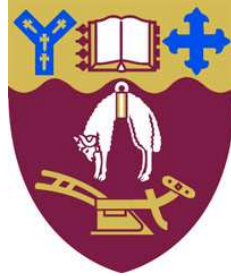University of Canterbury
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Constructive Notions of Compactness in Apartness Spaces

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Abstract

We present three criteria for compactness in the context of apartness spaces and Bishop-style constructive mathematics. Each of our three criteria can be summarised as requiring that there is a positive distance between any two disjoint closed sets. Neat locatedness and the product apartness give us three variations on this theme. We investigate how our three criteria relate to one another and to several existing compactness criteria, namely classical compactness, completeness, total boundedness, the anti-Specker property, and Diener's neat compactness.

Acknowledgements

A year ago I would have had difficulty imagining what this thesis has now become: I had resolved to finally learn what this constructive mathematics business was all about. As such, I had no idea what constructive topology would involve, yet alone how compactness would fit into it.

Over the past year I have developed a working understanding of and healthy appreciation for constructive mathematics and apartness spaces. Moreover, I have learnt some genuinely interesting results about the computational aspects of compactness. And, despite my initial confusion, I have thoroughly enjoyed my experience. This is in no small part due to the help and support of many people.

I cannot name everyone who has contributed directly or indirectly to this thesis, but I would like to acknowledge my family, my friends, and the department as a whole for providing a supportive environment. My supervisor Douglas Bridges deserves special mention. He has been enormously helpful from the beginning to the very end. Whatever I was doing—finding reading material, exploring ideas, writing, being stuck, or needeading some help proof reading—he was always willing to help. I could not have done any of this without his guidance and advice.

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Outline

Chapters 1 and 2 briefly introduce constructive mathematics and apartness spaces respectively. These are fascinating areas of research that can reveal elaborate mathematical structures that are lost classically. Neither subject can be done justice in the space we afford it. So we refer the reader to [9], [10], [19], and [27] for a more complete exposition.

Chapter 3 discusses the problems faced when framing the notion of compactness in a constructive setting. We discuss several existing criteria, which provide the backdrop for Chapter 4.

In Chapter 4 we introduce a three new conditions that capture various aspects of compactness in an apartness space. We then investigate these conditions and how they relate to one another and to the existing criteria. We show that these criteria characterise similar notions to those in Chapter 3. The differences between our criteria highlight the importance of locatedness and the need for something like the product apartness.

Finally, we conclude in Chapter 5.

Contents

Chapter 1

Constructive Mathematics

"The interesting thing about this book is that it reads essentially like ordinary mathematics, yet it is entirely algorithmic in nature if you look between the lines." —Donald Knuth on Errett Bishop's *Foundations of Constructive Mathematics* [6] (in [20])

What is constructive mathematics and why is it interesting? In short, constructive mathematics is the result of demanding more from proofs. The reason it is interesting is because (i) stronger proofs are philosophically satisfying, (ii) it exposes a rich structure that is not otherwise visible, and (iii) it has connections to other areas of mathematics, most notably recursive analysis. In this chapter we give a brief introduction to the principles of constructive mathematics, which we use throughout this thesis.

What is mathematical truth? When one classically asserts a statement P, it means that P is a tautology—that is, P must be true with respect to any *reasonable* truth assignment. However, if one asserts P constructively, one interprets that as "I can find a proof of P". The constructive interpretation is stronger than the classical interpretation—any constructively true statement is classically true, but not all classically true statements satisfy the constructive interpretation.

Let us look at an example to clarify this distinction. The following proposition is given

with a classical proof.

Proposition 1.1. *There exist irrational real numbers a and b such that a^b is rational.*

Proof. Note that $\sqrt{2}$ is irrational. Consider $\sqrt{2}^{\sqrt{2}}$. If this is rational, set $a = b = \sqrt{2}$ and we are done. If not, set $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$, whence we are also done. ∎

The above proof shows that a and b must exist, but it does not give explicit values for them. The problem is that we don't know whether or not $\sqrt{2}^{\sqrt{2}}$ is rational. A constructivist is therefore not satisfied by this proof.

What is the motivation for this interpretation of truth? There are two primary reasons. First, this interpretation means that truth and provability are equivalent, in the sense that there is no *a priori* notion of truth, only one of provability. The second reason is a practical consequence of the first; any constructive proof gives rise to an algorithm. This means that, at least in theory, constructive mathematics has applications to areas such as recursive analysis, algorithms, and numerical analysis; see [12], [16], [23], [8], [7], and [29] for more details.

1.1   Bishop-style Mathematics

There are actually many variants of constructive mathematics, some of which we will discuss. We use **Bishop-style mathematics** (abbreviated as **BISH**). BISH uses intuitionistic logic.[1]

Let A and B be statements, X a set and P a unary predicate. The intuitionistic interpretations of the basic logical connectives as follows.

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[1]Aside from intuitionistic logic, which we describe here, BISH also requires a formal set- or type-theoretic foundation. The standard foundations are Aczel-Myhill set theory [1, 2, 26] and Martin-Löf's type theory [24, 25]. We use **Intuitionistic Set Theory (IZF)**, as it has full separation; although there is strong evidence that **Constructive Set Theory (CZF)** can also be used, see [9] Chapter 2 and [17].

$\neg A$ means that we can derive a contradiction from A.

$A \wedge B$ means we can find a proof of A and a proof B.

$A \vee B$ means we can find a proof of A or we can find a proof of B. Note that this implies that we can decide which of the two holds. Thus $A \vee B$ is stronger than $\neg (\neg A \wedge \neg B)$.

$A \Rightarrow B$ means that, given a proof of A, we can find a proof of B. Essentially, this statement says that there is an algorithm for converting a proof of A into a proof of B.

$\forall x \in X \ P(x)$ means that, given x and a proof that $x \in X$, we can find a proof that x satisfies P; in other words, we have an algorithm which, applied to an object x and the data arising from a proof that $x \in X$, shows that $P(x)$ holds.

$\exists x \in X \ P(x)$ means that we can construct an object x which is in X and which satisfies P.

These interpretations form the building blocks of Bishop-style mathematics. Careful thought should make it clear which deductions can be made constructively and which deductions are not constructive. For example,

$$\neg \exists x \in X \ P(x) \Rightarrow \forall x \in X \ \neg P(x),$$

is constructively derivable, but

$$\neg \forall x \in X \ P(x) \Rightarrow \exists x \in X \ \neg P(x)$$

is not. Proving an existence statement is more difficult constructively, as we need to be able to describe the object; it is not sufficient to prove that an object cannot fail to exist.

Note that, contrary to popular belief, there is room for proof by contradiction in constructive mathematics. However, we can only use it to prove negative statements. The following is an example of its use.

Proposition 1.2. *The real number $\sqrt{2}$ is not rational.*

Proof. Suppose that $\sqrt{2}$ is rational. Then choose integers a and b with $(a/b)^2 = 2$. Assume, without loss of generality, that a and b are coprime. Now $a^2 = 2b^2$, whence a must be even. Hence, $4(a/2)^2 = 2b^2$ and $2(a/2)^2 = b^2$. So b is also even—a contradiction. ∎

Constructive definitions are also slightly different to their classical counterparts. For example, we might take irrational to mean more than just "not rational"; instead we demand that an **irrational** number is not equal to any rational number—that is, x is irrational means[2]

$$\forall a, b \in \mathbf{Z} \left(b > 0 \implies \frac{a}{b} \neq x \right).$$

This is classically equivalent to not being rational, but in the constructive setting this is a stronger criterion. The following proposition shows how we can work with this definition.

Proposition 1.3. *The real number $\sqrt{2}$ is irrational.*

Proof. Let a and b be integers with $b > 0$. Proposition 1.2 shows that $a^2 \neq 2b^2$. Consider the case where $b \leq a \leq 2b$. We have

$$\left| \sqrt{2} - \frac{a}{b} \right| 4 \geq \left| \sqrt{2} - \frac{a}{b} \right| \left| \sqrt{2} + \frac{a}{b} \right| = \left| 2 - \left(\frac{a}{b} \right)^2 \right| = \frac{|2b^2 - a^2|}{b^2} \geq \frac{1}{b^2},$$

whence $\sqrt{2} \neq a/b$. If $a < b$, then $a/b < 1 < \sqrt{2}$. And, if $a > 2b$, then $a/b > 2 > \sqrt{2}$. ∎

1.2 Equality, Inequality, and Counting

Terms such as "finite" and "countable" take slightly different meanings in the constructive setting.

- A set S is called **finitely enumerable** if $S = \{s_0, s_1, \cdots, s_n\}$ for some $n \in \mathbf{N}$.

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[2]See the next section for a more precise definition of inequality.

- A set S is called **finite** if $S = \{s_0, s_1, \cdots, s_n\}$ for some $n \in \mathbf{N}$ and $s_i \neq s_j$ whenever $i \neq j$. We take the empty set to be finite.

- A set S is called **countable** if $S = \{s_0, s_1, \cdots\}$ for some sequence $(s_n)_{n \in \mathbf{N}}$.

- A set S is called **denumerable** if S is finite or $S = \{s_0, s_1, \ldots\}$ for some sequence $(s_n)_{n \in \mathbf{N}}$ such that, for each $i, j \in \mathbf{N}$ with $i \neq j$, $s_i \neq s_j$.

Equality and inequality are separate notions constructively. Let X and Y be sets and $f : X \to Y$ a function. We always assume that X and Y have an equality relation $=$ and that f is **extensional**—that is, if $a = b$, then $f(a) = f(b)$. We also require that, for any set S, if $x = y$, then $x \in S$ if and only if $y \in S$. In BISH a set sometimes comes equipped with an inequality relation $\neq$ distinct from the denial of equality and satisfying the two properties

$$\forall x, y \in X \quad (x \neq y \Rightarrow \neg(x = y)),$$
$$\forall x, y \in X \quad (x \neq y \Rightarrow y \neq x).$$

An inequality relation $\neq$ on a set X is not generally **decidable**, in the sense that for each $x, x' \in X$, either $x = x'$ or $x \neq x'$. Indeed, the **denial inequality**, defined by

$$\forall x, y \in X \, (x \neq y \iff \neg(x = y)),$$

on the set $\mathbf{R}$ of real numbers is not decidable. However, denumerable sets, such as $\mathbf{Z}$ and $\mathbf{Q}$, do have decidable inequalities.

An inequality relation $\neq$ on a set X is said to be **tight** if, for each $x, y \in X$,

$$\neg(x \neq y) \Rightarrow x = y,$$

For example, the standard inequality on a metric space (X, ρ) is given by

$$x \neq y \iff \rho(x, y) > 0$$

and is tight.

In the presence of inequality relations on the domain X and codomain of a function f, we say that f is **strongly extensional** if

$$\forall x, y \in X \ (f(x) \neq f(y) \Rightarrow x \neq y).$$

We define the standard inequalities on real numbers and sets as follows.

$$\forall x, y \in \mathbf{R} \ (x \neq y \iff \exists n \in \mathbf{N} \ (|x - y| \geq 2^{-n}).$$
$$\forall S, T \ (S \neq T \iff \exists x \in S(x \notin T) \vee \exists x \in T(x \notin S)).$$

In light of this, we prefer to refer to a set S as being **inhabited** if $S \neq \emptyset$, rather than using the double negative term nonempty.

1.3 Non-constructive Principles

There are a number of statements that are known not to be derivable in BISH. The canonical example is the **law of excluded middle (LEM)**—the assertion that, for any statement P, P or its negation holds.

Let us investigate why LEM is not constructive. Classically, every statement is assigned a truth value and its negation is assigned the opposite value; thus it is impossible for both P and $\neg P$ to be assigned the value false; likewise, it is impossible for $P \vee \neg P$ to be assigned the value false, whence the statement must be true. However, the constructive interpretation of LEM is

Given an arbitrary statement P, I can find a proof of P *or I can find a proof of* $\neg P$.

This assertion is clearly unreasonable, as Gödel sentences, the continuum hypothesis, and the Riemann hypothesis are all examples of statements for which, in some sense,

we cannot find proofs or counter-proofs.

We can formally show that a statement is non-constructive by providing a model of BISH in which it is provably false. Alternatively, we can use a **Brouwerian counterexample** which reduces the statement in question to a known non-constructive statement. For example, the following proves that the axiom of choice is non-constructive using LEM.

Since, in our current constructive model we do not accept LEM, we must also reject the **axiom of choice**, as it implies LEM. The axiom of choice states that, for any binary predicate P and sets X and Y,

$$\left(\forall x \in X \; \exists y \in Y \; P(x,y)\right) \implies \left(\exists f \in Y^X \; \forall x \in X \; P(x, f(x))\right).$$

To derive LEM from this, let P be an arbitrary statement. Let[3]

$$a = \{0\} \cup \{1 : P\}, \quad b = \{0 : P\} \cup \{1\},$$

and

$$X = \{a, b\}, \quad Y = \{0, 1\}.$$

Then, for every $x \in X$, there exists $y \in Y$ such that $y \in x$: for if $x = a$, then $0 \in x$, and, if $x = b$, then $1 \in x$. So, by the axiom of choice, there exists a function $f : X \to Y$ such that, for every $x \in X$, $f(x) \in x$. Now, we can decide whether $f(a) = f(b)$ or $f(a) \neq f(b)$, as the values of f belong to $\{0, 1\}$. If $f(a) = f(b)$, then $f(a) \in a \cap b$, whence P holds. On the other hand, if $f(a) \neq f(b)$, then $\neg (a = b)$ and P is false. This proves that LEM holds.

