbb{F}_4\}$$

for $x, y, z \in \mathbb{F}_4$. Conversely, every Laguerre near-plane of order 4 can be uniquely described in this way by such a map.

A Laguerre near-plane $\mathcal{L}(f)$ can be uniquely extended to the Miquelian Laguerre plane of order 4 by adjoining the points of one generator if and only if one of the following holds.

1. *$f + f(0, 0, 0)$ is additive;*
2. *the circle set $\{(x, y, z, f(x, y, z)) \mid x, y, z \in \mathbb{F}_4\}$ (i.e., the graph of f) forms an affine subspace of $\mathbb{F}_4^4$ over the Galois field $\mathbb{F}_2$ of order 2;*
3. *its describing map f is of degree at most 3.*

Up to isomorphism, there are precisely five Laguerre near-planes of order 4. These planes are described by the maps

$$\begin{aligned} f_0(x, y, z) &= x + y + z, \\ f_1(x, y, z) &= (x^2 + x)(z^2 + z) + x + y + z, \\ f_2(x, y, z) &= (x^2 + x)(y^2 + y) + (y^2 + y)(z^2 + z) + (x^2 + x)(z^2 + z) + x + y + z, \\ f_3(x, y, z) &= (x^2 + x)(y^2 + y)(z^2 + z) + x + y + z, \\ f_4(x, y, z) &= (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z) + (x^2 + \omega^2 x)(y^2 + \omega^2 y) \\ &\quad + (x^2 + \omega^2 x)(z^2 + \omega^2 z) + (y^2 + \omega y)(z^2 + \omega z) + x + y + z. \end{aligned}$$

The automorphism group $\Gamma(f_i)$ of $\mathcal{L}(f_i)$ has order $2^{10} \cdot 3^2$, 2^9, $2^{10} \cdot 3$, $2^7 \cdot 3$ and 2^7 for $i = 0, 1, 2, 3, 4$, respectively. Moreover, $\Gamma(f_0)$ and $\Gamma(f_2)$ are transitive on the collection of all incident point-circle pairs; in particular, these groups act transitively on the point set, the set of circles and the set of generators. The automorphism groups of $\mathcal{L}(f_1)$ and $\mathcal{L}(f_4)$ are circle-transitive but not transitive on the sets of generators (and thus not point-transitive); $\Gamma(f_1)$ fixes no generator whereas $\Gamma(f_4)$ fixes two generators. $\Gamma(f_3)$ is neither point- nor circle-transitive but is transitive on the set of generators.

All Laguerre near-planes $\mathcal{L}(f_i)$, except possibly $\mathcal{L}(f_4)$, can be obtained from $\mathcal{L}(f_0)$ by the process of geometric substitution, cf. 4.1, where certain circles of $\mathcal{L}(f_0)$ are replaced by new circles obtained from the old ones by changing their points of intersection with one generator.

By the above theorem there are essentially five interpolating systems of rank 3. Clearly there is only one interpolating system of rank 4. An interpolating system of rank 2 corresponds to an affine plane, so that there is only one interpolating system of rank 2.

With the exception of section 2 we deal with Laguerre near-planes of order 4 exclusively and sometimes omit order 4 when speaking of Laguerre near-planes.

2. Laguerre near-planes of even order

In this section we give an example that shows that a Laguerre near-plane of even order may be extended in more than one way to a Laguerre plane of the same order.

Consider the ovoidal Laguerre plane over an oval $\mathcal{O}$ in $\mathrm{PG}(2, 2^m)$. The tangents of $\mathcal{O}$ pass through a common point ν, the *nucleus of* $\mathcal{O}$, so that $\mathcal{O} \cup \{\nu\}$ becomes a hyperoval; cf. [3, Lemma 12.10] or [2, §8.1]. We can now remove any point of $\mathcal{O} \cup \{\nu\}$ and obtain again an oval. Hence, if we delete a generator from the ovoidal Laguerre over $\mathcal{O}$, we obtain a Laguerre near-plane of order 2^m. But now we can either add the deleted generator or a generator formed from the line through the vertex and the nucleus of $\mathcal{O}$. In both cases we obtain a Laguerre plane. In general, the two Laguerre planes are not isomorphic. Substituting a point of a conic by its nucleus yields a translation oval which is not a conic unless $m \leq 2$. Hence one extension is the Miquelian Laguerre plane whereas another extension is an ovoidal non-Miquelian Laguerre plane. In coordinates, let $\mathbb{F}_{2^m} = \mathrm{GF}(2^m)$ be the Galois field of order 2^m. We consider the following Laguerre near-plane of order 2^m with point set $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, generators being the verticals $\{c\} \times \mathbb{F}_{2^m}$ for $c \in \mathbb{F}_{2^m}$ and circles being of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_{2^m}\}$$

for $a, b, c \in \mathbb{F}_{2^m}$. We call this the *parabola model* since the circles are the graphs of parabolae and lines. We extend this Laguerre near-plane by a generator $\{\infty\} \times \mathbb{F}_{2^m}$. A circle described by $a, b, c \in \mathbb{F}_{2^m}$ as above is adjoined the point (∞, a). This yields the Miquelian Laguerre plane of order 2^m. If we adjoin the point (∞, b) however we obtain an ovoidal non-Miquelian Laguerre plane of order 2^m if $m \geq 3$. (For $m = 1$ or 2 we obtain again the Miquelian Laguerre plane.) Explicitly, let ϕ be the permutation of $(\mathbb{F}_{2^m} \cup \{\infty\}) \times \mathbb{F}_{2^m}$ defined by $\phi(x, y) = (x^2, y)$. A circle

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_{2^m}\} \cup \{(\infty, b)\}$$

is taken under ϕ to

$$\{(u, bu^{2^{m-1}} + au + c) \mid u \in \mathbb{F}_{2^m}\} \cup \{(\infty, b)\}.$$

This gives the familiar representation of the ovoidal Laguerre plane $\mathcal{L}(2^{m-1})$ over the translation oval $\{(x, x^{2^{m-1}}) \mid x \in \mathbb{F}_{2^m}\} \cup \{(\infty)\}$ in the Desarguesian plane over $\mathbb{F}_{2^m}$.

The above example shows that it is possible for a Laguerre near-plane to be extended to two non-isomorphic Laguerre planes. Moreover, it is also possible that two non-isomorphic Laguerre near-planes can be extended to essentially the same Laguerre plane. To see this consider the following Laguerre near-plane of order $q = 2^m$, $m \geq 3$, whose circles are the sets $\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\}$ for $a, b, c \in \mathbb{F}_q$. Adjoining the point (∞, a) to such a circle yields essentially the Laguerre plane $\mathcal{L}(2^{m-1})$ from above. (The map

$$(x,y) \mapsto \begin{cases} (x^2, xy), & \text{if } x \in \mathbb{F}_q, x \neq 0, \\ (\infty, y), & \text{if } x = 0, \\ (0, y), & \text{if } x = \infty, \end{cases}$$

takes the set $\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\} \cup \{(\infty, a)\}$ to the set

$$\{(u, cu^{2^{m-1}} + bu + a) \mid u \in \mathbb{F}_q\} \cup \{(\infty, c)\}$$

so that one obtains the circles of $\mathcal{L}(2^{m-1})$. Note that $x^{q-1} = 1$ for $x \in \mathbb{F}_q, x \neq 0$.) As seen above, this Laguerre plane also is the extension the Laguerre near-plane obtained from the Miquelian Laguerre plane by deleting one generator. However, the two Laguerre near-planes are not isomorphic. Under the map

$$(x,y) \mapsto \begin{cases} (x, xy), & \text{if } x \in \mathbb{F}_q, x \neq 0, \\ (0, y), & \text{if } x = 0, \end{cases}$$

the set $\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\}$ is taken to the set

$$\{(x, bx^2 + cx + a) \mid x \in \mathbb{F}_q, x \neq 0\} \cup \{(0, c)\}$$

so that one almost has an isomorphism if it were not for the points on the generator $\{(0, y) \mid y \in \mathbb{F}_q\}$.

3. Linear Laguerre near-planes of order 4

In the following we look at the case of order 4 in more detail. Every projective plane $\mathcal{P}$ of order 4 is Desarguesian, cf. the remark following Satz 12.4.14 in [5], and every hyperoval in the Desarguesian projective plane of order 4 is obtained from a conic by adjoining its nucleus, see [9]. Furthermore, deleting a point from a hyperoval results in a conic. Since circles of a Laguerre plane not passing through the point p induce parabolic curves in the derived affine plane at p, circles are described by parabolae in the affine plane obtained from $\mathcal{P}$ by removing the line at infinity. However, such a configuration of lines and parabolae does not necessarily yield a Laguerre plane since Lemma 2.8 from [11] does no longer hold. In fact, we do not have to go through the process employed in [11] of extending the traces of circles to ovals or hyperovals etc. for $q = 4$. The point set of an internal incidence structure $\mathcal{I}_p$ at a point p can be identified with $(\mathbb{F}_4 \setminus \{0\}) \times \mathbb{F}_4$ and generators being the verticals $\{c\} \times \mathbb{F}_4$. For each circle its trace in $\mathcal{I}_p$ has three points and determines a unique polynomial of degree at most 2.

3.1 A representation of the circles of a Laguerre near-planes of order 4. One can expand on this idea as follows. The point set of a Laguerre near-planes of order 4 can be identified with $\mathbb{F}_4 \times \mathbb{F}_4$ and generators being the verticals $\{c\} \times \mathbb{F}_4$. Each circle has four points and determines a unique polynomial of degree at most 3, that is, each circle is described by some $(c_3, c_2, c_1, c_0) \in \mathbb{F}_4^4$ as

$$C_{c_3,c_2,c_1,c_0} = \{(x, c_3x^3 + c_2x^2 + c_1x + c_0) \mid x \in \mathbb{F}_4\}.$$

Figure 1 below gives a schematic representation of the generators and the circle $C_{1,\omega,0,1}$ where we write $\mathbb{F}_4$ as $\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$ with $\omega^2 + \omega + 1 = 0$.

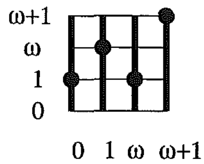

Figure 1

There are 256 vectors in $\mathbb{F}_4^4$ and 64 of them describe circles as above. We denote the collection of all circle describing vectors again by $\mathcal{C}$. In this representation axioms (P), (G) and (C) of a Laguerre near-plane are clearly satisfied. Axiom (J) is equivalent to saying that for any three mutually distinct $x_1, x_2, x_3 \in \mathbb{F}_4$ and any three $y_1, y_2, y_3 \in \mathbb{F}_4$ the system of linear equations

$$(c_3, c_2, c_1, c_0) \cdot \begin{pmatrix} x_1^3 & x_2^3 & x_3^3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{pmatrix} = (y_1, y_2, y_3)$$

has a unique solution (c_3, c_2, c_1, c_0) in $\mathcal{C}$. The kernel of the above matrix is spanned by

$$(1, x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_3x_1, x_1x_2x_3).$$

Since there are four such unordered triples (x_1, x_2, x_3), we obtain four lines through each point of the affine space $\mathbb{F}_4^4$. Then the condition on $\mathcal{C}$ is that it intersects each of these lines in exactly one point. In other words, $\mathcal{C}$ is a hypersurface of $\mathbb{F}_4^4$ that can be represented by a function from $\mathbb{F}_4^3$ to $\mathbb{F}_4$ in each of the four directions given by the above kernels.

3.2. Linear Laguerre near-planes of order 4. A nice class of examples are the *linear Laguerre near-planes of order 4*. In this case, $\mathcal{C}$ is a linear subspace of $\mathbb{F}_4^4$ – in fact, one can look more generally at affine subspaces of $\mathbb{F}_4^4$, that is, $\mathcal{C}$ is of the form

$$\mathcal{C} = \{(c_3, c_2, c_1, c_0) \in \mathbb{F}_4^4 \mid a_3c_3 + a_2c_2 + a_1c_1 + a_0c_0 = b\}$$

for some $a_3, a_2, a_1, a_0, b \in \mathbb{F}$, $(a_3, a_2, a_1, a_0) \neq (0,0,0,0)$, but we shall see that nothing is gained in so doing. (The linear Laguerre near-planes are those with $b = 0$.) For $\mathcal{C}$ to be a hypersurface as described above we have to require that (a_3, a_2, a_1, a_0) is linearly independent with any three of the four vectors $(x^3, x^2, x, 1)$ for $x \in \mathbb{F}_4$, or equivalently,

$$a_3 + (x_1 + x_2 + x_3)a_2 + (x_1x_2 + x_2x_3 + x_3x_1)a_1 + x_1x_2x_3a_0 \neq 0$$

for any three mutually distinct $x_1, x_2, x_3 \in \mathbb{F}_4$, that is, the entries of

$$(a_3, a_2, a_1, a_0) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \omega & \omega+1 \\ 0 & 1 & \omega+1 & \omega \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are all non-zero. There are 81 such vectors. Furthermore, because (a_3, a_2, a_1, a_0) and $\lambda(a_3, a_2, a_1, a_0)$ for $\lambda \in \mathbb{F}$, $\lambda \neq 0$, describe the same linear Laguerre near-plane, we have at most 27 potentially different linear Laguerre near-planes of order 4. These are described by the following vectors which we normalized so that they have a leading 1.

	a_3	a_2	a_1	a_0		a_3	a_2	a_1	a_0
1	0	0	1	1	2	0	0	1	ω
3	0	0	1	$\omega+1$	4	0	1	0	1
5	0	1	0	ω	6	0	1	0	$\omega+1$
7	1	0	0	0	8	1	0	0	ω
9	1	0	0	$\omega+1$	10	1	1	ω	0
11	1	1	ω	ω	12	1	1	ω	$\omega+1$
13	1	1	$\omega+1$	0	14	1	1	$\omega+1$	ω
15	1	1	$\omega+1$	$\omega+1$	16	1	ω	1	0
17	1	ω	1	ω	18	1	ω	1	$\omega+1$
19	1	ω	ω	0	20	1	ω	ω	ω
21	1	ω	ω	$\omega+1$	22	1	$\omega+1$	1	0
23	1	$\omega+1$	1	ω	24	1	$\omega+1$	1	$\omega+1$
25	1	$\omega+1$	$\omega+1$	0	26	1	$\omega+1$	$\omega+1$	ω
27	1	$\omega+1$	$\omega+1$	$\omega+1$					

The parabola model of a Laguerre near-plane of order 4 is of this form. More precisely, it can be obtained for $(a_3, a_2, a_1, a_0) = (1,0,0,0)$ and $b = 0$. In fact, all affine subspaces essentially yield the same model as we shall see in the following section.

3.3. Isomorphisms of Laguerre near-planes of order 4. In order to make the last statement precise we have to look at isomorphisms of Laguerre near-planes. The obvious definition of an isomorphism between Laguerre near-planes of the same order is that we have a bijection between the point sets that takes generators to generators and circles to circles. Using the representation 3.1 for Laguerre near-planes of order 4, every isomorphism is of the form

$$\mathbb{F}_4^4 \to \mathbb{F}_4^4 : (x, y) \mapsto (\alpha(x), \beta_x(y))$$

where α and β_x are permutations of $\mathbb{F}_4$ for each $x \in \mathbb{F}_4$. The collection of all permutations of $\mathbb{F}_4 \times \mathbb{F}_4$ as above forms a group Γ of order $24^4 = 2^{15} \cdot 3^5$.

Clearly, the group of permutations of $\mathbb{F}_4$ is the symmetric group S_4. Every even permutation can be written as $x \mapsto ax + b$ for some $a, b \in \mathbb{F}_4$, $a \neq 0$. The automorphism $x \mapsto x^2$ of $\mathbb{F}_4$ is an odd permutation of $\mathbb{F}_4$ – in fact, a transposition – and every odd permutation of $\mathbb{F}_4$ is of the form $x \mapsto ax^2 + b$ for some $a, b \in \mathbb{F}_4$. We give a set of generators for Γ as permutations of $\mathbb{F}_4 \times \mathbb{F}_4$ and determine how circles are transformed.

(1) $(x, y) \mapsto (x, y + t_3x^3 + t_2x^2 + t_1x + t_0)$ for $t_3, t_2, t_1, t_0 \in \mathbb{F}_4$. These permutations take C_{c_3,c_2,c_1,c_0} to $C_{c_3+t_3,c_2+t_2,c_1+t_1,c_0+t_0}$.

(2) $(x, y) \mapsto (x + t, y)$ for $t \in \mathbb{F}_4$. These permutations take C_{c_3,c_2,c_1,c_0} to C_{d_3,d_2,d_1,d_0} where

$$(d_3, d_2, d_1, d_0) = (c_3, c_2, c_1, c_0) \cdot \begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 0 & t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) $(x, y) \mapsto (rx, y)$ for $r \in \mathbb{F}_4$, $r \neq 0$. These permutations take C_{c_3,c_2,c_1,c_0} to C_{c_3,rc_2,r^2c_1,c_0}.

