Exact Capacity Distributions for MIMO Systems With Small Numbers of Antennas

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***Abstract*—It is well known that multiple input multiple output (MIMO) systems offer the promise of achieving very high spectrum efficiencies (many tens of bit/s/Hz) in a mobile environment. The gains in MIMO capacity are sensitive to the presence of spatial correlation introduced by the radio environment. In this letter we consider the capacity outage performance of MIMO systems in correlated environments. For systems with large numbers of antennas Gaussian approximations are very accurate. Hence, we concentrate on systems with small numbers of antennas and derive exact densities and distribution functions for the capacity, which are simple and rapid to compute.**

***Index Terms*—Information rates, MIMO systems, wireless channel models.**

I. Introduction

Since the pioneering work of Foschini and Gans [1] and Telatar [2], multiple input multiple output (MIMO) systems have received considerable attention in recent years as they have the potential to provide quantum leaps in capacity. The capacity of MIMO systems has been intensively studied. For uncorrelated Rayleigh fading there are many exact results available: the mean capacity [2], capacity variance [3], the characteristic function [4]. In addition there are many useful asymptotic results [5] including a Gaussian approximation [3]. For the non-Rayleigh case, simulations are usually required. For the correlated Rayleigh case, many results are currently emerging [6]–[9].

Our contribution is for small numbers of antennas where Gaussian approximations to the capacity become less useful. We have not found any exact results for capacity distributions in the literature. Hence, in Section III we derive the exact density and distribution function for MIMO capacity in a correlated Rayleigh channel. Firstly, in Section II we describe the system model and some link capacity results. After the derivations we give results and conclusions in Section IV.

II. System Model and Link Capacity

In this letter, we assume a single-user MIMO system with no channel state information (CSI) at the transmitter, perfect CSI at the receiver, and employing equal power transmission over a flat Rayleigh fading channel. For such a system with n_T transmit and n_R receive antennas the received signal is

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{r}$ is the $n_R \times 1$ received signal vector, $\boldsymbol{s}$ is the complex $n_T \times 1$ transmitted signal vector and $\boldsymbol{H}$ is an $n_R \times n_T$ complex channel gain matrix. The AWGN vector $\boldsymbol{n}$ consists of n_R independent noise components of modulus variance normalized to 1. The capacity of such a system is now very well known [1], [2] and is given by

$$C = \log_2\left[\det\left(\boldsymbol{I}_m + \frac{\rho}{n_T}\boldsymbol{W}\right)\right] \tag{2}$$

where $\boldsymbol{I}_m$ is the $m \times m$ identity matrix, ρ is the average signal-to-noise ratio (SNR) per receive antenna, $m = \min(n_R, n_T)$, $n = \max(n_R, n_T)$, and the $m \times n$ matrix $\boldsymbol{W}$ is given by

$$\boldsymbol{W} = \begin{cases} \boldsymbol{H}\boldsymbol{H}^\dagger, & \text{for } n_R \le n_T \\ \boldsymbol{H}^\dagger\boldsymbol{H}, & \text{for } n_T < n_R \end{cases} \tag{3}$$

where $(\cdot)^\dagger$ denotes the conjugate transpose.

For correlated Rayleigh channels a separable correlation model is often assumed [7], [10], where $\boldsymbol{H}$ is written as

$$\boldsymbol{H} = \boldsymbol{\Psi}_R \boldsymbol{U} \boldsymbol{\Psi}_T \tag{4}$$

and $\boldsymbol{\Psi}_R, \boldsymbol{\Psi}_T$ represent the correlations induced at the receiver and transmitter, respectively. The matrix $\boldsymbol{U}$ is an $n_R \times n_T$ complex channel gain matrix containing i.i.d. complex Gaussian entries with unit magnitude variance. Various measurements have been presented in the literature which support the accuracy of this form of the correlation matrix, for example, [6]. Hence, correlation may be present at either or both ends of the MIMO link. Here we consider channels with correlation at one end only, and we denote these channels as "semi-correlated" following [9]. Note that recent measurements conducted in downtown Helsinki show that the semi-correlated channel model is valid for certain urban environments [8]. In addition, we assume that the correlation is present at the end of the link with more antennas. This is true for SIMO or MISO systems and includes the most interesting special case $n_R = n_T$.