Despite the constructive failure of the full axiom of choice, there are two weaker forms that are normally accepted by practitioners of BISH.[4]

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[3] We use $\{a : P\}$ to denote $\{x : x = a \wedge P\}$.

[4] See, however, the work of Richman on choice-free constructive mathematics [28].

- The axiom of **countable choice**: for any binary predicate P and set Y,

$$(\forall n \in \mathbf{N} \; \exists y \in Y \; P(n, y)) \implies \left(\exists f \in Y^{\mathbf{N}} \; \forall n \in \mathbf{N} \; P(n, f(n)) \right).$$

- The axiom of **dependent choice**: for any binary predicate P and set A,

$$(\forall a \in A \; \exists a' \in A \; P(a, a')) \implies$$
$$\forall a \in A \; \exists f \in A^{\mathbf{N}} (f(0) = a \wedge \forall n \in \mathbf{N} \; P(f(n), f(n+1))).$$

The axiom of choice and LEM are two highly non-constructive statements. There are many weaker statements, often trivially true classically, that cannot be proved constructively. A partial list is the following.

WLEM: **The weak law of excluded middle:** for any statement P, $\neg P \vee \neg\neg P$.

LPO: **The limited principle of omniscience:** for any binary sequence a, either $a_n = 0$ for each n or there exists n such that $a_n = 1$; in symbols,

$$\forall a \in \{0, 1\}^{\mathbf{N}} \left(\forall n \, (a_n = 0) \vee \exists n \, (a_n = 1) \right)$$

LLPO: **The lesser limited principle of omniscience:** for any binary sequence a such that $a_j a_k = 0$ whenever $j \neq k$, either $a_{2n} = 0$ for each $n \in \mathbf{N}$ or $a_{2n+1} = 0$ for each $n \in \mathbf{N}$.

MP: **Markov's principle:** for any binary sequence a for which it is false that $a_n = 0$ for each n, there exists $n \in \mathbf{N}$ such that $a_n = 1$.

It is worth noting that LPO is equivalent to the statement

$$\forall x \in \mathbf{R} \; (x = 0 \vee x \neq 0).$$

Similarly, MP is equivalent to

$$\forall x \in \mathbf{R} \; (\neg(x = 0) \implies x \neq 0).$$

Clearly, LPO implies LLPO and MP and LEM implies WLEM and LPO. See [10] for a further discussion on the constructive properties of the constructive real numbers.

1.4   Models of BISH

Models of BISH are systems in which we can prove at least as much as we can prove in BISH. These are helpful for two reasons. Firstly, results in a model give us intuition about what to expect in BISH. And, secondly, if we can disprove a statement in a model of BISH, then we know that that statement cannot be proved in BISH.

1.4.1   CLASS

Classical mathematics (abbreviated **CLASS**) is a model of BISH. This is simply because any statement that holds in the constructive sense is also true classically. Classical logic is BISH with LEM added.

1.4.2   RUSS

Russian constructivism (abbreviated **RUSS**) [10, 30, 22] is also known as the **recursive model**. RUSS attempts to capture recursive analysis in logical form. RUSS adds two main axioms to BISH. The first is MP and the second is

CPF: There is an enumeration $\varphi_1, \varphi_2, \cdots$ of the set of partial functions from $\mathbf{N}$ to $\mathbf{N}$ with countable domains.

Note that a **partial function** f from X to Y is a function from a subset dom(f) of X (called the domain of f) to Y. This is a form of the **Church-Markov-Turing thesis**. CPF can be interpreted as asserting that all functions are computable. Note that CPF is provably false in CLASS, by a diagonalisation argument.

The "spirit" of RUSS is that everything can be represented by a natural number. Every statement in RUSS is to be interpreted as a statement about computability.

RUSS is inconsistent with CLASS. In particular, LPO and LLPO are provably false in it. Indeed, LPO corresponds to the halting problem. This justifies our earlier assertion that LPO and LLPO are non-constructive.

Another interesting result in RUSS is Specker's theorem. This theorem is important in our later study of compactness. It essentially states that $[0, 1]$ is not compact in RUSS in a very strong way.

Theorem 1.4 (Specker). *In RUSS, there exists a strictly increasing sequence $(r_n)_{n \in \mathbf{N}}$ in $\mathbf{Q} \cap [0, 1]$ that is eventually bounded away from every point in $[0, 1]$—that is, for every $x \in [0, 1]$, there exist $\delta > 0$ and $N \in \mathbf{N}$ such that, for every $n \geq N$, $|x - r_n| \geq \delta$.*

For a proof of Specker's theorem see [10] Chapter 3, Theorem 3.1. The basic idea behind the proof is that the "limit" of the sequence is not a computable real number.

Additionally, in RUSS all functions from $\mathbf{R}$ to $\mathbf{R}$ are continuous (though this requires slightly more than just MP and CPF to prove) and the intermediate value theorem is false.

1.4.3 INT

The last model we discuss is **Brouwer's intuitionism** (abbreviated **INT**) [10, 30, 14]. Again, we obtain it by adding two main principles to BISH. First, we must give several definitions.

We define a metric ρ on $\mathbf{N}^{\mathbf{N}}$ by

$$\rho(a, b) \equiv \inf \left\{ 2^{-n} : \forall i \leq n \ a_i = b_i \right\}.$$

Relative to this metric $\mathbf{N}^{\mathbf{N}}$ is a complete, separable metric space.

First we add the **principle of continuous choice**:

CC Any function from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$ is continuous. And, if $P \subset \mathbf{N}^{\mathbf{N}} \times \mathbf{N}$ and, for each $a \in \mathbf{N}^{\mathbf{N}}$, there exists $n \in \mathbf{N}$ with $(a, n) \in P$, then there is a **choice function** $f : \mathbf{N}^{\mathbf{N}} \to \mathbf{N}$ such that $(a, f(a)) \in P$ for each $a \in \mathbf{N}^{\mathbf{N}}$.

Note that CC is incompatible with CPF ([10] Chapter 5, Theorem 2.2). Thus INT and RUSS are incompatible. Also, LPO and LLPO are incompatible with CC.

We say that a set S is **detachable** if, for any x, either $x \in S$ or $x \notin S$. For any set S, let S^* be the set of all finite sequences in S and let $S^{\mathbf{N}}$ be the set of all infinite sequences in S. A detachable subset σ of $\{0, 1\}^*$ is called a **binary fan** if, for each $(a_0, a_1, \cdots, a_n) \in \sigma$ with $n > 0$, the restriction $(a_0, a_1, \cdots, a_{n-1})$ is also in σ. An infinite sequence $a = (a_0, a_1, \cdots) \in \{0, 1\}^{\mathbf{N}}$ is called a **path** in the binary fan σ if, for each $n \in \mathbf{N}$, $(a_0, a_1, \cdots, a_n) \in \sigma$. A subset B of a binary fan σ is called a **bar** for σ if every path a in σ has a **prefix** in B—that is, $(a_0, a_1, \cdots, a_n) \in B$ for some $n \in \mathbf{N}$. A bar B for a binary fan σ is called **uniform** if there exists $N \in \mathbf{N}$ such that, for every path a in σ, there exists $n \in \mathbf{N}$ with $n \leq N$ and $(a_0, a_1, \cdots, a_n) \in B$.

The second principle we add is called the **fan theorem**:

FT Every detachable bar of a binary fan is uniform.

The fan theorem is a contrapositive form of the classical König's lemma [21]. Thus it is also true in CLASS. Note that the name "fan theorem" is a misnomer, as we consider it to be an axiom, rather than a theorem.

In fact FT is equivalent to the following result [10, 5, 18].

Theorem 1.5. *Every uniformly continuous $f : [0, 1] \to (0, \infty)$ has positive infimum.*

Theorem 1.5 is, however, false in RUSS via an explicit counterexample [10]. Thus Theorem 1.5 is independent of BISH.

We can also add the following axiom to INT to make MP provably false.

Kripke's Schema: For each proposition P there exists an increasing binary sequence $a \in \{0,1\}^{\mathbf{N}}$ such that P holds if and only if $a_n = 1$ for some $n \in \mathbf{N}$.

This concludes our introduction to the principles of constructive mathematics. In the next chapter we build a framework for topology in BISH.

Chapter 2

Apartness and Uniformity

"Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size."—Errett Bishop ([6], page 63)

Apartness spaces provide a constructive framework for topology. They were developed by Bridges and Vîţă [9] in 2000. Before giving the axioms for those spaces, we introduce some notions of complement for a subset S of a set X with an inequality. We have

the **logical complement**
$$\neg S \equiv \{x \in X : \neg (x \in S)\},$$

the **complement**
$$\sim S \equiv \{x \in X : \forall s \in S \, (x \neq s)\},$$

and the **apartness complement**

$$-S \equiv \{x \in X : \{x\} \bowtie S\},$$

where $\bowtie$ is an apartness relation as introduced below.

Definition 2.1. *A apartness space is an inhabited set X and a binary relation $\bowtie$ on subsets of X satisfying the following for $A, B, C \subset X$.*

B0 $A \bowtie B \implies B \bowtie A$.

B1 $X \bowtie \emptyset$.

B2 $-A \subset\sim A$.

B3 $A \bowtie (B \cup C) \iff (A \bowtie B \wedge A \bowtie C)$.

B4 $-A \subset\sim B \implies -A \subset -B$.

B5 $\forall x \in -A \ \exists D \subset X \ (x \in -D \wedge X = -A \cup D)$.

Note that we will not discuss non-symmetric apartness spaces (those without B0) or pre-apartness spaces (those lacking B5). If $A \bowtie B$, we say that A is apart from B or that A and B are apart. For $x \in X$ and $S \subset X$, we write $x \bowtie S$ rather than $\{x\} \bowtie S$.

An apartness space $(X, \bowtie)$ induces a topology on X in which the apartness complements form a base of open sets. Classically, given a topological space (X, τ), one can define an apartness $\bowtie$ on X by

$$\forall A, B \subset X \ \left(A \bowtie B \iff \bar{A} \cap \bar{B} = \emptyset \right). \tag{2.1}$$

However, an apartness space is designed to capture more information than just the topology of the space. Two sets being apart should capture the notion of them having a positive distance between them. In this sense, the definition given in (2.1) is not very useful. For example, in $\mathbf{R}^2$, the sets

$$\left\{ (x, y) \in \mathbf{R}^2 : xy = 0 \right\},$$
$$\left\{ (x, y) \in \mathbf{R}^2 : xy = 1 \right\}$$

have disjoint closures, but they do not have a positive distance between them.

2.1 Uniform Spaces

Abstractions of metric or distance notions are usually defined in the context of uniform spaces. We will give a constructive definition of a uniform space and show how this induces an apartness. First we introduce some notation connected with subsets of the Cartesian product of a set X with itself.

The **diagonal** of X^2 is

$$\Delta \equiv \{(x, x) : x \in X\}.$$

For $W, W' \subset X^2$,

$$W \circ W' \equiv \left\{(x, z) \in X^2 : \exists y \in X \ ((x, y) \in W \wedge (y, z) \in W')\right\},$$

$$W^1 \equiv W, \text{ and } W^{n+1} \equiv W^n \circ W \quad (n \in \mathbf{N}),$$

and

$$W^{-1} \equiv \left\{(x, y) \in X^2 : (y, x) \in W\right\}.$$

We call W **symmetric** if $W = W^{-1}$. If $S \subset X$, then

$$W[S] \equiv \{y \in X : \exists s \in S \ (s, y) \in W\}.$$

If $S = \{x\}$ with $x \in X$, we write

$$W[x] \equiv W[\{x\}] = \{y \in X : (x, y) \in W\}.$$

We recall here that an inhabited set $\mathcal{F}$ of inhabited subsets of X is a **filter** if

- the intersection of two sets in $\mathcal{F}$ belongs to $\mathcal{F}$, and

- supersets of sets in $\mathcal{F}$ belong to $\mathcal{F}$.

Definition 2.2. *A **uniform space** is an inhabited set X equipped with an inequality relation $\neq$ and a set $\mathcal{U}$ of subsets of X^2 such that the following hold.*

U1 $\mathcal{U}$ is a filter on X^2.

U2 For each $x, y \in X$, $x \neq y$ if and only if there exists $U \in \mathcal{U}$ such that $(x, y) \in \neg U$.

U3 For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

U4 For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $X^2 = U \cup \neg V$.

U5 For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \subset U^{-1}$.

A member of $\mathcal{U}$ is called an **entourage**. A **base of entourages** is a set $\mathcal{B} \subset \mathcal{U}$ such that, for each $U \in \mathcal{U}$ there exists $V \in \mathcal{B}$ with $V \subset U$. A uniform space has an associated **uniform topology**, in which the sets of the form $U[x]$ with $U \in \mathcal{U}$ form a base of neighbourhoods of the point $x \in X$.

Any metric space (X, ρ) is also a uniform space in which

$$\left\{ \rho^{-1} \left(\left[0, \frac{1}{n+1} \right] \right) : n \in \mathbf{N} \right\}$$

is a base of entourages.