(4) $(x, y) \mapsto (x^2, y)$. This permutation takes C_{c_3,c_2,c_1,c_0} to C_{c_3,c_1,c_2,c_0}.

(5) $(x, y) \mapsto \begin{cases} (x, y), & \text{if } x \neq u, \\ (x, ry), & \text{if } x = u, \end{cases}$ for $r, u \in \mathbb{F}_4$, $r \neq 0$.

In order to describe how these permutations transform circles let x_1, x_2, x_3, x_4 be the four elements of $\mathbb{F}_4$. Each polynomial

$$p_{x_4}(X) = (X - x_1)(X - x_2)(X - x_3)$$

vanishes at x_1, x_2 and x_3 and has value 1 at x_4 because $(x_4-x_1)(x_4-x_2)(x_4-x_3)$ equals the product of all non-zero elements in $\mathbb{F}_4$. Expanding we obtain the coefficient-vector $(1, x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_3x_1, x_1x_2x_3)$ which is the same as the spanning vector for the kernel from above. Explicitly, we have the following four polynomials

$$\begin{aligned} p_0(X) &= X^3 + 1, \\ p_1(X) &= X^3 + X^2 + X, \\ p_\omega(X) &= X^3 + \omega X^2 + (\omega + 1)X, \\ p_{\omega+1}(X) &= X^3 + (\omega + 1)X^2 + \omega X. \end{aligned}$$

Using these four polynomials, the above permutation takes the points on the circle C_{c_3,c_2,c_1,c_0} to the points $(x, c_3x^3 + c_2x^2 + c_1x + c_0)$ for $x \neq u$ and $(u, r(c_3u^3+c_2u^2+c_1u+c_0))$ whose interpolating polynomial is $c_3X^3+c_2X^2+c_1X + c_0 + (r+1)(c_3u^3 + c_2u^2 + c_1u + c_0)p_u(X)$. Expanding we find that C_{c_3,c_2,c_1,c_0} is taken to C_{d_3,d_2,d_1,d_0} where $(d_3, d_2, d_1, d_0) = (c_3, c_2, c_1, c_0) \cdot M$ with

$$M = \begin{pmatrix} 1+(r+1)u^3 & (r+1)u & (r+1)u^2 & 0 \\ (r+1)u^2 & 1+(r+1)u^3 & (r+1)u & 0 \\ (r+1)u & (r+1)u^2 & 1+(r+1)u^3 & 0 \\ r+1 & (r+1)u & (r+1)u^2 & 1 \end{pmatrix}.$$

(6) $(x,y) \mapsto \begin{cases} (x,y), & \text{if } x \neq u, \\ (x,y^2), & \text{if } x = u, \end{cases}$ for $u \in \mathbb{F}_4$. Following the same path as for isomorphisms of type (5) above we see that a circle C_{c_3,c_2,c_1,c_0} is taken to C_{d_3,d_2,d_1,d_0} where

$$(d_3, d_2, d_1, d_0) = (c_3, c_2, c_1, c_0) + \sum_{i=0}^{3} (c_i^2 u^{2i} + c_i u^i)(p_u^3, p_u^2, p_u^1, p_u^0)$$

and $p_u^3, p_u^2, p_u^1, p_u^0$ are the coeffiecients of the polynomial $p_u(X)$, i.e., $p_u(X) = p_u^3X^3 + p_u^2X^2 + p_u^1X + p_u^0$.

We return to linear Laguerre near-planes. The permutation

$$(x, y) \mapsto (x, y + t_3x^3 + t_2x^2 + t_1x + t_0),$$

$t_3, t_2, t_1, t_0 \in \mathbb{F}_4$, of type (1) takes $\mathcal{C}$ determined by $a_3, a_2, a_1, a_0, b \in \mathbb{F}$ to one with parameters $a_3, a_2, a_1, a_0, b + a_3t_3 + a_2t_2 + a_1t_1 + a_0t_0$. This shows that every affine Laguerre near-plane of order 4 is isomorphic to one with $b = 0$. For the remainder of this section we always make this assumption, i.e., we only consider linear Laguerre near-planes, and we denote the linear Laguerre near-plane with parameters a_3, a_2, a_1, a_0 by $\mathcal{L}(a_3, a_2, a_1, a_0)$. From the action of generators of Γ on circles as found above we readily obtain the following.

(1) Permutations of type (1) where $a_3t_3+a_2t_2+a_1t_1+a_0t_0 = 0$ take the Laguerre near-plane $\mathcal{L}(a_3, a_2, a_1, a_0)$ to itself.
(2) Permutations or type (2) take $\mathcal{L}(a_3, a_2, a_1, a_0)$ to $\mathcal{L}(b_3, b_2, b_1, b_0)$ where

$$(b_3, b_2, b_1, b_0) = (a_3, a_2, a_1, a_0) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2 & 0 & 1 & 0 \\ t^3 & t^2 & t & 1 \end{pmatrix}.$$

(3) Permutations or type (3) take $\mathcal{L}(a_3, a_2, a_1, a_0)$ to $\mathcal{L}(a_3, r^2a_2, ra_1, a_0)$.
(4) The permutation of type (4) takes $\mathcal{L}(a_3, a_2, a_1, a_0)$ to $\mathcal{L}(a_3, a_1, a_2, a_0)$.

(5) Permutations of type (5) take $\mathcal{L}(a_3, a_2, a_1, a_0)$ to $\mathcal{L}(a_3, a_2, a_1, \frac{1}{r}(a_3+a_0)+a_3)$ for $u = 0$ and to $\mathcal{L}(b_3, b_2, b_1, b_0)$ where $(b_3, b_2, b_1, b_0) = \frac{1}{r}(a_3, a_2, a_1, a_0) \cdot M$ with

$$M = \begin{pmatrix} 1 & (r+1)u^2 & (r+1)u & r+1 \\ (r+1)u & 1 & (r+1)u^2 & (r+1)u \\ (r+1)u^2 & (r+1u) & 1 & (r+1)u^2 \\ 0 & 0 & 0 & r \end{pmatrix}$$

for $u = 1, \omega, \omega+1$.

(6) Permutations of type (6) do not take ($\mathbb{F}_4$-) linear subspaces of $\mathbb{F}_4^4$ to linear subspaces. (They do however permute $\mathbb{F}_2$-linear subpaces where $\mathbb{F}_2 = \{0, 1\}$ is the field of order 2.) We therefore can ignore them as far as linear Laguerre near-planes are concerned.

Applying the above tranformations and combinations thereof to $\mathcal{L}(1, 0, 0, 0)$ direct computation shows that all possible 27 linear Laguerre near-planes occur. For example,

$$\begin{array}{cccccccccccccccc}
25 & \xrightarrow{3} & 10 & \xrightarrow{2} & 22 & \xrightarrow{3} & 19 & \xrightarrow{3} & 13 & \xrightarrow{5} & 14 & \xrightarrow{4} & 23 & \xrightarrow{3} \\
20 & \xrightarrow{2} & 9 & \xrightarrow{5} & 8 & \longrightarrow & 18 & \xrightarrow{5} & 16 & \xrightarrow{5} & 17 & \xrightarrow{2} & 3 & \xrightarrow{3} \\
2 & \xrightarrow{4} & 5 & \xrightarrow{2,\lambda} & 15 & \xrightarrow{4} & 24 & \xrightarrow{3} & 21 & \xrightarrow{2,\lambda} & 1 & \xrightarrow{4} & 4 & \xrightarrow{3,\lambda} \\
6 & \xrightarrow{2,\lambda} & 11 & \xrightarrow{3} & 26 & \xrightarrow{5} & 27 & \xrightarrow{5} & 12 & \xrightarrow{5} & 7 & & &
\end{array}$$

where the numbers refer to the vectors as listed in 3.2 and the the numbers above the arrows refer to the type of permutation used; furthermore, λ indicates that one has to multiply by some non-zero element of $\mathbb{F}_4$ the obtain the respective vector with a leading 1. Hence, we have the following result.

Proposition 3.4. *Every linear Laguerre near-plane of order 4 is isomorphic to the Laguerre near-plane $\mathcal{L}(1, 0, 0, 0)$ obtained from the Miquelian Laguerre plane of order 4 by deleting one generator.*

4. A representation of Laguerre near-planes of order 4

As for models of Laguerre near-planes of order 4 that are not isomorphic to $\mathcal{L}(1, 0, 0, 0)$ we begin with a closer description of circles of $\mathcal{L}(1, 0, 0, 0)$. They are of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_4\}$$

for $a, b, c \in \mathbb{F}_4$ and they fall into three classes. First, there are the graphs of the four constant polynomials obtained for $a = b = 0$. Then there are the graphs of the 24 permutation polynomials obtained for $a = 0$, $b \neq 0$ and $a \neq 0$, $b = 0$. Third, there are the graphs of the remaining 36 polynomials obtained for $a, b \neq 0$; these polynomials take on exactly two values and each of these values occurs exactly twice.

Note that the same picture emerges if we delete a different generator from the Miquelian Laguerre plane of order 4 because the automorphism group of this plane is transitive on the point set.

Example 4.1. We now modify the above model to obtain a new Laguerre near-plane of order 4. To this end, we consider the circles that are entirely contained in

$$S = \mathbb{F}_4 \times \{\omega, \omega + 1\},$$

see the shaded areas in Figure 2 where $\mathbb{F}_4 = \{0, 1, \omega, \omega+1\}$ as before and $\omega^2+\omega+1 = 0$. There are 8 such circles, two of the first kind and six of the third kind, see Figure 2 for a schematic representation of these circles.

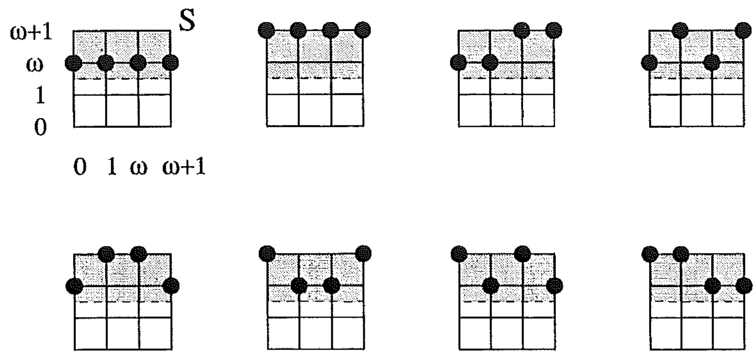

Figure 2

These 8 circles cover 32 *admissible* triples of points, that is, triples of points such that no two of the points are on the same generator. We now replace these circles by 8 new circles covering the same 32 admissible triples of points. From this property it will be clear that we again obtain a Laguerre near-plane of order 4. The new circles are obtained as the images of the 8 old circles under the map

$$\phi : (x, y) \mapsto \begin{cases} (x, y), & \text{if } x \neq 0, \\ (x, y^2), & \text{if } x = 0; \end{cases}$$

that is, the points $(0, \omega)$ and $(0, \omega + 1)$ are swapped and all other points remain unchanged, see Figure 3 for a schematic representation of these new circles.

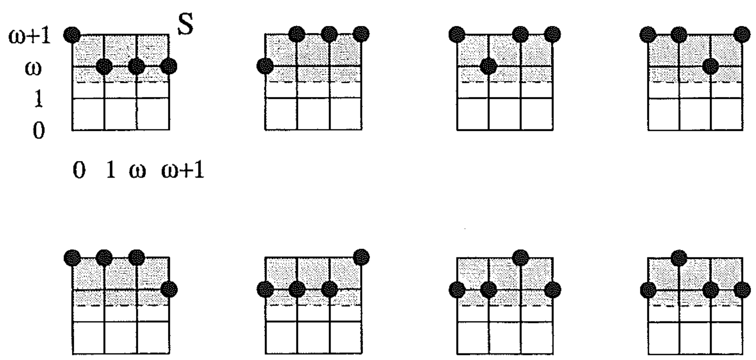

Figure 3

However, since we still have circles of all three types and the new circles, this Laguerre near-plane cannot be obtained from a Laguerre plane of order 4 by deleting one generator.

In terms of the representation developed in 3.2 we replace the eight polynomials

$$\begin{array}{ll} \omega, & \omega+1, \\ X^2+X+\omega, & \omega X^2+(\omega+1)X+\omega, \\ (\omega+1)X^2+\omega X+\omega, & (\omega+1)X^2+\omega X+\omega+1, \\ \omega X^2+(\omega+1)X+\omega+1, & X^2+X+\omega+1 \end{array}$$

(in the order of Figure 2) by the cubic polynomials

$$\begin{array}{ll} X^3+\omega+1, & X^3+\omega, \\ X^3+X^2+X+\omega+1, & X^3+\omega X^2+(\omega+1)X+\omega+1, \\ X^3+(\omega+1)X^2+\omega X+\omega+1, & X^3+(\omega+1)X^2+\omega X+\omega, \\ X^3+\omega X^2+(\omega+1)X+\omega, & X^3+X^2+X+\omega, \end{array}$$

respectively, obtained from the former polynomials by adding the polynomial X^3+1. Obviously, the collection of these eight cubic polynomials plus the remaining polynomials of degree at most 2 is not closed under addition, i.e., it is no $\mathbb{F}_2$-linear subspace of $\mathbb{F}_4^4$. Therefore this Laguerre near-plane cannot be isomorphic to $\mathcal{L}(1,0,0,0)$ since every element of the group Γ preserves $\mathbb{F}_2$-linearity, see the list in 3.3.

Cearly, the above construction generalises as follows, see also the example in [10, Section 4]. Let U and V be two 2-subsets of $\mathbb{F}_4$ and let U' and V' be the complements of U and V, respectively. Let $W=V$ or $W=V'$. We then replace all the 8 circles that are entirely contained in $S=U\times V\cup U'\times W$ by the circles obtained thereof by swapping two points in S on one fixed generator. We can even repeat the construction for different sets U and V as long as eight circles are contained in S and obtain a Laguerre near-plane of order 4 in each step. In fact, since we no longer have graphs of polynomials of degree at most 2, we can apply this construction to any subset $S\subset\mathbb{F}_4\times\mathbb{F}_4$ that contains exactly two points on each generator provided that exactly eight circles are entirely contained in it. When applying the method just described to a Laguerre near-plane $\mathcal{L}_1$ once or several times and obtaining a Laguerre near-plane $\mathcal{L}_2$, we say that $\mathcal{L}_2$ *is obtained from* $\mathcal{L}_1$ *by geometric substitution.*

4.2. A representation of Laguerre near-planes of order 4 in terms a single map. We use the description of $\mathcal{C}$ as a hypersurface of $\mathbb{F}_4^4$ developed in 3.1. In order to simplify the notation let

$$T=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & \omega & \omega+1 & 0 \\ 1 & \omega+1 & \omega & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ be the standard basis vectors of $\mathbb{F}_4$. Then $\mathcal{C}' = \mathcal{C} \cdot T^{-1}$, where

$$T^{-1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & \omega+1 & \omega & 0 \\ 1 & \omega & \omega+1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

is a hypersurface of $\mathbb{F}_4^4$ which intersects each line $v + \mathbb{F}_4 e_i$ for $i = 1,2,3,4$, $v \in \mathbb{F}_4^4$, in exactly one point. Hence, we can write $\mathcal{C}'$ in the form

$$\mathcal{C}' = \{(x,y,z,f(x,y,z)) \mid x,y,z \in \mathbb{F}_4\}$$

for some map $f : \mathbb{F}_4^3 \to \mathbb{F}_4$. Furthermore, for fixed $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $f_{y_0,z_0} : x \mapsto f(x,y_0,z_0)$, $f_{x_0,z_0} : y \mapsto f(x_0,y,z_0)$ and $f_{x_0,y_0} : z \mapsto f(x_0,y_0,z)$ are permutations of $\mathbb{F}_4$. Conversely, every map f with this property describes a modified circle set $\mathcal{C}'$ and thus defines a Laguerre near-plane of order 4. We denote this plane by $\mathcal{L}(f)$.

With every map $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ we can associate a unique polynomial in X, Y and Z of degree at most 3 in each of the three variables, i.e.,

$$f(X,Y,Z) = \sum_{i,j,k=0}^{3} a_{i,j,k} X^i Y^j Z^k$$

for some $a_{i,j,k} \in \mathbb{F}_4$. Each of the above restricted maps then is described by a polynomial in one variable of degree at most 3. However, there are only a few such polynomials that define permutations of $\mathbb{F}_4$.

