Discrete vs. Continuous Orthogonal Moments for Image Analysis

R. Mukundan
Faculty of Information Science and Technology, Multimedia University,
Bukit Beruang, 75450 Malacca, Malaysia.

S. H. Ong
Institute of Mathematical Sciences, University of Malaya,
50603 Kuala Lumpur, Malaysia.

P.A. Lee
Faculty of Information Technology, Multimedia University,
63100 Cyberjaya, Malaysia.

Abstract – *Image feature representation techniques using orthogonal moment functions have been used in many applications such as invariant pattern recognition, object identification and image reconstruction. Legendre and Zernike moments are very popular in this class, owing to their feature representation capability with a minimal information redundancy measure. This paper presents a comparative analysis between these moments and a new set of discrete orthogonal moments based on Tchebichef polynomials. The implementation aspects of orthogonal moments are discussed, and experimental results using both binary and gray-level images are included to show the advantages of discrete orthogonal moments over continuous moments.*

Keywords: Orthogonal moment functions, Legendre moments, Zernike moments, Tchebichef polynomials.

1. Introduction

Moment functions of the two-dimensional image intensity distribution are used in a variety of applications, as descriptors of shape. Image moments that are invariant with respect to the transformations of scale, translation, and rotation find applications in areas such as pattern recognition, object identification and template matching. Orthogonal moments have additional properties of being more robust in the presence of image noise, and having a near-zero redundancy measure in a feature set. Zernike moments, which are proven to have very good image feature representation capabilities, are based on the orthogonal Zernike radial polynomials[1]. They are effectively used in pattern recognition since their rotational invariants can be easily constructed. Legendre moments form another orthogonal set, defined on the Cartesian coordinate space[2]. Orthogonal moments also permit the analytical reconstruction of an image intensity function from a finite set of moments, using the inverse moment transform.

Both Legendre and Zernike moments are defined as continuous integrals over a domain of normalized coordinates. The implementations of such moment functions therefore involve the following sources of errors: (i) the discrete approximation of the continuous integrals, and (ii) the transformation of the image coordinate system into the domain of the orthogonal polynomials. The discrete approximation of integrals affects the analytical properties that the moment functions are intended to satisfy, such as invariance and orthogonality. The coordinate space normalization increases the computational complexity.

The use of discrete orthogonal polynomials[4,5] as basis functions for image moments, eliminates the aforesaid problems associated with the continuous moments. A few fundamental properties of discrete Tchebichef (sometimes also written as Chebyshev) polynomials and their moment functions can be found in [3] . This paper studies in detail, the implementation aspects of Zernike, Legendre and Tchebichef moments. The paper also analyses the functional characteristics of Legendre and Tchebichef moments, in the light of the close relationship between the corresponding polynomials.

2. Discrete Approximation

The discrete approximation errors that are introduced while implementing continuous orthogonal moments can be very significant in many applications. As an example, we illustrate below, how the computed values of Zernike moments differ from their theoretical values for a circular disc. Given an intensity distribution $f(r,\theta)$ in the polar coordinate space, the Zernike moments Z_{nl} of order n, and repetition l are defined as

$$Z_{nl} = \frac{(n+1)}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} R_{nl}(r) e^{-\hat{j}l\theta} f(r,\theta) r\, dr\, d\theta$$

$$|l| \le n, \quad n-|l| \text{ is even.} \tag{1}$$

$R_{nl}(r)$ denotes the Zernike radial polynomials of degree n, and $\hat{j} = \sqrt{-1}$. For a square image of size N given by $f(i, j)$, $i, j = 0, 1, 2,\ldots, N-1$, a discrete approximation of (1) is

$$Z_{nl} = \frac{2(n+1)}{\pi\,(N-1)^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} R_{nl}(r_{ij}) e^{-\hat{j}l\theta_{ij}} f(i,j) \tag{2}$$

where the image coordinate transformation is given by

$$r_{ij} = \sqrt{(c_1 i + c_2)^2 + (c_1 j + c_2)^2}, \quad c_1 = \frac{\sqrt{2}}{N-1},$$

$$c_2 = -\frac{1}{\sqrt{2}}, \quad \theta_{ij} = \tan^{-1}\left(\frac{c_1 j + c_2}{c_1 i + c_2}\right). \tag{3}$$