The Cartesian product

$$X = \prod_{i \in I} X_i$$

of a family $((X_i, \mathcal{U}_i))_{i \in I}$ of uniform spaces has a natural uniform structure, the **product uniformity**, in which a base of entourages consists of all sets of the form

$$\left\{ (\mathbf{x}, \mathbf{y}) \in X^2 : \forall i \in F \left((x_i, y_i) \in U_i \right) \right\}$$

with F a finitely enumerable subset of I and $U_i \in \mathcal{U}_i$ for each $i \in F$.

Given a uniform space $(X, \mathcal{U})$ we define a binary relation $\bowtie$ between subsets A, B of X by

$$A \bowtie B \iff \exists U \in \mathcal{U} \ (A \times B \subset \neg U). \tag{2.2}$$

It can be shown that this relation is an apartness (we call it the **uniform apartness**)

on X, and that, for each $A \subset X$,

$$-A = \{x \in X : \exists U \in \mathcal{U} \ (U[x] \subset \neg A)\}.$$

This implies that the topology induced by the apartness is the same as the uniform topology on X.

2.2 Product Apartnesses

There are two natural categorical notions in apartness spaces: the product of two apartness spaces and subspaces.

A subspace of an apartness space is defined in the obvious manner and behaves *mostly* as expected. Note that some properties do not pass immediately to subspaces. For example, a subspace of a separable space is not necessarily separable. Regularity conditions such as weak or neat locatedness (which we define later) will ensure that a subspace inherits more properties. See [9] for more details.

The product of two apartness spaces is more interesting for our purposes.

Definition 2.3. *Let* $(X, \bowtie_X)$ *and* $(Y, \bowtie_Y)$ *be apartness spaces. Then the* ***product apartness*** $\bowtie_{X \times Y}$ *on* $X \times Y$ *is defined as follows. Let* S, T *be subsets of* $X \times Y$. *Then* $S \bowtie T$ *if and only if there exist* $m, n \in \mathbf{N}$,

$$A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n \subset X$$

and

$$A_1', A_2', \cdots, A_m', B_1', B_2', \cdots, B_n' \subset Y$$

such that

$$S \subset \bigcup_{i=1}^{m} A_i \times A_i', \quad T \subset \bigcup_{j=1}^{n} B_j \times B_j'$$

and, for each i, j *with* $1 \le i \le m$ *and* $1 \le j \le n$, *either* $A_i \bowtie_X B_j$ *or* $A_i' \bowtie_Y B_j'$. *The pair* $(X \times Y, \bowtie_{X \times Y})$ *is called the* ***product of the apartness spaces*** $(X, \bowtie_X)$

and $(Y, \bowtie_Y)$.

Applying B5 we see that, for each $A \subset X \times Y$,

$$-A = \{(x, y) \in X \times Y : \exists E \subset X \ \exists F \subset Y \ ((x, y) \in -E \times -F \subset \sim A)\}.$$

It follows from this that the product apartness induces the usual product topology. However, the product apartness is somewhat irregular, in the sense that it does not coincide with the product of uniform spaces. Consider, for example, $\mathbf{R}^2$. Let

$$\begin{aligned} S &= \{(z, z) : z \in \mathbf{R}\}, \\ T &= \left\{(x, y) \in \mathbf{R}^2 : |x - y| \geq 1\right\}. \end{aligned}$$

Considered as subsets of the metric space $\mathbf{R}^2$, S and T are apart, since for each $(z, z) \in S$ and $(x, y) \in T$,

$$\max\{|x - z|, |y - z|\} \geq \frac{1}{2}(|x - z| + |z - y|) \geq \frac{1}{2}|x - y| \geq \frac{1}{2}.$$

However, if S and T are considered as subsets of the product apartness space $\mathbf{R} \times \mathbf{R}$, they are not apart. To see this, suppose that

$$S \subset \bigcup_{i=0}^{m} A_i \times A_i' \subset \mathbf{R}^2.$$

Then, for each $n \in \mathbf{N}$, choose $i_n \in \{0, 1, \cdots, m\}$ such that $(n, n) \in A_{i_n} \times A_{i_n}'$. Since i_n can only take finitely many values over infinitely many indices $n \in \mathbf{N}$, there exist $n, n' \in \mathbf{N}$ with $n \neq n'$ and $i_n = i_{n'}$—this is an application of the **pigeonhole principle**, which states that

$$\forall n \in \mathbf{N} \ \forall f \in \{0, 1, \cdots, n\}^{\{0, 1, \cdots, n+1\}} \ \exists i, j \in \{0, 1, \cdots, n+1\} \ (i \neq j \wedge f(i) = f(j)).$$

Now, $|n - n'| \geq 1$, so

$$(n, n') \in T \cap \left(A_{i_n} \times A_{i_n}'\right).$$

This shows that $\neg(S \bowtie T)$. Note that we now have two different apartnesses on $\mathbf{R}^2$ that induce the same topology.

2.3   Total Boundedness

We now discuss some notions that are closely connected with the compactness properties that are the main object of our investigations.

Definition 2.4. *Let $(X, \mathcal{U})$ be a uniform space. We say that X is **weakly totally bounded** if, for each $U \in \mathcal{U}$, there exist $n \in \mathbf{N}$ and $A_1, \cdots, A_n \subset X$ such that $X = \bigcup_{i=1}^n A_i$ and such that, for each $i \in \mathbf{N}$ with $1 \le i \le n$, $A_i \times A_i \subset U$. If also A_i is inhabited for each applicable i, then X is said to be **strongly totally bounded**.*

Note that these conditions are classically equivalent, and that, constructively, a strongly totally bounded space is weakly totally bounded.

Given a strongly totally bounded uniform space, we can recover the uniformity from the apartness it induces: Let $(X, \bowtie)$ be an apartness space. We say that $A \subset X$ is **well-contained** in $B \subset X$ if there exists $C \subset X$ such that $B \cup C = X$ and $C \bowtie A$; we write $A \ll B$ to denote this. Let $n \in \mathbf{N}$ and $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_n \subset X$ satisfy $\bigcup_{i=1}^n A_i = X$ and $A_i \ll B_i$ for each applicable i. If the apartness $\bowtie$ on X is induced by a uniform structure $\mathcal{U}$, then $\bigcup_{i=1}^n B_i \times B_i$ is an entourage of $\mathcal{U}$. Moreover, entourages of this form with each A_i inhabited form a base of entourages if and only if the uniform space is strongly totally bounded. See [9] Propositions 3.9.13 and 3.9.14 for more details.

2.4   Continuity Properties

Definition 2.5. *Let $(X, \bowtie_X)$ and $(Y, \bowtie_Y)$ be apartness spaces, and $f : X \to Y$ a function. We say that f is*

- *topologically continuous if $f^{-1}(U)$ is open in X whenever U is an open subset of Y;*

- *continuous if, for each $x \in X$ and $A \subset X$,*

$$f(x) \in -_Y f(A) \Rightarrow x \in -_X A$$

—that is to say $f^{-1}(-_Y f(A)) \subset -_X A$;

- *strongly continuous if, for each $A, B \subset X$, $f(A) \bowtie_Y f(B)$ implies that $A \bowtie_X B$.*

Clearly, strong continuity implies continuity. Moreover, in a metric space, continuity corresponds to the usual ε-δ definition of continuity. Continuity and topological continuity are equivalent if the range Y has the so-called **weak nested neighbourhoods property**, which asserts that

$$\forall A \subset Y \ \forall x \in -A \ \exists B \subset Y \ (x \in -B \ \wedge \ \neg B \subset -A).$$

Strong continuity is related to the well-studied notion of uniform continuity, which we now introduce. Let f be a function from a uniform space $(X, \mathcal{U})$ to a uniform space $(Y, \mathcal{V})$, and define $f \times f : X \times X \to Y \times Y$ by

$$(f \times f)(x, x') \equiv (f(x), f(x')) .$$

We say that f is **uniformly continuous** if $(f \times f)^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}$. Uniform continuity implies strong continuity. It can be shown that f is uniformly continuous if and only if $f \times f$ is strongly continuous with respect to the product apartnesses on $X \times X$ and $Y \times Y$; see [9], Proposition 3.3.4.

2.5 Locatedness Properties

In a metric space (X, ρ), we say that $S \subset X$ is **located** if

$$\rho(x, S) \equiv \inf \{\rho(x, y) : y \in S\}$$

exists for each $x \in X$—that is, for each $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$, either there exists $y \in S$ with $\rho(x, y) < \beta$ or $\rho(x, y) > \alpha$ for each $y \in S$. Classically, every subset of a metric space is located. However, constructively this is not always the case: for any statement P, the inhabited set

$$S = \{0\} \cup \{1 : P\}$$

is located subset of the discrete metric space $\{0, 1\}$ if and only if $P \vee \neg P$ holds. Locatedness is a very useful condition in the constructive study of metric spaces, so we would like an analogue for locatedness in apartness spaces.

There is at least one natural analogue of the metric property of locatedness in the context of a uniform space $(X, \mathcal{U})$. A subset S of X is called **almost located** [11] if, for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that, for every $x \in X$, either $S \cap U[x] \neq \emptyset$ or $S \cap V[x] = \emptyset$. Although there is no obvious analogue of locatedness for subsets of a general apartness space, there are two useful locatedness notions therein.

Definition 2.6. *Let $(X, \bowtie)$ be an apartness space, and $S \subset X$. We say that S is* ***weakly located*** *if, for each $x \in X$ and $A \subset X$ with $x \in -A$, either $S \cap -A \neq \emptyset$ or $x \in -S$.*

Weak locatedness is strictly weaker than locatedness. A stronger alternative is given in

Definition 2.7. *Let $(X, \bowtie)$ be an apartness space. We say that an ordered pair (A, B) of subsets of X is a* ***neat cover*** *of X if there exist $A', B' \subset X$ such that $A \cup A' = B \cup B' = X$ and $A' \bowtie B'$. We say that $S \subset X$ is* ***neatly located*** *if for any neat cover (A, B) of X, either $A \cap S \neq \emptyset$ or $S \subset B$.*

Intuitively, a neat cover is a pair of sets covering the whole space and with a "positive overlap". Note that, if (A, B) is a neat cover of an apartness space X, then $A \cup B = X$ and (B, A) is also a neat cover. Classically, all sets are neatly located. In a metric space, neat locatedness implies locatedness. Constructively, in an arbitrary apartness space, neat locatedness does not necessarily imply weak locatedness; however, Lemma 2.9, for which we introduce the next definition, shows that under a certain separation condition, this is the case.

Definition 2.8. *Let $(X, \bowtie)$ be an apartness space. We say that X has the nested neighbourhoods property if*

$$\forall x \in X \; \forall A \subset X \; (x \in -A \implies \exists B \subset X \; (x \in -B \wedge \neg B \bowtie A)).$$

Any uniform space has the nested neighbourhoods property.

Lemma 2.9. *Let X be an apartness space with the nested neighbourhoods property. Then any neatly located subset of X is weakly located.*

Proof. Let $S \subset X$ be neatly located. Let $A \subset X$ and $x \in -A$. Then, by B5, there exists $B \subset X$ such that $x \in -B$ and $B \cup -A = X$. By the nested neighbourhoods property, there exists $C \subset X$ such that $x \in -C$ and $\neg C \bowtie B$, whence $-C \bowtie B$. Again, by B5, there exists $D \subset X$ such that $x \in -D$ and $D \cup -C = X$. Then $(-A, D)$ is a neat cover, as $-A \cup B = D \cup -C = X$ and $B \bowtie -C$. So either $S \cap -A \neq \emptyset$ or $S \subset D$ and $x \in -D \subset -S$. ∎

Lemma 2.10. *A strongly totally bounded subset of a uniform space is neatly located.*

Proof. Let $(X, \mathcal{U})$ be a uniform space and S a strongly totally bounded subset thereof. Let (A, B) be a neat cover of X and choose $A', B' \subset X$ and $U \in \mathcal{U}$ such that $A \cup A' = B \cup B' = X$ and $A' \times B' \subset \neg U$. Choose a finitely enumerable $F \subset S$ such that $S \subset U[F]$. Since $A \cup A' = X$ and F is finite, either $F \cap A \neq \emptyset$ or $F \subset A'$. In the former case we are done, as $S \cap A \supset F \cap A \neq \emptyset$. Suppose, on the other hand, that $F \subset A'$. Let $y \in S$. Then there exists $x \in F$ with $y \in U[x]$. If $y \in B'$, then $(x, y) \in A' \times B' \cap U = \emptyset$ — a contradiction. Thus $S \subset \neg B' \subset B$. ∎

2.6   The Hausdorff Property

The following defines a very useful regularity condition for apartness spaces, namely the ability to separate distinct points by open sets.

Definition 2.11. *Let $(X, \bowtie)$ be an apartness space. We say that X is **Hausdorff** if, for every $x, y \in X$ with $x \neq y$, there exist $U, V \subset X$ such that $x \in -U$, $y \in -V$ and $-U \cap -V = \emptyset$.*

Any uniform space is Hausdorff. A simple consequence of a space being Hausdorff is that, for every $x, y \in X$, $x \neq y$ if and only if $\{x\} \bowtie \{y\}$.

2.7   Sequences, Nets, and Completeness

Next we discuss nets, convergence, and completeness in apartness spaces.