Lemma 4.3. *A polynomial $p(X) = \sum_{i=0}^{3} a_i X^i$ of degree at most 3 over $\mathbb{F}_4$ defines a permutation of $\mathbb{F}_4$ if and only if $a_3 = 0$ and either $a_2 = 0$ or $a_1 = 0$. The latter condition is equivalent to $a_2^3 + a_1^3 = 1$.*

Proof. The polynomial $p(X)$ defines a permutation of $\mathbb{F}_4$ if and only if the evaluation map $p : \mathbb{F}_4 \to \mathbb{F}_4 : x \mapsto p(x)$ is one-to-one. Since translations and homotheties are permutations, we may assume that the leading coefficient of $p(X)$ is 1 and that the constant term equals 0. Suppose that $p(X)$ has degree 3 so that $p(X) = X^3 + a_2X^2 + a_1X$. If $a_1 = 0$, then $p(0) = p(a_2) = 0$ and p is not injective for $a_2 \neq 0$. If $a_2 = a_1 = 0$, then $p(1) = p(w) = 1$ and again p is not injective. We now assume that $a_1 \neq 0$. Since $p(X)$ defines a permutation of $\mathbb{F}_4$ if and only if $p(\frac{1}{a_1}X)$ defines a permutation of $\mathbb{F}_4$, we may assume that $a_1 = 1$ so that $p(X) = X^3 + a_2X^2 + X$. But then $p(1) = p(a_2) = a_2$ and p is not injective. This proves that $p(X)$ has degree at most 2, i.e., $a_3 = 0$.

Suppose that $p(X)$ has degree 2. As before we may assume that $a_2 = 1$ and $a_0 = 0$. Then $p(0) = p(a_1) = 0$ and p is not injective for $a_1 \neq 0$. Clearly, $x \mapsto x^2$ is a permutation of $\mathbb{F}_4$ so that a quadratic polynomial defines a permutation of $\mathbb{F}_4$ if and only if the linear term equals 0.

Since $u^3 = 0$ or 1 for $u = 0$ or $u \neq 0$, respectively, it readily follows that $u^3 + v^3 = 1$ for $u, v \in \mathbb{F}_4$ if and only if either $u = 0$, $v \neq 0$ or $u \neq 0$, $v = 0$. □

Note that maps of the form $x \mapsto a_2x^2 + a_1x$ for $a_2, a_1 \in \mathbb{F}_4$ are additive, that is, they are linear of $\mathbb{F}_2$. Hence each such permutation of $\mathbb{F}_4$ represents an element of $\mathrm{GL}(2,2)$, the group of all invertible 2×2 matrices over the field $\mathbb{F}_2$. This group has order 6 and obviously the maps for $a_2 \neq 0$, $a_1 = 0$ and $a_2 = 0$, $a_1 \neq 0$ belong to it. Therefore we must have already covered all permutations of this form.

From this point of view one further obtains the inverse of $x \mapsto a_2x^2 + a_1x$ for $a_2, a_1 \in \mathbb{F}_4$, $a_2^3 + a_1^3 = 1$, in closed form. Let

$$a_1x + a_2x^2 = u;$$

then

$$a_2^2x + a_1^2x^2 = u^2.$$

In matrix notation we have

$$\begin{pmatrix} a_1 & a_2 \\ a_2^2 & a_1^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ x^2 \end{pmatrix} = \begin{pmatrix} u \\ u^2 \end{pmatrix}.$$

Since the coefficient matrix of this system of linear equations has determinant $a_1^3 + a_2^3 = 1$, one finds

$$x = a_1^2u + a_2u^2.$$

We now consider the partial map $z \mapsto f(x, y, z)$ which is described by a polynomial in Z. By the preceding lemma the coefficient of Z^3 must be 0. Therefore

$$\sum_{i,j=0}^{3} a_{i,j,3}x^iy^j = 0$$

for all $x, y \in \mathbb{F}_4$. Hence the polynomial $p(X,Y) = \sum_{i,j=0}^{3} a_{i,j,3}X^iY^j$ vanishes identically. This implies $a_{i,j,3} = 0$ for all i and j. Considering the other partial maps, we similarly find that $a_{i,3,k} = a_{3,j,k} = 0$ for all i, j and k from 0 to 3. Therefore $f(X,Y,Z)$ reduces to $f(X,Y,Z) = \sum_{i,j,k=0}^{2} a_{i,j,k}X^iY^jZ^k$ for some $a_{i,j,k} \in \mathbb{F}_4$.

Furthermore,

$$\Big(\sum_{i,j=0}^{2} a_{i,j,2}x^iy^j\Big)^3 + \Big(\sum_{i,j=0}^{2} a_{i,j,1}x^iy^j\Big)^3 = 1$$

for all $x, y \in \mathbb{F}_4$. In particular, for $x = y = 0$ we obtain $a_{0,0,2}^3 + a_{0,0,1}^3 = 1$ so that either $a_{0,0,2} = 0$ or $a_{0,0,1} = 0$ and the respective other term being non-zero.

By looking at the other partial maps we obtain the following characterisation.

Proposition 4.4. *$f(X,Y,Z)=\sum_{i,j,k=0}^{2} a_{i,j,k}X^iY^jZ^k$ describes a Laguerre near-plane if and only if*

$$\begin{aligned}(\sum_{i,j=0}^{2} a_{i,j,2}x^iy^j)^3+(\sum_{i,j=0}^{2} a_{i,j,1}x^iy^j)^3&=1\\(\sum_{i,k=0}^{2} a_{i,2,k}x^iz^k)^3+(\sum_{i,k=0}^{2} a_{i,1,k}x^iz^k)^3&=1\\(\sum_{j,k=0}^{2} a_{2,j,k}y^jz^k)^3+(\sum_{j,k=0}^{2} a_{1,j,k}y^jz^k)^3&=1\end{aligned}$$

for all $x,y,z\in\mathbb{F}_4$. In particular, $a_{0,0,2}^3+a_{0,0,1}^3=a_{0,2,0}^3+a_{0,1,0}^3=a_{2,0,0}^3+a_{1,0,0}^3=1$.

It seems that the corresponding polynomial identities cannot be algebraically used in general to simplify the form of f a great deal although we shall come back to them later on. However note that the above conditions do not involve the coefficient $a_{0,0,0}$. In fact, up to isomorphism, we can always assume that $a_{0,0,0}=0$, see 5.1.

Corollary 4.5. *The inverses of the partial maps with respect to x, y and z are given by*

$$\begin{aligned}f_{y,z}^{-1}(x)=&f_2^x(y,z)x^2+f_1^x(y,z)^2x+f_0^x(y,z)^2f_2^x(y,z)+f_0^x(y,z)f_1^x(y,z)^2,\\f_{x,z}^{-1}(y)=&f_2^y(x,z)y^2+f_1^y(x,z)^2y+f_0^y(x,z)^2f_2^y(x,z)+f_0^y(x,z)f_1^y(x,z)^2,\\f_{x,y}^{-1}(z)=&f_2^z(x,y)z^2+f_1^z(x,y)^2z+f_0^z(x,y)^2f_2^z(x,y)+f_0^z(x,y)f_1^z(x,y)^2,\end{aligned}$$

respectively, where $f_2^x(y,z)$, $f_1^x(y,z)$, $f_0^x(y,z)$, $f_2^y(x,z)$, $f_1^y(x,z)$, $f_0^y(x,z)$, $f_2^z(x,y)$, $f_1^z(x,y)$ and $f_0^z(x,y)$ are the respective coefficient functions, i.e.,

$$\begin{aligned}f(x,y,z)=&f_2^x(y,z)x^2+f_1^x(y,z)x+f_0^x(y,z)\\=&f_2^y(x,z)y^2+f_1^y(x,z)y+f_0^y(x,z)\\=&f_2^z(x,y)z^2+f_1^z(x,y)z+f_0^z(x,y).\end{aligned}$$

Proof. Let $f(x,y,z)=f_2^z(x,y)z^2+f_1^z(x,y)z+f_0^z(x,y)$. We can write the inverse $f_{x,y}$ of the partial map with respect to z in the form $f_{x,y}(z)=g_2(x,y)z^2+g_1(x,y)z+g_0(x,y)$. Expanding the identity $f_{x,y}(f(x,y,z))=z$ one finds

$$\begin{aligned}g_2(x,y)f_1^z(x,y)^2+g_1(x,y)f_2^z(x,y)&=0,\\g_2(x,y)f_2^z(x,y)^2+g_1(x,y)f_1^z(x,y)&=1,\\g_2(x,y)f_0^z(x,y)^2+g_1(x,y)f_0^z(x,y)+g_0(x,y)&=0.\end{aligned}$$

This is a system of linear equations for $g_2(x,y)$, $g_1(x,y)$ and $g_0(x,y)$. Since its determinant is $f_2^z(x,y)^3 + f_1^z(x,y)^3 = 1$ by Proposition 4.5, this system has a unique solution and one readily finds

$$\begin{aligned}
g_2(x,y) &= f_2^z(x,y),\\
g_1(x,y) &= f_1^z(x,y)^2,\\
g_0(x,y) &= f_0^z(x,y)^2 f_2^z(x,y) + f_0^z(x,y) f_1^z(x,y)^2.
\end{aligned}$$

The inverses of the other partial maps are found likewise. □

Examples 4.6.

(1) The linear Laguerre near-plane $\mathcal{L}(1,0,0,0)$ is represented in the form $\mathcal{L}(f)$ with $f(x,y,z) = x+y+z$.

(2) Let $f(x,y,z) = (x^2+x)(z^2+z)+x+y+z$. Then f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto (w^2+w+y^2+y)(z^2+z)+w+y+z^2$, $w \mapsto (x^2+x)(z^2+z)+w+x+z$, $w \mapsto (w^2+w+y^2+y)(x^2+x)+w+x^2+y$, respectively, where $w = f(x,y,z)$.

(3) Let $f(x,y,z) = (x^2+x)(y^2+y)+(y^2+y)(z^2+z)+(x^2+x)(z^2+z)+x+y+z$. Then f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto f(w,y^2,z^2)$, $w \mapsto f(x^2,w,z^2)$, $w \mapsto f(x^2,y^2,w)$, respectively, where $w = f(x,y,z)$.

(4) In order to represent the Laguerre near-plane in 4.1 by a function f one computes the transforms of the eight polynomials of degree at most 2 and of the eight cubic polynomials they are replaced with. One finds that adding $(0,0,0,1)$ brings one from one set to the other, that is, f is obtained from $(x,y,z) \mapsto x+y+z$ by adding a function $g(x,y,z)$ that vanishes on all the vectors corresponding to the polynomials not replaced and has constant value 1 on the eight vectors corresponding to the polynomials that are replaced. The latter eight vectors are (u,v,w) where $u,v,w \in \{\omega, \omega+1\}$, i.e. $(u,v,w,u+v+w)$ is the transform of a coefficient vector of a poynomial that is replaced. Now $g(x,y,z)$ can be found as

$$\begin{aligned}
g(x,y,z) =& p_\omega(x)p_\omega(y)p_\omega(z) + p_\omega(x)p_\omega(y)p_{\omega+1}(z)\\
&+ p_\omega(x)p_{\omega+1}(y)p_\omega(z) + p_\omega(x)p_{\omega+1}(y)p_{\omega+1}(z)\\
&+ p_{\omega+1}(x)p_\omega(y)p_\omega(z) + p_{\omega+1}(x)p_\omega(y)p_{\omega+1}(z)\\
&+ p_{\omega+1}(x)p_{\omega+1}(y)p_\omega(z) + p_{\omega+1}(x)p_{\omega+1}(y)p_{\omega+1}(z)\\
=&(p_\omega(x)+p_{\omega+1}(x))(p_\omega(y)+p_{\omega+1}(y))(p_\omega(z)+p_{\omega+1}(z))\\
=&(x^2+x)(y^2+y)(z^2+z)
\end{aligned}$$

Therefore

$$f(x,y,z) = (x^2+x)(y^2+y)(z^2+z)+x+y+z.$$

Again, it is easily verified that f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto f(w,y,z)$, $w \mapsto f(x,w,z)$, $w \mapsto f(x,y,w)$, respectively, where $w = f(x,y,z)$.

5. The Classification

We return to the permutations listed in 3.3 and examine how they transform a Laguerre near-plane $\mathcal{L}(f)$ given by a map f as in 4.2.

5.1 Isomorphisms and normal form. Using the formulae for the action of generators of Γ as given in 3.3 we 'conjugate' by the transformation matrix T from 4.2 to find how a Laguerre near-plane $\mathcal{L}(f)$ is transformed. (For example, in types (1) to (5) the coefficients are linearly transformed by a matrix S, then $(x, y, z, f(x, y, z))$ is taken to $(x, y, z, f(x, y, z))TST^{-1}$. This gives us the new map f is transformed into.) One obtains the following.

(1) $(x, y) \mapsto (x, y + t_3x^3 + t_2x^2 + t_1x + t_0)$ for $t_3, t_2, t_1, t_0 \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where $f'(x, y, z) = f(x + s_3, y + s_2, z + s_1) + s_0$ with

$$\begin{aligned}(s_3, s_2, s_1, s_0) &= (t_3, t_2, t_1, t_0) \cdot T^{-1} \\ &= (t_3 + t_2 + t_1 + t_0, t_3 + (\omega + 1)t_2 + \omega t_1 + t_0, \\ &\qquad t_3 + \omega t_2 + (\omega + 1)t_1 + t_0, t_0).\end{aligned}$$

Writing $p(x) = t_3x^3 + t_2x^2 + t_1x + t_0$ we have $p(0) = s_0$, $p(1) = s_3$, $p(\omega) = s_2$ and $p(\omega + 1) = s_1$. Hence $(x, y) \mapsto (x, y + p(x))$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where $f'(x, y, z) = f(x + p(1), y + p(\omega), z + p(\omega + 1)) + p(0)$. In particular, the permutation

$$(x, y) \mapsto (x, y + tp_u(x)) = \begin{cases} (x, y), & \text{if } x \neq u, \\ (x, y + t), & \text{if } x = u, \end{cases}$$

for $t, u \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$\begin{aligned} f'(x, y, z) &= f(x + tp_u(1), y + tp_u(\omega), z + tp_u(\omega + 1)) + tp_u(0) \\ &= \begin{cases} f(x, y, z) + t, & \text{if } u = 0, \\ f(x + t, y, z), & \text{if } u = 1, \\ f(x, y + t, z), & \text{if } u = \omega, \\ f(x, y, z + t), & \text{if } u = \omega + 1. \end{cases}\end{aligned}$$

(2) $(x, y) \mapsto (x + t, y)$ for $t \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where $f' = f$ for $t = 0$ and f' is an inverse of a partial map of f with the other two variables exchanged given by $f'(f(x, y, z), z, y) = x$, $f'(z, f(x, y, z), x) = y$ and $f'(y, x, f(x, y, z)) = z$ for $t = 1$, ω and $\omega + 1$, respectively, that is, the maps $(x, y, z) \mapsto f^{-1}_{z,y}(x)$, $(x, y, z) \mapsto f^{-1}_{z,x}(y)$ and $(x, y, z) \mapsto f^{-1}_{y,x}(z)$, respectively.

(3) $(x, y) \mapsto (rx, y)$ for $r \in \mathbb{F}_4$, $r \neq 0$, takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x, y, z) = \begin{cases} f(x, y, z), & \text{if } r = 1, \\ f(y, z, x), & \text{if } r = \omega, \\ f(z, x, y), & \text{if } r = \omega + 1. \end{cases}$$

(4) $(x,y) \mapsto (x^2,y)$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where $f'(x,y,z) = f(x,z,y)$.

(5) $(x,y) \mapsto \begin{cases} (x,y), & \text{if } x \neq u, \\ (x,ry), & \text{if } x = u, \end{cases}$ for $r,u \in \mathbb{F}_4$, $r \neq 0$, takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} rf(x,y,z), & \text{if } u = 0, \\ f(r^2x,y,z), & \text{if } u = 1, \\ f(x,r^2y,z), & \text{if } u = \omega, \\ f(x,y,r^2z), & \text{if } u = \omega + 1. \end{cases}$$

(6) $(x,y) \mapsto \begin{cases} (x,y), & \text{if } x \neq u, \\ (x,y^2), & \text{if } x = u, \end{cases}$ for $u \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} f(x,y,z)^2, & \text{if } u = 0, \\ f(x^2,y,z), & \text{if } u = 1, \\ f(x,y^2,z), & \text{if } u = \omega, \\ f(x,y,z^2), & \text{if } u = \omega + 1. \end{cases}$$

Note that permutations of type (1) can be used to yield a map that takes (0,0,0) to 0 whereas permutations of type (3) and (4) allow us to obtain any permutation of the coordinates x, y and z. Permutations of type (5) and (6) allow us to replace any of the coordinates by a fixed multiple or by its square, respectively. This can be applied to obtain some normalizations for some of the coefficients of f. In particular, applying an isomorphism of type (1), we can achieve that $a_{0,0,0} = 0$. This means that $C_{0,0,0,0}$ is then a circle in our Laguerre near-plane. Furthermore, since $a_{0,0,2}^3 + a_{0,0,1}^3 = a_{0,2,0}^3 + a_{0,1,0}^3 = a_{2,0,0}^3 + a_{1,0,0}^3 = 1$, see 4.4, we can use isomorphisms of type (6), if necessary, to achieve $a_{0,0,2} = a_{0,2,0} = a_{2,0,0} = 0$. Finally, using isomorphisms of type (5), if necessary, we can further assume that $a_{0,0,1} = a_{0,1,0} = a_{1,0,0} = 1$. Then $f(x,0,0) = x$, $f(0,y,0) = y$ and $f(0,0,z) = z$ for all $x,y,z \in \mathbb{F}_4$. We say that f is in *normal form* if the above identities are satisfied. All examples 4.6 are in normal form. With this notation we have proved the following.