With these correlation assumptions we define $\boldsymbol{V} = \boldsymbol{U}^\dagger$, $\boldsymbol{\Psi} = \boldsymbol{\Psi}_T^\dagger$ for $n_R \le n_T$, and $\boldsymbol{V} = \boldsymbol{U}$, $\boldsymbol{\Psi} = \boldsymbol{\Psi}_R$ for $n_R > n_T$ which gives

$$\boldsymbol{W} = \boldsymbol{V}^\dagger \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \boldsymbol{V} = \boldsymbol{X}^\dagger \widetilde{\boldsymbol{\Gamma}} \boldsymbol{X}. \tag{5}$$

In (5) we have factorized the hermitian matrix $\boldsymbol{\Psi}^\dagger\boldsymbol{\Psi}$ into $\boldsymbol{\Phi}^\dagger\widetilde{\boldsymbol{\Gamma}}\boldsymbol{\Phi}$, where $\widetilde{\boldsymbol{\Gamma}}$ is a diagonal matrix containing the nonnegative eigen-

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values of $\boldsymbol{\Psi}^\dagger\boldsymbol{\Psi}$. Since $\boldsymbol{\Phi}$ is unitary, the matrix $\boldsymbol{X} = \boldsymbol{\Phi V}$ has the same statistics as $\boldsymbol{V}$. Hence, we have

$$\begin{aligned} C &= \log_2 \left[\det\left(\boldsymbol{I}_m + \frac{\rho}{n_T}\boldsymbol{X}^\dagger\widetilde{\boldsymbol{\Gamma}}\boldsymbol{X}\right)\right] \\ &= \log_2 \left[\det\left(\boldsymbol{I}_m + \boldsymbol{X}^\dagger\boldsymbol{\Gamma}\boldsymbol{X}\right)\right] \end{aligned} \tag{6}$$

where we have absorbed the constant ρ/n_T into $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$. If we define x_i, $i = 1, \ldots, m$ to be the eigenvalues of $\boldsymbol{X}^\dagger\boldsymbol{\Gamma}\boldsymbol{X}$, we can rewrite (6) as

$$C = \log_2 \left[\prod_{i=1}^{m}(1 + x_i)\right]. \tag{7}$$

III. Derivations

To derive the density of C, we need the following results. The joint density of elements of $\boldsymbol{X}^\dagger\boldsymbol{\Gamma}\boldsymbol{X}$ is given by [11]

$$\pi^{-m(m-1)/2}\frac{\Delta_1}{\Delta_2\Delta_3} \tag{8}$$

where Δ_1 is defined by the determinant

$$\begin{vmatrix} 1 \cdots \gamma_1^{n-m-1} & \gamma_1^{n-m-1}e^{-x_1/\gamma_1} & \cdots & \gamma_1^{n-m-1}e^{-x_m/\gamma_1} \\ \vdots \cdots \quad \vdots & \vdots & \cdots & \vdots \\ 1 \cdots \gamma_n^{n-m-1} & \gamma_n^{n-m-1}e^{-x_1/\gamma_n} & \cdots & \gamma_n^{n-m-1}e^{-x_m/\gamma_n} \end{vmatrix}$$

$\Delta_2 = \prod_{i>j}(x_i - x_j)$, and $\Delta_3 = \prod_{i>j}(\gamma_i - \gamma_j)$. Here, the x_i's are *ordered* eigenvalues. The Jacobian J of the transformation from elements of $\boldsymbol{X}^\dagger\boldsymbol{\Gamma}\boldsymbol{X}$ to eigenvalues can be written [12]

$$J = \frac{\pi^{m(m-1)/2}\Delta_2^2}{\prod_{i=1}^{m}(m-i)!}. \tag{9}$$

Therefore, the joint distribution of the ordered eigenvalues ($0 \le x_1 \le x_2 \le \cdots \le x_m$) has the form

$$f(x_1, \ldots, x_m) = \frac{\Delta_1\Delta_2}{\prod_{i=1}^{m}(m-i)!\Delta_3}. \tag{10}$$

Note that the denominator is independent of $x_1, \ldots, x_m$. Finally, for the *unordered* eigenvalues with $x_i \ge 0$ for $i = 1, \ldots, m$, the joint distribution is

$$f(x_1, \ldots, x_m) = \frac{\Delta_1\Delta_2}{\prod_{i=0}^{m}(m-i)!\Delta_3}. \tag{11}$$