Definition 2.12. *A **directed set** consists of an inhabited set $\mathcal{D}$ and a binary relation $\succeq$ on $\mathcal{D}$ such that*

- *$n \succeq n$ for each $n \in \mathcal{D}$;*

- *if $l, m, n \in \mathcal{D}$, $l \succeq m$, and $m \succeq n$, then $l \succeq n$; and*

- *for each $m, n \in \mathcal{D}$ there exists $l \in \mathcal{D}$ such that $l \succeq m \wedge l \succeq n$.*

*A **net** in a space X consists of a directed set $(\mathcal{D}, \succeq)$ and a function $x : \mathcal{D} \to X$; we denote such a net by $(x_n)_{n \in \mathcal{D}}$. A **subnet** of a net $(x_n)_{n \in \mathcal{D}}$ is a net $(n_k)_{k \in \mathcal{E}}$ in $\mathcal{D}$ with the property that for each $n \in \mathcal{D}$, there exists $k \in \mathcal{E}$ such that $n_{k'} \succeq n$ whenever $k' \in \mathcal{E}$ and $k' \succeq k$; this subnet is denoted by $(x_{n_k})_{k \in \mathcal{E}}$.*

Definition 2.13. *A net $(x_n)_{n \in \mathcal{D}}$ in an apartness space X is said to **converge** to $x \in X$ if, for each $A \subset X$ with $x \in -A$, there exists $m \in \mathcal{D}$ such that $x_n \in -A$ whenever $n \in \mathcal{D}$ and $n \succeq m$. The point x is then called a **limit** of the net.*

Cauchy sequences play an important role in the theory of metric spaces, and, in particular, in compact ones. We now define analogues of Cauchyness for nets, and then of completeness, in uniform and apartness spaces.

Definition 2.14. *Let $(X, \mathcal{U})$ be a uniform space and $x = (x_n)_{n \in \mathcal{D}}$ a net in X. We say that x is a **Cauchy net** if, for each $U \in \mathcal{U}$, there exists $m \in \mathcal{D}$ such that $x_n \in U[x_m]$ whenever $n \in \mathcal{D}$ and $n \succeq m$.*

Definition 2.15. *Let $(X, \bowtie)$ be an apartness space and $x = (x_n)_{n \in \mathcal{D}}$ a net in X. We say that x is **totally Cauchy** if, for each $S, T \subset \mathcal{D}$ such that $x(S) \bowtie x(T)$, there exists $m \in \mathcal{D}$ such that it is impossible for there to be $n, n' \in \mathcal{D}$ with $n \succeq m$, $n \in S$, $n' \succeq m$, and $n' \in T$.*

Definition 2.16. *A uniform space or an apartness space X is said to be **complete** (respectively, **totally complete**) if every Cauchy (respectively, totally Cauchy) net converges to a limit in X.*

Note that every sequence is a net. By restricting ourselves to sequences rather than nets, we can define **sequentially complete** and **sequentially totally complete** by making the obvious modifications in Definition 2.16.

It is clear that in a uniform space, a Cauchy net is totally Cauchy, and hence that a totally complete uniform space is complete. In a strongly totally bounded uniform space, a totally Cauchy net is Cauchy. A difficult argument shows that a totally Cauchy sequence in a uniform space is Cauchy; see [9], Theorem 3.5.12.

What is the motivation for studying apartness spaces? In terms of structure, apartness spaces lie between topological spaces and uniform spaces. An apartness space allows one to define strong continuity, which cannot be defined on an arbitrary topological space. Also, different apartnesses may induce the same topology. So this shows that an apartness space has strictly more structure than a topological space. On the other hand, an apartness space lacks any axiom similar to the powerful U3 axiom of uniform spaces and it also lacks a natural analogue to locatedness, which indicates that it has strictly less structure than a uniform space. This already makes apartness spaces very interesting, and has led to extensive research. A classical exposition, based on the

notion of proximity rather than that of apartness, is given in [27, 3]; a constructive exposition will appear as [9].

Constructively, apartness spaces provide significant computational information; the apartness of two sets is a much stronger property than the mere disjointness of the sets, or even their closures. For example, if an apartness space is derived from a totally bounded uniform space, then the uniformity can be recovered from the apartness [9], which is not true of a mere toplogy. And, classicaly, every apartness space with the Efremovič property is uniformisable [27]. This makes apartness spaces a strong foundation upon which to build a constructive theory of topology.

Chapter 3

Compactness Properties for Apartness Spaces

"The classic theorem of Heine-Borel-Lebesgue asserts that every open cover of a closed and bounded subset of the space of real numbers has a finite subcover. This theorem has extraordinarily profound consequences, and, like most good theorems, its conclusion has become a definition."—John L. Kelley ([19], page 135)

Classically, compactness is a very strong regularity condition; it has nice categorical properties and has numerous applications. In this chapter we will take a look at classical compactness and then previous attempts to develop a constructive version thereof.

Definition 3.1. *Let* (X, τ) *be a topological space. Then* X *is said to be compact, if every open cover has a finitely enumerable subcover—that is to say, for each* $C \subset \tau$ *with* $\bigcup C = X$, *there exists a finitely enumerable* $F \subset C$ *with* $\bigcup F = X$.

This definition, unfortunately, is not very useful constructively, because it is too strong.[1] We cannot even prove that $[0, 1]$ is compact. Note that we cannot demand a finite subcover, rather than a finitely enumerable one: otherwise the compactness of $\{0, 1\}$ is equivalent to LEM.

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[1]Some constructive formal topologists, however, find this definition acceptable.

A classically equivalent characterisation of compactness is that every net in X has a convergent subnet. And, in a uniform space, compactness is classically equivalent to the space being complete and totally bounded. The product of compact spaces is a compact space; a closed subset of a compact space is compact; and the continuous image of a compact space is compact. An important theorem about compact uniform spaces is the so-called uniform continuity theorem, which states that a continuous function from a compact uniform space to a uniform space is uniformly continuous.

Constructively, completeness and total boundedness are very useful notions. We can prove that, say, $[0, 1]$ possesses both of these properties. We can also show that they are both preserved by countable products and by closed almost located subspaces. And total boundedness (in either the strong or weak form) and sequential completeness are preserved by uniformly continuous functions.

So it seems that "complete and totally bounded" is a satisfactory constructive criterion for compactness. However, this definition requires the structure of a uniform space. We would like to have a definition which only requires the structure of an apartness space.

We now discuss several candidate criteria from the literature.

3.1   Total Completeness

Since we can define total completeness in an apartness space (see Definition 2.16), it seems that this would make a good approximation to completeness. However, it turns out that total completeness is classically equivalent to compactness; see [9], Section 3.5. Have we found a constructive criterion for compactness? To an extent yes, but, unfortunately, there are still problems with total completeness.

The problem arises firstly from the fact that total completeness doesn't "look" like a compactness condition and secondly from the fact that it is a very strong condition. To derive compactness from total completeness we need ultrafilters, which require heavy

use of the axiom of choice; it seems unlikely that there is a direct or constructive proof. We are also unable to prove that $[0,1]$ is totally complete; we are, however, able to show that $\mathbf{R}$ is totally sequentially complete.

3.2 Anti-Specker Properties

The alternative approach to defining compactness—via convergent subnets or subsequences—also has some problems. In the recursive model of constructive mathematics we can show that $[0,1]$ is not compact—this is the result of Specker's theorem (Theorem 1.4).

Specker's theorem implies that we cannot constructively prove that an increasing rational sequence in $[0,1]$ converges. However, the antithesis of Specker's theorem is a useful compactness criterion.

Definition 3.2. *Let $(X, \bowtie)$ be an apartness space. Then we say that X has the **weak anti-Specker property** if it is impossible for there to be a sequence $(x_n)_{n \in \mathbf{N}}$ in X that is eventually bounded away from each point in X—that is, for every $x \in X$, there exists $N \in \mathbf{N}$ such that*

$$x \in -\left\{x_n : n \in \mathbf{N} \wedge n \geq N\right\}.$$

*We also say that X has the **(strong) anti-Specker property** if, for any sequence $(x_n)_{n \in \mathbf{N}}$ in $X \cup \{\infty\}$ (where ∞ is bounded away from X) that is eventually bounded away from every point in X, there exists $n \in \mathbf{N}$ such that $x_n = \infty$.*

Note that the strong anti-Specker property implies the weak anti-Specker property and the converse is true if one assumes Markov's principle. Classically, the anti-Specker property is implied by sequential compactness and the converse is true if the space is first-countable.

It can be shown that the strong anti-Specker property for $[0,1]$ is equivalent to a form

of Brower's fan theorem and that the product of two anti-Specker spaces is anti-Specker under the assumption of BD-N; see [4].

3.3   Neat Compactness

There is one more approach to defining compactness that is worth discussing. This approach is due to Diener [13].

Definition 3.3. *Let $(X, \bowtie)$ be an apartness space. We say that*

- *X is **neatly compact** if X is neatly located and it is impossible that both LPO hold and there is a sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X such that $\bigcup_{n \in \mathbb{N}} U_n = X$ and, for each $n \in \mathbb{N}$, $U_n \subset U_{n+1}$ and $\neg U_n \neq \emptyset$;*

- *a net $(x_n)_{n \in \mathcal{D}}$ in X is **neatly Cauchy** if, for any finitely enumerable collection $\{(S_j, T_j) : j \in F\}$ of neat covers of X, there exists $N \in \mathcal{D}$ such that either $x_N \in T_j$ for each $j \in F$ or there exists $k \in F$ such that $x_n \in S_k$ for all $n \in \mathcal{D}$ with $n \succeq N$;*

- *a net $(x_n)_{n \in \mathcal{D}}$ in X **converges neatly** to a point $x \in X$ if, for any finitely enumerable collection $\{(S_j, T_j) : j \in F\}$ of neat coverings of X, either $x \in T_j$ for each $j \in F$ or there exist $k \in F$ and $N \in \mathcal{D}$ such that $x \in S_k$ and $x_n \in S_k$ for each $n \in \mathcal{D}$ with $n \succeq N$; and*

- *X is **neatly complete** if every neatly Cauchy sequence in X converges neatly to a limit in X.*

Neat compactness implies total boundedness in a separable uniform space, and, in a uniform space, neat completeness implies completeness. Conversely, a complete and totally bounded uniform space with a countable base of entourages is neatly compact, neatly complete, and, of course, separable. Moreover, if f is a strongly continuous and topologically continuous function from a neatly compact apartness space to an apartness space, then the range of f is neatly compact. This implies that, if f is a

strongly continuous function from a neatly compact apartness space X to the real line, then $\sup f(X)$ exists.

Neat compactness and completeness seem to be very good criteria for compactness, as they have desirable categorical properties and they are more-or-less equivalent to the space being totally bounded and complete. However, the definitions are very unwieldy and neat compactness is mostly a negative condition.

Chapter 4

Compactness Criteria

"Just because something doesn't do what you planned it to do doesn't mean it's useless."—Thomas Edison [15]

It already seems that we will have difficulty finding one compactness criterion that is as universally accepted as Definition 3.1 is classically. So we will work with several criteria. All of the criteria discussed in the previous chapter, apart from neat locatedness and neat completeness, used only the topology of the space; none of them refer to apartness between sets. We will make use of the extra structure of apartness spaces. We have come up with three criteria which are based on the observation that, in a classical compact uniform space, disjoint closed sets have a positive distance between them. Our three criteria differ in their use of neat locatedness and the product apartness.

As we have mentioned before, different apartnesses can induce the same topology. As our criteria depend on the apartness, they are more sensitive to the structure of the space than those discussed in Chapter 3.

Definition 4.1. *Let* $(X, \bowtie)$ *be an apartness space. We say that* X *is*

CC1 *if, for any neatly located* $S, T \subset X$ *with* $-S \cup -T = X$, $S \bowtie T$;

CC2 *if, for any neatly located* $S \subset X^2$ *with* $-S \cup -\Delta = X^2$, $S \bowtie \Delta$ *in the product apartness; and*

CC3 *if, for any* $S \subset X^2$ *such that* $-S \cup -\Delta = X^2$, $S \bowtie \Delta$ *in the product apartness.*

We immediately note that any CC3 space is also a CC2 space. Moreover, Lemma 4.2 shows that a Hausdorff CC3 space is CC1 under either a uniformity or LEM. And, classically, CC2 and CC3 are equivalent. Thus the above conditions are roughly ordered by strength. Also CC1, CC2 and CC3 are all preserved by strong homeomorphisms.

Lemma 4.2. *Let* $(X, \bowtie)$ *be a Hausdorff apartness space. If* X *is CC3, then it is also CC1 if we assume that either*

   (i) X *is a uniform space[1] or*

   (ii) LEM holds.

Proof. Let X be a CC3 apartness space and $S, T \subset X$ with $-S \cup -T = X$. Take an arbitrary $(x, y) \in X^2$. If $x \in -S$ or $y \in -T$, then $(x, y) \in -(S \times T)$. Suppose, on the other hand, that $x \in -T$ and $y \in -S$. By B5 there exists $A \subset X$ such that $x \in -A$ and $-T \cup A = X$. If $y \in -T$, then $(x, y) \in -(S \times T)$. So we suppose instead that $y \in A$, whence $x \in -\{y\}$ and $x \neq y$. Since X is Hausdorff there exist $U, V \subset X$ such that $x \in -U$, $y \in -V$ and $-U \cap -V = \emptyset$. Thus $(x, y) \in -U \times -V \subset \Delta$ and $(x, y) \in -\Delta$. So $-(S \times T) \cup -\Delta = X^2$, whence, by CC3, $S \times T \bowtie \Delta$. Now choose $m, n \in \mathbf{N}$ and $A_1, A_2, \cdots, A_m, A_1', A_2', \cdots, A_m', B_1, B_2, \cdots, B_n, B_1', B_2', \cdots, B_n' \subset X$ such that

$$S \times T \subset \bigcup_{i=1}^{m} A_i \times A_i', \quad \Delta \subset \bigcup_{j=1}^{n} B_j \times B_j',$$

and, for each $i, j \in \mathbf{N}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$, either $A_i \bowtie B_j$ or $A_i' \bowtie B_j'$.