Proposition 5.2. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to a Laguerre near-plane $\mathcal{L}(f')$ where f' is in normal form.*

Note that inverses of partial maps can be obtained as a composition of permutations of types (2), (3) and (4). More precisely, the inverse of the partial map with respect to x, y and z can be found after the permutation $(x,y) \mapsto (x^2+1,y)$, $(x,y) \mapsto (\omega(x^2+1),y)$ and $(x,y) \mapsto ((\omega+1)(x^2+1),y)$, respectively.

A map f in normal form can obviously be written in the form $f(x,y,z) = g(x,y,z) + x + y + z$ where the polynomial $g(X,Y,Z)$ corresponding to g has no pure terms X^i, Y^j or Z^k. From this representation it follows that every Laguerre near-plane of order 4 can be obtained from the parabola model of a Laguerre plane of order 4 by replacing some of the circles by graphs of cubic polynomials. To be more precise, since

$$(x,y,z,f(x,y,z)) = (x,y,z,x+y+z) + (0,0,0,g(x,y,z))$$

and

$$(0,0,0,g(x,y,z))T = (g(x,y,z),0,0,g(x,y,z)),$$

the polynomial $g(x,y,z)(X^3+1)$ is added to the polynomial of degree at most 2 corresponding to $(x,y,z,x+y+z)$ (with coefficient vector $(x,y,z,x+y+z)T$). If $(x,y,z,x+y+z)T = (0,a,b,c)$ so that $(x,y,z,x+y+z)$ corresponds to the polynomial $q(X) = aX^2+bX+c$, then $x = q(1)$, $y = q(\omega)$ and $z = q(\omega+1)$. This correspondence between (x,y,z) and (a,b,c) defines a permutation of $\mathbb{F}_4$. Under this correspondence $g(x,y,z)$ becomes a function $h : \mathbb{F}_4^3 \to \mathbb{F}_4$ of a, b and c. Then the graph of aX^2+bX+c is replaced by the graph of $h(a,b,c)X^3+aX^2+bX+c+h(a,b,c)$. Furthermore, since we add a multiple of X^3+1, this process can be viewed as swapping two points on the generator $\{0\}\times\mathbb{F}_4$. In particular, every Laguerre near-plane of order 4 is isomorphic to one obtained from the parabola model by altering some circles by moving their points on the generator $\{0\}\times\mathbb{F}_4$. Note that this may not necessarily be done by geometric substitution.

Example 5.3. The Laguerre near-plane in Example 4.6.2 is in normal form and $g(x,y,z) = (x^2+x)(z^2+z)$ is non-zero if and only if x and z are in $\{\omega,\omega+1\}$ and $g(x,y,z) = 1$ in this case. Hence $2\cdot 2\cdot 4 = 16$ circles are replaced. Moreover, the points $(0,0)$ and $(0,1)$ and the points $(0,\omega)$ and $(0,\omega+1)$ are swapped on $\{0\}\times\mathbb{F}_4$. Circles that are replaced pass through $\{1,\omega+1\}\times\{\omega,\omega+1\}$. Since $q(0) = c = x+y+z = y$ or $y+1$ in this case, all the circles that are replaced also pass through either $\{0,\omega\}\times\{0,1\}$ or $\{0,\omega\}\times\{\omega,\omega+1\}$. Hence we proceed in two steps with respect to the sets $S_1 = \mathbb{F}_4\times\{\omega,\omega+1\}$ and $S_2 = \{0,\omega\}\times\{0,1\}\cup\{1,\omega+1\}\times\{\omega,\omega+1\}$, see Figure 4.

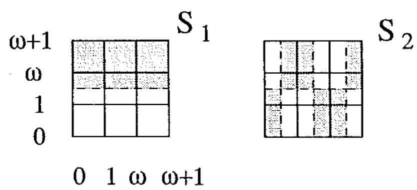

Figure 4

The first set yields Example 4.1, that is, we add

$$g_1(x,y,z) = (x^2+x)(y^2+y)(z^2+z),$$

see Example 4.6.4. In the second step we add

$$\begin{aligned} g_2(x,y,z) =& p_\omega(x)p_0(y)p_\omega(z) + p_\omega(x)p_1(y)p_\omega(z) \\ &+ p_\omega(x)p_{0(y)p_\omega+1}(z) + p_\omega(x)p_1(y)p_{\omega+1}(z) \\ &+ p_{\omega+1}(x)p_0(y)p_\omega(z) + p_{\omega+1}(x)p_1(y)p_\omega(z) \\ &+ p_{\omega+1}(x)p_0(y)p_{\omega+1}(z) + p_{\omega+1}(x)p_1(y)p_{\omega+1}(z) \\ =&(p_\omega(x)+p_{\omega+1}(x))(p_0(y)+p_1(y))(p_\omega(z)+p_{\omega+1}(z)) \\ =&(x^2+x)(y^2+y+1)(z^2+z). \end{aligned}$$

Then $g_1 + g_2 = g$ as required.

Proposition 5.4. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to the Laguerre near-plane obtained from the Miquelian Laguerre plane of order 4 by deleting one generator if and only if $f + f(0,0,0)$ is additive, that is,*

$$f(x,y,z) = a_2x^2 + a_1x + b_2y^2 + b_1y + c_2z^2 + c_1z + d$$

for some $a_2, a_1, b_2, b_1, c_2, c_1, d \in \mathbb{F}_4$, $a_2^3 + a_1^3 = b_2^3 + b_1^3 = c_2^3 + c_1^3 = 1$.

A Laguerre near-plane $\mathcal{L}(f)$ with f in normal form is isomorphic to the parabola model of a Laguerre near-plane of order 4 if and only if $f(x,y,z) = x + y + z$.

Proof. As we have seen in 3.3, each permutation of the form (1) to (6) takes an affine subspace of the affine space $\mathbb{F}_4^4$ over the prime field $\mathbb{F}_2$ to such a subspace. Furthermore, each such subspace that meets each parallel of the coordinates axes in exactly one point can be described by a function f as in the proposition. Hence every Laguerre near-plane isomorphic to $\mathcal{L}(1,0,0,0)$ can be represented in this form.

Conversely, $\mathcal{L}(f)$ with f as above is isomorphic to $\mathcal{L}(f')$ for some f' in normal form. Furthermore, f' still has the same overall form. Hence $f'(x,y,z) = x+y+z$, that is, $\mathcal{L}(f')$ is the parabola model. □

In the proof of Proposition 5.4 we found another characterization of the parabloa model.

Corollary 5.5. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to the parabola model of a Laguerre near-plane of order 4 if and only if the graph of f is an affine subspace of $\mathbb{F}_4^4$ over $\mathbb{F}_2$.*

Let $f(x,x,y) = \sum_{i,j,k=0}^{2} a_{i,j,k}x^iy^jz^k$ for some $a_{i,j,k} \in \mathbb{F}_4$. We say that f *has degree n* if $n = \max\{i+j+k \mid a_{i,j,k} \neq 0\}$. Note that isomorphisms of types (1), (3), (4) and (5) do not change the degree. In the following we discuss the degrees from 1 to 4 separately and do a computer search for degrees 5 and 6.

Degree 1. In this case, we clearly have $f(x,y,z) = ax + by + cz + d$ for some $a, b, c, d \in \mathbb{F}_4$, $a, b, c \neq 0$, and $\mathcal{L}(f)$ is isomorphic to $\mathcal{L}(1,0,0,0)$ by Proposition 5.4.

Degree 2. In this case, $f(x,y,z) = a_{2,0,0}x^2 + a_{0,2,0}y^2 + a_{0,0,2}z^2 + a_{1,1,0}xy + a_{1,0,1}xz + a_{0,1,1}yz + a_{1,0,0}x + a_{0,1,0}y + a_{0,0,1}z + a_{0,0,0}$ for some $a_{i,j,k} \in \mathbb{F}_4$ where at least one coefficient $a_{i,j,k}$ is non-zero for $i+j+k = 2$. Rewriting f as a partial map in x we have

$$\begin{aligned} f(x,y,z) = &a_{2,0,0}x^2 \\ &+ (a_{1,1,0}y + a_{1,0,1}z + a_{1,0,0})x \\ &+ a_{0,2,0}y^2 + a_{0,0,2}z^2 + a_{0,1,1}yz + a_{0,1,0}y + a_{0,0,1}z + a_{0,0,0}. \end{aligned}$$

By Propsition 4.4 we have either $a_{2,0,0} = 0$ or $a_{1,1,0}y + a_{1,0,1}z + a_{1,0,0} = 0$ for all $y, z \in \mathbb{F}_4$. If $a_{2,0,0} \neq 0$, then we obtain $a_{1,1,0} = a_{1,0,1} = a_{1,0,0} = 0$; if $a_{2,0,0} = 0$, then $a_{1,1,0}y + a_{1,0,1}z + a_{1,0,0} \neq 0$ for any $y, z \in \mathbb{F}_4$ and one finds $a_{1,1,0} = a_{1,0,1} = 0$. Therefore $a_{1,1,0} = a_{1,0,1} = 0$ in any case. Considering the partial map in y one likewise obtains $a_{0,1,1} = 0$. Hence $a_{1,1,0} = a_{1,0,1} = a_{0,1,1} = 0$ and $\mathcal{L}(f)$ is isomorphic to $\mathcal{L}(1,0,0,0)$ by Proposition 5.4.

Degree 3. Writing f as a partial map in z we have

$$\begin{aligned} f(x,y,z) =& (a_{1,0,2}x + a_{0,1,2}y + a_{0,0,2})z^2 \\ & + (a_{2,0,1}x^2 + a_{1,1,1}xy + a_{0,2,1}y^2 + a_{1,0,1}x + a_{0,1,1}y + a_{0,0,1})z \\ & + a_{2,1,0}x^2y + a_{1,2,0}xy^2 + a_{2,0,0}x^2 + a_{1,1,0}xy + a_{0,2,0}y^2 \\ & \quad + a_{1,0,0}x + a_{0,1,0}y + a_{0,0,0} \end{aligned}$$

for some $a_{i,j,k} \in \mathbb{F}_4$ where at least one coefficient $a_{i,j,k}$ is non-zero for $i + j + k = 3$.

Equating the coefficient of z^2 to 0 describes a line unless $a_{1,0,2} = a_{0,1,2} = 0$. Equating the coefficient of z to 0 describes a conic or a line unless $a_{2,0,1} = a_{1,1,1} = a_{0,2,1} = a_{1,0,1} = a_{0,1,1} = 0$. Since a line has four points and a conic has at most eight points (in case of a degenerate conic representing two parallel lines), a line and a conic or line cannot cover all 16 points of $\mathbb{F}_4^2$. Therefore, both coefficients of z^2 and z must be constant so that $f(x,y,z) = a_{0,0,2}z^2 + a_{0,0,1}z + a_{2,1,0}x^2y + a_{1,2,0}xy^2 + a_{2,0,0}x^2 + a_{0,2,0}y^2 + a_{1,1,0}xy + a_{1,0,0}x + a_{0,1,0}y + a_{0,0,0}$. In particular, $a_{1,0,2} = a_{0,1,2} = a_{2,0,1} = a_{1,1,1} = a_{0,2,1} = 0$.

A simimlar argument for the partial map in x shows that $a_{2,1,0} = a_{1,2,0} = 0$ – a contradiction to f being of degree 3. Hence degree 3 cannot occur.

Since example 4.6.2 gives a map f of degree 4, and by Proposition 5.4 and the above we have the following characterization.

Proposition 5.6. *A Laguerre near-plane describing map f cannot have degree 3. Furthermore, $\mathcal{L}(f)$ is isomorphic to $\mathcal{L}(1,0,0,0)$ if and only if f has degree at most 3.*

Degree 4. Let

$$\begin{aligned} f(x,y,z) =& a_{2,2,0}x^2y^2 + a_{0,2,2}y^2z^2 + a_{2,0,2}x^2z^2 \\ & + a_{2,1,1}x^2yz + a_{1,2,1}xy^2z + a_{1,1,2}xyz^2 \\ & + a_{2,1,0}x^2y + a_{2,0,1}x^2z + a_{1,2,0}xy^2 + a_{0,2,1}y^2z + a_{1,0,2}xz^2 + a_{0,1,2}yz^2 \\ & + a_{1,1,1}xyz \\ & + a_{2,0,0}x^2 + a_{0,2,0}y^2 + a_{0,0,2}z^2 + a_{1,1,0}xy + a_{0,1,1}yz + a_{1,0,1}xz \\ & + a_{1,0,0}x + a_{0,1,0}y + a_{0,0,1}z + a_{0,0,0} \end{aligned}$$

for some $a_{i,j,k} \in \mathbb{F}_4$ where at least one coefficient $a_{i,j,k}$ is non-zero for $i + j + k = 4$. Since isomorphisms of type (1) do not change the degree of f, we may assume that $a_{0,0,0} = 0$. There are two types of terms of degree 4, one involving all three variables (like in xyz^2) and the other involving only two variables (like in x^2y^2).

We now suppose that terms of the first type occur. Using isomorphisms of types (3) and (4), we may assume that $a_{1,1,2} \neq 0$. Rewriting f as a partial map in z we have

$$\begin{aligned} f(x,y,z) =&(a_{2,0,2}x^2 + a_{1,1,2}xy + a_{0,2,2}y^2 + a_{1,0,2}x + a_{0,1,2}y + a_{0,0,2})z^2 \\ &+ (a_{2,1,1}x^2y + a_{1,2,1}xy^2 + a_{2,0,1}x^2 + a_{1,1,1}xy + a_{0,2,1}y^2 \\ &\quad + a_{1,0,1}x + a_{0,1,1}y + a_{0,0,1})z \\ &+ a_{2,2,0}x^2y^2 + a_{2,1,0}x^2y + a_{1,2,0}xy^2 + a_{2,0,0}x^2 + a_{1,1,0}xy + a_{0,2,0}y^2 \\ &\quad + a_{1,0,0}x + a_{0,1,0}y. \end{aligned}$$

Equating the coefficient of z^2 to 0 describes a nondegenerate quadric which has at most five points or a pair of intersecting lines which have seven points. Equating the coefficient of z to 0 yields at most ten points unless this coefficient is identically 0. (The equation $(a_{1,2,1}x + a_{0,2,1})y^2 + (a_{2,1,1}x^2 + a_{1,1,1}x + a_{0,1,1})y + a_{2,0,1}x^2 + a_{1,0,1}x + a_{0,0,1} = 0$ has at most two solutions y for each x unless $a_{1,2,1}X + a_{0,2,1}$ is a common factor of $a_{2,1,1}X^2 + a_{1,1,1}X + a_{0,1,1}$ and $a_{2,0,1}X^2 + a_{1,0,1}X + a_{0,0,1}$ in which case one may have $4 + 3 \cdot 2 = 10$ solutions; if the coefficient of y^2 is identically 0, then a similar consideration shows that at most 10 solutions can occur.) Since we must cover 16 points the coefficient of z^2 must describe a pair of intersecting lines and equating the coefficient of z to 0 must yield nine points. To get this number of points we must have $a_{1,2,1} \neq 0$ and $a_{1,2,1}X + a_{0,2,1}$ must be a common factor of $a_{2,1,1}X^2 + a_{1,1,1}X + a_{0,1,1}$ and $a_{2,0,1}X^2 + a_{1,0,1}X + a_{0,0,1}$. Hence all points on the vertical line $\{a_{2,1,1}^{-1}a_{1,0,1}\} \times \mathbb{F}_4$ are solutions. However, such a line intersects at least one of the two non-parallel lines determined by the coefficient of z^2 – a contradiction to the fact that the two sets of zeros must be disjoint.

Finally, if the coefficient of z is identically 0, then the coefficient of z^2 must never become 0 – a contradiction. This shows that terms of the first type cannot occur.