A. SIMO or MISO Systems

For $m = 1$, corresponding either to a single input multiple output (SIMO) system or a multiple input single output (MISO) system, the capacity (6) reduces to

$$C = \log_2 \left[1 + \sum_{i=1}^{n}\gamma_i|x_i|^2\right] \tag{12}$$

where the $|x_i|^2$'s are independent unit exponentially distributed. If we set $Y = \sum_{i=1}^{n}\gamma_i|x_i|^2$, for unequal γ_i's we have [13]

$$f_Y(y) = \sum_{j=1}^{n}\left[\prod_{k\neq j}(\gamma_j - \gamma_k)^{-1}\right]\gamma_j^{n-2}e^{-y/\gamma_j}, \qquad y > 0.$$

Therefore, for $c > 0$ the density and distribution of C are

$$f_C(c) = 2^c \ln 2 \sum_{j=1}^{n}\left[\prod_{k\neq j}(\gamma_j - \gamma_k)^{-1}\right]\gamma_j^{n-2}e^{(1-2^c)/\gamma_j} \tag{13}$$

$$F_C(c) = \sum_{j=1}^{n}\left[\prod_{k\neq j}(\gamma_j - \gamma_k)^{-1}\right]\gamma_j^{n-1}\left[1 - e^{(1-2^c)/\gamma_j}\right]. \tag{14}$$

Note that the assumption that the γ_i's are unequal is valid for practical wireless channels. Nevertheless, the distribution for the case of subsets of equal γ_i's can easily be derived.

B. Dual Input or Dual Output Systems

Here we have $m = 2$, corresponding to either $n_T = 2$ or $n_R = 2$. We assume $n \ge 2$ since the single antenna case was given above. The density (11) has the specific form

$$\begin{aligned} f(x_1, x_2) &= \frac{(x_2 - x_1)}{2\prod_{i>j}(\gamma_i - \gamma_j)} \\ &\times \begin{vmatrix} 1 & \gamma_1 & \cdots & \gamma_1^{n-3} & \gamma_1^{n-3}e^{-x_1/\gamma_1} & \gamma_1^{n-3}e^{-x_2/\gamma_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \gamma_n & \cdots & \gamma_n^{n-3} & \gamma_n^{n-3}e^{-x_1/\gamma_n} & \gamma_n^{n-3}e^{-x_2/\gamma_n} \end{vmatrix}. \end{aligned} \tag{15}$$

Note that (15) reduces to the well-known result [2] for uncorrelated fading when the γ_i's are all equal. However, demonstrating this requires successive application of Cauchy's mean value theorem and multiple differentiation of the numerator and denominator in (15). This is beyond the scope of this letter.

The determinant in (15) can be rewritten as

$$\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}(\gamma_i\gamma_j)^{n-3}(-1)^{i+g(j)-1}D_{ij}\exp\left(-\frac{x_2}{\gamma_i} - \frac{x_1}{\gamma_j}\right)$$

where

$$g(j) = \begin{cases} j, & \text{for } j < i \\ j-1, & \text{for } j > i \end{cases}$$

and D_{ij} is the determinant of the matrix without the last two columns and the i-th and j-th rows. For $n = 2$, we have $D_{ij} = 1$. Thus, the density can be reduced to the form

$$f(x_1, x_2) = \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\alpha_{ij}(x_2 - x_1)e^{-(x_2/\gamma_i + x_1/\gamma_j)} \tag{16}$$

where the α_{ij}'s are consolidated weights. Applying the required transformations, the density and distribution of

$$C = \log_2[(1 + x_1)(1 + x_2)]$$

for $c > 0$ are given by

$$\begin{aligned} f_C(c) = &2^c \ln 2 \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\alpha_{ij}\left\{\exp\left(\frac{1}{\gamma_i} + \frac{1}{\gamma_j}\right)\frac{\gamma_i - \gamma_j}{\gamma_i}\right. \\ &\times \int_1^{2^c}\exp\left(-\frac{v}{\gamma_i} - \frac{2^c}{\gamma_j v}\right)dv \\ &\left. + \gamma_j\left[e^{(1-2^c)/\gamma_j} - e^{(1-2^c)/\gamma_i}\right]\right\} \end{aligned} \tag{17}$$

$$F_C(c) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \alpha_{ij}\gamma_j \Bigg\{ \exp\left(\frac{1}{\gamma_i} + \frac{1}{\gamma_j}\right) \times \frac{\gamma_j - \gamma_i}{\gamma_i} \int_1^{2^c} v \exp\left(-\frac{v}{\gamma_i} - \frac{2^c}{\gamma_j v}\right) dv + \gamma_i(\gamma_i - \gamma_j) - \gamma_j e^{(1-2^c)/\gamma_j} + [\gamma_i + (\gamma_j - \gamma_i)(\gamma_i + 2^c)] e^{(1-2^c)/\gamma_i} \Bigg\}. \quad (18)$$