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[1] Note that a uniform space is necessarily Hausdorff.

(i) Suppose that $(X, \mathcal{U})$ is a uniform space. Then, for each $i, j \in \mathbf{N}$ with $1 \le i \le m$ and $1 \le j \le n$, we can choose $U_{i,j} \in \mathcal{U}$ such that either $A_i \times B_j \cap U_{i,j} = \emptyset$ or $A_i' \times B_j' \cap U_{i,j} = \emptyset$. Let $U = \bigcap_{i,j} U_{i,j}$ and suppose that $(s,t) \in S \times T \cap U$. Choose i such that $(s,t) \in A_i \times A_i'$ and choose j such that $(s,s) \in B_j \times B_j'$. Since $(s,s) \in A_i \times B_j \cap U_{i,j}$, we must have $A_i' \times B_j' \cap U_{i,j} = \emptyset$. This contradiction shows that $S \times T \cap U = \emptyset$ and thus $S \bowtie T$.

(ii) Instead suppose that LEM holds. Fix $j \in \mathbf{N}$ with $1 \le j \le n$ and choose $I_j, I_j' \subset \{1, 2, \cdots m\}$ with $I_j \cap I_j' = \emptyset$ and $I_j \cup I_j' = \{1, 2, \cdots m\}$ such that, for every $i \in I_j$, $A_i \bowtie B_j$ and, for every $i \in I_j'$, $A_i' \bowtie B_j'$. Let $C_j = \bigcup_{i \in I_j} A_i$ and $C_j' = \bigcup_{i \in I_j'} A_i'$. Then, by B3, $B_j \bowtie C_j$ and $B_j' \bowtie C_j'$. We also have

$$S \times T \subset (C_j \times X) \cup (X \times C_j'),$$

whence, by LEM, either $S \subset C_j$ or $T \subset C_j'$. Otherwise choose $s \in S \backslash C_j$ and $t \in T \backslash C_j'$; then $(s,t) \in S \times T \backslash ((C_j \times X) \cup (X \times C_j'))$—a contradiction. Thus either $S \bowtie B_j$ or $T \bowtie B_j'$.

Now choose $J, J' \subset \{1, 2, \cdots n\}$ with $J \cap J' = \emptyset$ and $J \cup J' = \{1, 2, \cdots n\}$ such that, for every $j \in J$, $S \bowtie B_j$ and, for every $j \in J'$, $T \bowtie B_j'$. Let $D = \bigcup_{j \in J} B_j$ and $D' = \bigcup_{j \in J'} B_j'$. Then $S \bowtie D$, $T \bowtie D'$, and $X \subset D \cup D'$. Now $T \subset D$, whence $S \bowtie T$. $\blacksquare$

We will now investigate these definitions further. Some of the following results are not constructive; this is either because they deal with nonconstructive ideas or because we have not yet been able to find a constructive version. Nonconstructive results are marked with a †.

The following diagram summarizes the results of this chapter.
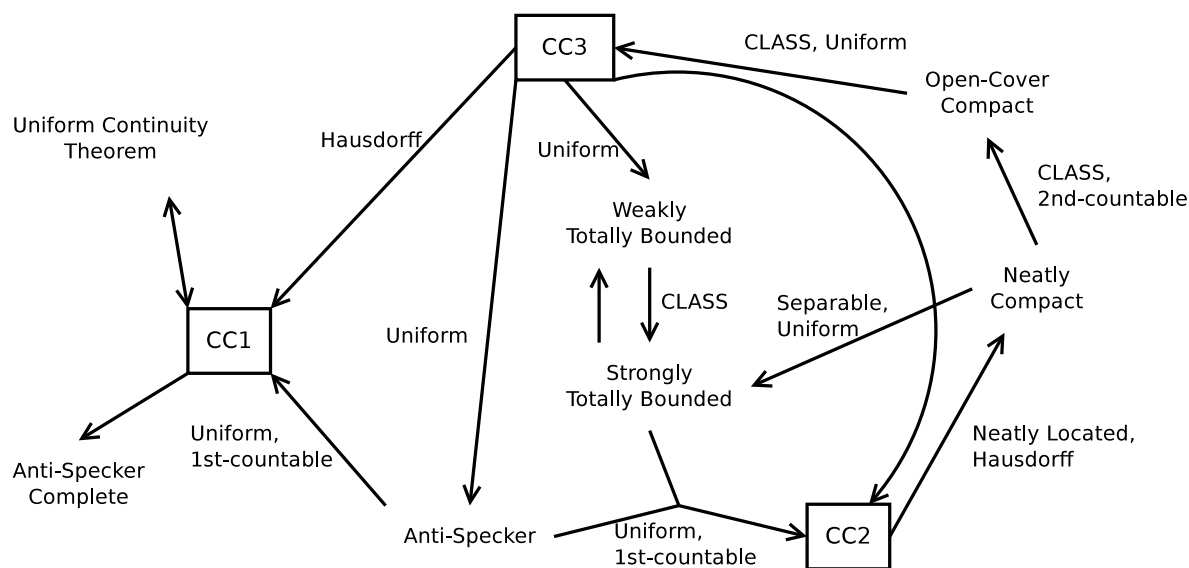
The highlights of this chapter are as follows.

- CC2 and CC3 are classically equivalent to open-cover compactness in a separable metric space.

- CC2 and CC3 can be connected to the anti-Specker property and total boundedness.

- CC1 is weaker than compactness, but it implies completeness.

- CC1 can be characterised in terms of an analogue of the uniform continuity theorem.

4.1 CC1

First we relate CC1 to the compactness notions discussed in Chapter 3. Then we give a characterisation of CC1 in terms of an analogue to the uniform continuity theorem.

We begin with some lemmas.

Lemma 4.3. *Let $(X, \mathcal{U})$ be a uniform space and $S \subset X$. If S is neatly located, then S is almost located.*

Proof. Choose an arbitrary $U \in \mathcal{U}$. Pick symmetric $W, W', V \in \mathcal{U}$ such that $U \cup \neg W = X^2$, $W' \circ W' \subset W$, and $W' \cup \neg V = X^2$. Then, for any $x \in X$, $(U[x], \neg V[x])$ is a neat cover of X. To see this, let $x \in X$ be arbitrary. Given $y \in X$, either $(x, y) \in U$ or $(x, y) \notin W$. Thus $U[x] \cup \neg W[x] = X^2$. Similarly, $W'[x] \cup \neg V[x] = X^2$. If $(y, z) \in (\neg W[x]) \times W'[x] \cap W'$, then $y \in (W' \circ W')[x] \subset W[x]$, which is impossible. So $\neg W[x] \bowtie W'[x]$. Now either $U[x] \cap S \neq \emptyset$ or $S \subset \neg V[x]$. ∎

Lemma 4.4. *Let $(X, \mathcal{U})$ be a uniform space and $S \subset X$ almost located. Then, for any symmetric $U \in \mathcal{U}$, there exists a symmetric $V \in \mathcal{U}$ such that $(U[S], \neg V[S])$ is a neat cover of X.*

Proof. Choose symmetric $W, W' \in \mathcal{U}$ such that, for any $x \in X$, either $U[x] \cap S \neq \emptyset$ or $W[x] \cap S = \emptyset$ and $W' \circ W' \subset W$. For an arbitrary $x \in X$, either $x \in U[S]$ or $x \notin W[S]$. So $U[S] \cup \neg W[S] = X$. If $(x, y) \in (\neg W[S]) \times W'[S] \cap W'$, then $x \in (W' \circ W')[S] \subset W[S]$, which is impossible. So $\neg W[S] \bowtie W'[S]$. Again, by the almost locatedness of S, we can choose a symmetric $V \in \mathcal{U}$ such that $W[S] \cup \neg V[S] = X$. This shows that $(U[S], \neg V[S])$ is a neat cover. ∎

Proposition 4.5. *Let $(X, \mathcal{U})$ be a uniform space with a countable base of entourages and the strong anti-Specker property. Then X is CC1.*

Proof. Let $S, T \subset X$ be neatly located and satisfy $-S \cup -T = X$. By Lemma 4.3, S is almost located. By Lemma 4.4, we can choose an countable base of entourages $\{U_n : n \in \mathbf{N}\}$ such that, for each $n \in \mathbf{N}$, $U_{n+1}^2 \subset U_n = U_n^{-1}$ and $(U_n[S], \neg U_{n+1}[S])$ is a neat cover of X.

As T is neatly located, for each $n \in \mathbf{N}$, either $U_n[S] \cap T \neq \emptyset$ or $U_{n+1} \cap T = \emptyset$. We can choose a sequence $(x_n)_{n \in \mathbf{N}}$ in $X \cup \{\infty\}$ such that

$$\forall n \in \mathbf{N} \ (x_n \in U_n[S] \cap T \vee (x_n = \infty \wedge U_{n+1}[S] \cap T = \emptyset)).$$

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Let $x \in X$ be arbitrary. If $x \in -T$, then, as $T \supset \{x_n : n \in \mathbf{N} \wedge x_n \neq \infty\}$, $(x_n)_{n \in \mathbf{N}}$ is eventually bounded away from x. Suppose, on the other hand, that $x \in -S$. Then there exists $N \in \mathbf{N}$ such that $U_N[x] \subset \neg S$. If $n > N$ and $x_n \in U_{N+1}[x]$, then, as $x_n \in U_n[S]$, $x \in (U_{N+1} \circ U_n)[S]$—a contradiction. So $(x_n)_{n \in \mathbf{N}}$ is eventually bounded away from x.

By the anti-Specker property, there exists $N \in \mathbf{N}$ such that $x_N = \infty$. Thus $U_N[S] \cap T = \emptyset$ and $S \bowtie T$. ∎

Proposition 4.6. *Let $(X, \mathcal{U})$ be a CC1 uniform space and let $(x_n)_{n \in \mathbf{N}}$ be a sequence in $X \cup \{\infty\}$. Suppose that whenever the sequence falls in X it is a Cauchy sequence and that the sequence is eventually bounded away from every point in X. Then there exists $n \in \mathbf{N}$ such that $x_n = \infty$.*

Proof. By applying dependent choice and passing to a subsequence, we may ensure that
$$\forall n \in \mathbf{N} \; x_n \in -\{x_k : k \in \mathbf{N} \wedge k \geq n\}.$$
Let
$$S = \{x_{2n} : n \in \mathbf{N}\} \text{ and } T = \{x_{2n+1} : n \in \mathbf{N}\}.$$
Take an arbitrary $x \in X$. Choose $N \in \mathbf{N}$ and a symmetric $U \in \mathcal{U}$ such that $(x, x_n) \notin U$ for each $n \geq N$ and $(x_i, x_j) \notin U^2$ for each $i, j < N$ with $i \neq j$. If $(x, x_i), (x, x_j) \in U$ for some $i, j \in \mathbf{N}$, then $i, j < N$ and $(x_i, x_j) \in U^2$, whence $i = j$. Choose a symmetric $V \in \mathcal{U}$ with $U \cup \neg V = X^2$. Now, either $(x, x_n) \in U$ for exactly one $n \in \mathbf{N}$ or $(x, x_n) \notin V$ for each $n \in \mathbf{N}$. Either way, either $x \in -S$ or $x \in -T$. Since $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, S and T are totally bounded and, therefore, neatly located. This implies that $S \bowtie T$. However, since $(x_n)_{n \in \mathbf{N}}$ is Cauchy, this is impossible. ∎

Proposition 4.6 immediately implies that, classically, a CC1 uniform space with a countable base of entourages is complete.

Propositions 4.5 and 4.6 show that CC1 lies somewhere between completeness and compactness. However, the following examples show that it is equivalent to neither compactness nor completeness.

Example 4.7. *The integers are a CC1 metric space, but they are not weakly totally bounded and, therefore, not compact.*

Proof. Let $S, T \subset \mathbf{Z}$ be neatly located and satisfy $-S \cup -T = \mathbf{Z}$. If $(x, y) \in S \times T$, then $x \in -T \subset \sim \{y\}$, whence $x \neq y$ and $|x - y| \geq 1$. Thus $S \bowtie T$. So $\mathbf{Z}$ is CC1. There is no finite cover of $\mathbf{Z}$ with sets of diameter at most $\frac{1}{2}$, as at least one set would contain two distinct integers. ∎

It is clear that CC1 cannot be equivalent to classical compactness: Equation 2.1 shows how, classically, we can find a CC1 apartness for *any* topological space.