We thus have $a_{1,1,2} = a_{1,2,1} = a_{2,1,1} = 0$. Thus the isomorphism of type (6)that substitutes z^2 for z does not change the degree. Moreover, applying the same argument as before to the map $(x, y, z) \to f(x, y, z^2)$, we see that $a_{1,1,1} = 0$. Using isomorphisms of types (3) and (4) we may assume that $a_{2,0,2} \neq 0$ and using an isomorphism of type (6), we may further assume that $a_{0,0,2} = 0$. Note that the degree has not chaged. Considering f as a partial map in z as before we see that equating the coefficients of z^2 and z to 0 describes a quadric or a quadric/line, respectively. Such a configuration can cover 16 points if and only if we have degenerate quadrics describing two parallel lines in both cases; together, one has a full bundle of parallel lines. Using isomorphisms of type (5), we may assume that $a_{2,0,2} = 1$ and $a_{0,2,2} = 0$ or 1. Let $a = a_{0,2,2}$. Since $x^2 + ay^2 + a_{1,0,2}x + a_{0,1,2}y = 0$ describes a pair of parallel lines, we must have $a_{1,0,2} \neq 0$ and we can achieve that $a_{1,0,2} = 1$ by isomorphisms of type (5). But then $a_{0,1,2} = a$. Thus the coefficient of z^2 now has the form

$$x^2 + ay^2 + x + ay = (x + ay)^2 + x + ay.$$

(Note that $a = 0$ or 1 so that $a^2 = a$.) Since the coefficient of z must represent the two remaining parallel lines in the bundle, one finds that

$$a_{2,0,1}x^2 + a_{0,2,1}y^2 + a_{1,0,1}x + a_{0,1,1}y + a_{0,0,1} = a_{0,0,1}(x^2 + ay^2 + x + ay)$$

and $a_{0,0,1} \neq 0$. Using isomorphisms of type (5), we may assume that $a_{0,0,1} = 1$.

We now rewrite f as a partial map in x. We find

$$\begin{aligned} f(x,y,z) =& (z^2 + z + a_{2,2,0}y^2 + a_{2,1,0}y + a_{2,0,0})x^2 \\ &+ (z^2 + z + a_{1,2,0}y^2 + a_{1,1,0}y + a_{1,0,0})x \\ &+ ay^2z^2 + ayz^2 + ay^2z + a_{0,2,0}y^2 + a_{0,1,1}yz + z + a_{0,1,0}y. \end{aligned}$$

After applying an isomorphism of type (6), if necessary, we may assume that $a_{2,0,0} = 0$ and $a_{1,0,0} \neq 0$. Note that the substitution of x by x^2 does not change the degree. Equating the coefficients of x^2 and x to 0 gives us again two quadrics and these must describe four parallel lines between them. Let $b = a_{2,1,0}$. Then $a_{2,2,0} = b^2$ and $z^2 + z + a_{1,2,0}y^2 + a_{1,1,0}y + a_{1,0,0} = a_{1,0,0}(z^2 + z + b^2y^2 + by)$ as before. Hence $a_{1,0,0} = 1$, $a_{1,1,0} = b$ and $a_{1,2,0} = b^2$.

Finally, rewriting f as a partial map in y, we find

$$\begin{aligned} f(x,y,z) =& (az^2 + az + b^2x^2 + b^2x + a_{0,2,0})y^2 \\ &+ (az^2 + az + bx^2 + bx + a_{0,1,0})y \\ &+ (x^2 + x)(z^2 + z) + x + z. \end{aligned}$$

After applying an isomorphism of type (6), if necessary, we may assume that $a_{0,2,0} = 0$ and $a_{0,1,0} \neq 0$. Note that the substitution of y by y^2 does not change the degree. The same arguments as before yield for f the following form:

$$f(x,y,z) = (x^2 + x)(z^2 + z) + a(y^2 + y)(z^2 + z) + b(x^2 + x)(y^2 + y) + x + y + z$$

where $a^2 = a$ and $b^2 = b$.

Clearly, $(a,b) = (0,1)$ and $(a,b) = (1,0)$ yield isomorphic Laguerre near-planes. $(a,b) = (0,0)$ yields the plane in Example 4.5.2. Furthermore, in this case, the inverse of the partial map with respect to x essentially is the above map with $(a,b) = (0,1)$. Hence $(a,b) = (0,1)$ and $(a,b) = (0,0)$ yield isomorphic Laguerre near-planes of order 4. In summary, we have obtained the following result.

Proposition 5.7. *A Laguerre near-plane $\mathcal{L}(f)$ with f of degree 4 is isomorphic to a Laguerre near-plane $\mathcal{L}(f_1)$ or $\mathcal{L}(f_2)$ where f_1 and f_2 are the maps defined by*

$$f_1(x,y,z) = (x^2 + x)(z^2 + z) + x + y + z$$

and

$$f_2(x,y,z) = (x^2 + x)(y^2 + y) + (y^2 + y)(z^2 + z) + (x^2 + x)(z^2 + z) + x + y + z,$$

respectively.

Note that both Laguerre near-planes $\mathcal{L}(f_1)$ and $\mathcal{L}(f_2)$ can be obtained by geometric substitution. For $\mathcal{L}(f_1)$ see Example 5.3.

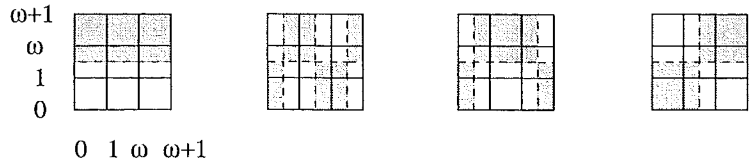

Figure 5

In order to obtain $\mathcal{L}(f_2)$ one applies geometric substitution for all the sets indicated in Figure 5 by the shaded areas and by swapping points on the leftmost generator $\{(0,v) \mid v \in \mathbb{F}_4\}$. Note that these areas define mutually disjoint circle sets so that gemetric substitution can be carried out in each step. Of course, one can also start with $\mathcal{L}(f_1)$ and then adding

$$(x,y,z) \mapsto (x^2+x)(y^2+y)(z^2+z+1)$$

(third square) and

$$(x,y,z) \mapsto (x^2+x+1)(y^2+y)(z^2+z)$$

(forth square).

Degrees 5 and 6. For these last two remaining cases we did a computer search for functions f. In fact, we searched for all functions f in normal form, not necessarily of degree 5 or 6. Let

$$\begin{aligned} f(x,y,z) =& a_{2,2,2}x^2y^2z^2 + a_{2,2,1}x^2y^2z + a_{2,1,2}x^2yz^2 + a_{1,2,2}xy^2z^2 + \\ & a_{2,2,0}x^2y^2 + a_{0,2,2}y^2z^2 + a_{2,0,2}x^2z^2 \\ & + a_{2,1,1}x^2yz + a_{1,2,1}xy^2z + a_{1,1,2}xyz^2 \\ & + a_{2,1,0}x^2y + a_{2,0,1}x^2z + a_{1,2,0}xy^2 + a_{0,2,1}y^2z + a_{1,0,2}xz^2 + a_{0,1,2}yz^2 \\ & + a_{1,1,1}xyz \\ & + a_{2,0,0}x^2 + a_{0,2,0}y^2 + a_{0,0,2}z^2 + a_{1,1,0}xy + a_{0,1,1}yz + a_{1,0,1}xz \\ & + a_{1,0,0}x + a_{0,1,0}y + a_{0,0,1}z + a_{0,0,0} \end{aligned}$$

for some $a_{i,j,k} \in \mathbb{F}_4$. Since we are only interested in Laguerre near-planes up to isomorphism we can assume that f is in normal form, that is,

$$a_{2,0,0} = a_{0,2,0} = a_{0,0,2} = a_{0,0,0} = 0 \text{ and } a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1.$$

Furthermore, under an isomorphism of type (5) we can replace f by $(x,y,z) \mapsto \omega^2 f(\omega x, \omega y, \omega z)$ or $(x,y,z) \mapsto \omega f(\omega^2 x, \omega^2 y, \omega^2 z)$. Then the coefficients $a_{2,2,2}$ of $X^2Y^2Z^2$ and $a_{1,1,1}$ of XYZ are replaced by $\omega^2 a_{2,2,2}$ or $\omega a_{2,2,2}$, and $\omega^2 a_{1,1,1}$ or $\omega a_{1,1,1}$, respectively. Hence we may further assume that

$$a_{2,2,2} = 0 \text{ or } 1 \text{ and } a_{1,1,1} = 0 \text{ or } 1 \text{ if } a_{2,2,2} = 0$$

We proceed in three steps.
Step 1: We determine all coefficients $b_{i,j} \in \mathbb{F}_4$, $i = 0, 1, 2$, $j = 1, 2$, such that $(b_{2,2}x^2 + b_{1,2}x + b_{0,2})^3 + (b_{2,1}x^2 + b_{1,1}x + b_{0,1})^3 = 1$ for all $x \in \mathbb{F}_4$. We found 96 solution vectors $b = (b_{2,2}, b_{1,2}, b_{0,2}, b_{2,1}, b_{1,1}, b_{0,1})$. Note that one can restrict the search to $b_{0,2} = 0$ and $b_{0,1} = 1$; this yields 16 solution vectors. These vectors are

$b_{2,2}$	$b_{1,2}$	$b_{0,2}$	$b_{2,1}$	$b_{1,1}$	$b_{0,1}$
0	0	0	0	0	1
0	0	0	1	ω	1
0	0	0	1	$\omega+1$	1
0	0	0	ω	1	1
0	0	0	ω	ω	1
0	0	0	$\omega+1$	1	1
0	0	0	$\omega+1$	$\omega+1$	1
1	1	0	1	1	1
1	ω	0	ω	$\omega+1$	1
1	$\omega+1$	0	$\omega+1$	ω	1
ω	1	0	$\omega+1$	ω	1
ω	ω	0	1	1	1
ω	$\omega+1$	0	ω	$\omega+1$	1
$\omega+1$	1	0	ω	$\omega+1$	1
$\omega+1$	ω	0	$\omega+1$	ω	1
$\omega+1$	$\omega+1$	0	1	1	1

All other solution vectors are then obtained by multiplication by ω and ω^2 and by exchanging the roles of $b_{i,2}$ and $b_{i,1}$, i.e., multiplying b by $\begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}$ where 0_3 and I_3 denotes the 3×3 zero and identity matrix, respectively.
Step 2: We determine all coefficients $b_{i,j,k} \in \mathbb{F}_4$, $i = 0, 1, 2$, $j = 0, 1, 2$, $k = 1, 2$, where $b_{2,2,2} = 0, 1$ and $b_{1,1,1} = 0, 1$ if $b_{2,2,2} = 0$ such that $(b_{2,2,2}x^2y^2 + b_{2,1,2}x^2y + b_{2,0,2}x^2 + b_{1,2,2}xy^2 + b_{1,1,2}xy + b_{1,0,2}x + b_{0,2,2}y^2 + b_{0,1,2}y + b_{0,0,2})^3 + (b_{2,2,1}x^2y^2 + b_{2,1,1}x^2y + b_{2,0,1}x^2 + b_{1,2,1}xy^2 + b_{1,1,1}xy + b_{1,0,1}x + b_{0,2,1}y^2 + b_{0,1,1}y + b_{0,0,1})^3 = 1$ for all $x, y \in \mathbb{F}_4$.

Fixing y we obtain four identities in x, namely

$$(b_{2,0,2}x^2 + b_{1,0,2}x + b_{0,0,2})^3 + (b_{2,0,1}x^2 + b_{1,0,1}x + b_{0,0,1})^3 = 1$$

$(y = 0)$,

$$((b_{2,2,2} + b_{2,1,2} + b_{2,0,2})x^2 + (b_{1,2,2} + b_{1,1,2} + b_{1,0,2})x + b_{0,2,2} + b_{0,1,2} + b_{0,0,2})^3 +$$

$$((b_{2,2,1} + b_{2,1,1} + b_{2,0,1})x^2 + (b_{1,2,1} + b_{1,1,1} + b_{1,0,1})x + b_{0,2,1} + b_{0,1,1} + b_{0,0,1})^3 = 1$$

$(y = 1)$,

$$((b_{2,2,2}(\omega+1) + b_{2,1,2}\omega + b_{2,0,2})x^2 + (b_{1,2,2}(\omega+1) + b_{1,1,2}\omega + b_{1,0,2})x$$
$$+ b_{0,2,2}(\omega+1) + b_{0,1,2}\omega + b_{0,0,2})^3 +$$

$$((b_{2,2,1}(\omega+1)+b_{2,1,1}\omega+b_{2,0,1})x^2+(b_{1,2,1}(\omega+1)+b_{1,1,1}\omega+b_{1,0,1})x$$
$$+b_{0,2,1}(\omega+1)+b_{0,1,1}\omega+b_{0,0,1})^3=1$$

$(y=\omega)$ and

$$((b_{2,2,2}\omega+b_{2,1,2}(\omega+1)+b_{2,0,2})x^2+(b_{1,2,2}\omega+b_{1,1,2}(\omega+1)+b_{1,0,2})x$$
$$+b_{0,2,2}\omega+b_{0,1,2}(\omega+1)+b_{0,0,2})^3+$$

$$((b_{2,2,1}\omega+b_{2,1,1(\omega+1)}+b_{2,0,1})x^2+(b_{1,2,1}\omega+b_{1,1,1}(\omega+1)+b_{1,0,1})x$$
$$+b_{0,2,1}\omega+b_{0,1,1}(\omega+1)+b_{0,0,1})^3=1$$

$(y=\omega+1)$ for all $x\in\mathbb{F}_4$. Hence the vectors

$$\begin{aligned}
b_0 =&(b_{2,0,2},b_{1,0,2},b_{0,0,2},b_{2,0,1},b_{1,0,1},b_{0,0,0})\\
b_1 =&(b_{2,2,2}+b_{2,1,2}+b_{2,0,2},b_{1,2,2}+b_{1,1,2}+b_{1,0,2},b_{0,2,2}+b_{0,1,2}+b_{0,0,2},\\
&b_{2,2,1}+b_{2,1,1}+b_{2,0,1},b_{1,2,1}+b_{1,1,1}+b_{1,0,1},b_{0,2,1}+b_{0,1,1}+b_{0,0,1})\\
b_\omega =&(b_{2,2,2}(\omega+1)+b_{2,1,2}\omega+b_{2,0,2},b_{1,2,2}(\omega+1)+b_{1,1,2}\omega+b_{1,0,2},\\
&b_{0,2,2}(\omega+1)+b_{0,1,2}\omega+b_{0,0,2},b_{2,2,1}(\omega+1)+b_{2,1,1}\omega+b_{2,0,1},\\
&b_{1,2,1}(\omega+1)+b_{1,1,1}\omega+b_{1,0,1},b_{0,2,1}(\omega+1)+b_{0,1,1}\omega+b_{0,0,1})\\
b_{\omega+1} =&(b_{2,2,2}\omega+b_{2,1,2}(\omega+1)+b_{2,0,2},b_{1,2,2}\omega+b_{1,1,2}(\omega+1)+b_{1,0,2},\\
&b_{0,2,2\omega}+b_{0,1,2}(\omega+1)+b_{0,0,2},b_{2,2,1}\omega+b_{2,1,1(\omega+1)}+b_{2,0,1},\\
&b_{1,2,1}\omega+b_{1,1,1}(\omega+1)+b_{1,0,1},b_{0,2,1}\omega+b_{0,1,1}(\omega+1)+b_{0,0,1})
\end{aligned}$$

must appear in the list found in Step 1. Furthermore,

$$b_0+b_1+b_\omega+b_{\omega+1}=0.$$

Thus, going through the list found in Step 1 one looks for triples (b_0,b_1,b_ω) such that $b_0+b_1+b_\omega$ is also in this list. The coefficients $b_{i,j,k}$ are then determined by

$$\begin{aligned}
(b_{2,2,2},b_{2,1,2},b_{2,0,2}) =&(b_0^1,b_1^1,b_\omega^1)\cdot S\\
(b_{2,2,1},b_{2,1,1},b_{2,0,1}) =&(b_0^4,b_1^4,b_\omega^4)\cdot S\\
(b_{1,2,2},b_{1,1,2},b_{1,0,2}) =&(b_0^2,b_1^2,b_\omega^2)\cdot S\\
(b_{1,2,1},b_{1,1,1},b_{1,0,1}) =&(b_0^5,b_1^5,b_\omega^5)\cdot S\\
(b_{0,2,2},b_{0,1,2},b_{0,0,2}) =&(b_0^3,b_1^3,b_\omega^3)\cdot S\\
(b_{0,2,1},b_{0,1,1},b_{0,0,1}) =&(b_0^6,b_1^6,b_\omega^6)\cdot S
\end{aligned}$$

where b_c^i denotes the ith entry of b_c, $c=0,1,\omega$, and

$$S=\begin{pmatrix}\omega+1 & \omega & 1\\ \omega & \omega+1 & 0\\ 1 & 1 & 0\end{pmatrix}.$$

A total of 4056 solution vectors $(b_{i,j,k})_{i,j,k}$ were found.