Note that (17) and (18) are in closed form except for the integral

$$\int_1^{2^c} v^k \exp\left(-\frac{v}{\gamma_i} - \frac{2^c}{\gamma_j v}\right) dv. \quad (19)$$

The integral in (19) is proportional to the distribution function of a Generalized Inverse Gaussian (GIG) random variable [14]. There appears to be no established algorithms for computing this distribution function so we resort to numerical integration.

C. Larger Systems

For the case of $m = 3$ (with $n \geq 3$), we again begin with the joint eigenvalue density (11). Transforming from (x_1, x_2, x_3) to $U = (1 + x_1)(1 + x_2)(1 + x_3)$, $V = (1 + x_2)(1 + x_3)$, $W = (1 + x_3)$, and integrating out V and W, the distribution of $C = \log_2(U)$ is

$$F_C(c) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sum_{k=1, k\neq\{i,j\}}^{n} \alpha_{ijk}\gamma_k e^{(1/\gamma_i + 1/\gamma_j + 1/\gamma_k)} \times \int_1^{2^c} \int_w^{2^c} \left(\frac{1}{v} - \frac{1}{w^2}\right) A(v, w) \times \exp\left(-\frac{v}{w\gamma_j} - \frac{w}{\gamma_i}\right) dv dw \quad (20)$$

with

$$A(v, w) = v e^{-1/\gamma_k} \left[1 + v + \left(2\gamma_k - \frac{v}{w} - w\right)(1 + \gamma_k)\right] + \exp\left(-\frac{2^c}{v\gamma_k}\right) \left[\left(\frac{v}{w} + w - 2\gamma_k\right)(2^c + v\gamma_k) - \frac{2^{2c}}{v} - v^2\right]$$

$$\alpha_{ijk} = \frac{(\gamma_i\gamma_j\gamma_k)^{n-4}(-1)^{n+i+j+k+g(i,j,k)} D_{ijk}}{12\prod_{i>j}(\gamma_i - \gamma_j)}$$

where D_{ijk} is the determinant of the matrix in (15) without the last three columns and the i-th, jth and kth rows, and $g(i, j, k) = 0$ for $j < k < i$, $k < i < j$, $i < j < k$ and is 1, otherwise.

For larger systems, results such as (14), (18) and (20) are of limited use, since capacity outage probabilities require an $m - 1$ dimensional numerical integral and Monte Carlo simulation may be preferable. In addition, a central limit theorem for random determinants may be used to give a Gaussian approximation to the capacity [3], which is surprisingly accurate even for moderate numbers of antennas.

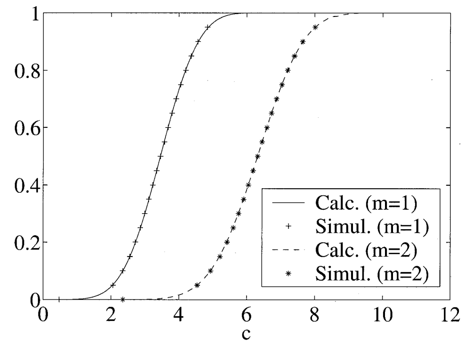

Fig. 1. Capacity distributions for $m = 1$ and 2 from analysis and simulation.

IV. Simulation Results and Conclusions

To verify our analysis, we have calculated the capacity distribution for a $\lambda/2$-spaced four-antenna transmitter with spatial correlation $\mathbf{\Psi}_T$ defined for the mobile in [10]. For the one-antenna and two-antennae receiver cases, Fig. 1 shows the capacity distributions, calculated from (14) and (18) with $\rho = 10$, along with corresponding Monte Carlo simulation results. Our theoretical distributions line up exactly with the simulation results. Hence, we have derived the exact density and distribution function for MIMO capacity in a correlated flat Rayleigh fading channel where $m \leq 3$.

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