If the image is a circular disc of radius R pixels, then the theoretical values of moments can be computed directly from (1) as follows:

$Z_{00} = \gamma^2$ $\quad (\gamma = c_1 R)$.

$Z_{20} = 3\gamma^2(\gamma^2-1)$

$Z_{40} = 5\gamma^2(\gamma^2-1)(2\gamma^2-1)$, etc. (4)

Table 1 provides a comparison of the values obtained using (2) and (4), with R=50 and N=101.

Table 1. Ideal and Computed Zernike Moments.

	Ideal Values	Computed Moments
Z_{00}	0.5	0.499428
Z_{20}	–0.75	–0.74991
Z_{40}	0	0.001429
Z_{60}	0.4375	0.437469

We give below another example of the effect of discrete approximation, using Legendre moments. The Legendre moment of order $m+n$ of a two-dimensional intensity distribution $f(x, y)$ is given by

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_m(x) P_n(y) f(x,y)\, dx\, dy \tag{5}$$

where $P_m(x)$ is the Legendre polynomial of degree m. The discrete approximation of (5) in the domain of the image space is

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{N^2} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} P_m(x_i) P_n(y_j) f(i,j) \tag{6}$$

where the image coordinate transformation is given by

$$x_i = \frac{2i-N+1}{N-1}; \quad y_j = \frac{2j-N+1}{N-1} \tag{7}$$

The discretization of continuous integrals as given in (6) affects the orthogonality property of the Legendre moments. If we define

$$e(m,n) = \sum_{i=0}^{N-1} P_m(x_i)P_n(x_i) \qquad (8)$$

in the discrete domain of the image space, then it is desired to have $e(m,n) = 0$ whenever $m \neq n$. However this condition is not satisfied especially for small values of N. A plot of the variation of e(2,4) with respect to N is in Fig. 1.

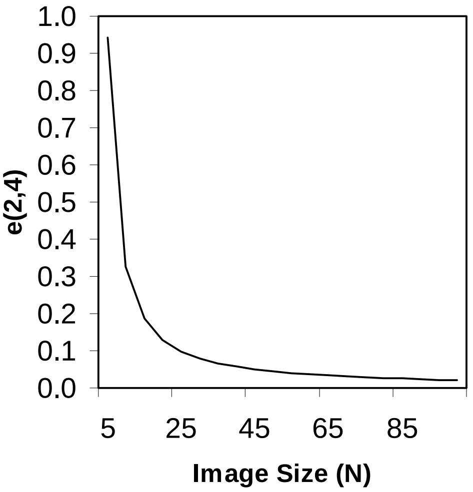

Fig. 1. Effect of discretization on the orthogonality property of Legendre polynomials.

3. Discrete Orthogonal Moments

In this section we introduce a new set of moments with discrete orthogonal polynomials as basis functions. Such discrete orthogonal moments will eliminate the need for numerical approximations, and will exactly satisfy the orthogonality property in the image coordinate space. An ideal choice of the basis functions would be the Tchebichef polynomials[3,4], which have the orthogonality relation

$$\sum_{i=0}^{N-1} t_m(i)t_n(i) = \rho(n,N)\delta_{mn} \qquad (9)$$

for $0 \le m, n \le N-1$. The function $\rho(n, N)$ is the squared norm given by

$$\rho(n, N) = (2n)! \binom{N+n}{2n+1}. \qquad (10)$$

The polynomial values $t_n(x)$ can be easily computed using the recurrence relation

$$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(N^2-n^2)t_{n-1}(x) = 0 \qquad (11)$$