Step 3: We determine all coefficients $a_{i,j,k} \in \mathbb{F}_4$ of a Laguerre near-plane describing map in normal form where $a_{2,2,2}$ equals 0 or 1 and $a_{1,1,1}$ equals 0 or 1 if $a_{2,2,2} = 0$. Note that $a_{2,2,2}$ and $a_{1,1,1}$ appear as $b_{2,2,2}$ and $b_{1,1,1}$ in each of the three identities associated with the three partial maps so that we can directly use the list found in Step 2. Looking at the partial map with respect to z we have $a_{i,j,k} = b_{i,j,k}$ for $i,j = 0,1,2$ and $k = 1,2$. For the partial map with respect to y we now have $a_{i,j,k} = b_{i,k,j}$ for $i,k = 0,1,2$ and $j = 1,2$. Finally, for the partial map with respect to x we obtain $a_{i,j,k} = b_{j,k,i}$ for $j,k = 0,1,2$ and $i = 1,2$. Hence we search through the list found in step 2 for triples of vectors $b^1 = (b^1_{i,j,k})$, $b^2 = (b^2_{i,j,k})$ and $b^3 = (b^3_{i,j,k})$ that show the following identities

$$\begin{array}{ll}
b^1_{2,2,2} = b^2_{2,2,2} = b^3_{2,2,2} & b^1_{2,1,1} = b^2_{1,2,1} = b^3_{1,1,2} \\
b^1_{2,2,1} = b^2_{2,1,2} = b^3_{1,2,2} & b^1_{1,2,1} = b^2_{2,1,1} = b^3_{1,2,1} \\
b^1_{2,1,2} = b^2_{2,2,1} = b^3_{2,1,2} & b^1_{1,1,2} = b^2_{1,1,2} = b^3_{2,1,1} \\
b^1_{1,2,2} = b^2_{2,1,2} = b^3_{2,2,1} & b^1_{1,1,1} = b^2_{1,1,1} = b^3_{1,1,1}
\end{array}$$

$$\begin{array}{lll}
b^1_{0,2,2} = b^2_{0,2,2} & b^1_{2,0,2} = b^3_{2,0,2} & b^2_{2,0,2} = b^3_{0,2,2} \\
b^1_{0,2,1} = b^2_{0,1,2} & b^1_{2,0,1} = b^3_{1,0,2} & b^2_{2,0,1} = b^3_{0,1,2} \\
b^1_{0,1,2} = b^2_{0,2,1} & b^1_{1,0,2} = b^3_{2,0,1} & b^2_{1,0,2} = b^3_{0,2,1} \\
b^1_{0,1,1} = b^2_{0,1,1} & b^1_{1,0,1} = b^3_{1,0,1} & b^2_{1,0,1} = b^3_{0,1,1}
\end{array}$$

Then

$$\begin{aligned}
a_{i,j,k} &= b^1_{i,k,j} \text{ for } i,k = 0,1,2 \text{ and } j = 1,2, \\
a_{i,j,0} &= b^2_{i,0,j} \text{ for } i = 0,1,2, \text{ and } j = 1,2.
\end{aligned}$$

Note that $a_{2,0,0} = 0$, $a_{1,0,0} = 1$ and $a_{0,0,0} = 0$ by our assumptions.

A total of 36 maps were found. Of these one is of degree 1, 21 are of degree 4 and 14 are of degree 6. The map of degree 1 is $(x,y,z) \mapsto x+y+z$. The maps f_1 and f_2 are among the 21 maps of degree 4 all of which can be transformed to either f_1 or f_2. This observation agrees with (and confirms) our previous results on maps of degrees at most 4. The 14 maps of degree 6 are listed in the table below where column i shows the coefficients of the map i.

Each of these can be transformed to f_3 (column 1) or f_4 (column 14), see the proposition below. In fact, colums 2 to 8 transform into column 1 and columns 9 to 13 transform into column 14. For example, one can use an isomorphism of type (6) to replace f by $(x,y,z) \mapsto f(x^2,y^2,z^2)^2$. This has the effect that each coefficient is squared so that 0 and 1 are fixed and ω and $\omega+1$ are exchanged. Hence the coefficent vectors in columns 10, 11 and 13 can be transformed into those in columns 14, 12 and 9, respectively. tTe coefficent vectors in columns 9 and 13 are transformed into those in column 14 by swapping x with y and x with z, respectively.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$a_{2,2,2}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$a_{2,2,1}$	1	1	1	1	1	1	1	1	ω	$\omega+1$	ω	$\omega+1$	$\omega+1$	ω
$a_{2,1,2}$	1	1	1	1	1	1	1	1	$\omega+1$	$\omega+1$	$\omega+1$	ω	ω	ω
$a_{1,2,2}$	1	1	1	1	1	1	1	1	ω	ω	$\omega+1$	ω	$\omega+1$	$\omega+1$
$a_{2,2,0}$	0	1	0	1	0	1	0	1	1	1	1	1	1	1
$a_{2,1,1}$	1	1	1	1	1	1	1	1	1	ω	1	1	1	$\omega+1$
$a_{2,0,2}$	0	0	0	0	1	1	1	1	1	1	1	1	1	1
$a_{1,2,1}$	1	1	1	1	1	1	1	1	$\omega+1$	1	1	1	ω	1
$a_{1,1,2}$	1	1	1	1	1	1	1	1	1	1	ω	$\omega+1$	1	1
$a_{0,2,2}$	0	0	1	1	0	0	1	1	1	1	1	1	1	1
$a_{2,1,0}$	0	1	0	1	0	1	0	1	$\omega+1$	ω	$\omega+1$	ω	ω	$\omega+1$
$a_{2,0,1}$	0	0	0	0	1	1	1	1	ω	ω	ω	$\omega+1$	$\omega+1$	$\omega+1$
$a_{1,2,0}$	0	1	0	1	0	1	0	1	$\omega+1$	$\omega+1$	ω	$\omega+1$	ω	$\omega+1$
$a_{1,1,1}$	1	1	1	1	1	1	1	1	ω	$\omega+1$	$\omega+1$	ω	$\omega+1$	ω
$a_{1,0,2}$	0	0	0	0	1	1	1	1	ω	ω	ω	$\omega+1$	$\omega+1$	$\omega+1$
$a_{0,2,1}$	0	0	1	1	0	0	1	1	$\omega+1$	$\omega+1$	ω	$\omega+1$	ω	ω
$a_{0,1,2}$	0	0	1	1	0	0	1	1	$\omega+1$	$\omega+1$	ω	$\omega+1$	ω	ω
$a_{2,0,0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$a_{1,1,0}$	0	1	0	1	0	1	0	1	ω	$\omega+1$	ω	$\omega+1$	$\omega+1$	ω
$a_{1,0,1}$	0	0	0	0	1	1	1	1	$\omega+1$	$\omega+1$	$\omega+1$	ω	ω	ω
$a_{0,2,0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$a_{0,1,1}$	0	0	1	1	0	0	1	1	ω	ω	$\omega+1$	ω	$\omega+1$	$\omega+1$
$a_{0,0,2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$a_{1,0,0}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$a_{0,1,0}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$a_{0,0,1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$a_{0,0,0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Proposition 5.8. *A Laguerre near-plane $\mathcal{L}(f)$ with f of degree > 4 is isomorphic to a Laguerre near-plane $\mathcal{L}(f_3)$ or $\mathcal{L}(f_4)$ where f_3 and f_4 are the maps defined by*

$$f_3(x,y,z) = (x^2+x)(y^2+y)(z^2+z) + x + y + z$$

and

$$\begin{aligned} f_4(x,y,z) =&(x^2+\omega^2 x)(y^2+\omega y)(z^2+\omega z) + (x^2+\omega^2 x)(y^2+\omega^2 y) \\ &+ (x^2+\omega^2 x)(z^2+\omega^2 z) + (y^2+\omega y)(z^2+\omega z) + x + y + z, \end{aligned}$$

respectively. There are no Laguerre near-planes $\mathcal{L}(f)$ with f of degree 5.

Proof. Examining the 36 solutions one finds that either all coefficients $a_{2,2,2}$, $a_{2,2,1}$, $a_{2,1,2}$, $a_{1,2,2}$, $a_{2,1,1}$, $a_{1,2,1}$, $a_{1,1,2}$ and $a_{1,1,1}$ are non-zero or they are all zero. It is clear that under an isomorphism of type (1), (3), (4), (5) or (6) this property is preserved. From Corollary 4.5 it follows that under an isomorphism of type (2) these coefficients are permuted among themselves and perhaps squared so that they remain all non-zero or all zero. Since for a map of degree 5 some of these coefficients would have to be zero and some others would have to be non-zero, the above argument shows that there are no Laguerre near-plane describing maps of degree 5. Furthermore, if f has degree 6 it must be tranformed into one of the maps of degree 6 found in Step 3. This proves the proposition. □

As seen in 4.5.4 the Laguerre near-plane $\mathcal{L}(f_3)$ can be obtained from the parabola model by geometric substitution. As for $\mathcal{L}(f_4)$ one readily sees that $\mathcal{L}(f_4)$ shares 36 circles with the parabola model (just find the number of solutions of $f_4(x,y,z) = x+y+z$). Hence 28 circles have to be replaced in $\mathcal{L}(f_0)$ in order to obtain $\mathcal{L}(f_4)$. However as 28 is not divisible by 8 the sets used in the process of geometric substitution cannot be disjoint and some circles must be changed back to parabolae if geometric substitution is possible. So far we have not found suitable sets for geometric substitution. We therefore conjecture that $\mathcal{L}(f_4)$ cannot be obtained from the parabola model by geometric substitution.

6. Automorphism groups

So far we have established that a Laguerre near-plane of order 4 is isomorphic to one of the Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0,1,2,3,4$. In order to show that in fact the latter five planes are mutually non-isomorphic we investigate the automorphism groups $\Gamma(f_i)$ of these planes, that is, the collection of all permutations in Γ that preserve the Laguerre near-plane. Recall from 3.3 that every automorphism of a Laguerre near-plane of order 4 is of the form

$$\mathbb{F}_4^4 \to \mathbb{F}_4^4 : (x,y) \mapsto (\alpha(x), \beta_x(y))$$

where α and β_x are permutations of $\mathbb{F}_4$ for each $x \in \mathbb{F}_4$. The collection of all permutations with $\alpha = id$ is a normal subgroup Δ. A composition of permutations of types (1), (5) and (6) yields elements in Δ. We further denote by Δ_1 the collection of all permutations in Δ whose accompanying field automorphisms on generators are the identity, i.e., permutations of the form $(x,y) \mapsto (x, a_x y + b_x)$ for $a_x, b_x \in \mathbb{F}_4$, $a_x \neq 0$. Clearly, Δ_1 is a normal subgroup of Δ. From 5.1 one readily obtains the following.

Lemma 6.1. *Let $\delta : \mathbb{F}_4^4 \to \mathbb{F}_4^4 : (x,y) \mapsto (x, \beta_x(y))$ be an element of Δ. Then δ takes the Laguerre near-plane $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where*

$$f'(x,y,z) = \beta_0(f(\tilde{\beta}_1(x), \tilde{\beta}_\omega(y), \tilde{\beta}_{\omega+1}(z)))$$

and where $\tilde{\beta}$ denotes the permutation obtained from β by squaring the leading coefficient of β, that is, if $\beta(x) = ax^m + b$ with $a,b \in \mathbb{F}_4$, $a \neq 0$, $m = 1,2$, then $\tilde{\beta}(x) = a^2x^m + b$ for $x \in \mathbb{F}_4$.

6.2. The automorphism group of the parabola model. Since $\mathcal{L}(f_0)$ extends to the Miquelian Laguerre plane, every automorphism of the Miquelian Laguerre plane that fixes a distinguished generator induces an automorphism of the Laguerre near-plane obtained by removing the distinguished generator. It is well known that the automorphism group of the Miquelian Laguerre plane of order 4 has order $2^9 \cdot 3^2 \cdot 5$ and acts transitively on the set of all incident point-circle pairs. In particular, this group is transitive on the set of generators. Hence the stabilizer of a generator has order $2^9 \cdot 3^2$. In terms of the isomorphisms from 3.3 the group induced by this stabilizer is generated by all permutations of types (2) and (3) and by the following permutations:

- $(x, y) \mapsto (x, y + t_2x^2 + t_1x + t_0)$ for $t_2, t_1, t_0 \in \mathbb{F}_4$ (type (1)),
- $(x, y) \mapsto (x^2, y^2)$ (types (4) and (6) combined) and
- $(x, y) \mapsto (x, ry)$ for $r \neq 0$ (type (5)).

However, $\mathcal{L}(f_0)$ also admits the permutation of type (4) as an automorphism. In fact, together they generate the entire automorphism group of $\mathcal{L}(f_0)$. From the transitivity properties of the automorphism group of the Miquelian Laguerre plane of order 4 we see that the group G_0 generated by the above automorphisms is transitive on the set of point-circles pairs ('flags') and induces the full symmetric group of degree 4 on the set of generators. Now let γ be an automorphism of $\mathcal{L}(f_0)$. Up to elements in G_0 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$, and that γ fixes the circle $\{(x, 0) \mid x \in \mathbb{F}_4\}$. Then $\gamma(x, y) = (x, \beta_x(y))$ where $\beta_x(y) = a_xy^{m_x}$ with $a_x \in \mathbb{F}_4$, $a \neq 0$, and $m_x = 1, 2$. Using the automorphisms $(x, y) \mapsto (x, y^2)$ and $(x, y) \mapsto (x, ry)$ in G_0, if necessary, we may further assume that β_0 is the identity. By Lemma 6.1 we then must have $f_0(x, y, z) = f_0(\tilde{\beta}_1(x), \tilde{\beta}_\omega(y), \tilde{\beta}_{\omega+1}(z)))$, that is,

$$a_1^2x^{m_1} + a_\omega^2y^{m_\omega} + a_{\omega+1}^2z^{m_{\omega+1}} = x + y + z$$

for all $x, y, z \in \mathbb{F}_4$. But this implies $a_1 = a_\omega = a_{\omega+1} = 1$ and $m_1 = m_\omega = m_{\omega+1} = 1$, that is, γ is the identity.

This shows that the automorphism group of $\mathcal{L}(f_0)$ is contained in G_0. In summary we obtain the following.

Proposition 6.3. *The automorphism group $\Gamma(f_0)$ of the Laguerre near-plane $\mathcal{L}(f_0)$ has order $2^{10} \cdot 3^2$. Furthermore, $\Gamma(f_0)$ acts transitively on the set of point-circles pairs of $\mathcal{L}(f_0)$ and induces the full symmetric group S_4 of degree 4 on the set of generators. In particular, $\Gamma(f_0)$ is point-transitive and circle-transitive.*

Although $\mathcal{L}(f_0)$ extends to the Miquelian Laguerre plane of order 4 not every automorphism of $\mathcal{L}(f_0)$ extends to an automorphism of the Laguerre plane.

6.4. The automorphism group of $\mathcal{L}(f_2)$. From 5.1 we find that the following permutations are automorphisms of $\mathcal{L}(f_2)$.

- $(x, y) \mapsto (x, y + t_1^2x^2 + t_1x + t_0)$ for $t_1 \in \mathbb{F}_4$, $t_0 \in \mathbb{F}_2$;
- $(x, y) \mapsto \begin{cases} (x + t, y), & \text{if } x \in \{0, t\} \\ (x + t, y^2), & \text{if } x \in \mathbb{F}_4 \setminus \{0, t\} \end{cases}$ for $t \in \mathbb{F}_4$, $t \neq 0$;
- $(x, y) \mapsto (rx, y)$ for $r \neq 0$;
- $(x, y) \mapsto (x^2, y)$;

– $(x, y) \mapsto (x, y^2)$;
– $(x, y) \mapsto \begin{cases} (x, y+\omega), & \text{if } x \in \{0, t\} \\ (x, y^2), & \text{if } x \in \mathbb{F}_4 \setminus \{0, t\} \end{cases}$ for $t \in \mathbb{F}_4$, $t \neq 0$.

These automorphisms generate a group G_2. By looking at the x-coordinates we see that every permutation of the set of generators can be obtained by an element of G_2.