Example 4.8. *The plane is a complete metric space, but it is not both CC1 and neatly located.*

Proof. Clearly, $(\mathbf{R}^2, d)$ is a complete metric space, where d is the Euclidean metric. Let

$$S = \left\{ (x, y) \in \mathbf{R}^2 : xy = 0 \right\}, \text{ and } T = \left\{ (x, y) \in \mathbf{R}^2 : xy = 1 \right\}.$$

We will show that $\mathbf{R}^2$ being neatly located implies that LPO holds. Both S and T are separable; thus, by LPO, they are neatly located. We will also show that $-S \cup -T = \mathbf{R}^2$ and $\neg(S \bowtie T)$, whence $\mathbf{R}^2$ cannot be CC1.

The $(a_n)_{n \in \mathbf{N}}$ be a binary sequence. Let

$$A = \bigcup_{n \in \mathbf{N} \wedge a_n = 1} B\left((n, n), \frac{1}{2}\right), \quad A' = \bigcap_{n \in \mathbf{N} \wedge a_n = 1} \neg B\left((n, n), \frac{1}{4}\right),$$

$$B' = \bigcup_{n \in \mathbf{N} \wedge a_n = 1} B\left((n, n), \frac{1}{8}\right), \quad B = \bigcap_{n \in \mathbf{N} \wedge a_n = 1} \neg B\left((n, n), \frac{1}{16}\right).$$

Pick an arbitrary $(x, y) \in \mathbf{R}^2$. Then either there exists $n \in \mathbf{N}$ with $d((x, y), (n, n)) < \frac{1}{2}$ or $d((x, y), (n, n)) > \frac{1}{4}$ for each $n \in \mathbf{N}$. So either $(x, y) \in A$ or $(x, y) \in A'$. Similarly, $B \cup B' = \mathbf{R}^2$. Now, if $(x, y) \in A'$ and $(x', y') \in B'$, then there exists $n \in \mathbf{N}$ with $a_n = 1$ and $d((x', y'), (n, n)) < \frac{1}{8}$, but $d((x, y), (n, n)) \geq \frac{1}{4}$, so $d((x, y), (x', y')) \geq \frac{1}{4}$. Thus (A, B) is a neat cover of $\mathbf{R}^2$. So either $A \neq \emptyset$ or $B = \mathbf{R}^2$. In the former case,

there exists $n \in \mathbf{N}$ with $a_n = 1$. On the other hand, if $B = \mathbf{R}^2$, then, for every $n \in \mathbf{N}$, $(n, n) \in B$, so $a_n = 0$. This shows that LPO must hold.

For any $\varepsilon > 0$, $(\frac{1}{\varepsilon}, 0) \in S$, $(\frac{1}{\varepsilon}, \varepsilon) \in T$ and

$$d\left((\frac{1}{\varepsilon}, 0), (\frac{1}{\varepsilon}, \varepsilon) \right) = \varepsilon.$$

Thus S and T are not apart.

Let $(x, y) \in \mathbf{R}^2$ be arbitrary. Now, let

$$\varepsilon = \frac{1}{4(1 + |x| + |y|)}.$$

Choose $(x', y') \in \mathbf{R}^2$ with $d((x, y), (x', y')) < \varepsilon$. Then $|x - x'| < \varepsilon$ and $|y - y'| < \varepsilon$, whence

$$|xy - x'y'| \leq |x||y - y'| + |x - x'||y'| \leq \varepsilon(|x| + |y| + \varepsilon) \leq \varepsilon(|x| + |y| + 1) = \frac{1}{4}.$$

Either $xy > \frac{1}{3}$ or $xy < \frac{2}{3}$. In the former case, if $(x', y') \in S$, then, as $|xy - x'y'| > \frac{1}{3}$,

$$d((x, y), (x', y')) \geq \frac{1}{4(1 + |x| + |y|)} > 0,$$

so $(x, y) \in -S$. Similarly, in the latter case, $(x, y) \in -T$. So $-S \cup -T = \mathbf{R}^2$. ∎

A more precise characterisation of CC1 is given by an analogue of the uniform continuity theorem. We first need to define the following two regularity and continuity conditions.

Definition 4.9. *We say that an apartness space X has the reverse-CC1 property if, for each weakly located $S, T \subset X$, $S \bowtie T$ implies that $-S \cup -T = X$.*

Note that, classically, every apartness space has the reverse-CC1 property. Constructively, any uniform space has the reverse-CC1 property, and, if we assume MP, every first-countable space has the reverse-CC1 property.

Definition 4.10. *Let X and Y be apartness spaces and $f : X \to Y$ a function. We say that f is almost strongly continuous if, for each neatly located $S, T \subset X$ with*

f(S) and f(T) weakly located, if f(S) ⋈ f(T), then S ⋈ T.

Now we can state an analogue of the uniform continuity theorem.

Proposition 4.11. *Let X and Y be apartness spaces and $f : X \to Y$ a continuous function. Suppose that X is CC1 and Y has the reverse-CC1 property. Then f is almost strongly continuous.*

Proof. Let $S, T \subset X$ be neatly located, with $f(S)$ and $f(T)$ weakly located. Suppose that $f(S) \bowtie f(T)$. By the reverse-CC1 property, $-f(S) \cup -f(T) = Y$. Since f is continuous, $f^{-1}(-f(S)) \subset -S$ and $f^{-1}(-f(T)) \subset -T$, whence $-S \cup -T = X$. CC1 now implies that $S \bowtie T$. ∎

The following result is a partial converse to Proposition 4.11.

Proposition 4.12. *Let X be an apartness space with the nested neighbourhoods property. Suppose that every continuous function from X to a reverse-CC1 apartness space is almost strongly continuous. Then X is CC1.*

Proof. Define a second apartness $\bowtie'$ on X by

$$\forall A, B \subset X \ \left(A \bowtie' B \iff -A \cup -B = X \right).$$

Clearly $\bowtie'$ is symmetric. Fix $A \subset X$. If $x \in -A$, then, by B5, there exists $B \subset X$ such that $x \in -B$ and $B \cup -A = X$, whence $-\{x\} \cup -A = X$ and $x \in -'A$, as, by symmetry, $B \subset -\{x\}$. If, on the other hand, $x \in -'A$, then $-\{x\} \cup -A = X$ and, as $x \notin -\{x\}$, $x \in -A$. Thus $-A = -'A$. This immediately verifies that $\bowtie'$ satisfies B2, B4 and B5. Clearly, it also satisfies B1. B3 follows from the observation that, for each $A, B \subset X$, $-(A \cup B) = -A \cap -B$.

Now let $f : (X, \bowtie) \to (X, \bowtie')$ be the canonical bijection. Then, as $f(-A) = -'f(A)$ for each $A \subset X$, f is a homeomorphism. By our supposition, f is almost strongly continuous. Let $S, T \subset X$ be neatly located and satisfy $-S \cup -T = X$. Then $f(S) \bowtie'$

$f(T)$. By Lemma 2.9, S and T are weakly located, whence $f(S)$ and $f(T)$ are weakly located. Thus, by almost strong continuity, $S \bowtie T$. ■

4.2 CC2

First we show that CC2 follows from the anti-Specker property and strong total boundedness. We then show that CC2 implies neat compactness, strong total boundedness, and, classically, compactness.

Lemma 4.13. *Let $(X, \mathcal{U})$ be a strongly totally bounded uniform space. Suppose that $S \subset X^2$ and there exists $U \in \mathcal{U}$ such that $S \cap U = \emptyset$. Then $S \bowtie \Delta$ in the product apartness.*

Proof. Choose $U_{-1}, U_0, U_1, U_2, U_3, U_4 \in \mathcal{U}$ such that $U_{-1} \subset U$, $U_4 = U_4^{-1}$ and, for each $n \in \{-1, 0, 1, 2, 3\}$, $U_n = U_n^{-1} \supset U_{n+1}^3$ and $U_n \cup \neg U_{n+1} = X^2$. Let $\{x_i : i \in F\}$ be a U_4-approximation to X, where F is a finite set. Then

$$\Delta \subset \bigcup_{k \in F} U_4[x_k] \times U_4[x_k].$$

Also, if $(x, y) \in S \subset \neg U_{-1}$, then there exist $i, j \in F$ such that $(x, y) \in U_4[x_i] \times U_4[x_j]$; if $(x_i, x_j) \in U_0$, then
$$(x, y) \in U_4 \circ U_0 \circ U_4 \subset U_0^3 \subset U_{-1},$$

which is impossible. So

$$S \subset \bigcup \left\{ U_4[x_i] \times U_4[x_j] : i, j \in F \land (x_i, x_j) \notin U_0 \right\}.$$

Note that $U_0 \cup \neg U_1 = X^2$. Choose a finite $A \subset F^2$ such that, if $(i, j) \in A$, then

$(x_i, x_j) \notin U_1$, and, if $i, j \in F$ and $(x_i, x_j) \notin U_0$, then $(i, j) \in A$. Then

$$S \subset \bigcup_{(i,j) \in A} U_4[x_i] \times U_4[x_j].$$

Now choose $(i, j) \in A$ and $k \in F$. Then $(x_i, x_j) \notin U_1$. If both (x_i, x_k) and (x_j, x_k) are in U_2, then $(x_i, x_j) \in U_2^2 \subset U_1$ — a contradiction. Since $U_2 \cup \neg U_3 = X^2$, either $(x_i, x_k) \notin U_3$ or $(x_j, x_k) \notin U_3$. Suppose that $(x_i, x_k) \notin U_3$. If $U_4[x_i] \times U_4[x_k] \cap U_4 \neq \emptyset$, then $(x_i, x_k) \in U_4^3 \subset U_3$, which is impossible. Thus $U_4[x_i] \bowtie U_4[x_k]$. Similarly, if $(x_j, x_k) \notin U_3$, then $U_4[x_j] \bowtie U_4[x_k]$. ∎

Lemma 4.14. *Let $(X, \mathcal{U})$ be a strongly totally bounded uniform space and let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $(U, \sim V)$ is a neat cover of X^2.*

The proof of Lemma 4.14 is similar to that of Lemma 4.13, but it is slightly more complicated; we have chosen to omit it.

Proposition 4.15. *Let $(X, \mathcal{U})$ be a uniform space with a countable base of entourages. Suppose that X is strongly totally bounded and has the strong anti-Specker property. Then X is CC2.*

Proof. Apply Lemma 4.14 to find a base $\{U_n : n \in \mathbf{N}\}$ for the uniformity such that, for each $n \in \mathbf{N}$, $(U_n, \sim U_{n+1})$ is a neat cover of X^2 and $U_n = U_n^{-1} \supset U_{n+1}^2$. Let $S \subset X^2$ be neatly located and satisfy $\Delta \subset -S$.

Now choose a sequence $((x_n, y_n))_{n \in \mathbf{N}}$ in $X^2 \cup \{(\infty, \infty)\}$ such that, for each $n \in \mathbf{N}$, if $(x_n, y_n) = (\infty, \infty)$, then $S \cap U_{n+1} = \emptyset$ and, if $(x_n, y_n) \in X^2$, then $(x_n, y_n) \in S \cap U_n$. Let $x \in X$ be arbitrary. Then $(x, x) \in \Delta \subset -S$, so there exists $N \in \mathbf{N}$ such that

$$U_N[x] \times U_N[x] \subset \sim S.$$

If $n \in \mathbf{N}$, $n > N$, and $x_n \in U_{N+1}[x]$, then, since $(x_n, y_n) \in U_n \subset U_{N+1}$,

$$(x_n, y_n) \in U_{N+1}[x] \times (U_{N+1} \circ U_{N+1})[x] \subset U_N[x] \times U_N[x]$$

—a contradiction. Thus $(x_n)_{n \in \mathbf{N}}$ is eventually bounded away from x.

By the anti-Specker property, there exists $n \in \mathbf{N}$ such that $x_n = \infty$, whence $S \cap U_{n+1} = \emptyset$. By Lemma 4.13, $S \bowtie \Delta$. ∎

Proposition 4.16. *Let $(X, \bowtie)$ be a neatly located Hausdorff CC2 apartness space. Then X is neatly compact.*

Proof. Suppose that LPO holds and $(U_n)_{n \in \mathbf{N}}$ is a sequence of open subsets of X such that $\bigcup_{n \in \mathbf{N}} U_n = X$ and, for each $n \in \mathbf{N}$, $U_n \subset U_{n+1}$ and $\neg U_n \neq \emptyset$. To complete our proof it suffices to find a contradiction. By applying dependent choice and passing to a subsequence, we may, without loss of generality, assume that there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that, for each $n \in \mathbf{N}$, $x_n \in U_{n+1} \cap \neg U_n$.

If $i, j \in \mathbf{N}$ and $i < j$, then
$$x_i \in U_{i+1} \subset U_j \subset \neg\{x_j\},$$
whence $\{x_i\} \bowtie \{x_j\}$. Let
$$S = \{(x_i, x_j) : i, j \in \mathbf{N} \wedge i \neq j\}.$$

By LPO, S is neatly located.