with the initial conditions

$$t_0(x) = 1,$$
$$t_1(x) = 2x-N+1. \qquad (12)$$

If we scale the Tchebichef polynomials by a factor β_n which is independent of x, then the squared norm of the scaled functions will be $\rho(n, N)/\beta_n^2$. The recurrence formula in (11) will also have to be suitably modified. Since a Tchebichef polynomial $t_n(x)$ of degree n is a function of x^n, $x=0,1,\ldots N-1$, a commonly used scale factor is

$$\beta_n = N^n \qquad (13)$$

The above selection is mainly done to avoid large variations in the dynamic range of polynomial values and numerical instabilities in the polynomial computation, when N is large.

Tchebichef moments of order $p+q$ of an image $f(i, j)$ are defined based on the scaled orthogonal Tchebichef polynomials $t_n(i)$, as

$$T_{mn} = \frac{1}{\rho(m,N)\rho(n,N)} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} t_m(i)t_n(j)f(i,j)$$
$$m, n = 0, 1, \ldots N-1, \qquad (14)$$

and has an exact image reconstruction formula (inverse moment transform),

$$f(i,j) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} T_{mn}t_m(i)t_n(j). \qquad (15)$$

The Legendre and Tchebichef polynomials are closely related to each other in the limiting case when N tends to infinity:

$$\lim_{N\to\infty} N^{-n}t_n(Nx) = P_n(2x-1), \quad x \in [0,1]. \qquad (16)$$

The above fact suggests an exhaustive comparative analysis between the two types of polynomials and the corresponding moment functions. A comparison plot of Tchebichef and Legendre polynomials of degree 5, is given in Fig. 2(a), with a scale factor β_n as in (13), and N=20. The corresponding plot for the polynomial of degree 10, is in Fig. 2(b).

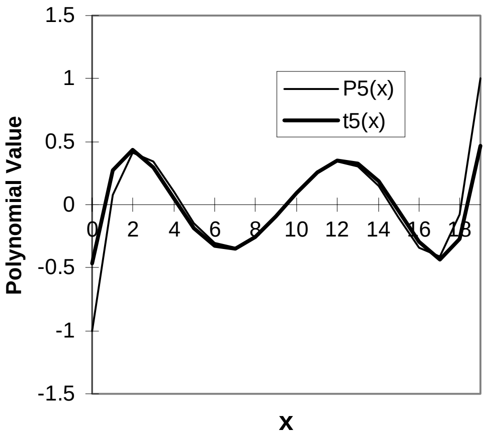

Fig. 2(a). Plot with n=5, and β_n as in (13).

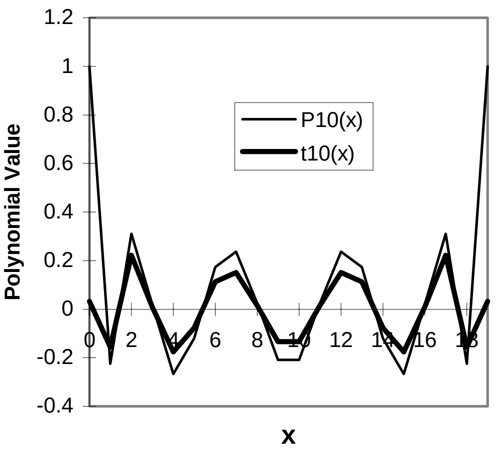

Fig. 2(b). Plot with n=10, and β_n as in (13).

Note that unlike the Legendre polynomials, the Tchebichef polynomial values are not constrained to the interval [–1,1]. Apart from (13), another important choice of β_n is given by

$$\beta_n = \sqrt{(N^2-1^2)(N^2-2^2)\cdots(N^2-n^2)}\,. \qquad (17)$$

With the above scale factor, the squared norm becomes $N/(2n+1)$ which is the same as that of the Legendre polynomials in the discrete domain. A plot showing the values of Legendre and Tchebichef polynomials of degree 10, with the above scale factor, is given in Fig. 3.