Let γ be an automorphism of $\mathcal{L}(f_2)$. Up to elements in G_2 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(x, y) = (x, \beta_x(y))$ where $\beta_x(y) = a_x y^{m_x} + b_x$ with $a_x, b_x \in \mathbb{F}_4$, $a \neq 0$, and $m_x = 1, 2$. Using the automorphisms $(x, y) \mapsto (x, y^2)$ and $(x, y) \mapsto \begin{cases} (x, y+\omega), & \text{if } x \in \mathbb{F}_2 \\ (x, y^2), & \text{if } x \in \{\omega, \omega+1\} \end{cases}$ in G_2, if necessary, we may further assume that $m_0 = m_{\omega+1} = 1$. By Lemma 6.1 we then must have $f_2(x, y, z) = \beta_0(f_2(\tilde{\beta}_1(x), \tilde{\beta}_\omega(y), \tilde{\beta}_{\omega+1}(z)))$, that is,

$$
\begin{aligned}
a_0[(a_1 x^{2m_1} + a_1^2 x^{m_1} + b_1^2 + b_1)(a_\omega y^{2m_\omega} + a_\omega^2 y^{m_\omega} + b_\omega^2 + b_\omega) \\
+(a_\omega y^{2m_\omega} + a_\omega^2 y^{m_\omega} + b_\omega^2 + b_\omega)(a_{\omega+1} z^2 + a_{\omega+1}^2 z + b_{\omega+1}^2 + b_{\omega+1}) \\
+(a_1 x^{2m_1} + a_1^2 x^{m_1} + b_1^2 + b_1)(a_{\omega+1} z^2 + a_{\omega+1}^2 z + b_{\omega+1}^2 + b_{\omega+1}) \\
+a_1^2 x^{m_1} + a_\omega^2 y^{m_\omega} + a_{\omega+1}^2 z + b_1 + b_\omega + b_{\omega+1}] + b_0 \\
= (x^2+x)(y^2+y) + (y^2+y)(z^2+z) + (x^2+x)(z^2+z) + x + y + z
\end{aligned} \tag{1}
$$

for all $x, y, z \in \mathbb{F}_4$. Looking at terms x^2 and x in (1) we find

$$a_0[(b_\omega^2 + b_\omega + b_{\omega+1}^2 + b_{\omega+1})(a_1 x^{2m_1} + a_1^2 x^{m_1}) + a_1^2 x^{m_1}] = x.$$

Since $b^2 + b \in \mathbb{F}_2$ for each $b \in \mathbb{F}_4$, we obtain two cases. Either $b_\omega^2 + b_\omega + b_{\omega+1}^2 + b_{\omega+1} = 1$ and then $m_1 = 2$, $a_1 = a_0^2$, or $b_\omega^2 + b_\omega + b_{\omega+1}^2 + b_{\omega+1} = 0$, and then $m_1 = 1$, $a_1 = a_0$. In both cases we have $a_1 = a_0^{m_1}$ and $a_1 x^{2m_1} + a_1^2 x^{m_1} = a_0^{m_1} x^{2m_1} + a_0^{2m_1} x^{m_1} = a_0 x^2 + a_0^2 x$. One similarly finds that $a_\lambda = a_0^{m_\lambda}$ for $\lambda = \omega, \omega+1$ and $a_\omega y^{2m_\omega} + a_\omega^2 y^{m_\omega} = a_0 y^2 + a_0^2 y$, $a_{\omega+1} z^2 + a_{\omega+1}^2 z = a_0 z^2 + a_0^2 z$. Comparing terms xy in (1) yields $a_0 = 1$ and thus $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.

Let $d_\lambda = b_\lambda^2 + b_\lambda$ for $\lambda = 1, \omega, \omega+1$. Then (1) becomes

$$
\begin{aligned}
(x^2+x+d_1)(y^2+y+d_\omega) + (y^2+y+d_\omega)(z^2+z+d_{\omega+1}) \\
+(x^2+x+d_1)(z^2+z+d_{\omega+1}) + x^{m_1} + y^{m_\omega} + z + b_0 + b_1 + b_\omega + b_{\omega+1} \\
= (x^2+x)(y^2+y) + (y^2+y)(z^2+z) + (x^2+x)(z^2+z) + x + y + z.
\end{aligned}
$$

Expanding the left-hand side, we see that

$$
\begin{aligned}
(d_\omega + d_{\omega+1})(x^2+x) + x^{m_1} &= x \\
(d_1 + d_{\omega+1})(y^2+y) + y^{m_\omega} &= y \\
(d_1 + d_\omega)(z^2+z) + z &= z
\end{aligned}
$$

and

$$d_1 d_\omega + d_\omega d_{\omega+1} + d_1 d_{\omega+1} + b_0 + b_1 + b_\omega + b_{\omega+1} = 0.$$

Thus $d_\omega = d_1$ and, as before, either $d_{\omega+1} = d_1$, $m_1 = 1$ or $d_{\omega+1} = d_1 + 1$, $m_1 = 2$, and $b_0 + b_1 + b_\omega + b_{\omega+1} = d_1$. Hence, we have two cases.
Case 1: $d_{\omega+1} = d_\omega = d_1 \in \mathbb{F}_2$, $m_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.
If $d_1 = 0$, then γ is the map $(x, y) \mapsto (x, y + t_1^2 x^2 + t_1 x + t_0)$ where $t_0 = b_0 \in \mathbb{F}_2$ and $t_1 = b_1 + (\omega + 1)b_\omega + \omega b_{\omega+1}$, see 5.1, that is, $\gamma \in G_2$. For $d_1 = 1$ we use the composition of the last automorphisms in the list above for each $t \in \mathbb{F}_4$, $t \neq 0$. This yields the automorphism $\gamma' : (x, y) \mapsto \begin{cases} (x, y + \omega), & \text{if } x \in \mathbb{F}_4 \setminus \{\omega\} \\ (x, y + \omega + 1), & \text{if } x = \omega \end{cases} = (x, y+\omega+p_\omega(x))$. Then $\gamma' \circ \gamma$ is an automorphism as before and thus again $\gamma \in G_2$.
Case 2: $d_{\omega+1} = d_1 + 1$, $d_\omega = d_1 \in \mathbb{F}_2$, $m_0 = m_{\omega+1} = 1$, $m_1 = m_\omega = 2$.
In this case we use the automorphism $(x, y) \mapsto \begin{cases} (x, y + \omega), & \text{if } x \in \{0, \omega + 1\} \\ (x, y^2), & \text{if } x \in \{1, \omega\} \end{cases}$ to obtain an automorphism as in the first case.

This shows that the automorphism group of $\mathcal{L}(f_2)$ is contained in G_2. Furthermore, G_2 is of order $2^{10}{\cdot}3$ since, from above, $G_2/\Delta \cong S_4$, $|\Delta/\Delta_1| = 8$ and $|\Delta_1| = 16$. Taking the first and last automorphisms in the list above one readily sees that G_2 is transitive on the generator $\{0\} \times \mathbb{F}_4$. Thus G_2 acts transitively on the point set $\mathbb{F}_4 \times \mathbb{F}_4$.

Using the first and last automorphisms in the list above we see that Δ is transitive on the generator $\{0\} \times \mathbb{F}_4$. The stabilizer $\Delta_{(0,0)}$ of $(0, 0)$ has order 2^5 and is generated by the following automorphisms:

- $(x, y) \mapsto (x, y + t^2 x^2 + tx)$ for $t \in \mathbb{F}_4$;
- $(x, y) \mapsto (x, y^2)$;
- $(x, y) \mapsto \begin{cases} (x, y), & \text{if } x \in \mathbb{F}_2 \\ (x, y^2 + x), & \text{if } x \in \mathbb{F}_4 \setminus \mathbb{F}_2 \end{cases}$;
- $(x, y) \mapsto \begin{cases} (x, y), & \text{if } x \in \{0, \omega\} \\ (x, y^2 + \omega + 1), & \text{if } x = 1 \\ (x, y^2 + \omega), & \text{if } x = \omega + 1 \end{cases}$

The last two automorphisms are obtained as a composition of two of the last automorphisms in the list at the beginning of this section (for $t = \omega, \omega + 1$ and $t = 1, \omega + 1$, respectively). The first and last automorphisms then show that $\Delta_{(0,0)}$ is transitive on the generator $\{1\} \times \mathbb{F}_4$. Finally, the stabilizer $\Delta_{(0,0),(1,0)}$ of $(0, 0)$ and $(1, 0)$ has order 2^3 and is generated by $(x, y) \mapsto (x, y + x^2 + x)$, $(x, y) \mapsto (x, y^2)$ and $(x, y) \mapsto \begin{cases} (x, y), & \text{if } x \in \mathbb{F}_2 \\ (x, y^2 + x), & \text{if } x \in \mathbb{F}_4 \setminus \mathbb{F}_2 \end{cases}$. It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\} \times \mathbb{F}_4$. Since each circle through $(0, 0)$ is uniquely determined by its intersection with the two generators $\{1\} \times \mathbb{F}_4$ and $\{\omega\} \times \mathbb{F}_4$, we see that $\Delta_{(0,0)}$ is transitive on the set of circles through $(0, 0)$. Hence, $(G_2)_{(0,0)}$ is transitive on the set of circles through $(0, 0)$, and because G_2 is point-transitive, we finally obtain that G_2 acts transitively on the set of point-circles pairs of $\mathcal{L}(f_2)$.

In summary we obtain the following.

Proposition 6.5. *The automorphism group $\Gamma(f_2)$ of the Laguerre near-plane $\mathcal{L}(f_2)$ has order $2^{10} \cdot 3$. Furthermore, $\Gamma(f_2)$ acts transitively on the set of point-circles pairs of $\mathcal{L}(f_2)$ and induces the full symmetric group S_4 on the set of generators. In particular, $\Gamma(f_2)$ is point-transitive and circle-transitive.*

6.6. The automorphism group of $\mathcal{L}(f_3)$. From 5.1 we find that the following permutations are automorphisms of $\mathcal{L}(f_3)$.

- $(x,y) \mapsto (x, y + t_1^2 x^2 + t_1 x + t_0)$ for $t_1 \in \mathbb{F}_4$, $t_0 \in \mathbb{F}_2$;
- $(x,y) \mapsto (x+t, y)$ for all $t \in \mathbb{F}_4$,
- $(x,y) \mapsto (rx, y)$ for $r \neq 0$;
- $(x,y) \mapsto (x^2, y)$;
- $(x,y) \mapsto (x, y^2)$.

These automorphisms generate a group G_3 of order $2^7 \cdot 3$. By looking at the x-coordinates we see that every permutation of the set of generators can be obtained by an element of G_3.

Let γ be an automorphism of $\mathcal{L}(f_3)$. Up to elements in G_3 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(x,y) = (x, a_x y^{m_x} + b_x)$ where $a_x, b_x \in \mathbb{F}_4$, $a \neq 0$, and $m_x = 1, 2$. Using the automorphism $(x,y) \mapsto (x, y^2)$ in G_3, if necessary, we may further assume that $m_0 = 1$. By Lemma 6.1 we then must have

$$(2) \qquad \begin{aligned} a_0[(a_1 x^{2m_1} + a_1^2 x^{m_1} + b_1^2 + b_1)(a_\omega y^{2m_\omega} + a_\omega^2 y^{m_\omega} + b_\omega^2 + b_\omega) \\ \cdot (a_{\omega+1} z^{2m_{\omega+1}} + a_{\omega+1}^2 z^{m_{\omega+1}} + b_{\omega+1}^2 + b_{\omega+1}) \\ + a_1^2 x^{m_1} + a_\omega^2 y^{m_\omega} + a_{\omega+1}^2 z^{m_{\omega+1}} + b_1 + b_\omega + b_{\omega+1}] + b_0 \\ = (x^2+x)(y^2+y)(z^2+z) + x + y + z \end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$. Looking at terms x^2 and x in (2) we find

$$a_0[(b_\omega^2 + b_\omega)(b_{\omega+1}^2 + b_{\omega+1})(a_1 x^{2m_1} + a_1^2 x^{m_1}) + a_1^2 x^{m_1}] = x.$$

Since $b^2 + b \in \mathbb{F}_2$ for each $b \in \mathbb{F}_4$, we obtain two cases. Either $b_\omega^2 + b_\omega = b_{\omega+1}^2 + b_{\omega+1} = 1$ and then $m_1 = 2$, $a_1 = a_0^2$, or $(b_\omega^2 + b_\omega)(b_{\omega+1}^2 + b_{\omega+1}) = 0$, and then $m_1 = 1$, $a_1 = a_0$. In both cases we have $a_1 = a_0^{m_1}$ and $a_1 x^{2m_1} + a_1^2 x^{m_1} = a_0 x^2 + a_0^2 x$ as in 6.6. One similarly finds that $a_\lambda = a_0^{m_\lambda}$ for $\lambda = \omega, \omega+1$ and $a_\omega y^{2m_\omega} + a_\omega^2 y^{m_\omega} = a_0 y^2 + a_0^2 y$, $a_{\omega+1} z^{2m_{\omega+1}} + a_{\omega+1}^2 z^{m_{\omega+1}} = a_0 z^2 + a_0^2 z$. Comparing terms $x^2 y^2 z^2$ in (2) yields $a_0 = 1$ and thus $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.

Now (2) becomes

$$\begin{aligned} (x^2 + x + b_1^2 + b_1)(y^2 + y + b_\omega^2 + b_\omega)(z^2 + z + b_{\omega+1}^2 + b_{\omega+1}) \\ + x^{m_1} + y^{m_\omega} + z^{m_{\omega+1}} + b_0 + b_1 + b_\omega + b_{\omega+1} \\ = (x^2+x)(y^2+y)(z^2+z) + x + y + z. \end{aligned}$$

Expanding the left-hand side, we see that $b_1^2 + b_1 = b_\omega^2 + b_\omega = b_{\omega+1}^2 + b_{\omega+1} = 0$ and $b_0 + b_1 + b_\omega + b_{\omega+1} = 0$; in particular, $b_\lambda \in \mathbb{F}_2$ for all $\lambda \in \mathbb{F}_4$. Furthermore, $m_\lambda = 1$

for all $\lambda \in \mathbb{F}_4$. But this implies that γ is the map $(x, y) \mapsto (x, y + t_1^2 x^2 + t_1 x + t_0)$ where $t_0 = b_0$ and $t_1 = b_1 + (\omega + 1)b_\omega + \omega b_{\omega+1}$, that is, $\gamma \in G_3$.

This shows that the automorphism group of $\mathcal{L}(f_3)$ is contained in G_3. Since $b^2 + b \in \mathbb{F}_2$ for each $b \in \mathbb{F}_4$, it readily follows that each of the generators of G_3 in the list above maps $\mathbb{F}_4 \times \mathbb{F}_2$ to itself. In fact, G_3 has the two orbits $\mathbb{F}_4 \times \mathbb{F}_2$ and $\mathbb{F}_4 \times \{\omega, \omega + 1\}$ in the point set. In particular, G_3 is neither point-transitive not circle transitive (a circle entirely contained in $\mathbb{F}_4 \times \mathbb{F}_2$ cannot be mapped to one having a point in the other point-orbit). In summary we obtain the following.

Proposition 6.7. *The automorphism group $\Gamma(f_3)$ of the Laguerre near-plane $\mathcal{L}(f_3)$ has order $2^7 \cdot 3$. Furthermore, $\Gamma(f_3)$ is neither point- nor circle-transitive but induces the full symmetric group S_4 on the set of generators.*

6.8. The automorphism group of $\mathcal{L}(f_1)$. From 5.1 we find that the following permutations are automorphisms of $\mathcal{L}(f_1)$.

- $(x, y) \mapsto (x, y + t_2 x^2 + t_1 x + t_0)$ for $t_2, t_1, t_0 \in \mathbb{F}_4$ such that $t_2 + t_1 + t_0 \in \mathbb{F}_2$ and $\omega t_2 + (\omega + 1)t_1 + t_0 \in \mathbb{F}_2$;
- $(x, y) \mapsto (x + \omega, y)$;
- $(x, y) \mapsto (\omega^2 x^2, y)$;
- $(x, y) \mapsto (x, y^2)$;
- $(x, y) \mapsto \begin{cases} (x, y + \omega^2 x^2), & \text{if } x \in \mathbb{F}_4 \setminus \{\omega + 1\} \\ (x, y^2 + \omega^2 x^2), & \text{if } x = \omega + 1 \end{cases}$;
- $(x, y) \mapsto \begin{cases} (x, y + x), & \text{if } x \in \mathbb{F}_4 \setminus \{1\} \\ (x, y^2 + x), & \text{if } x = 1 \end{cases}$

These automorphisms generate a group G_1. By looking at the x-coordinates we see that G_1 has two orbits $\{\{0\} \times \mathbb{F}_4, \{\omega\} \times \mathbb{F}_4\}$ and $\{\{1\} \times \mathbb{F}_4, \{\omega + 1\} \times \mathbb{F}_4\}$ on the set of generators.

We first show that the automorphism group $\Gamma(f_1)$ of $\mathcal{L}(f_1)$ cannot be transitive on the set of generators. Otherwise there is an automorphism γ that takes the generator $\{0\} \times \mathbb{F}_4$ to the generator $\{1\} \times \mathbb{F}_4$. Using the automorphism $(x, y) \mapsto (\omega^2 x^2, y)$, if necessary, we may assume that γ is of the form $(x, y) \mapsto (ax + 1, \beta_x(y))$ for some $a \in \mathbb{F}_4$, $a \neq 0$, and permutations β_x of $\mathbb{F}_4$. From 5.1 and 4.6.2 we see that the permutation $(x, y) \mapsto (ax + 1, y)$ takes $\mathcal{L}(f_1)$ to $\mathcal{L}(f)$ where

$$f(x, y, z) = \begin{cases} (x^2 + x + z^2 + z)(y^2 + y) + x + y^2 + z, & \text{if } a = 1, \\ (x^2 + x + y^2 + y)(z^2 + z) + x + y + z^2, & \text{if } a = \omega, \\ (y^2 + y + z^2 + z)(x^2 + x) + x^2 + y + z, & \text{if } a = \omega + 1. \end{cases}$$

But γ is a composition of this permutation and a permutation in Δ. Using Lemma 6.1 we now see that f_1 cannot be obtained in this way. This shows that $\Gamma(f_1)$ cannot be transitive on the set of generators. Hence, $\Gamma(f_1)$ has the same orbits as G_1 on the set of generators.