Let $(x, y) \in X^2$ be arbitrary and choose $N \in \mathbf{N}$ such that $x, y \in U_N$. Take $A, B \subset X$ such that $x \in -A \subset U_N$ and $y \in -B \subset U_N$. If $n \in \mathbf{N}$ and $n \geq N$, then $x_n \notin U_n \supset U_N$. Thus
$$(x, y) \in -A \times -B \subset \neg\{(x_i, x_j) : i, j \in \mathbf{N} \wedge i \neq j \wedge (i \geq N \vee j \geq N)\}.$$

Now let
$$F = \{(x_i, x_j) : i, j \in \mathbf{N} \wedge i \neq j \wedge i, j < N\}.$$
Take some $(x_i, x_j) \in F$. Then $\{x_i\} \bowtie \{x_j\}$, and, as X is Hausdorff, there exist $C, D \subset X$ such that $x_i \in -C$, $x_j \in -D$, and $-C \cap -D = \emptyset$. By applying B5 we get $C', D' \subset X$ such that $x_i \in -C'$, $x_j \in -D'$, and $-C \cup C' = -D \cup D' = X$. Now, if $(x, y) \in -C \times -D \subset \neg\Delta$, then $(x, y) \in -\Delta$. On the other hand, if $(x, y) \in C' \times X \cup X \times D'$, then $(x, y) \in -\{(x_i, x_j)\}$. Since F is finite, either $(x, y) \in -\Delta$ or $(x, y) \in -F$. In the latter case, $(x, y) \in -S$. Since (x, y) was arbitrary, $-S \cup \Delta = X$.

By CC2, $S \bowtie \Delta$. So choose $m \in \mathbf{N}$ and $A_1, A_2, \cdots, A_m \subset X$ such that

$$\Delta \subset \bigcup_{k=1}^{m} A_k \times A_k \subset \neg S.$$

By the pigeonhole principle, there exist $i, j, k \in \mathbf{N}$ with $i \neq j$ and $1 \leq k \leq m$ such that $x_i, x_j \in A_k$. But then $(x_i, x_j) \in S \cap A_k \times A_k$—a contradiction ∎

The proof of Proposition 4.16 is fairly loose. The neat locatedness assumption carries straight through from the hypotheses to conclusion and is not mentioned in the proof. Moreover, when deriving a contradiction, we only use LPO once to show that a countable set is neatly located. If we replace the hypothesis CC2 with CC3, we can prove this result without any application of LPO.

A converse to Proposition 4.16 seems unlikely, as neat compactness is a very negative condition, whereas CC2 is very positive.

Proposition 4.17. *If $(X, \mathcal{U})$ is a separable neatly located CC2 uniform space, then X is strongly totally bounded.*

Proof. Any uniform space is Hausdorff, so, by Proposition 4.16, X is neatly compact. Proposition 3.10.12 in [9] shows that a separable neatly compact uniform space is strongly totally bounded. ∎

Lemma 4.18 (†). *Let $(X, \bowtie)$ be an apartness space with a countable base of open sets. If X is neatly compact, then, classically, X is compact.*

Proof. Fix some countable base of open sets $\{U_n : n \in \mathbf{N}\}$. Let C be an arbitrary open cover of X. For each $W \in C$, there exists $B_W \subset \mathbf{N}$ such that $\bigcup_{n \in B_W} U_n = W$. Set $B = \bigcup_{W \in C} B_W$ and, for each $n \in \mathbf{N}$, $V_n = \bigcup_{k \in B \wedge k \leq n} U_k$. We see that $(V_n)_{n \in \mathbf{N}}$ is a sequence of open subsets of X such that $\bigcup_{n \in \mathbf{N}} V_n = X$ and, for each $n \in \mathbf{N}$, $V_n \subset V_{n+1}$. Since LPO holds classically and X is neatly compact, it is impossible that $\neg V_n \neq \emptyset$ for each $n \in \mathbf{N}$. Thus there exists $n \in \mathbf{N}$ such that $V_n = X$. However, $V_n = \bigcup_{k \in B \wedge k \leq n} U_k$ and $B = \bigcup_{W \in C} B_W$. So there exists a finite $F \subset B$ such that $\bigcup_{k \in F} U_k = X$. For each $k \in F$ there exists $W_k \in C$ such that $U_k \subset W_k$. Consequently, $\{W_k : k \in F\}$ is a finitely enumerable subcover of C. ∎

Proposition 4.19 (†). *Let $(X, \bowtie)$ be a Hausdorff apartness space with a countable base of open sets. If X is CC2, then it is compact.*

Proof. First note that, as we are working classically, X is neatly located. Suppose that X is CC2. Then Proposition 4.16 implies that X is neatly compact. Lemma 4.18 now shows that X is compact. ∎

4.3   CC3

We begin by showing that CC3 follows, classically, from compactness. Then we show that CC3 implies weak total boundedness and the anti-Specker property.

Lemma 4.20 (†). *Let $(X, \bowtie)$ be an apartness space and $S, T \subset X^2$ such that $\bar{S} \cap \bar{T} = \emptyset$. Suppose that X is classically compact. Then, classically, there exist $n \in \mathbf{N}$, $E_1, \cdots, E_n, F_1, \cdots, F_n \subset X$ such that $\bar{T} \subset \bigcup_{i=1}^{n} \bar{E}_i \times \bar{F}_i \subset \sim \bar{S}$.*

Proof. First note that $\sim \bar{S} = -S$ and $\sim \bar{T} = -T$. Take some $s \in \bar{S} \subset -\bar{T}$. Then there exist $A_s, B_s \subset X$ such that $s \in -A_s \times -B_s \subset \sim \bar{T}$. Now $\left\{ -A_s \times -B_s : s \in \bar{S} \right\}$ is an open cover of $\bar{S}$. Since $\bar{S}$ is a closed subset of a compact space, there is a finite $F \subset \bar{S}$ such that $\bar{S} \subset \bigcup_{s \in F} -A_s \times -B_s \subset \sim \bar{T}$.

Noting that $\sim (-A_s \times -B_s) = (\bar{A}_s \times X) \cup (X \times \bar{B}_s)$ and taking complements gives us

$$\bar{T} \subset \bigcap_{s \in F} (\bar{A}_s \times X) \cup (X \times \bar{B}_s) = \bigcup_{f \in \{0,1\}^F} \bigcap_{s \in F} C_{s,f(s)} \subset \sim \bar{S},$$

where $C_{s,0} = \bar{A}_s \times X$ and $C_{s,1} = X \times \bar{B}_s$. Then, for each $f \in \{0, 1\}^F$, set

$$E_f = \bigcap_{s \in F, f(s)=0} \bar{A}_s, \quad F_f = \bigcap_{s \in F, f(s)=1} \bar{B}_s.$$

To finish set $n = 2^{|F|}$ and associate $\{0, 1\}^F$ with $\{1, \cdots, n\}$. ∎

Proposition 4.21 (†). *Let X be a classically compact topological space and define*

an apartness on X by

$$\forall S, T \subset X \left(S \bowtie T \iff \bar{S} \cap \bar{T} = \emptyset \right).$$

Then, classically, X is CC3.

Proof. Let $S \subset X^2$ satisfy $-S \cup -\Delta = X^2$. Then $\bar{S} \cap \bar{\Delta} = \emptyset$. Apply Lemma 4.20 to find $n \in \mathbf{N}$, $E_1, \cdots, E_n, F_1, \cdots, F_n \subset X$ such that $\bar{\Delta} \subset R := \bigcup_{i=1}^n \bar{E}_i \times \bar{F}_i \subset \sim \bar{S}$. Now $\bar{R} \cap \bar{S} = \emptyset$, so we can apply Lemma 4.20 again to find $m \in \mathbf{N}$, $A_1, \cdots, A_m, B_1, \cdots, B_m \subset X$ such that $\bar{S} \subset Q := \bigcup_{j=1}^m \bar{A}_j \times \bar{B}_j \subset \sim \bar{R}$. Note that, for each $i, j \in \mathbf{N}$ with $1 \leq i \leq n$ and $1 \leq j \leq m$, $\bar{E}_i \times \bar{F}_i \cap \bar{A}_j \times \bar{B}_j \subset R \cap Q = \emptyset$ and, thus, $\bar{E}_i \bowtie \bar{A}_j$ or $\bar{F}_i \bowtie \bar{B}_j$. By the definition of the product apartness, $S \bowtie \Delta$. ∎

Lemma 4.22 (†). *Let $(X, \mathcal{U})$ be a uniform space. Suppose that the topology induced on X is compact in the classical sense. Then, classically, for any $S, T \subset X$,*

$$S \bowtie T \iff \bar{S} \cap \bar{T} = \emptyset.$$

Proof. Let $(X, \mathcal{U})$ be a compact uniform space and $S, T \subset X$. If $S \bowtie T$, then there exists a symmetric $U \in \mathcal{U}$ such that $S \times T \in \sim U^2$. Suppose that there exists some $x \in \bar{S} \cap \bar{T}$. Then there exist $s \in U[x] \cap S$ and $t \in U[x] \cap T$, whence $(s, t) \in S \times T \cap U^2$ — a contradiction.

Now suppose that $\bar{S} \cap \bar{T} = \emptyset$ and $\neg(S \bowtie T)$. Then, for every $U \in \mathcal{U}$, there exists $x_U = (s_U, t_U) \in S \times T \cap U$. Ordering $\mathcal{U}$ by reverse inclusion, we have a net $(x_U)_{U \in \mathcal{U}}$ in $X \times X$. Since X and, therefore, $X \times X$ are compact, there is a subnet $(x_{U_n})_{n \in \mathcal{D}}$ such that $(x_{U_n})_{n \in \mathcal{D}}$ converges to some $x = (s, t) \in X \times X$. Clearly, $s \in \bar{S}$ and $t \in \bar{T}$.

Now let $V \in \mathcal{U}$ be arbitrary. Since $(s_{U_n})_{n \in \mathcal{D}}$ converges to s and $(t_{U_n})_{n \in \mathcal{D}}$ converges to t, there exist $n_0, n_1, n_2 \in \mathcal{D}$ such that, for each $n \in \mathcal{D}$,

$$\left((n \succeq n_0 \implies s_{U_n} \in V[s]) \wedge (n \succeq n_1 \implies t_{U_n} \in V^{-1}[t]) \wedge (n \succeq n_2 \implies U_n \subset V) \right).$$

Choose $n \in \mathcal{D}$ such that $n \succeq n_0$, $n \succeq n_1$ and $n \succeq n_2$. Then $(s, s_{U_n}) \in V$, $(s_{U_n}, t_{U_n}) \in V$ and $(t, t_{U_n}) \in V^{-1}$. So $(s, t) \in V^3$. As V was arbitrary, we conclude that $s = t$. This is

a contradiction, as $s = t \in \bar{S} \cap \bar{T} = \emptyset$. ∎

Proposition 4.23 (†). *Let $(X,\mathcal{U})$ be a uniform space. Suppose that the topology induced on X is compact in the classical sense. Then, classically, X is CC3.*

Proof. Lemma 4.22 implies that the apartness induced on X by $\mathcal{U}$ is the same as the one used in Proposition 4.21. Thus, by said proposition, X is CC3. ∎

The following two theorems summarise the above classical results.

Theorem 4.24 (†). *Let $(X,\mathcal{U})$ be a separable uniform space with a countable base of entourages. Then X is compact if and only if it is CC3.*

Proof. First note that X has a countable base of open sets. Suppose that X is CC3. Then X is CC2 and Proposition 4.19 implies that X is compact. Conversely, Proposition 4.23 shows that, if X is compact, then it satisfies CC3. ∎

Corollary 4.25 (†). *A separable metric space is compact if and only if it is CC3.*

Proof. Apply Theorem 4.24. ∎

Theorem 4.26 (†). *Let $(X,\bowtie)$ be a Hausdorff apartness space with a countable base of open sets such that*

$$\forall S, T \subset X \ (S \bowtie T \iff \bar{S} \cap \bar{T} = \emptyset).$$

Then X is compact if and only if it is CC3.

Proof. Proposition 4.19 implies that if X is CC3, then it is compact. Conversely, Proposition 4.21 shows that, if X is compact, then it is CC3. ∎

Now we look at constructive results concerning CC3. First we show that a CC3 uniform space is weakly totally bounded. Note that Proposition 4.17 already shows that a separable, neatly located CC3 uniform space is strongly totally bounded; the following requires fewer assumptions. We also show that CC3 implies the anti-Specker property.

Lemma 4.27. *Let $(X, \mathcal{U})$ be a uniform space and $U, V \in \mathcal{U}$ symmetric with $U \cup \sim V = X^2$. Then, for each $x \in X$, $\sim\sim V[x] \subset U$.*

Proof. Let $y \in (\sim V)[x]$ and $z \in V[x]$. Then $(x, y) \in\sim V$ and $(x, z) \in V$, whence $y \neq z$. This shows that $(\sim V)[x] \subset\sim V[x]$, and $\sim (\sim V)[x] \supset\sim\sim V[x]$. Now let $y \in\sim\sim V[x] \subset\sim (\sim V)[x]$. If $(x, y) \in\sim V$, then $y \in (\sim V)[x]$, which is impossible. So $(x, y) \in U$ and $y \in U[x]$. ∎

Lemma 4.28. *Let $(X, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then $\sim U \subset -\Delta$.*

Proof. Choose symmetric $V, W \in \mathcal{U}$ such that $V^3 \subset U$, $W \subset V$ and $X^2 = V \cup \sim W$. Pick $(x, y) \in\sim U$. Then

$$
\begin{aligned}
(x, y) \quad \in \quad & - \sim W[x] \times - \sim W[y] \\
\subset \quad & V[x] \times V[y] \quad \text{(by Lemma 4.27)} \\
\subset \quad & \neg V \quad ((u, v) \in V[x] \times V[y] \cap V \implies (x, y) \in V^3 \subset U) \\
\subset \quad & \sim W.
\end{aligned}
$$

Thus $(x, y) \in -W \subset -\Delta$. ∎

Lemma 4.29. *Let $(X, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then $X^2 = -\Delta \cup - \sim U$.*

Proof. Choose symmetric $V, W \in \mathcal{U}$ such that $V^3 \subset U$, $W \subset V$ and $X^2 = V \cup \sim W$. Let $(x, y) \in W$. Clearly $(x, y) \in - \sim W[x] \times - \sim W[y]$. Let

$$
(u, v) \in - \sim W[x] \times - \sim W[y] \subset V[x] \times V[y]
$$

(note Lemma 4.27). Now $(x, u), (x, y), (y, u) \in V$, so $(u, v) \in V^3 \subset U \subset\sim\sim U$. So $(x, y) \in - \sim U$ and $W \subset - \sim U$.