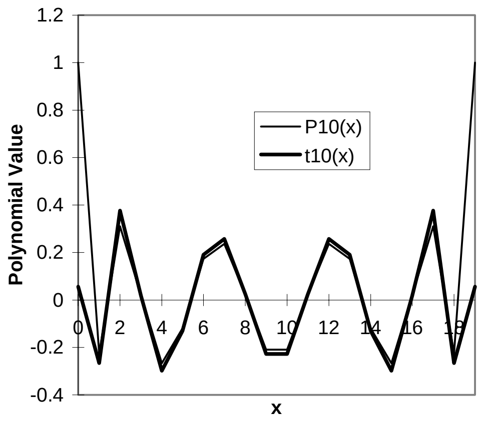

Fig. 3. Plot with n=10, and β_n as in (17).

Comparing Fig.2(b) and Fig. 3, it is obvious that equation (17) can be used to scale Tchebichef polynomials so as to effectively replace Legendre polynomials in the discrete domain.

4. Image Reconstruction

Orthogonal moments allow the intensity distribution to have a polynomial approximation, which can be used to reconstruct the image from a finite set of its computed moments. The image reconstruction error provides a measure of the feature representation capability of the moment functions.

An image can be reconstructed from its Legendre moments λ_{mn} using the formula,

$$f(i,j) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} P_m(x_i) P_n(y_j) \qquad (18)$$

where x_i , y_j are given by (7). The infinite series in (18) will have to be truncated to the maximum order of moments computed, to obtain the reconstructed image. Tchebichef moments, on the other hand, has a reconstruction formula from a finite set of moments, as given by (15). This section analyzes the reconstruction capabilities of Tchebichef moments and compares it with that of Legendre moments.

Fig. 4 shows a binary image of a Chinese character, having size 96x96 pixels. The Legendre (6) and Tchebichef moments (14) of this image were computed. The reconstructed images using a maximum order of moments 16, 20, 24, 28, and 32 respectively, are also shown in Fig. 4. The reconstruction error between the original image $f(i, j)$ and the reconstructed image $f'(i, j)$ can be defined as

$$\varepsilon = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left| f(i, j) - f'(i, j) \right|. \tag{19}$$

A plot of the reconstruction error with respect to the maximum order of moments, is given in Fig. 5(a). This plot shows only a marginal improvement of Tchebicef moments over Legendre moments. However Tchebichef moments are much more robust than Legendre moments in the presence of image noise.

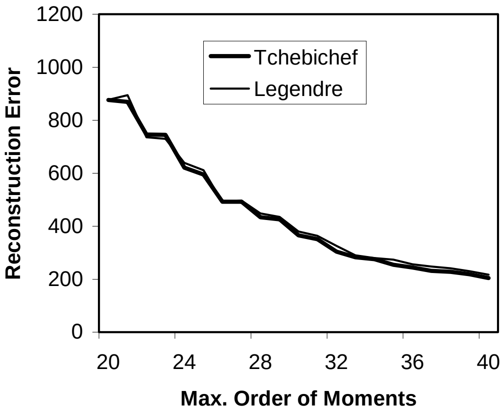

Fig. 5(a). Plot of reconstruction errors without image noise.

When the comparative analysis was repeated after adding 5% random noise to the binary image, the plot of reconstruction errors shows a significant difference between both the approaches (Fig. 5(b)).