Let γ be an automorphism of $\mathcal{L}(f_1)$. Up to elements in G_1 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(x, y) = (x, a_x y^{m_x} + b_x)$ where $a_x, b_x \in \mathbb{F}_4$, $a \neq 0$, and $m_x = 1, 2$. Using the last three automorphisms in G_1 in the list above, if necessary, we may further assume that $m_0 = m_1 = m_{\omega+1} = 1$. By Lemma 6.1 we then must have

$$\begin{aligned} a_0[(a_1x^2 + a_1^2x + b_1^2 + b_1)(a_{\omega+1}z^2 + a_{\omega+1}^2z + b_{\omega+1}^2 + b_{\omega+1}) \\ +a_1^2x + a_\omega^2y^{m_\omega} + a_{\omega+1}^2z + b_1 + b_\omega + b_{\omega+1}] + b_0 \\ = (x^2+x)(z^2+z) + x + y + z \end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$. As before in 6.4 and 6.6 we see that $m_\omega = 1$, $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$, $b_1^2 + b_1 = b_{\omega+1}^2 + b_{\omega+1} = 0$, that is $b_1, b_{\omega+1} \in \mathbb{F}_2$ and $b_1 + b_\omega + b_{\omega+1} + b_0 = 0$. But then γ is of the first type in the list above.

This shows that the automorphism group of $\mathcal{L}(f_1)$ is contained in G_1. Furthermore, G_1 has order 2^9 since, from above, $|G_1/\Delta| = 4$, $|\Delta/\Delta_1| = 8$ and $|\Delta_1| = 16$. Using the first automorphisms in the list above we see that Δ is transitive on the generator $\{0\} \times \mathbb{F}_4$. The stabilizer $\Delta_{(0,0)}$ of $(0,0)$ has order 2^5 and is generated by the following automorphisms:

- $(x, y) \mapsto (x, y + t_2x^2 + t_1x)$ for $t_2, t_1 \in \mathbb{F}_4$ such that $t_2 + t_1 \in \mathbb{F}_2$ and $\omega t_2 + (\omega+1)t_1 \in \mathbb{F}_2$;
- $(x, y) \mapsto (x, y^2)$;
- $(x, y) \mapsto \begin{cases} (x, y + \omega^2x^2), & \text{if } x \in \mathbb{F}_4 \setminus \{\omega+1\} \\ (x, y^2 + \omega^2x^2), & \text{if } x = \omega+1 \end{cases}$;
- $(x, y) \mapsto \begin{cases} (x, y + x), & \text{if } x \in \mathbb{F}_4 \setminus \{1\} \\ (x, y^2 + x), & \text{if } x = 1 \end{cases}$

The first and third automorphisms then show that $\Delta_{(0,0)}$ is transitive on the generator $\{1\} \times \mathbb{F}_4$. Finally, the stabilizer $\Delta_{(0,0),(1,0)}$ of $(0,0)$ and $(1,0)$ has order 2^3 and is generated by $(x, y) \mapsto (x, y + s(x^2 + x))$ for $s \in \mathbb{F}_4$ and $(x, y) \mapsto (x, y^2)$. It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\} \times \mathbb{F}_4$. Since each circle is uniquely determined by its intersection with the three generators $\{0\} \times \mathbb{F}_4$, $\{1\} \times \mathbb{F}_4$ and $\{\omega\} \times \mathbb{F}_4$, we see that Δ is transitive on the set of circles. In summary we obtain the following.

Proposition 6.9. *The automorphism group $\Gamma(f_1)$ of the Laguerre near-plane $\mathcal{L}(f_1)$ has order 2^9. Furthermore, $\Gamma(f_1)$ is circle-transitive and induces a group of order 4 (the noncyclic group $\mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the cyclic group of order 2) on the set of generators and has precisely two orbits of length two each on this set. In particular, $\Gamma(f_1)$ is not point-transitive.*

6.10. The automorphism group of $\mathcal{L}(f_4)$. From 6.1 and 5.1 we find that the following permutations are automorphisms of $\mathcal{L}(f_4)$.

- $(x, y) \mapsto (x, y + t_1x^2 + t_1x + t_0)$ for $t_1 \in \mathbb{F}_4$, $t_0 \in \{0, \omega+1\}$ such that $t_1 + t_0 \in \{0, \omega\}$;
- $(x, y) \mapsto \begin{cases} (x, y + \omega + 1), & \text{if } x \in \{0, \omega\}, \\ (x, y), & \text{if } x = 1, \\ (x, \omega y^2), & \text{if } x = \omega + 1; \end{cases}$

– $(x,y) \mapsto \begin{cases} (x, y+\omega+1), & \text{if } x \in \{0, \omega+1\}, \\ (x,y), & \text{if } x = 1, \\ (x, \omega y^2), & \text{if } x = \omega; \end{cases}$

– $(x,y) \mapsto \begin{cases} (x, y+\omega), & \text{if } x \in \{0, \omega\}, \\ (x,y), & \text{if } x = \omega+1, \\ (x, (\omega+1) y^2), & \text{if } x = 1; \end{cases}$

– $(x,y) \mapsto \begin{cases} (x, \omega y^2), & \text{if } x = 0, \\ (x,y+x), & \text{if } x \in \{1, \omega\}, \\ (x, y), & \text{if } x = \omega+1; \end{cases}$

– $(x,y) \mapsto (x^2, y)$.

These automorphisms generate a group G_4 of order 2^7.

Let γ be an automorphism of $\mathcal{L}(f_4)$. We first assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(x,y) = (x, a_x y^{m_x} + b_x)$ where $a_x, b_x \in \mathbb{F}_4$, $a \neq 0$, and $m_x = 1, 2$. Using the automorphisms in G_4 in the list above, if necessary, we may further assume that $m_0 = m_1 = m_\omega = m_{\omega+1} = 1$. By Lemma 6.1 we then must have

$$\begin{aligned} & a_0[(a_1x^2 + \omega^2 a_1^2 x + b_1^2 + \omega^2 b_1)(a_\omega y^2 + \omega a_\omega^2 y + b_\omega^2 + \omega b_\omega) \\ & \qquad \cdot (a_{\omega+1} z^2 + \omega a_{\omega+1}^2 z + b_{\omega+1}^2 + \omega b_{\omega+1}) \\ & + (a_1x^2 + \omega^2 a_1^2 x + b_1^2 + \omega^2 b_1)(a_\omega y^2 + \omega^2 a_\omega^2 y + b_\omega^2 + \omega^2 b_\omega) \\ & + (a_1x^2 + \omega^2 a_1^2 x + b_1^2 + \omega^2 b_1)(a_{\omega+1} z^2 + \omega^2 a_{\omega+1}^2 z + b_{\omega+1}^2 + \omega^2 b_{\omega+1}) \\ & + (a_\omega y^2 + \omega a_\omega^2 y + b_\omega^2 + \omega b_\omega)(a_{\omega+1} z^2 + \omega a_{\omega+1}^2 z + b_{\omega+1}^2 + \omega b_{\omega+1}) \\ & + a_1^2 x + a_\omega^2 y + a_{\omega+1}^2 z + b_1 + b_\omega + b_{\omega+1}] + b_0 \\ & = (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z) + (x^2 + \omega^2 x)(y^2 + \omega^2 y) \\ & + (x^2 + \omega^2 x)(z^2 + \omega^2 z) + (y^2 + \omega y)(z^2 + \omega z) + x + y + z \end{aligned} \tag{3}$$

for all $x, y, z \in \mathbb{F}_4$. Comparing terms in which each of x, y and z occurs, i.e.,

$$\begin{aligned} & a_0(a_1 x^2 + \omega^2 a_1^2 x)(a_\omega y^2 + \omega a_\omega^2 y)(a_{\omega+1} z^2 + \omega a_{\omega+1}^2 z) \\ & \qquad = (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z), \end{aligned}$$

we obtain

$$a_0 = a_1 = a_\omega = a_{\omega+1} = 1.$$

Looking at terms involving both y and z but no x in (3) we find

$$b_1^2 + \omega^2 b_1 = 0, \text{ i.e., } b_1 \in \{0, \omega+1\}.$$

Now (3) becomes

$$\begin{aligned}
&(x^2+\omega^2x)(y^2+\omega y+b_\omega^2+\omega b_\omega)(z^2+\omega z+b_{\omega+1}^2+\omega b_{\omega+1})\\
&+(x^2+\omega^2x)(y^2+\omega^2y+b_\omega^2+\omega^2b_\omega)+(x^2+\omega^2x)(z^2+\omega^2z+b_{\omega+1}^2+\omega^2b_{\omega+1})\\
&+(y^2+\omega y+b_\omega^2+\omega b_\omega)(z^2+\omega z+b_{\omega+1}^2+\omega b_{\omega+1})+b_1+b_\omega+b_{\omega+1}+b_0\\
&=(x^2+\omega^2x)(y^2+\omega y)(z^2+\omega z)\\
&+(x^2+\omega^2x)(y^2+\omega^2y)+(x^2+\omega^2x)(z^2+\omega^2z)+(y^2+\omega y)(z^2+\omega z)
\end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$.

By looking at terms z^2 and y^2, respectively, one similarly finds that

$$\begin{aligned}
&b_\omega^2+\omega b_\omega=0, \text{ i.e., } b_\omega\in\{0,\omega\},\\
&b_{\omega+1}^2+\omega b_{\omega+1}, \text{ i.e., } b_{\omega+1}\in\{0,\omega\}.
\end{aligned}$$

Now $x=y=z=0$ yields $b_1+b_\omega+b_{\omega+1}+b_0=0$. Finally, by looking at the term x^2 one obtains

$$(b_\omega+b_{\omega+1})^2+\omega^2(b_\omega+b_{\omega+1})=0, \text{ i.e., } b_\omega+b_{\omega+1}\in\{0,\omega+1\}.$$

The last three conditions then imply that

$$b_\omega=b_{\omega+1}\in\{0,\omega\}$$

and hence

$$b_0=b_1\in\{0,\omega+1\}.$$

But now γ is of the form $(x,y)\mapsto(x,y+t_1x^2+t_1x+t_0)$ where $t_0=b_0\in\{0,\omega+1\}$ and $t_1+t_0=b_\omega\in\{0,\omega\}$.

Using the first and fifth automorphisms in the list above we see that Δ is transitive on the generator $\{1\}\times\mathbb{F}_4$. The third and fourth automorphisms then show that $\Delta_{(1,0)}$ is transitive on the generator $\{0\}\times\mathbb{F}_4$.

The stabilizer $\Delta_{(1,0),(0,0)}$ of $(1,0)$ and $(0,0)$ contains the following automorphisms:

- $(x,y)\mapsto(x,y+\omega(x^2+x))$;
- $(x,y)\mapsto\begin{cases}(x,y), & \text{if } x\in\mathbb{F}_2,\\ (x,(\omega+1)y^2+\omega+1), & \text{if } x=\omega,\\ (x,\omega y^2+\omega+1), & \text{if } x=\omega+1.\end{cases}$

  (This is the composition of the second and third automorphisms.)

It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\}\times\mathbb{F}_4$. Since each circle is uniquely determined by its intersection with the three generators $\{0\}\times\mathbb{F}_4$, $\{1\}\times\mathbb{F}_4$ and $\{\omega\}\times\mathbb{F}_4$, we see that Δ is transitive on the set of circles.

We finally show that every automorphism of $\mathcal{L}(f_4)$ fixes the generators $\{0\}\times\mathbb{F}_4$ and $\{1\}\times\mathbb{F}_4$. Using the transitivity of Δ on the circle set and the sixth automorphism in the list at the beginning of this section, if necessary, we may assume that we have an automorphism γ of the form $(x,y) \mapsto (sx+t, a_x y^{m_x})$ where $s, a_x \in \mathbb{F}_4 \setminus \{0\}$ and $m_x \in \{1,2\}$ for $x \in \mathbb{F}_4$. We now write γ as the composition of the permutations

$$\begin{aligned}
\gamma_1 :&(x,y) \mapsto (x+t, y),\\
\gamma_2 :&(x,y) \mapsto (sx, y),\\
\gamma_3 :&(x,y) \mapsto (x, a_x y^{m_x}),
\end{aligned}$$

as $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$. Then γ_3 takes $\mathcal{L}(f_4)$ to the same Laguerre near-plane as $(\gamma_1 \circ \gamma_2)^{-1} = \gamma_2^{-1} \circ \gamma_1$.

For example, if $t = 1$ and $s = \omega + 1$, and using Corollary 4.5 and 5.1.(2) and (3) we find that

$$a_0 f_4(a_1^2 x^{m_1}, a_\omega^2 y^{m_\omega}, a_{\omega+1}^2 z^{m_{\omega+1}})^{m_0} = g(y, z, x)$$

for all $x, y, z \in \mathbb{F}_4$, where g is the inverse of the partial map f_4 with respect to x, that is,

$$\begin{aligned}
g(x,y,z) =&(x^2+\omega^2 x)(y^2+\omega y)(z^2+\omega z) + (x^2+\omega^2 x)(y^2+\omega^2 y)\\
&+ (x^2+\omega^2 x)(z^2+\omega^2 z) + \omega^2(y^2+\omega y)(z^2+\omega z)\\
&+ x + \omega y^2 + \omega z^2
\end{aligned}$$

Explicitly one obtains

$$\begin{aligned}
a_0[(a_1 x^{2m_1} + \omega^2 a_1^2 x^{m_1})(a_\omega y^{2m_\omega} + \omega a_\omega^2 y^{m_\omega})(a_{\omega+1} z^{2m_{\omega+1}} + \omega a_{\omega+1}^2 z^{m_{\omega+1}})&\\
+(a_1 x^{2m_1} + \omega^2 a_1^2 x^{m_1})(a_\omega y^{2m_\omega} + \omega^2 a_\omega^2 y^{m_\omega})&\\
+(a_1 x^{2m_1} + \omega^2 a_1^2 x^{m_1})(a_{\omega+1} z^{2m_{\omega+1}} + \omega^2 a_{\omega+1}^2 z^{m_{\omega+1}})&\\
+(a_\omega y^{2m_\omega} + \omega a_\omega^2 y^{m_\omega})(a_{\omega+1} z^{2m_{\omega+1}} + \omega a_{\omega+1}^2 z^{m_{\omega+1}})&\\
+a_1^2 x^{m_1} + a_\omega^2 y^{m_\omega} + a_{\omega+1}^2 z^{m_{\omega+1}}]^{m_0}&\\
= (y^2+\omega^2 y)(z^2+\omega z)(x^2+\omega x) + (y^2+\omega^2 y)(z^2+\omega^2 z)&\\
+(y^2+\omega^2 y)(x^2+\omega^2 x) + \omega^2(z^2+\omega z)(x^2+\omega x) + y + \omega z^2 + \omega x^2&
\end{aligned} \tag{4}$$

for all $x, y, z \in \mathbb{F}_4$. Comparing terms x^2, y and z^2 in (4) we find that

$$m_0 m_\omega = 1,\ m_0 m_1 = m_0 m_{\omega+1} = 2,$$

$$a_\omega = a_0^{m_0},\ a_1 = a_{\omega+1} = \omega^{2m_0} a_0^{m_0}.$$

But then the coefficient of $x^2y^2z^2$ on the left-hand side in (4) becomes ω^2 whereas on the right-hand side it is 1 - a contradiction. This shows that $t = 1$, $s = \omega + 1$ is not possible. Similar arguments yield that all the other combinations for s and t except $s = 1$, $t = 0$ are not possible. From the list of automorphisms at the beginning of this section we see that only the last automorphism moves some generators. In particular, the generators $\{0\} \times \mathbb{F}_4$ and $\{1\} \times \mathbb{F}_4$ are fixed and $\{\omega\} \times \mathbb{F}_4$ and $\{\omega + 1\} \times \mathbb{F}_4$ can be exchanged. In summary we obtain the following.

Proposition 6.11. *The automorphism group $\Gamma(f_4)$ of the Laguerre near-plane $\mathcal{L}(f_4)$ has order 2^7. Furthermore, $\Gamma(f_4)$ is circle-transitive and induces on the set of generators a group of order 2 that fixes two generators.*

Looking at the orders of the automorphism groups or their transitivity properties we obtain the following.

Corollary 6.12. *The Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0, 1, 2, 3, 4$, are mutually non-isomorphic.*

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