Choose $W' \in \mathcal{U}$ such that $X^2 = W \cup \sim W'$. By Lemma 4.28, $-\Delta \supset\sim W'$, so $X^2 = - \sim U \cup -\Delta$. ∎

Proposition 4.30. *Let $(X, \mathcal{U})$ be a CC3 uniform space. Then X is weakly totally bounded.*

Proof. Let $U \in \mathcal{U}$ be arbitrary and choose a symmetric $V \in \mathcal{U}$ such that $X^2 = U \cup \sim V$. Then $\neg \sim V \subset U$. By Lemma 4.29, $X^2 = -\Delta \cup - \sim V$. Thus, by CC3 and the definition of the product apartness, there exist $n, m \in \mathbf{N}$ and

$$B_{1,0}, \cdots, B_{m,0}, B_{1,1}, \cdots, B_{m,1}, C_{1,0}, \cdots, C_{n,0}, C_{1,1}, \cdots, C_{n,1} \subset X$$

such that

$$\Delta \subset \bigcup_{i=1}^{m} B_{i,0} \times B_{i,1}, \ \ \sim V \subset \bigcup_{j=1}^{n} C_{j,0} \times C_{j,1},$$

and, for each $i, j \in \mathbf{N}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$, there exists $k \in \{0, 1\}$ such that $B_{i,k} \bowtie C_{j,k}$. Thus, for each $i, j \in \mathbf{N}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$, $B_{i,0} \times B_{i,1} \subset \neg C_{j,0} \times C_{j,1}$ and

$$B_{i,0} \times B_{i,1} \subset \bigcap_{j'=1}^{n} \neg C_{j',0} \times C_{j',1} \subset \neg \bigcup_{j'=1}^{n} C_{j',0} \times C_{j',1} \subset \neg \sim V \subset U.$$

For each $i \in \mathbf{N}$ with $1 \leq i \leq m$, set $A_i = B_{i,0} \cap B_{i,1}$, whence $A_i \times A_i \subset B_{i,0} \times B_{i,1} \subset U$. And, for each $x \in X$, there exists $i \in \mathbf{N}$ such that $1 \leq i \leq m$ and $(x, x) \in B_{i,0} \times B_{i,1}$, which implies that $x \in A_i$. $\blacksquare$

Proposition 4.31. *Let $(X, \mathcal{U})$ be a CC3 uniform space. Then X has the strong anti-Specker property.*

Proof. Let $(x_n)_{n \in \mathbf{N}}$ be a sequence in $X \cup \{\infty\}$ that is eventually bounded away from each point in X. By applying dependent choice and passing to a subsequence, we may, without loss of generality, assume that, for each $n \in \mathbf{N}$,

$$x_n \in -\{x_k : k \in \mathbf{N} \wedge k \neq n\}.$$

Now let

$$S = \{(x_i, x_j) : i, j \in \mathbf{N} \wedge i \neq j \wedge x_i \neq \infty \wedge x_j \neq \infty\}.$$

Now let $(x, y) \in X^2$. Choose a symmetric $U \in \mathcal{U}$ and $N \in \mathbf{N}$ such that $x_n \in$

$\neg U[x] \cap \neg U[y]$ for each $n \geq N$ and $(x_i, x_j) \notin U^3$ for each $i, j \in \mathbf{N}$ with $i, j < N$. Choose a symmetric $V \in \mathcal{U}$ such that $U \cup \sim V^2 = X^2$. Suppose that $(x, y) \in U$. If there exists $i, j \in \mathbf{N}$ with $i \neq j$ and $(x_i, x_j) \in U[x] \times U[y]$, then $i, j < N$ and, as $(x_i, x), (x, y), (y, x_j) \in U = U^{-1}$, $(x_i, x_j) \in U^3$, which is impossible. Thus $U[x] \times U[y] \cap S = \emptyset$ and $(x, y) \in -S$. Suppose, on the other hand, that $(x, y) \in \sim V^2$. If $V[x] \times V[y] \cap \Delta \neq \emptyset$, then $(x, y) \in V^2$, which is impossible. Thus $(x, y) \in -\Delta$. As $(x, y) \in X^2$ was arbitrary, we conclude that $X^2 = -S \cup -\Delta$.

By CC3, $S \bowtie \Delta$. There exists a symmetric $U \in \mathcal{U}$ such that, for each $(x, y) \in S$ and $(z, z) \in \Delta$, either $(x, z) \in \sim U$ or $(y, z) \in \sim U$, whence $S \in \sim U$. Proposition 4.30 allows us to choose $m \in \mathbf{N}$ and $A_1, A_2, \cdots, A_m \subset X$ such that

$$X = A_1 \cup A_2 \cup \cdots \cup A_m \text{ and } A_1 \times A_1, A_2 \times A_2, \cdots, A_m \times A_m \subset U.$$

We now apply the pigeonhole principle. There are $(m+2)^2 - (m+2)$ values of $i, j \in \mathbf{N}$ with $1 \leq i, j \leq m+2$ and $i \neq j$; this is at least $m+1$ values. Suppose that $x_i \neq \infty$ for each $i \in \mathbf{N}$ with $1 \leq i \leq m+2$. Then there exist $i, j, k \in \mathbf{N}$ with $1 \leq i, j \leq m+2$, $i \neq j$, $1 \leq k \leq m$ and $x_i, x_j \in A_k$; this is impossible as it implies that $(x_i, x_j) \in U$. So our supposition is false and there exists $i \in \mathbf{N}$ such that $1 \leq i \leq m+2$ and $x_i = \infty$. $\blacksquare$

This concludes our investigation of CC1, CC2, and CC3. We now have a good understanding of how these criteria relate to other compactness conditions; we refer the reader back to the summary diagram at the beginning of this chapter. The next chapter speculates about extending this investigation.

Chapter 5

Further Work

"What's past is prologue"—William Shakespeare

The three criteria we have proposed in Chapter 4 all capture some aspect of compactness. Most crucially, they make use of set-set apartness rather than just point-set apartness like the criteria in Chapter 3. We have demonstrated how our criteria relate to other notions of compactness, even though our results are not tight.

Our results could be strengthened and expanded. This might clarify which of our three criteria is most useful; at the moment it seems that CC1 and CC2 are too weak to classify compactness, while CC3 is too strong. Perhaps further exploration will lead to a criterion of intermediate strength. The crucial issue seems to be locatedness: neat locatedness is too strong, but, as we mentioned in Section 2.5, there is no natural analogue to locatedness in apartness spaces.

We have not explored the categorical properties of compactness: subspaces, products, and continuous images. Perhaps products will be the most interesting of these, as the product apartness plays an important role in the definitions of CC2 and CC3. The differences between CC1 and both CC2 and CC3 stem from the product apartness; this leads one to suspect that an analogue to Tychonoff's theorem [19] would be difficult to prove. However, the crucial feature of the product apartness is the finiteness

requirement it contains, not the fact that it produces a different space. The reason we needed the product apartness was because, as we have already remarked, classically, any topological space can be equipped with a CC1 apartness; the product is our way of avoiding this problem. However, because of this almost all of our results resort to working in uniform spaces.

CC1 captures completeness in an interesting way. Normally, to show that a space is complete, we must be able to construct a certain point from a Cauchy sequence or net. This can be difficult, and CC1 does not require the construction of a point. This may also make CC1 interesting from a point-free perspective.

We have no results that link our compactness criteria to RUSS and INT. In particular, does $[0, 1]$ satisfy CC1 in either of these models? Results of this form will give us a much better understanding of the meaning of these criteria.

Lastly, it remains to choose more creative names for CC1, CC2, and CC3.

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Bibliography

[1] P. Aczel and M. Rathjen: *Notes on Constructive Set Theory*, Report No. 40, Institut Mittag-Leffler, Royal Swedish Academy of Sciences, 2001.

[2] P. Aczel and M. Rathjen: *Constructive Set Theory*, in preparation.

[3] H.L. Bentley, H. Herrlich, and M. Hušek: *The historical development of uniform, proximal, and nearness concepts in topology*, in *Handbook of the History of General Topology,* Vol. 2 (C.E. Aull and R. Lowen, eds.), 577–629, Kluwer, Amsterdam, The Netherlands, 1998.

[4] Josef Berger and Douglas Bridges: *A fan-theoretic equivalent of the antithesis of Specker's theorem*, Indagationes Mathematicae **18**(2), 195–202, 2007.

[5] J. Berger and D.S. Bridges: *The fan theorem and positive-valued uniformly continuous functions on compact intervals*, New Zealand J. Math. **38**, 129–135, 2008.

[6] Errett Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[7] D.S. Bridges: *On the foundations of best approximation theory*, J. Approx. Theory **28**(3), 273–292, 1980.

[8] D.S. Bridges: *A constructive analysis of the Remes algorithm*, J. Approx. Theory **32**(4), 257–270, 1981.

[9] D.S. Bridges and L.S. Vîţă: *Apartness and Uniformity: A Constructive Development*, CiE Series *Theory and Applications of Computability*, Springer Verlag, Heidelberg, 2011.

[10] D. Bridges and F. Richman: *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.

[11] D.S. Bridges, H. Ishihara, R. Mines, F. Richman, P.M. Schuster, and L.S. Vîţă: *Almost locatedness in uniform spaces*, Czech. Math. J. **57**(1), 1–12, 2007.

[12] R.L. Constable et al.: *Implementing Mathematics with the Nuprl Proof Development System*, Prentice-Hall, Englewood Cliffs, New Jersey, 1986.

[13] H. Diener: *Generalising Compactness*, Math. Logic Quart. **51**(1), 49–57, 2008.

[14] M.A.E. Dummett: *Elements of Intuitionism* (Second Edition), Oxford Logic Guides **39**, Clarendon Press, Oxford, 2000.

[15] Christine Finn: *Artifacts : An Archaeologist's Year in Silicon Valley*, p. 90 2001.

[16] S. Hayasi and H. Nakano: *PX: A Computational Logic*, MIT Press Cambridge, Mass., 1988.

[17] A. Hedin: *Towards set-presentable apartness spaces*, in A. Hedin, *Contributions to Pointfree Topology and Apartness Spaces*, doctoral thesis, Uppsala University, Sweden, 2011.

[18] W.H. Julian and F. Richman: *A uniformly continuous function on* $[0, 1]$ *that is everywhere different from its infimum*, Pacific J. Math. **111**, 333–340, 1984.

[19] John L. Kelley: *General Topology*, van Nostrand, 1955.

[20] Donald E. Knuth: *Algorithmic Thinking and Mathematical Thinking*, The American Mathematical Monthly **92**(3), 170–181, 1985.

[21] D. König: *Sur les correspondances multivoques des ensembles*, Fundamenta Mathematicae **8**, 114–134, 1926.

[22] B.A. Kushner: *Lectures on Constructive Mathematical Analysis*, Amer. Math. Soc., Providence RI, 1985.

[23] P. Martin-Löf: *Constructive mathematics and computer programming*, in *Logic, Methodology, and Philosophy of Science VI* (J.J. Cohen, J. Loś, H. Pfeiffer, K-P. Podewski, eds.), 153–175, North-Holland, Amsterdam, 1982.

[24] P. Martin-Löf: *Intuitionistic Type Theory*, Notes by G. Sambin, Bibliopolis, Napoli, 1984.

[25] P. Martin-Löf: *An intuitionistic theory of types*, in *Twenty-five Years of Constructive Type Theory* (G. Sambin, J. Smith, eds.), Clarendon Press, Oxford, 1998.

[26] J. Myhill: *Some properties of Intuitionistic Zermelo-Fraenkel set theory*, in *Proceedings of the 1971 Cambridge Summer School in Mathematical Logic* (A.R.D. Mathias, and H. Rogers eds.), 206–231, Lecture Notes in Mathematics 337, Springer, Berlin, 1973.

[27] S.A. Naimpally and B.D. Warrack: *Proximity Spaces*, Cambridge University Press, 1970.

[28] F. Richman: *Constructive mathematics without choice*, in *Reuniting the Antipodes—Constructive and Nonstandard Views of the Continuum* (P.M. Schuster, U. Berger, and H. Osswald, eds.), Kluwer, Synthese Library **306**, 199–205, 2001.

[29] H.A. Schwichtenberg: *Program extraction in constructive analysis*, in *Logicism, Intuitionism, and Formalism—What has become of them?* (S. Lindström, E. Palmgren, K. Segerberg, and V. Stoltenberg-Hansen, eds.), 255–275, Synthese Library **341**, Springer Verlag, Berlin, 2009.

[30] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics: An Introduction* (two volumes), North Holland, Amsterdam, 1988.