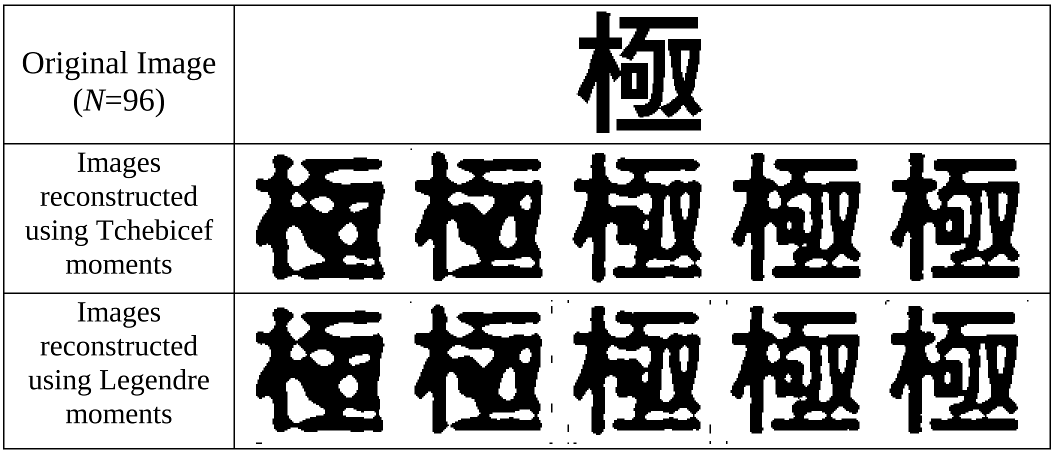

Fig. 4. Binary image reconstruction using Tchebichef and Legendre moments.

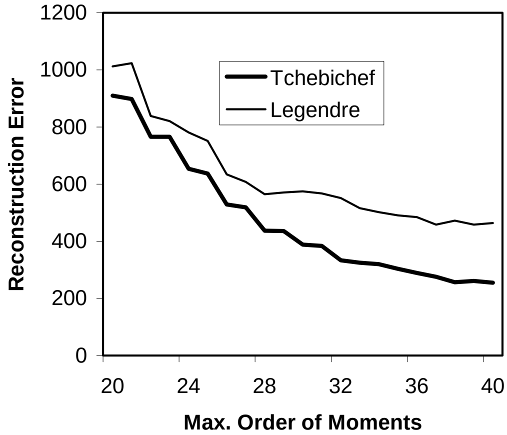

Fig. 5(b). Plot of reconstruction errors with image noise.

Fig. 6 shows a gray-level image of size 200x200 pixels, used for the performance comparison between Tchebichef and Legendre moments. Moments up to a maximum order of 100 were computed from the image, and the results of reconstruction are also given in Fig. 6. The corresponding plot showing the variation of the error with the number of moments used, is in Fig. 7. The noise sensitivity of Legendre moments is evident from this experiment

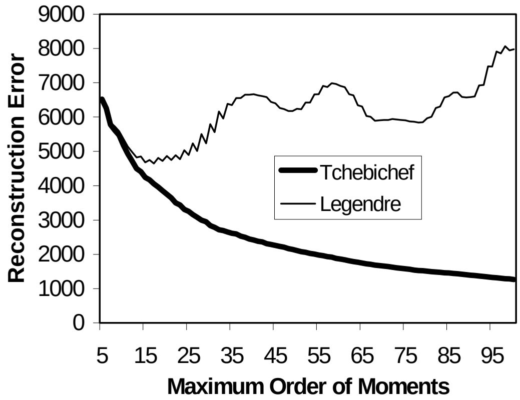

Fig. 7. Plot of reconstruction errors for the gray-level image.

5. Conclusions

This paper discussed the problems related to the implementation of continuous orthogonal moments, such as the effects of discrete approximation errors, and the need for a transformation from the image coordinate space to a normalized coordinate space. The discrete orthogonal moments based on Tchebichef polynomials are compared with Legendre moments. Tchebichef moments have superior feature representation capability over continuous moments, as demonstrated through the process of image reconstruction.

Original Image (N=200)	Image reconstructed using Tchebichef moments	Image reconstructed using Legendre moments
		

Fig. 6. Gray-level image reconstruction using Tchebichef and Legendre moments.

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